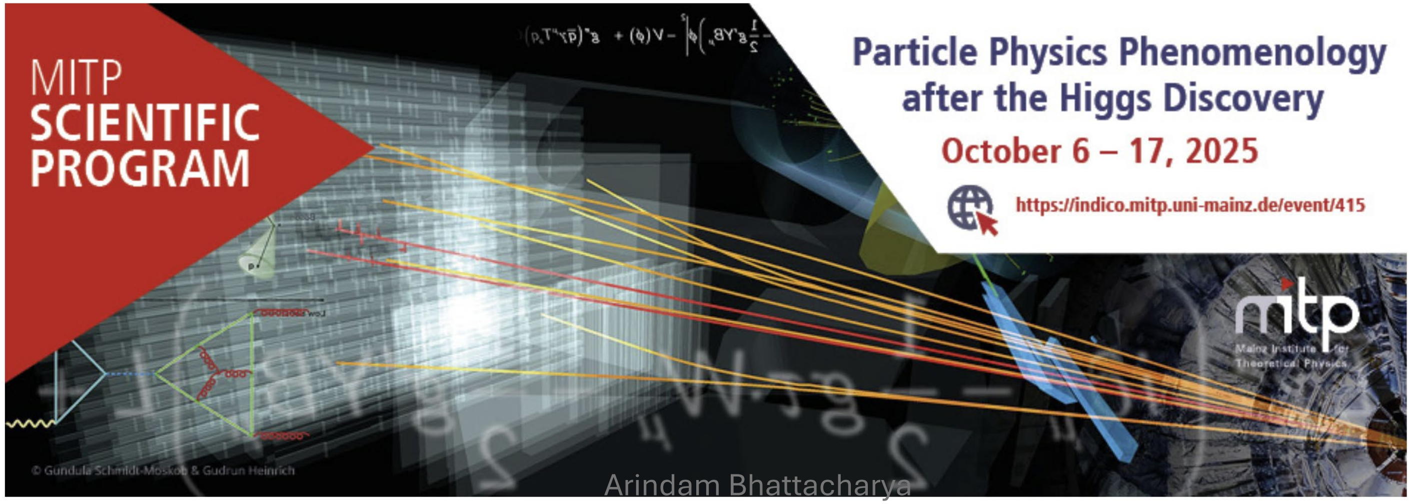
Renormalons as Saddle Points

Arindam Bhattacharya

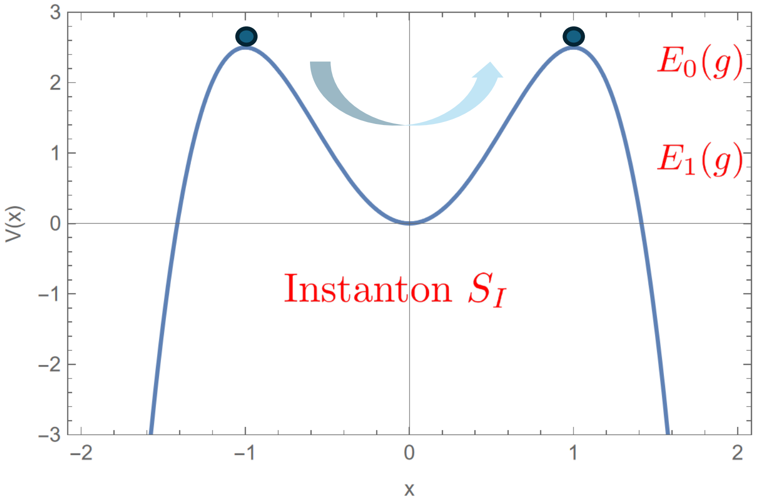
Based on work with Jordan Cotler, Aurélien Dersy, Matthew D Schwartz

arXiv: 2410.07351



Outline

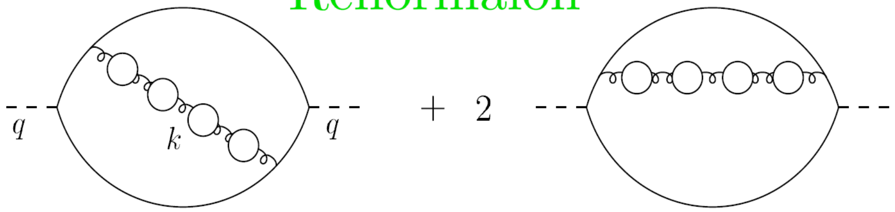
- Asymptotic Series in QFT and QM
- Borel Resummation
- Saddle Points : Instantons
- Renormalons
- Borel Action Correspondence
- Renormalons as Saddle Points



$$E_0(g) \sim \sum_n g^n (2S_I)^{-n} n! + \cdots$$

$$E_1(g) \sim e^{-S_I/g} \sum_n g^n (2S_I)^{-n} n! + \cdots$$

Renormalon



$$D(Q^2) \sim \int \frac{dk^2}{k^2}\, F(k^2)\, \alpha(\mu)^{n+1}\, \beta_0^n \ln^n \frac{\mu^2}{k^2} \quad \Longrightarrow \quad S_R \stackrel{?}{=} \frac{2}{\beta_0}$$

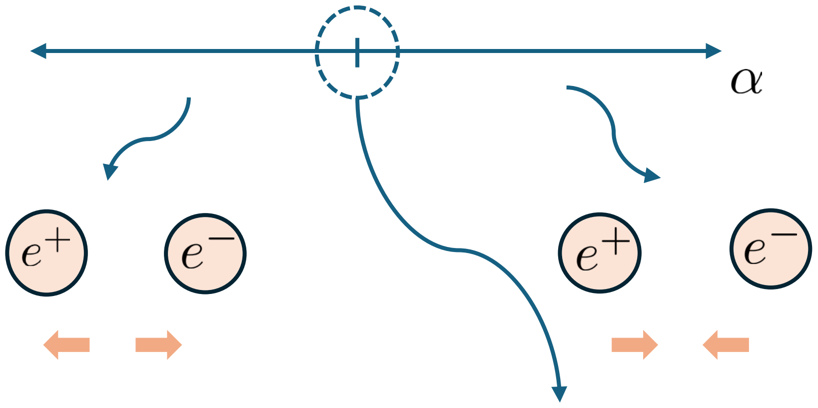
$$\sim \alpha(\mu)^{n+1} \left(\frac{\beta_0}{2}\right)^n n! + \cdots$$

Perturbation Theory gives Asymptotic Series

- Perturbation Theory is ubiquitous in QFT and QM

$$\text{Physical Quantity}(\alpha) = c_{-1} + \sum_{n=0}^{\infty} c_n \alpha^{n+1}$$

- Series does not converge, and is asymptotic at best (Dyson, 1952)



Very different physics on change of sign! ⟶ No finite radius of convergence

Perturbation Theory gives Asymptotic Series

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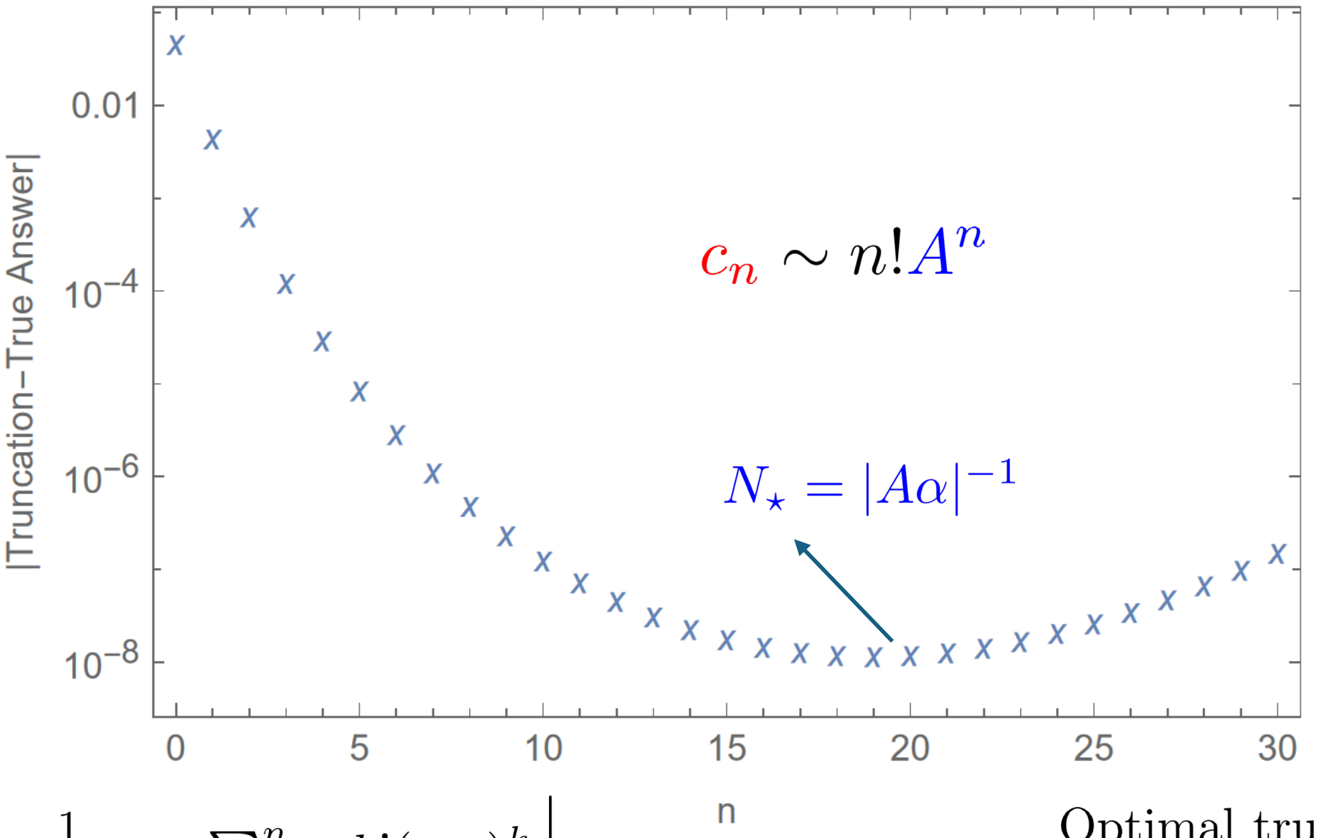


$$\left| \frac{1}{\alpha} \int_0^\infty dt \; e^{-t/\alpha} \frac{1}{1+t} - \sum_{k=0}^{n} k!(-\alpha)^k \right|$$

$$\alpha = 0.05$$

Optimal truncation intuitive, but not rigorous

How do we ‘resum’ these series ?

