Gradient flow on and off the lattice

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MITP discussion · October 16, 2025

QCD gradient flow [Narayanan, Neuberger JHEP 0603(2006)064] [Lüscher CMP 293(2010)899][JHEP 1008(2010)071]

$$\frac{\partial}{\partial \tau} B_\mu(\tau) = \mathcal{D}_\nu(\tau) G_{\nu\mu}(\tau) \qquad \text{with} \qquad B_\mu(\tau = 0) = A_\mu$$

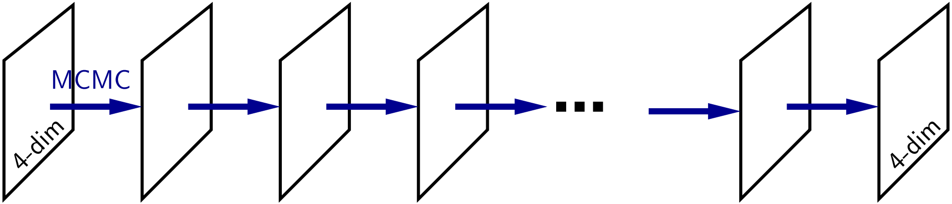
- Ordinary differential equation (ODE)
- Covariant derivative defined in terms of the flow field $B_\mu(\tau)$ and the Yang-Mills action $S_{YM}(B)$

Lattice gradient (gauge) flow

$$\frac{\partial}{\partial \tau} V_\tau(x,\mu) = -g_0^2 \left\{\partial_{x,\mu} S_{YM}\right\} V_\tau(x,\mu) \quad \text{with} \quad V_\tau(x,\mu)\Big|_{\tau=0} = U(x,\mu)$$

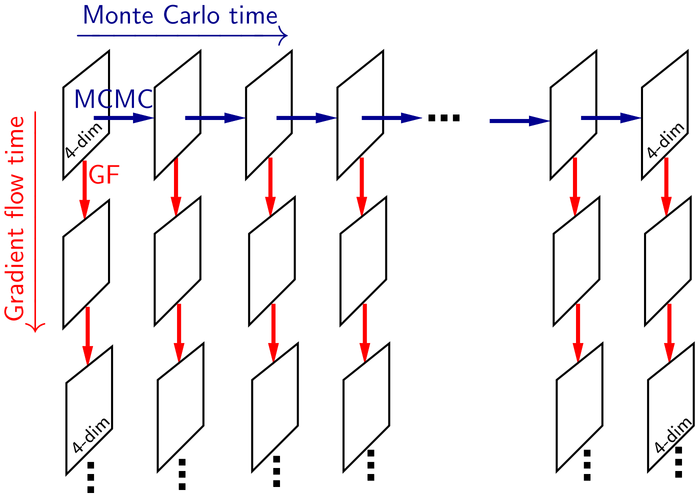
Lattice gradient flow

- Gauge field configurations are generated using a Markov Chain Monte Carlo (MCMC) simulations
  $\rightarrow$ Configurations are evolved along a Monte Carlo time



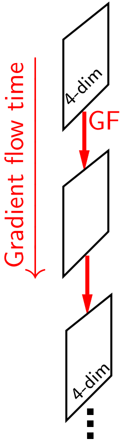
- Matrix elements are commonly evaluated on sufficiently independent configurations
- Similarly we apply gradient flow to a set of independent configurations

Lattice gradient flow

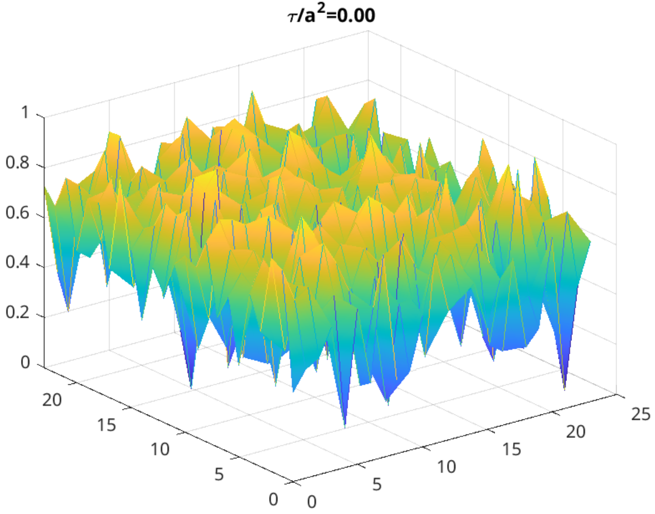


- Gauge field configurations are saved as binary checkpoints
- Gradient flow is cheap
  → Calculated "on the fly"
- ODE are numerically calculated using standard Runge-Kutta methods
  → Numerically stable only in forward direction
  → Backward flow requires tricks
- Evaluate matrix elements at the same GF time