Borel Resummation

$$f(\alpha) = \sum_{n=0}^{\infty} c_n \alpha^{n+a} \qquad \mathcal{B}[f](t) = B_f(t) = \sum_{n=0}^{\infty} c_n \frac{t^{n+a}}{\Gamma(n+a+1)}$$

Divide out by factorial growth

Formal agreement order by order in α

Inverse Borel Transform

$$\boxed{\tilde{f}(\alpha) = \frac{1}{\alpha} \int_0^{\infty} dt\; e^{-t/\alpha} B_f(t)}$$

If the integral exists, it 'assigns' a meaning to the sum of the asymptotic series

Converges for convergent series

Example:

$$c_n \sim n!(-A)^{-n}\ ,\ A > 0,\ a = 0 \quad \longrightarrow \quad B(t) = \frac{1}{1+At} \quad \longrightarrow \quad \tilde{f}(\alpha) = \frac{e^{\frac{1}{A\alpha}}}{A}\Gamma\left[0, \frac{1}{A\alpha}\right]$$

Alternating factorial blowups can be resummed

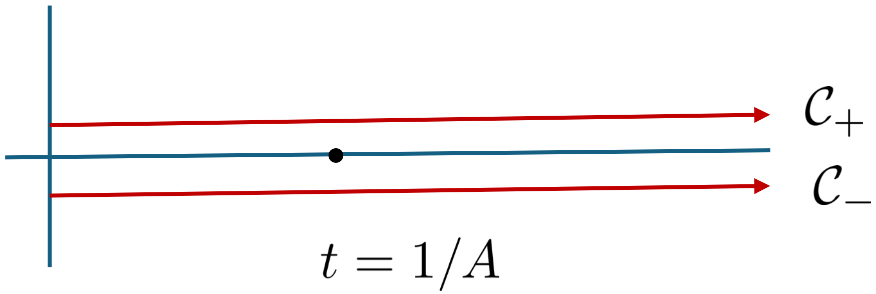
Borel Resummation: Obstruction

- What if one cannot resum it using the Borel transform?

Example:

$$c_n \sim n! A^{-n}\ ,\ \ A > 0,\ \ a = 0 \quad \Longrightarrow \quad B(t) = \frac{1}{1 - At} \quad \Longrightarrow \quad \tilde{f}(\alpha) = \frac{1}{\alpha} \int_0^{\infty} dt\ e^{-t/\alpha} \frac{1}{1 - At}$$

Pole obstructs resummation



- Resummation is ambiguous, upto terms of the form

$$\frac{1}{\alpha} \oint_{t=1/A} dt\ e^{-t/\alpha} \frac{1}{1 - At} = \boxed{-\frac{2\pi i}{\alpha A}\ e^{-1/A\alpha}}$$

- Does not admit a series
- Non perturbative contributions (Instantons, $\propto \Lambda_{\mathrm{QCD}}$ if $A = \beta_0$)

Sources of Factorial Growth

- Number of Feynman diagrams is generically order $n!$ at α^n

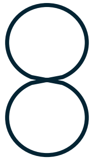
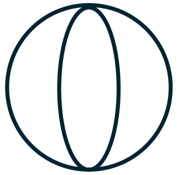
Example 1d Integral

$$Z(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\; e^{-\frac{z^2}{2} - \frac{\alpha z^4}{4!}} = \sqrt{\frac{3}{2\pi\alpha}}\, e^{\frac{3}{4\alpha}} K_{\frac{1}{4}}\left(\frac{3}{4\alpha}\right) \;,\; \alpha > 0$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \alpha^n \, \frac{(-6)^{-n}\Gamma\left(2n+\frac{1}{2}\right)}{n!} \sim \frac{1}{\sqrt{2\pi}} \sum_n \alpha^n \left(-\frac{3}{2}\right)^{-n} n!$$

Propagator $= 1$


Interaction $= -\alpha$

$Z(\alpha) = 1 +$  $+$  $+ \cdots$

Symmetry factor ≤ 1, so number of diagrams $\sim n!$

- In field theory, existence of instantons implies combinatorial blowup of number of diagrams (Lipatov 1976)

Sources of Factorial Growth : Instantons

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$$S_E[\phi] = \int d^4x \left[-\frac{1}{2}\phi(x)\Box\phi(x) - \frac{1}{4!}\phi(x)^4\right]$$

$$Z(\lambda) = \int \mathcal{D}\phi \; e^{-S[\phi]/\lambda} = \sum_n c_n \lambda^n$$

$$c_n = \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda^{n+1}} Z(\lambda) \;\; = \frac{1}{2\pi i}\int d\lambda \; \mathcal{D}\phi \; e^{-S[\phi]/\lambda-(n+1)\ln\lambda}$$

Saddle point approximation in ϕ and λ

$$\frac{\delta S}{\delta \phi} = 0$$

$$\phi_I(x) = 4\sqrt{3}\frac{R}{R^2+(x+x_0)^2}$$

$$S[\phi_I] = 16\pi^2$$

$$\lambda_* = \frac{S_I}{n+1}$$

$$\implies c_n \approx (n+1)!(S_I)^{-n-1}$$

Using Stirling's approx.

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$$S_E[\phi] = \int d^4x \left[-\frac{1}{2}\phi(x)\Box\phi(x) - \frac{1}{4!}\phi(x)^4 \right]$$

$$Z(\lambda) = \int \mathcal{D}\phi \; e^{-S[\phi]/\lambda} \underset{\phi=0 \text{ Expansion}}{=} \sum_n c_n \lambda^n$$

$$\frac{\delta S_E}{\delta \phi} = 0 \implies \phi_I(x) = 4\sqrt{3}\frac{R}{R^2 + (x+x_0)^2}$$

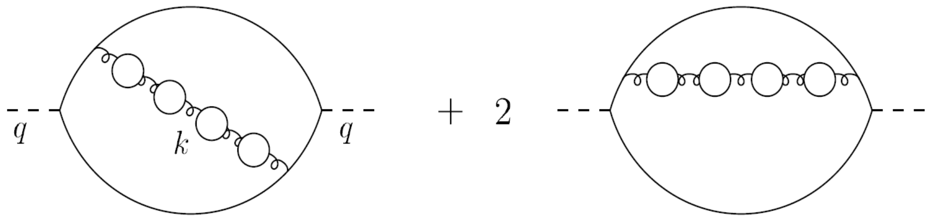
$$\boxed{S[\phi_I] = 16\pi^2} \implies \boxed{c_n \sim n!(S_I)^{-n}}$$

$$\boxed{\frac{\delta S_E}{\delta \phi} = 0 \implies \phi_{\text{Instanton}} \implies Z(\lambda) \sim \sum_{n=0}^{\infty} \lambda^n \; n! \; S_I^{-n}}$$

Sources of Factorial Growth : Renormalons

- Is that all of the factorial growth? No! **Renormalons** (t'Hooft 1977, Lautrup 1977, Gross & Neveu 1974)

- Subclass of diagrams lead to faster factorial growth than the total number



$$D(Q^2) \sim \int_0^{\mu^2} \frac{dk^2}{k^2}\, k^{2a}\, \alpha(\mu)^{n+1}\, (-N_f)^n \ln^n \frac{\mu^2}{k^2}$$

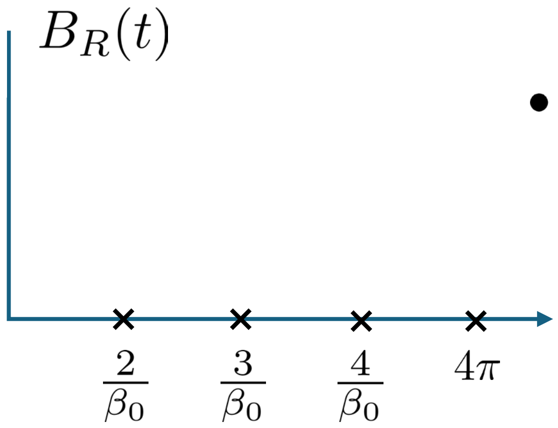
$$= \alpha(\mu)^{n+1} \boxed{(-aN_f)^n \;\; n!}\, \frac{\mu^{2a}}{a}$$

- Adding up other loops that **renormalize** the coupling turns N_f to β_0 (Naive non-abelianization)

$$\alpha(\mu)^{n+1} \boxed{(a\beta_0)^n}\; n!\, \frac{\mu^{2a}}{a} \qquad\qquad \frac{d}{d\ln\mu^2}\frac{1}{\alpha(\mu)} = \boxed{\beta_0} + \beta_1\alpha(\mu) + \cdots$$

- How do these two factorial growths look in the Borel plane?

Renormalons

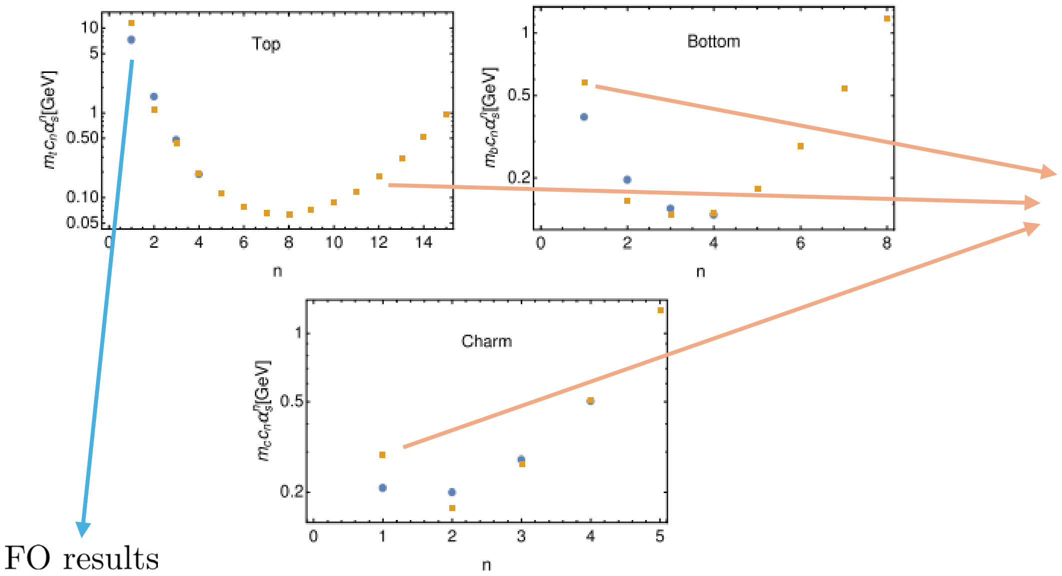


- Renormalons are closer to the origin than instantons. Phenomenologically more relevant!

Examples

- Heavy quark masses suffer from a pole mass renormalon giving an ambiguity $\sim \Lambda_{\mathrm{QCD}}$

$$m_{\text{Pole, Heavy Quark}} = m_{\overline{\mathrm{MS}}}(\mu)\left[1 + \sum_{n=0}^{\infty} \alpha(\mu)^{n+1} c_n(\mu)\right]$$



- Better scheme choices to remove the leading renormalon (MSR Hoang et.al., Potential Subtracted Beneke, $\cdots$)

$$\sim \frac{C_F}{\pi}\mu \sum_n (2\beta_0)^n n! \alpha(\mu)^{n+1}$$

- Precision theory predictions for measurements of strong coupling constant α_s, quark masses, etc. need to account for renormalon subtraction to ensure due convergence

Beneke, Eur. Phys. J. Spec. Top., 2021

Factorial Growth $\underset{?}{\Longleftrightarrow}$ Saddle Points

$$\frac{\delta S_E}{\delta \phi} = 0 \qquad \Longrightarrow \phi_{\text{Instanton}} \qquad \Longrightarrow \quad Z(\alpha) \sim \sum_{n=0}^{\infty} \alpha^n \; n! \; S_I^{-n}$$

Saddle point of an action $\longrightarrow$ Series

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Saddle point of an action ⟵ Series

$$\frac{\delta S_?}{\delta \phi} = 0 \qquad \underset{?}{\Longleftarrow} \phi_{\text{Renormalon}} \qquad \underset{?}{\Longleftarrow} \mathcal{O}(\alpha) = \sum_{n=0}^{\infty} \alpha^n n! (\beta_0 / k)^n$$

$$S_? = k/\beta_0$$

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- Which S? Which renormalon?

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- Which S? Which renormalon?
- Spoiler : S_{eff} with an operator insertion
- Let's build up to it

Saddle Point Approximation : Why Borel transforms work

Exponential Laplace integrals are common (Euclidean path integrals)

Obtain perturbative series by expanding around saddle point of exponential

1D Integrals

Saddle point approximation around $S'(z_\star) = 0$

$$f(\alpha) = \int dz\; e^{-S(z)/\alpha}$$

$$= e^{-S(z_\star)/\alpha} \int dz\; e^{-\frac{1}{\alpha}\left[S^{(2)}(z_\star)\frac{(z-z_\star)^2}{2} + S^{(4)}(z_\star)\frac{(z-z_\star)^4}{24} + \cdots\right]}$$

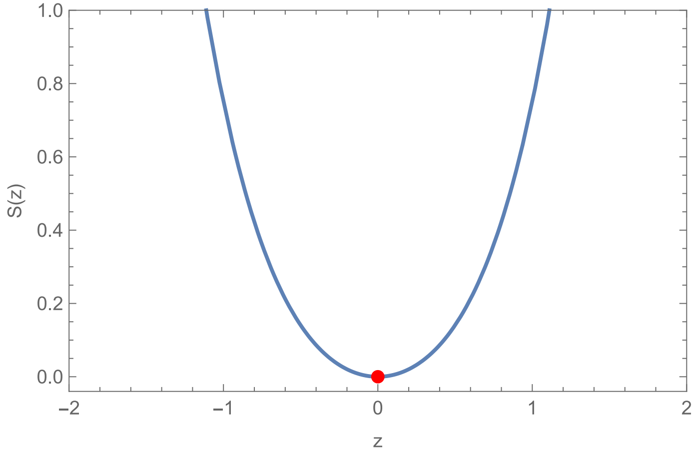
$$= \sqrt{\frac{\alpha}{S^{(2)}(z_\star)}}\; e^{-S(z_\star)/\alpha} \int dz\; e^{-\frac{z^2}{2} - \alpha\frac{z^4}{24(S^{(2)}(z_\star))^2} + \cdots}$$

Asymptotic series, which we Borel transform

Gaussian integration *but* along the path $\mathrm{Re}[S(z)]$ increases rapidly → Thimble

Thimbles : Why Borel transforms work

$$S(z) = \frac{z^2}{2} + \frac{z^4}{4}$$



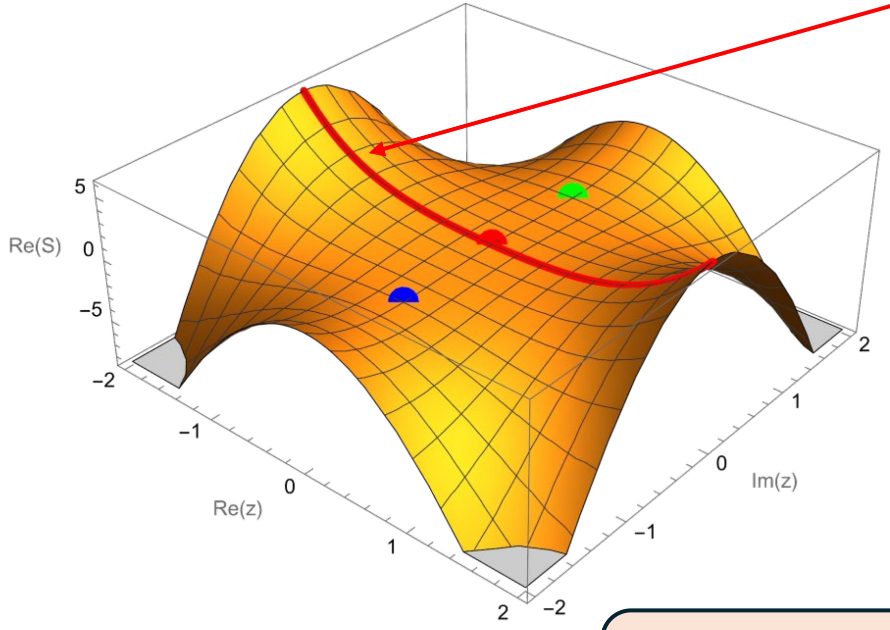
$$\sim \sum_{n=0}^{\infty} \sqrt{2\alpha}(-\alpha)^n \frac{\Gamma\left(\frac{1}{2}+2n\right)}{n!}$$

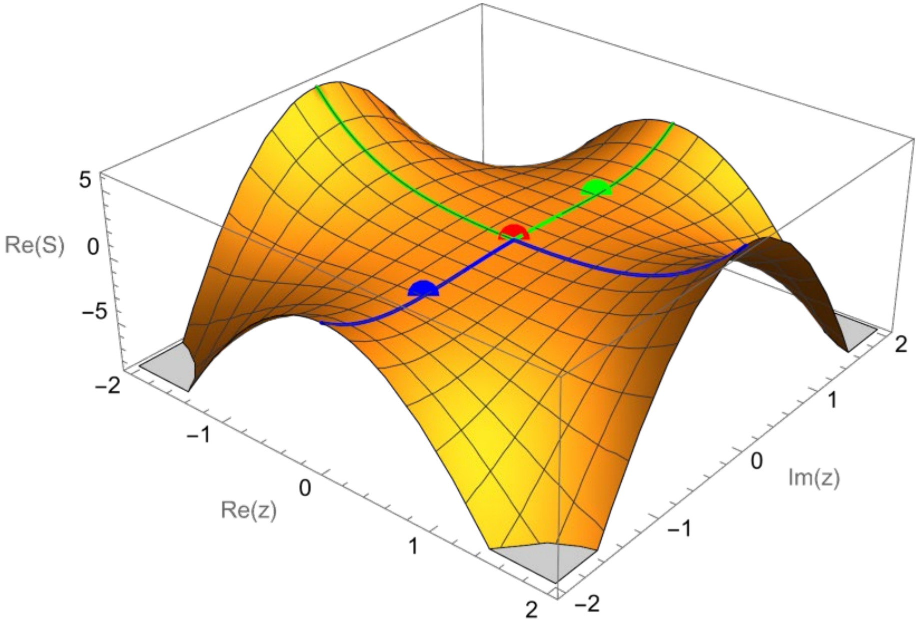
Borel summable, alternating series

$$S'(z^\star) = 0, \;\; z^\star = 0, \pm i$$

$$\int_{-\infty}^{\infty} dz \; e^{-S(z)/\alpha} = \frac{1}{\sqrt{2}} \, e^{\frac{1}{8\alpha}} \, \mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8\alpha}\right) .$$

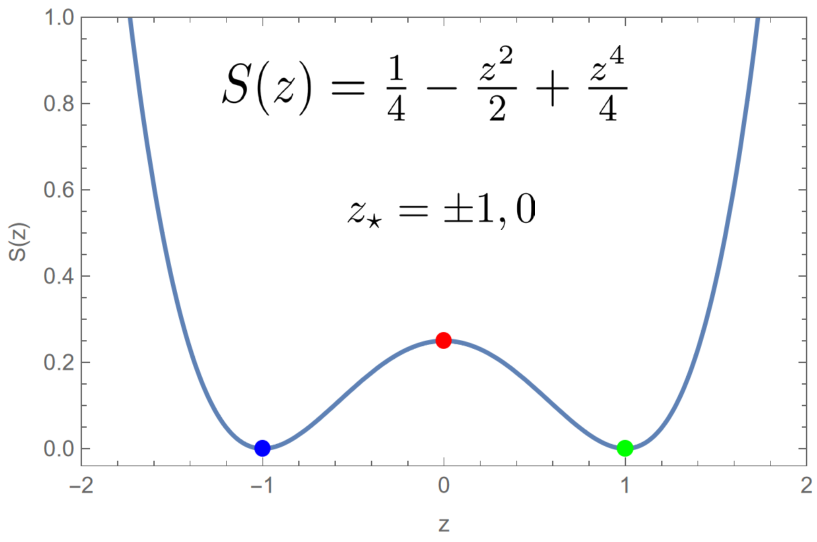
Thimble





Theorem : Borel resummation from expansion along a saddle gives the resultant integral along the thimble passing through it

Thimbles : Why Borel transforms work



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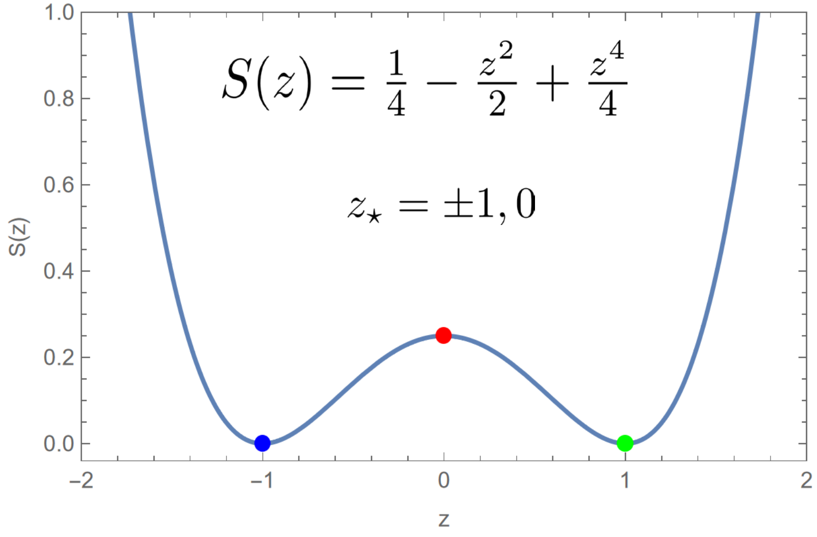