Lattice gradient flow



- Gradient flow acts as a "smearing" procedure
  $\rightarrow$ Infinitesimal stout-smearing
- Gradually removing UV fluctuations
- $N_f = 4$
  24^4, $\beta_b = 4.30$
  Wilson flow $\epsilon = 0.01$



Gradient flow scale setting

- Define gradient flow renormalized coupling using the energy density E

$$g^2_{GF}(\tau) = \mathcal{N}\tau^2\langle E(\tau)\rangle / C(\tau, L)$$

$\rightarrow$ $\tau^2\langle E(\tau)\rangle$ is dimensionless $\Rightarrow$ τ has mass dimension -2

$\rightarrow$ Normalization to match tree-level $\overline{\text{MS}}$: $\mathcal{N} = 128\pi^2/(3N_c^2 - 3)$

$\rightarrow$ Estimate $E(\tau)$ using plaquette, Symanzik, clover, ...operator

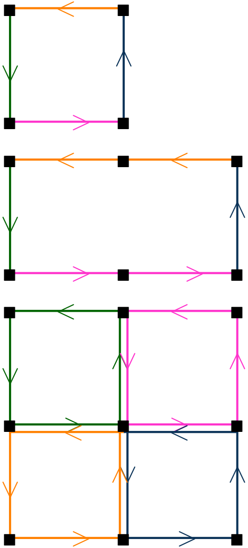
$\rightarrow$ $C(\tau, L)$ tree-level improvement coefficient, [Fodor et al. JHEP09(2014)018]
correct for zero modes of periodic boundaries [Fodor et al. JHEP11(2012)007]

- Determine gradient flow scale t_0 [Lüscher JHEP 1008(2010)071]

$$\left\{\tau^2\langle E(\tau)\rangle\right\}_{\tau=t_0} = 0.3$$

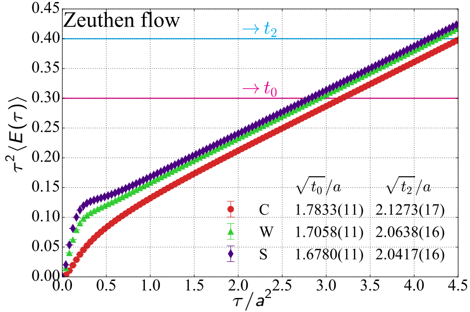
- Alternative gradient flow scale w_0 [Borsanyi et al. JHEP09 (2012)010]

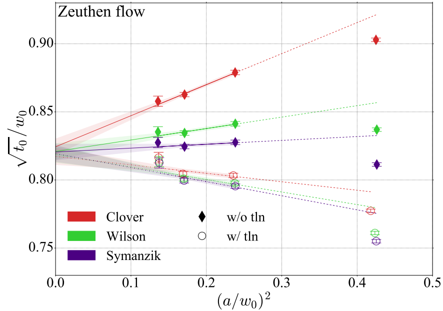
$$\left\{\tau\frac{d}{d\tau}\left[\tau^2\langle E(\tau)\rangle\right]\right\}_{\tau=w_0} = 0.3$$



Gradient flow scales

[Schneider, Hasenfratz, OW arXiv:2211.12406]





- $\sqrt{t_0}/a$ and w_0/a are lattice scale and have a continuum limit

  $\rightarrow$ Use other scales/hadronic quantities to get continuum limit $\sqrt{t_0}$ or w_0 in physical units

  [FLAG 2024] (2+1) $\sqrt{t_0} = 0.14474(57)$ fm [RQCD][RBC/UKQCD][BMWc][CLS]

  $w_0 = 0.17355(92)$ fm [RBC/UKQCD][BMWc][HotQCD]

  $\rightarrow$ "Shortcut" to turn $\sqrt{t_0}/a$ or w_0 into determinations of a [fm]

Real-space Renormalization Group (RG) flow

- RG flow: change of (bare) parameters and coarse graining (blocking)

- Gradient flow is a continuous transformation
  → Define real-space RG blocked quantities by incorporating coarse graining as part of calculating expectation values [Carosso, Hasenfratz, Neil PRL 121 (2018) 201601]