Saddle point Expansion around $z = -1$

$$\sum_{n=0}^{\infty} \alpha^{n+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + 2n\right)}{n!}$$

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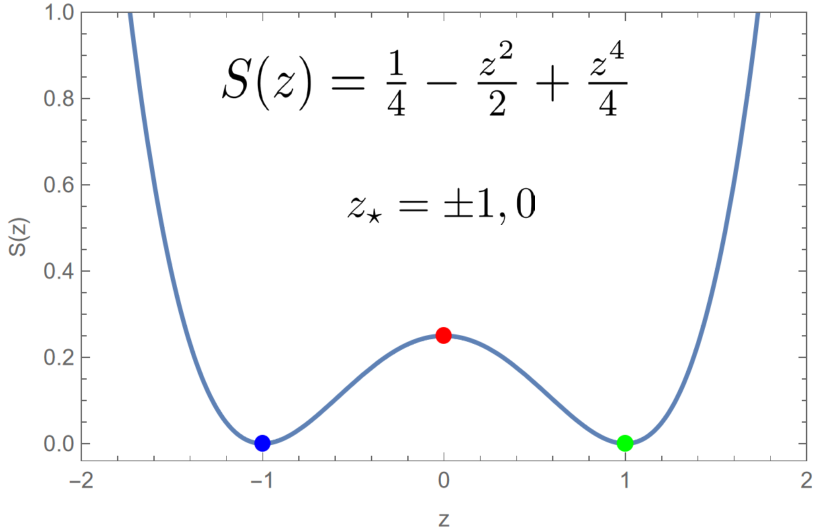
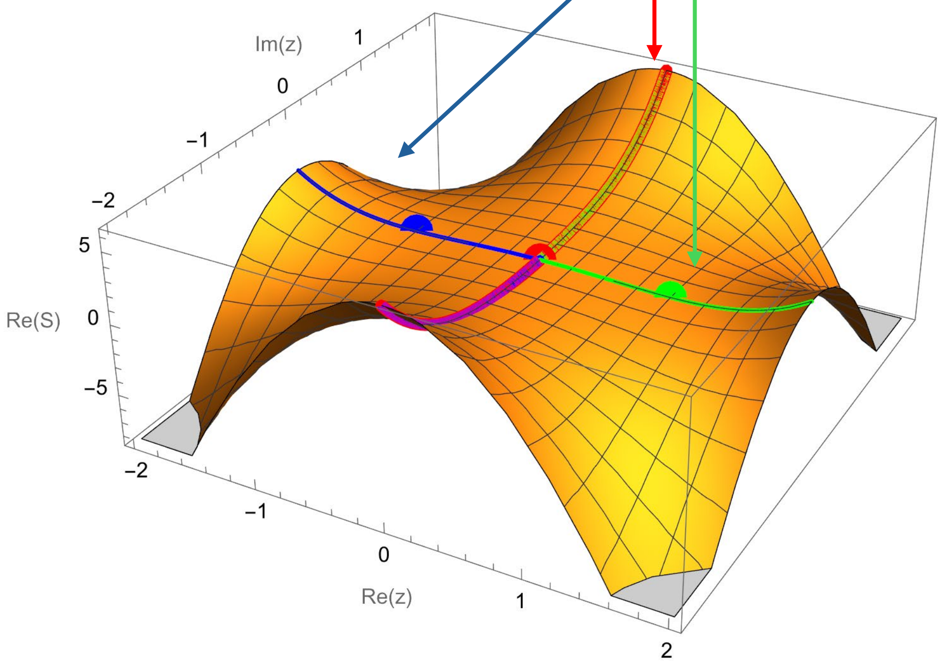


Re$[S(z)]$ increases, Thimble

Im$[S(z)]$ is constant



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$$\sum_{n=0}^{\infty} \alpha^{n+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}+2n\right)}{n!}$$

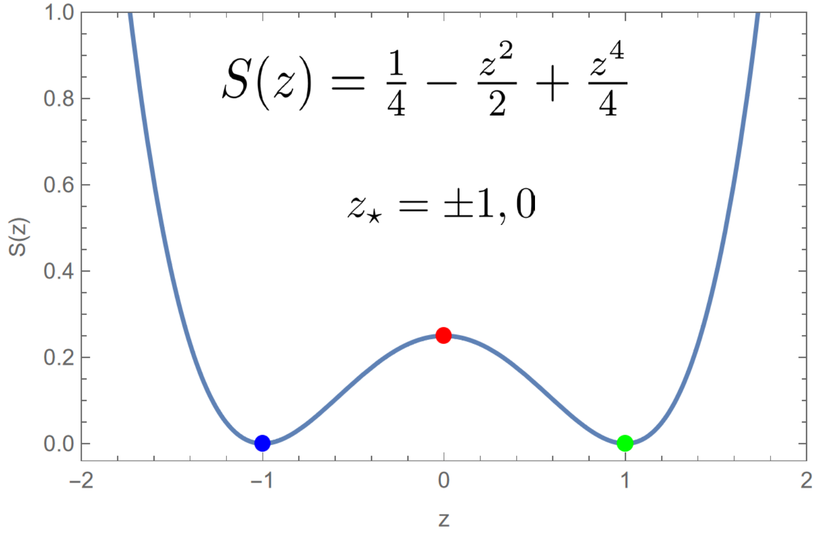
Borel transform

$$\sqrt{2 - 2\sqrt{1-4t \pm i\epsilon}}$$

Resum

Reproduce the integral along the thimble through $z = -1$

Thimbles : Why Borel transforms work



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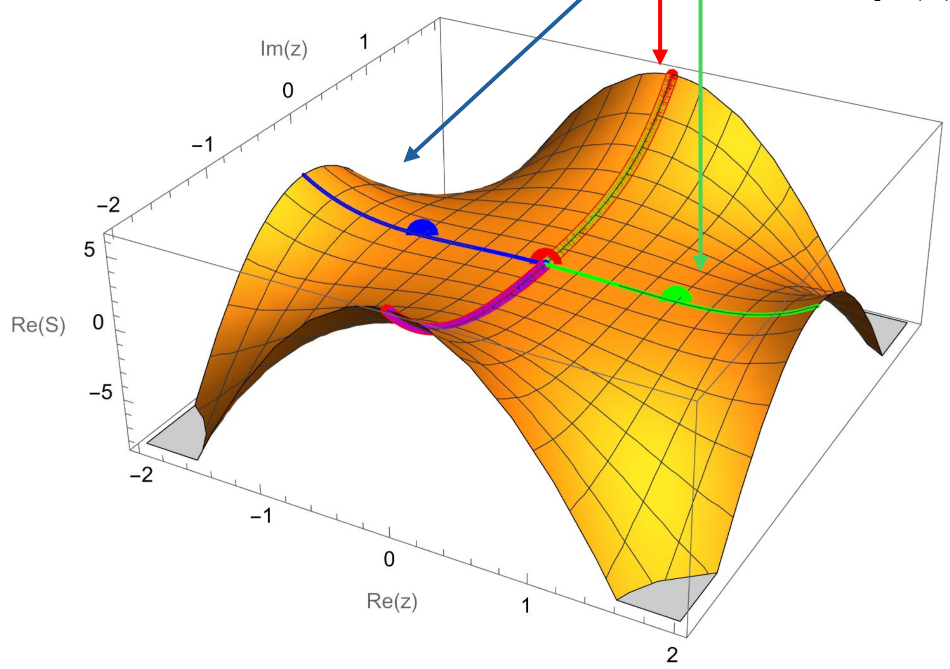

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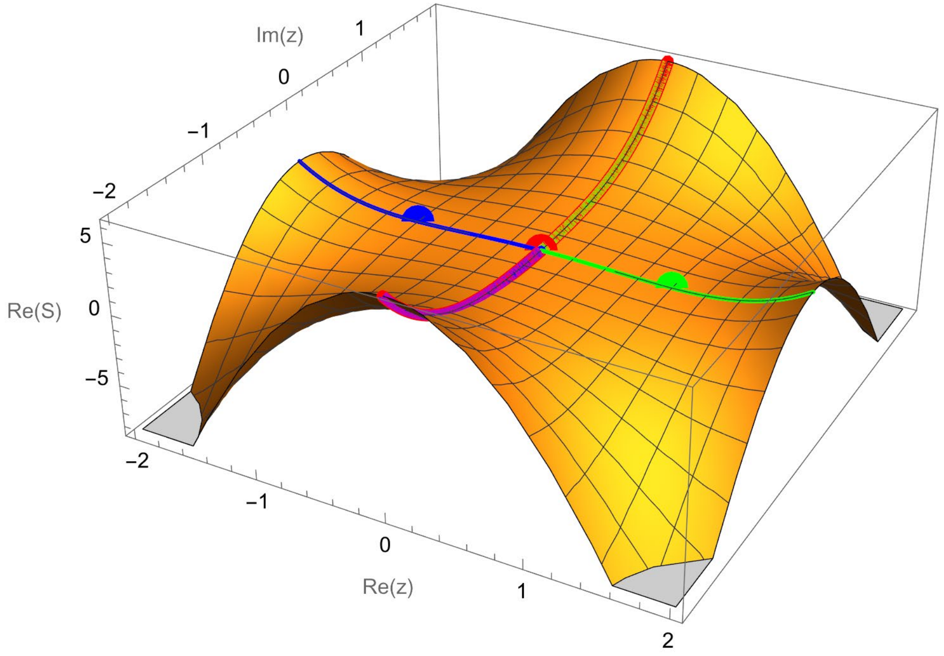
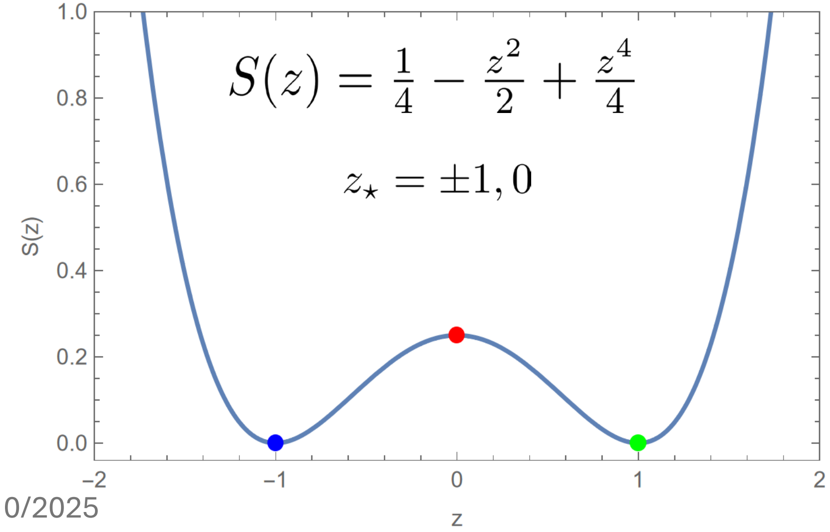
How does the Borel pole show up in thimbles?

Thimbles : Why Borel transforms work



How does the Borel pole show up in thimbles?

If thimble from one saddle intersects the other. Stokes phenomena



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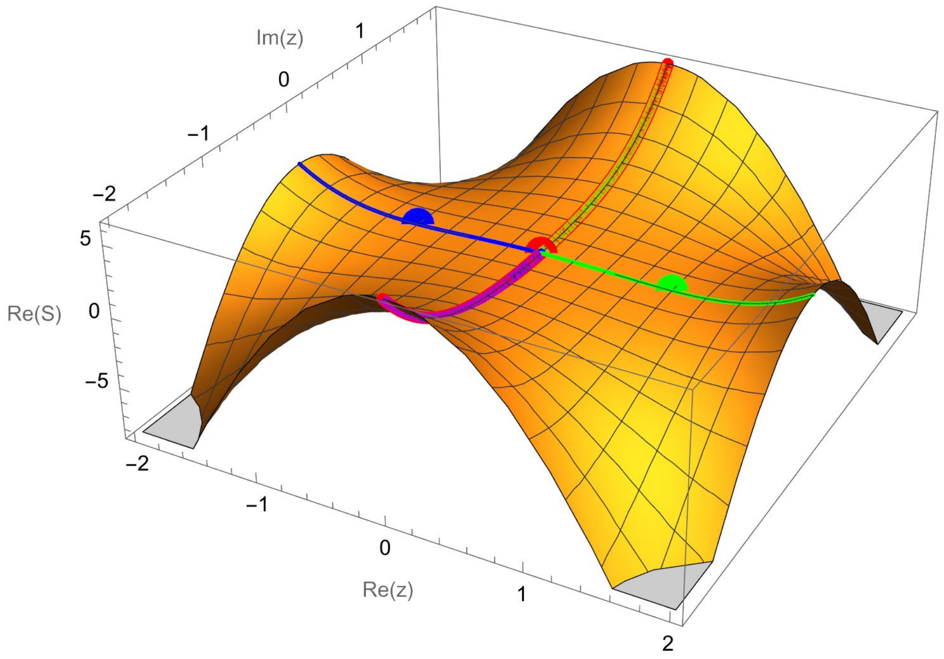

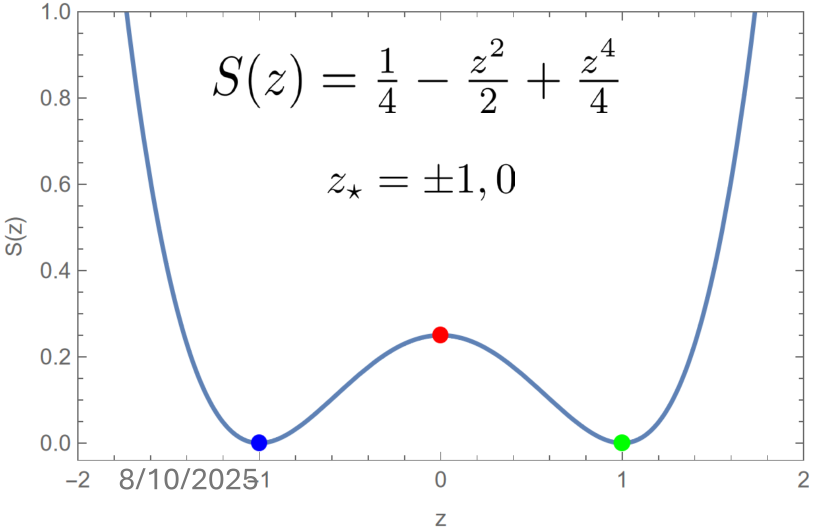

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Write original integration contour as sum of thimbles

$$\mathcal{C} = \sum n_j \mathcal{C}_j$$

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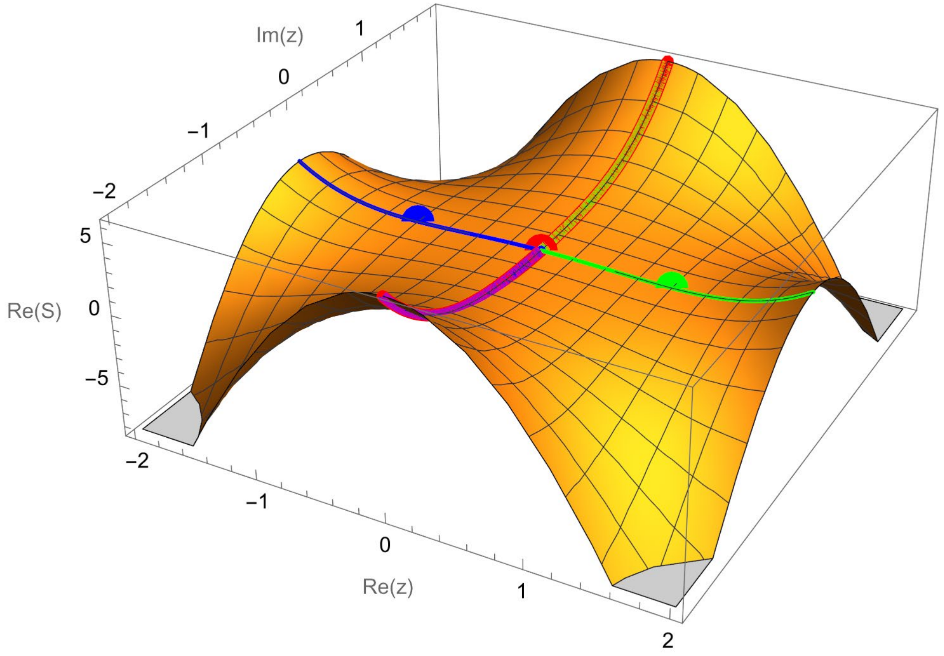

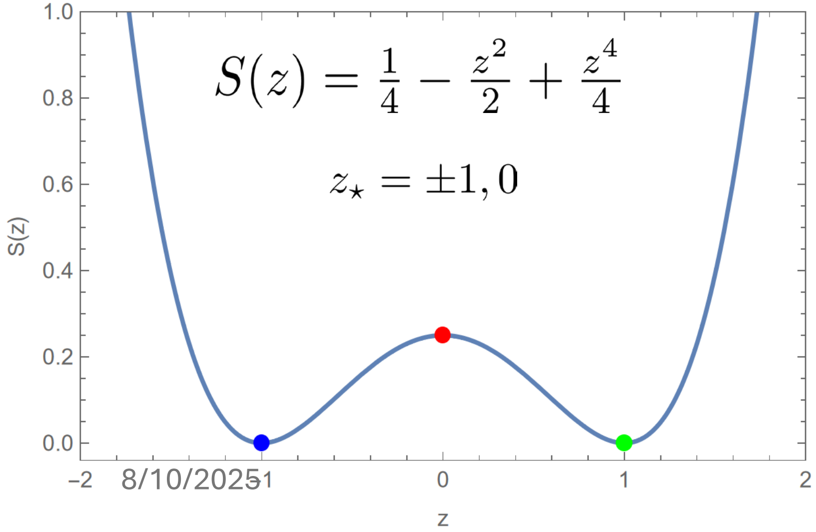

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Sum of inverse Borel transforms recovers the integral

$$B_{\pm 1}(t) = \sqrt{2 - 2\sqrt{1 - 4t \pm i\epsilon}}$$

$$B_0(t) = \pm 2i\sqrt{\sqrt{1+4t} - 1}$$

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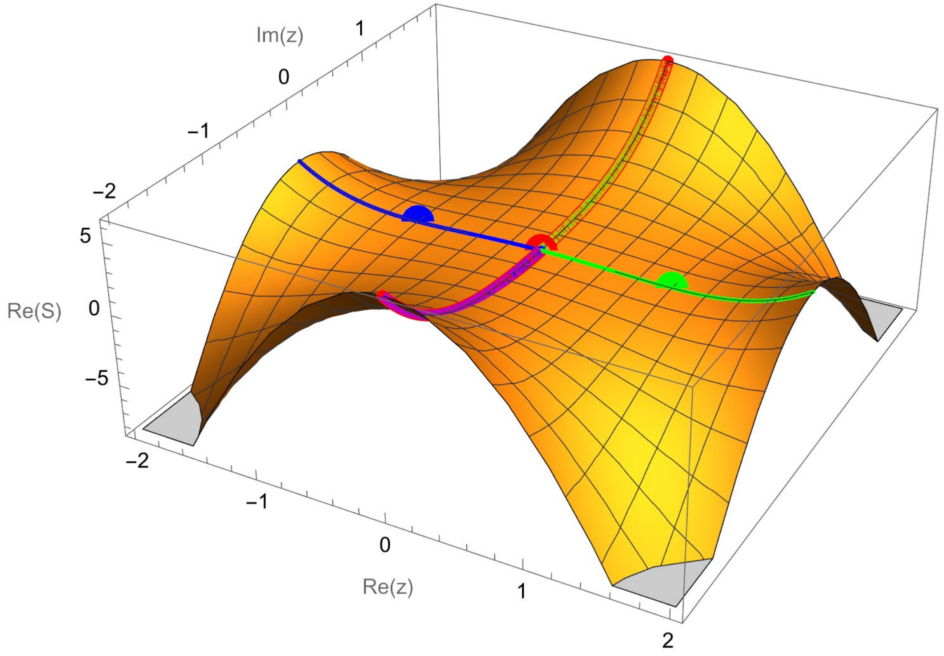