- Relate GF time τ/a^2 to RG scale change $b \propto \sqrt{\tau/a^2}$
  → Quantities at flow time τ/a^2 describe physical quantities at energy scale $\mu \propto 1/\sqrt{\tau}$
  → Local operator with non-vanishing expectation value can be used to define running coupling
  ⇝ Simplest choice: $\tau^2\langle E(\tau)\rangle$ [Lüscher JHEP 1008 (2010) 071]

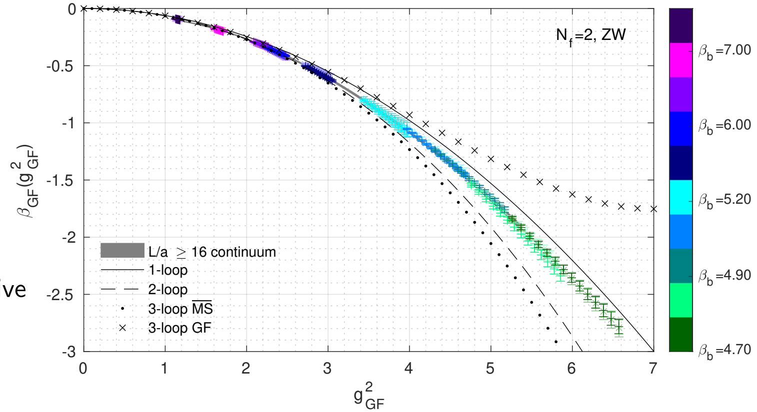
- Continuous RG β function

$$\beta(g^2_{GF}) = \mu^2 \frac{dg^2_{GF}}{d\mu^2} = -\tau \frac{dg^2_{GF}}{d\tau}$$

Example: QCD

- QCD: SU(3) gauge theory with two light flavors in the fundamental representation
- Fast "running" coupling
  $\rightsquigarrow$ Confinement
- Plot: Comparison of nonperturbative and perturbative determinations
  $\rightarrow$ 3-loop order in the GF scheme
  [Harlander, Neumann JHEP06(2016)161]
  $\rightarrow$ Details presented by Robert

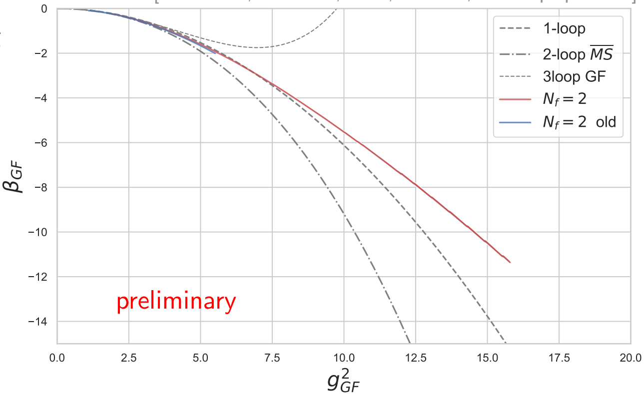
[Hasenfratz, OW PRD101(2020)034514]



Example: QCD

- Extended data to stronger coupling
  $\rightarrow$ Confining region, $g^2_{\mathrm{GF}} \sim 16$
- At strong coupling RG β function is highly linear
  $\rightarrow$ Nonperturbative phenomenon
- Integrate β function to obtain Λ_{GF}
  $\rightarrow$ g_m^2 GF renormalized coupling at energy scale $\mu = 1/\sqrt{8t_0}$

[Hasenfratz, Monahan, Rizik, Shindler, OW in preparation]



$$\Lambda_{GF} = \mu (b_0 g_m^2)^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{1}{2b_0 g_m^2}\right) \exp\left[-\int_0^{g_m^2} \mathrm{d}g^2 \left(\frac{1}{\beta(g^2)} + \frac{1}{b_0 g^4} - \frac{b_1}{b_0^2 g^2}\right)\right]$$

$\Rightarrow \Lambda^{\mathrm{prelim}}_{\overline{\mathrm{MS}}} = 326(13)$ MeV (stat. error only) compare to ALPHA: f_K: 310(20) MeV

[Fritzsch et al. NPB865(2012)397]

Gauge flow [Narayanan, Neuberger JHEP 0603(2006)064] [Lüscher CMP 293(2010)899][JHEP 1008(2010)071]

$$\boxed{\frac{\partial}{\partial \tau} B_\mu(\tau) = \mathcal{D}_\nu(\tau) G_{\nu\mu}(\tau) \qquad \text{with} \qquad B_\mu(\tau = 0) = A_\mu}$$

- Covariant derivative defined in terms of the flow field $B_\mu(\tau)$ and the Yang-Mills action $S_{YM}(B)$