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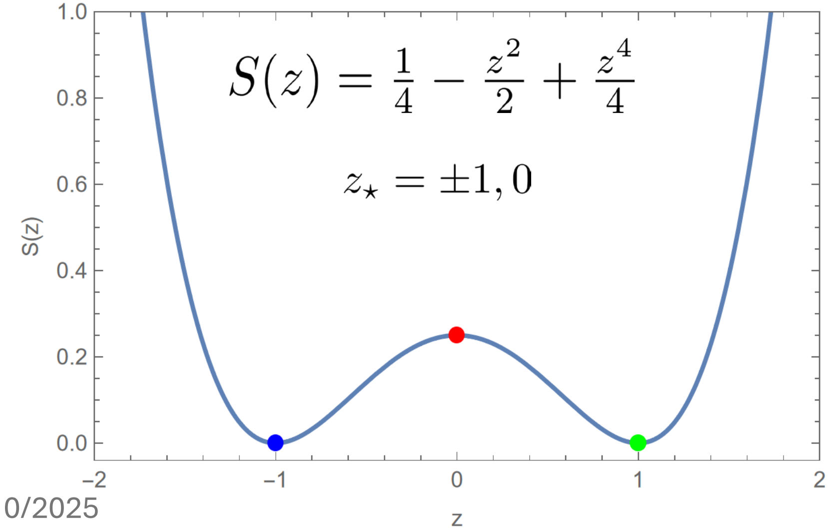

$$\frac{1}{\alpha}\int_0^\infty dt\, e^{-t/\alpha}\left[B_1(t) + B_{-1}(t)\right] + \frac{1}{\alpha} e^{-\frac{1}{4\alpha}} \int_0^\infty dt\, e^{-t/\alpha} B_0(t) = \int_{-\infty}^{\infty} dz\, e^{-S(z)/\alpha}$$

Imaginary parts from lateral Borel resummation cancel between thimbles yielding a real result

Lateral Borel Resummation works since original integral has a thimble decomposition

Borel Action Correspondence

General observation
$$f(\alpha) = \int_{-\infty}^{\infty} dz\ e^{-S(z)/\alpha}$$

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$$\implies \quad \frac{f(\alpha)}{\alpha} = \frac{1}{\alpha}\int_{0}^{\infty} dt\ e^{-t/\alpha}\ \underbrace{\int_{-\infty}^{\infty} dz\ \delta(t - S(z))}_{\mathcal{B}[f(\alpha)/\alpha](t)}$$

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$$\mathcal{B}\left[\frac{f(\alpha)}{\alpha}\right](t) = \frac{d}{dt}\mathcal{B}\left[f(\alpha)\right](t) = \sum_{S(z)=t} \frac{1}{S'(z)} = \sum_{S(z)=t}\left|\frac{dz}{dS}\right|$$

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$$f(\alpha) = \int_{-\infty}^{\infty} dz\ e^{-S(z)/\alpha}$$

$$\implies \quad \frac{f(\alpha)}{\alpha} = \frac{1}{\alpha}\int_0^{\infty} dt\ e^{-t/\alpha}\ \underbrace{\int_{-\infty}^{\infty} dz\ \delta(t - S(z))}_{\mathcal{B}[f(\alpha)/\alpha](t)}$$

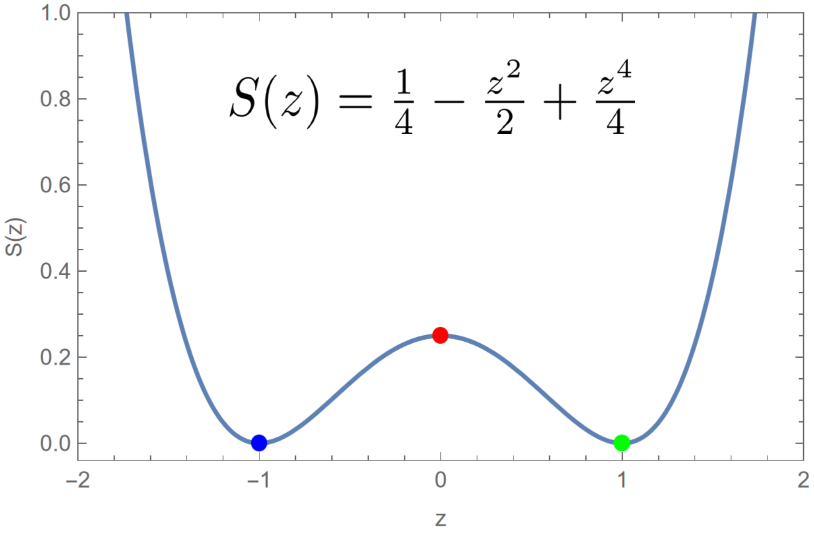
$$\mathcal{B}\left[\frac{f(\alpha)}{\alpha}\right](t) = \frac{d}{dt}\mathcal{B}\left[f(\alpha)\right](t) = \sum_{S(z)=t} \frac{1}{S'(z)} = \sum_{S(z)=t} \left|\frac{dz}{dS}\right|$$

Integrate

$$B(t) = \sum_{S(z)=t} \pm\ z \qquad\qquad t = S(z)$$

Borel Action correspondence

Borel Action Correspondence



$$B(t) = \sum_{S(z)=t} \pm\, z \qquad\qquad t = S(z)$$

Two solutions for $t > 1/4$ $\quad z = \pm\sqrt{1 + 2\sqrt{t}}$

Four solutions for $t < 1/4$ $\quad z = \pm\sqrt{1 + 2\sqrt{t}} \quad z = \pm\sqrt{1 - \sqrt{2}\sqrt{t}}$

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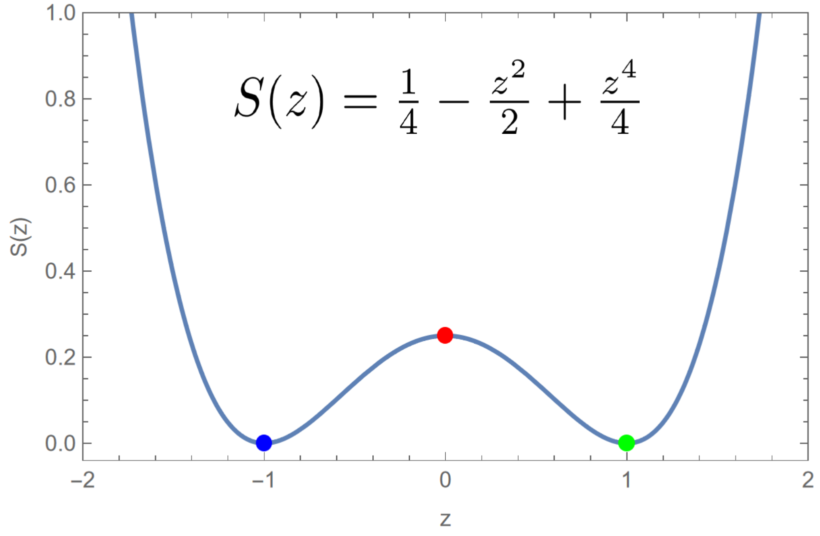

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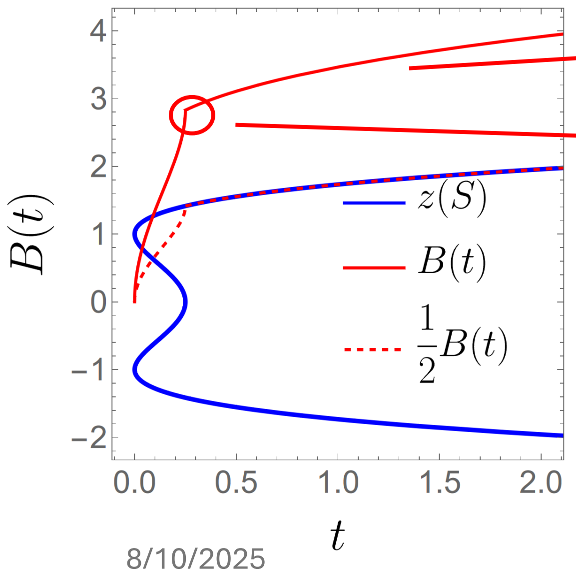

$$B(t) = -2\sqrt{1-2\sqrt{t}}\;\theta\left(\tfrac{1}{4} - t\right) + 2\sqrt{1+2\sqrt{t}}$$

Equivalent to $B_{-1}(t) + B_1(t) + B_0(t)$

Non analytic behaviour at $t = 1/4$

$$B_{\pm 1}(t) = \sqrt{2 - 2\sqrt{1 - 4t \pm i\epsilon}}$$

Expansion about $z = 1$ indicates the presence of saddle at $S'(0) = 0$

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$$1\text{D} : B(t) = \int dz\ \theta(t - S(z))$$

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$B'(t)$ is *density of fields with action* $\leq t$

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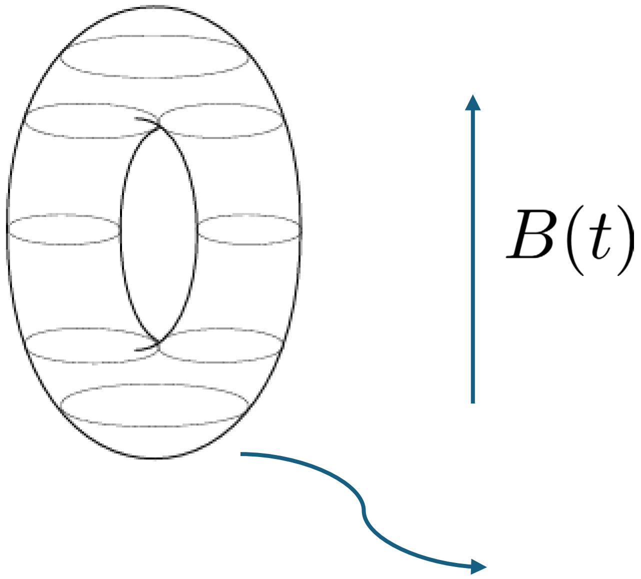

Map $S(\vec{z})$ as a height function of the manifold of $\vec{z}$

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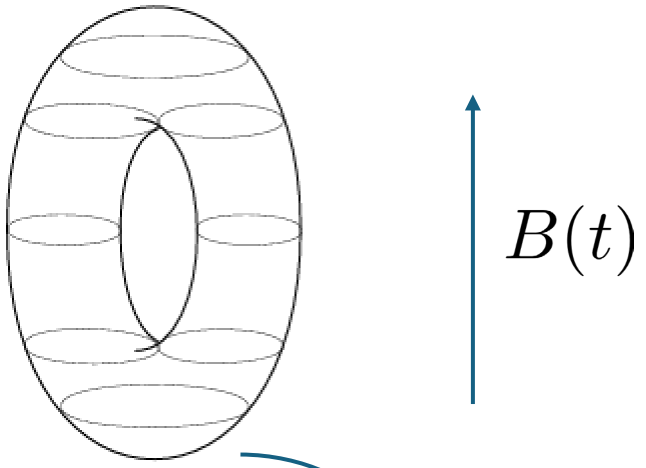

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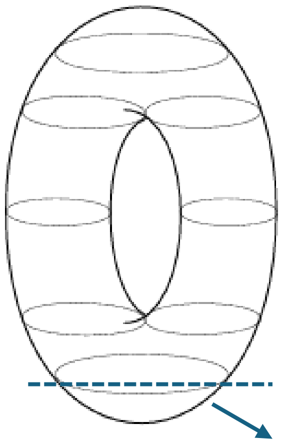

Volume increases continuously

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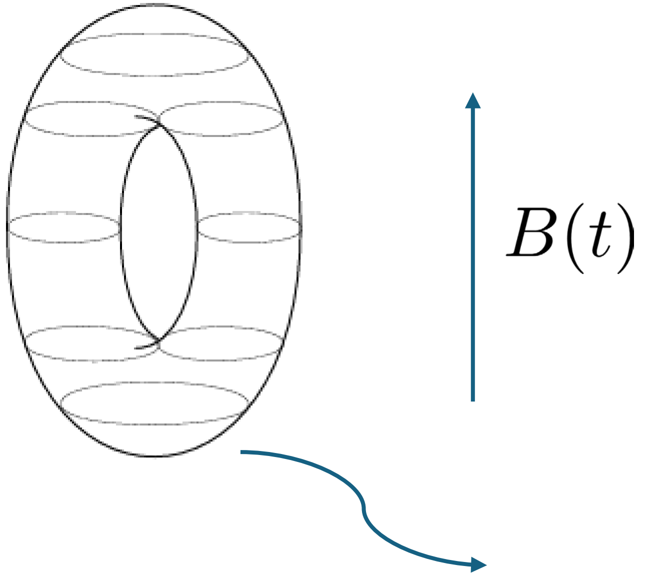

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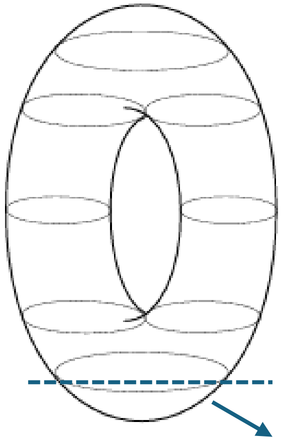

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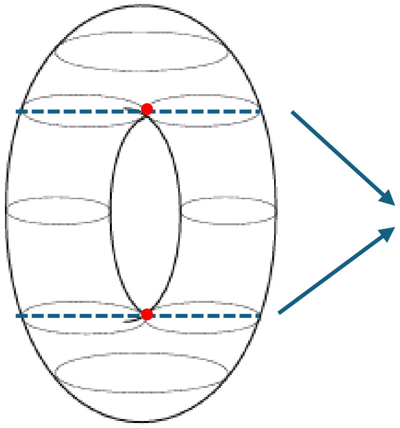


Level set of volume 'splits'

$B(t)$ has non-analytic behaviour, discontinuities

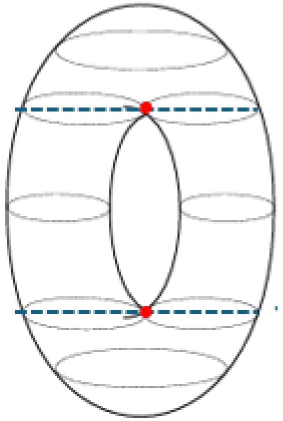
• ⟶ Saddle points of action $S(\vec{z})$

Borel Singularities

Borel Action correspondence shows how $B(t)$ can blow up

$$B(t) = \int d^n\vec{z}\; \theta(t - S(\vec{z}))$$

- Case 1: $B(t)$ has a branch cut, $S'(z) = 0$ at finite z



$$\frac{\delta S_E}{\delta \phi} = 0 \implies \phi_{\text{Instanton}} \implies Z(\alpha) \sim \sum_{n=0}^{\infty} \alpha^n\, n!\, S_I^{-n}$$

Ex) Fubini instanton in $\lambda\phi^4$ theory,
Anharmonic oscillator energy levels in QM, etc.

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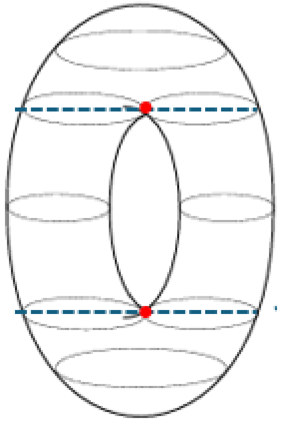

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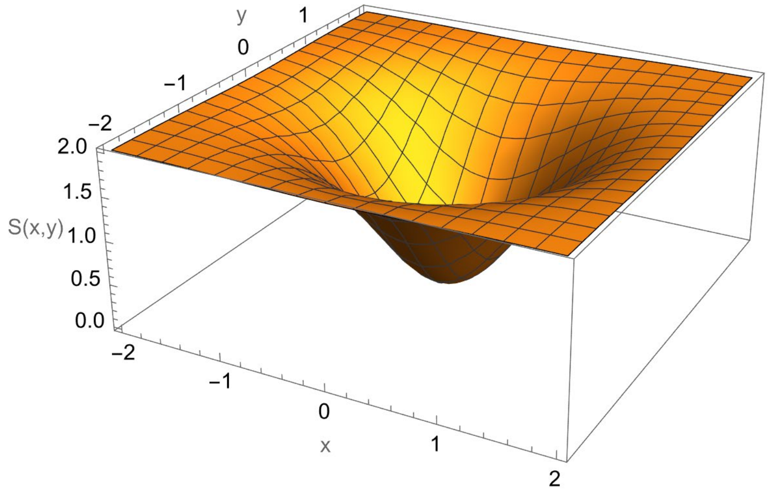
- Case 2: $S(z) \to$ constant as $z \to \infty$, $B(t)$ diverges

$$S(z) = 2S_I(1 - e^{-z})$$
$$\Downarrow$$
$$B(t) = -\ln\left(1 - \frac{t}{2S_I}\right)$$



Ex) Double well energy levels,
Instanton Anti Instanton pair in QCD, etc.

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Finite z solution, and $B(t)$ blows up! Renormalon at $t = t_R$

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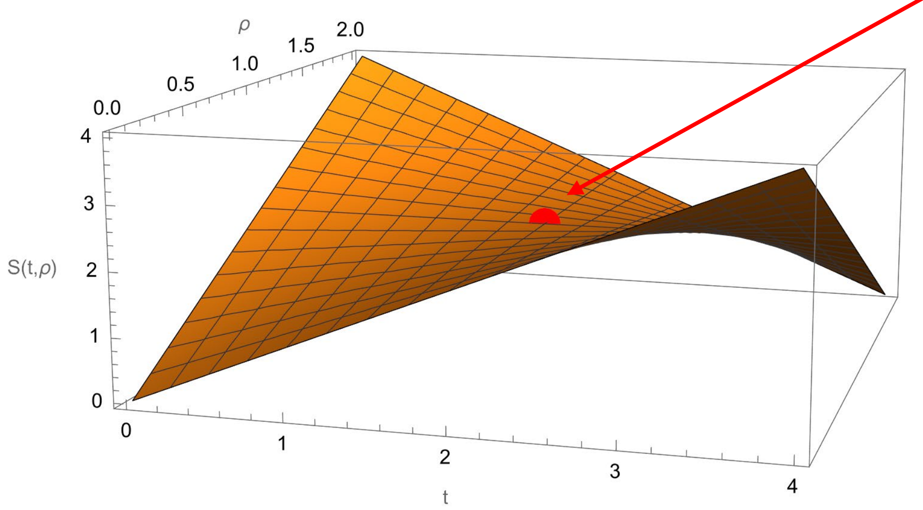

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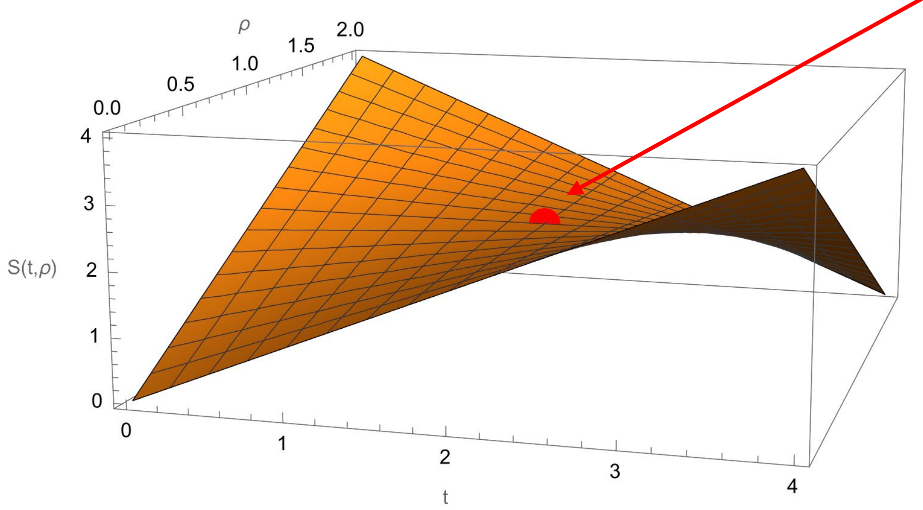

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ρ is the breaking of dilatation symmetry, one-loop running

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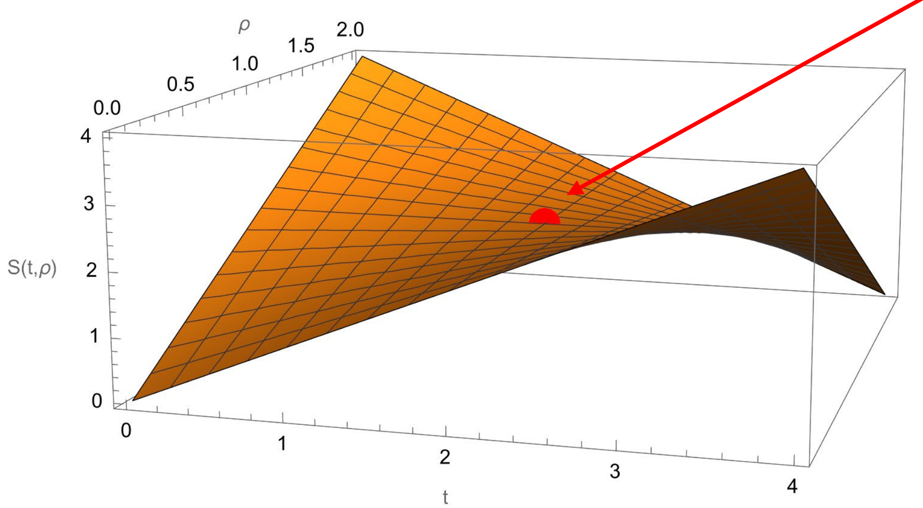