Fermion flow [Lüscher JHEP04(2013)123]

$$\boxed{\frac{\partial}{\partial \tau} \chi(\tau, x) = \mathcal{D}^2(\tau) \chi(\tau, x) \quad \text{with} \quad \chi(\tau = 0, x) = q(x)}$$

- Derivative given by Laplace operator acting on flowed fermion field $\chi(\tau, x)$

Renormalized quark masses [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

- GF renormalized partially conserved axial current (PCAC) relation (Ward-Identity)

$$\left(m_{\mathrm{GF}}^{(r)}(\tau)+m_{\mathrm{GF}}^{(s)}(\tau)\right)=M_{\mathrm{PS}}^{(rs)}\bar{R}_{rs}^{\mathcal{O}}(\tau) \qquad \text{with} \quad \bar{R}_{rs}^{\mathcal{O}}(t;\tau)=\lim_{t\to\infty}-\frac{\langle A_0(t;\tau)\mathcal{O}(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)\mathcal{O}(t=0;\tau=0)\rangle_{rs}}$$

→ M_{PS} pseudoscalar meson with flavors r and s
→ $m_{\mathrm{GF}}^{(r)}(\tau)$ GF renormalized quark mass for flavor r
→ Smearing radius must remain small compared to Euclidean time: $\sqrt{8\tau}\ll t$

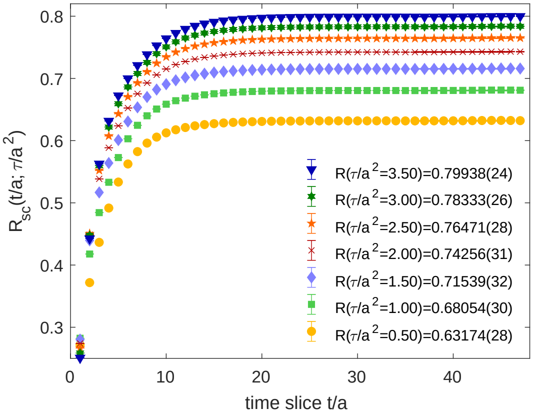
- Match to the $\overline{\mathrm{MS}}$ scheme

$$m_{\overline{\mathrm{MS}}}(\mu_{\mathrm{UV}})=\lim_{\tau\to 0}\ \zeta_{AP}^{-1}(\mu_{\mathrm{UV}},\tau)m_{\mathrm{GF}}(\tau)$$

→ $\zeta_{AP}^{-1}(\mu_{UV},\tau)$ perturbatively calculated short-flow-time expansion (SFTX) coefficient
→ Details presented by Robert

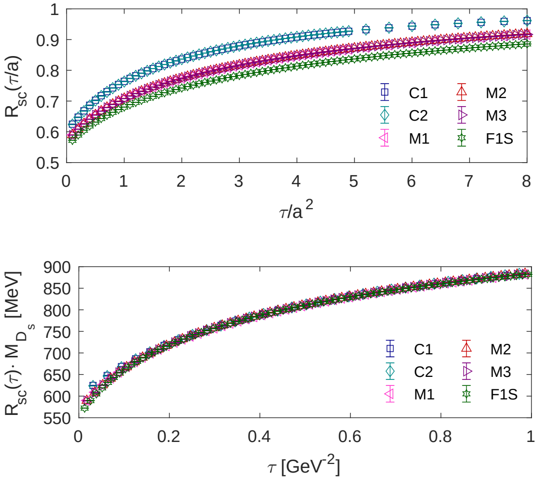
Example: D_s meson [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
  $\rightarrow$ F1S: $a^{-1} = 2.785$ GeV, $M_\pi = 267$ MeV



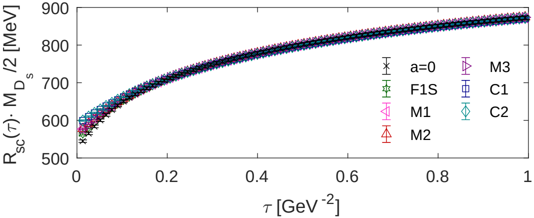
Example: D_s meson [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

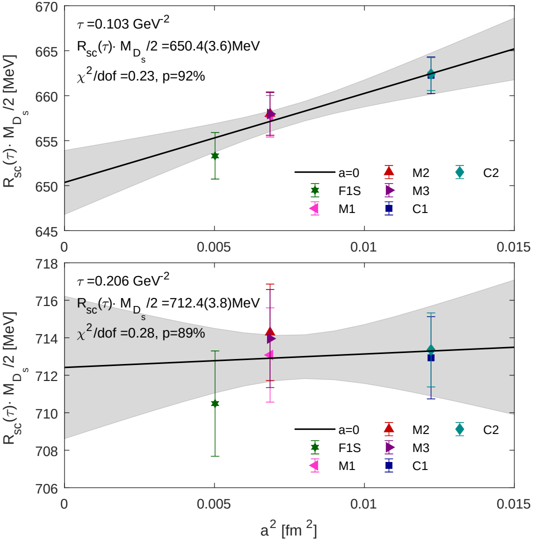
- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
- Results for flow time in lattice units
  $\rightarrow$ No sea light quark mass dependence
- Convert to "physical" flow time in GeV^{-2}
  $\rightarrow$ Mild continuum limit



Example: D_s meson [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

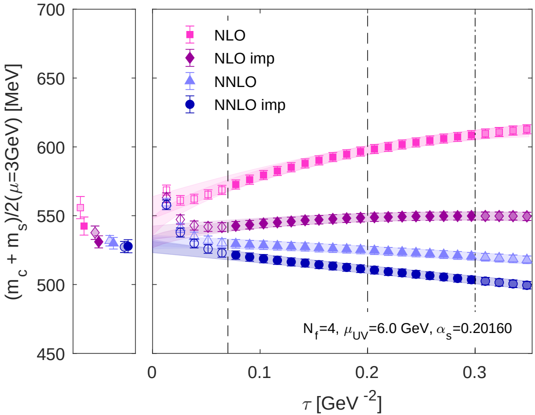
- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
- Results for flow time in lattice units
- Convert to "physical" flow time in GeV^{-2}
- Take $a \to 0$ continuum limit for $\tau > 0$



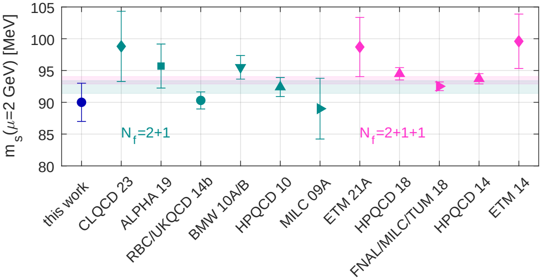


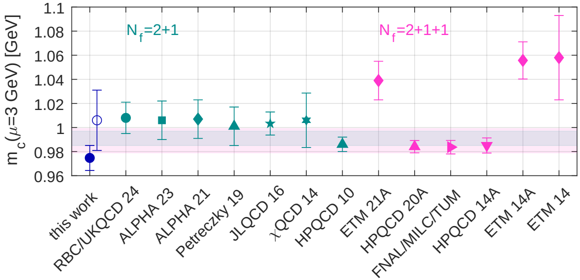
Example: D_s meson [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
- Results for flow time in lattice units
- Convert to "physical" flow time in GeV^{-2}
- Take $a \to 0$ continuum limit for $\tau > 0$
- Multiply ζ_{AP}^{-1} to match to $\overline{\text{MS}}$ scheme and take $\tau \to 0$ limit
- $$\frac{m_c + m_s}{2} = 528(5)\text{ MeV} \quad \text{at } \mu = 3\,\text{GeV}$$



Strange and charm quark masses [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]





[FLAG 2024 arXiv:2411.04268] [χQCD PRD92(2015)034517] [ETM PRD104(2021)074515] [FNAL/MILC/TUM PRD98(2018)054517] [HPQCD PRD91(2015)054508] [ETM NPB887(2014)19] [CQCD PRD109(2024)054507] [Alpha EPJC80(2020)169] [RBC-UKQCD PRD93(2016)074505] [HPQCD PRD82(2010)034512] [BMW JHEP08(2011)146] [FNAL/MILC PoS CD09(2009)007] [HPQCD PRD98 (2018) 014513] [Alpha EPJC84(2024)506] [Alpha JHEP05(2021)288] [Petreczky,Weber PRD100(2019)034519] [JLQCD PRD94(2016)054507] [HPQCD PRD102(2020)054511] [HPQCD PRD102(2020)054511] [RBC-UKQCD PRD110(2024)054512]

- $m_s^{\eta_s}(\mu = 2\ \text{GeV}) = 90(3)$ MeV
- $m_c^{\eta_c}(\mu = 3\ \text{GeV}) = 1006(25)$ MeV
- $m_c(\mu = 3\ \text{GeV}) = 975(10)$ MeV (D_s and η_s)
- $\frac{m_c}{m_s} = 12.0(4)$