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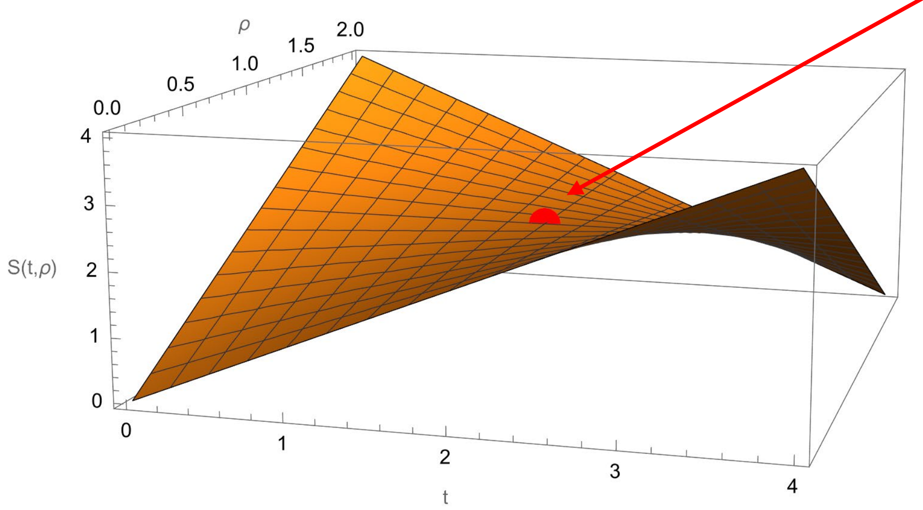

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$$\langle \mathcal{O}(0) \rangle = \frac{1}{Z} \int [d\phi]\, e^{-\frac{1}{\alpha} S_0[\phi]} \mathcal{O}[\phi(0)]\,.$$

Example operator matrix element appearing in OPE

Scalar field theory example, but general. $[\mathcal{O}] = \Delta_{\mathcal{O}}$

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Use basis of fields ‘factoring’ out translation and dilatation zero modes

$$\phi(x) = \sum_{n=1}^{\infty} a_n \phi_n \left(\frac{x - x_0}{R} \right)$$

arXiv:1707.08124 Andreassen, Frost, Schwartz

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Integration over translation modes,
scale invariant measure with operator dimension

One loop running fixed by RG

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Integrate over IR modes in R,
energy scales $\leq 1/R_0$

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One loop running fixed by RG

Similar to Balitsky Babansky (2000) but do not need to thread along instanton-anti-instanton valley $t = S(A_I + A_{\bar{I}})$

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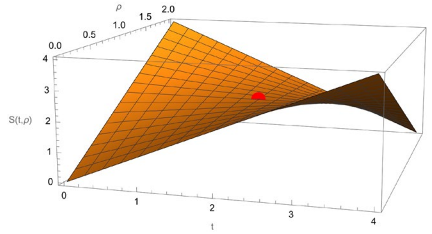

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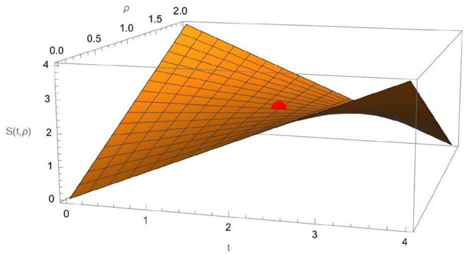


Diagrammatic renormalon is thus a saddle of the effective action

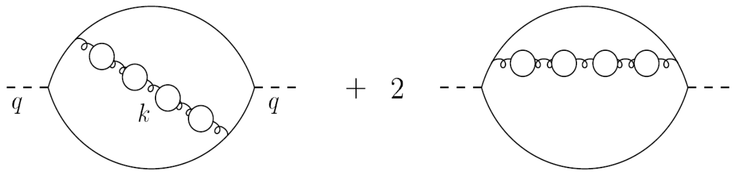
$$t_R = \frac{\Delta_{\mathcal{O}}}{2\beta_0} \qquad \rho = \frac{1}{2\beta_0 \alpha(\mu)}$$

$$R = \frac{1}{\Lambda_{\mathrm{QCD}}}$$

$$\frac{\delta S_{\text{eff}}}{\delta \phi} = 0 \Leftarrow \phi_{\text{Renormalon}} \Leftarrow \mathcal{O}(\alpha) = \sum_{n=0}^{\infty} \alpha^n n! (\beta_0/k)^n$$

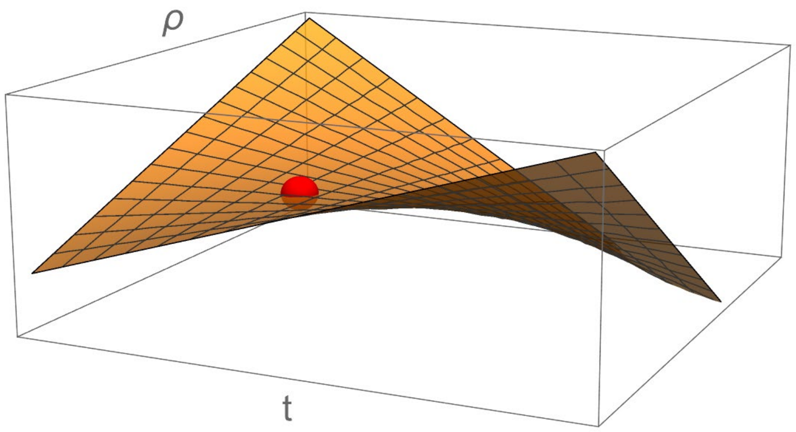
Example

$\mathcal{O} = \mathrm{Tr}(G^2)$, $\Delta_{\mathcal{O}} = 4$ gives $t_R = 2/\beta_0$



Leading renormalon in Adler function, R ratio

Renormalon Thimble



Thimbles in higher dimensional complex space are middle real dimensional complex cycles that have Im$[S]$ constant

$$S(t,\rho) = \frac{t}{\alpha(\mu)} + (t_R - 2\beta_0 t)\rho$$

$$t = x + iw$$

$$\rho = y + iw$$

4d complex space, 2d real middle dimension

Action increases in a real direction and in an imaginary direction

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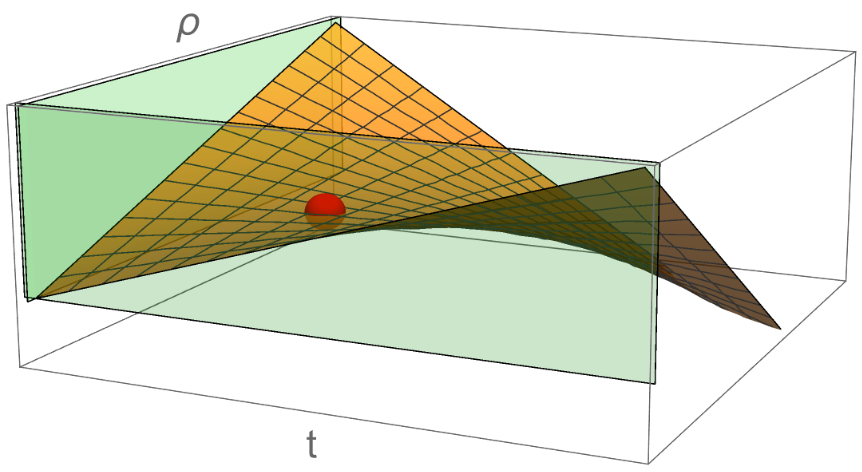

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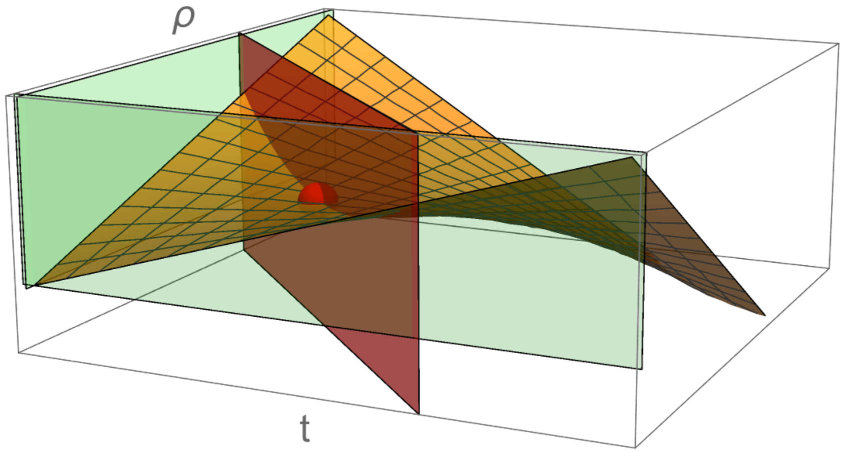

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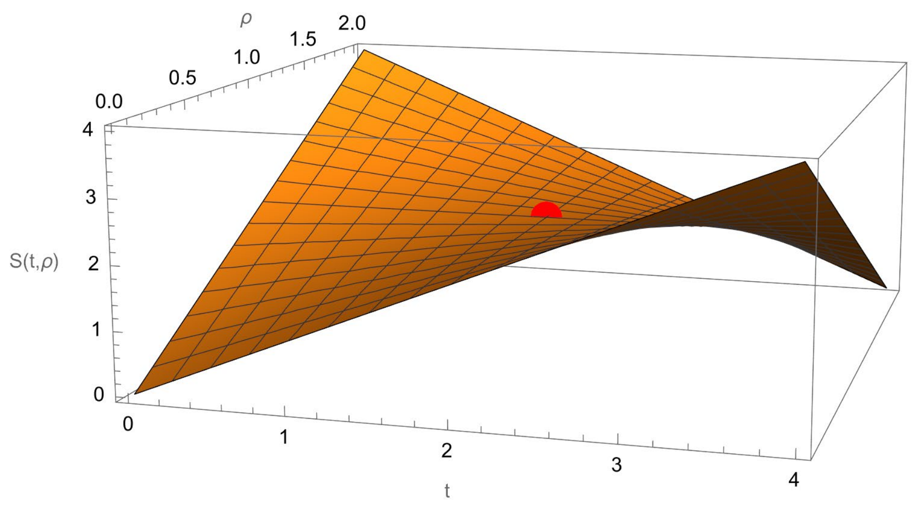
Thimble through renormalon saddle $t = t_R,\ \rho = \frac{1}{2\beta_0\alpha(\mu)}$ $\quad w \in \mathbb{R},\ -\infty \leq x - t_R = \frac{1}{\alpha} - y \leq \infty$

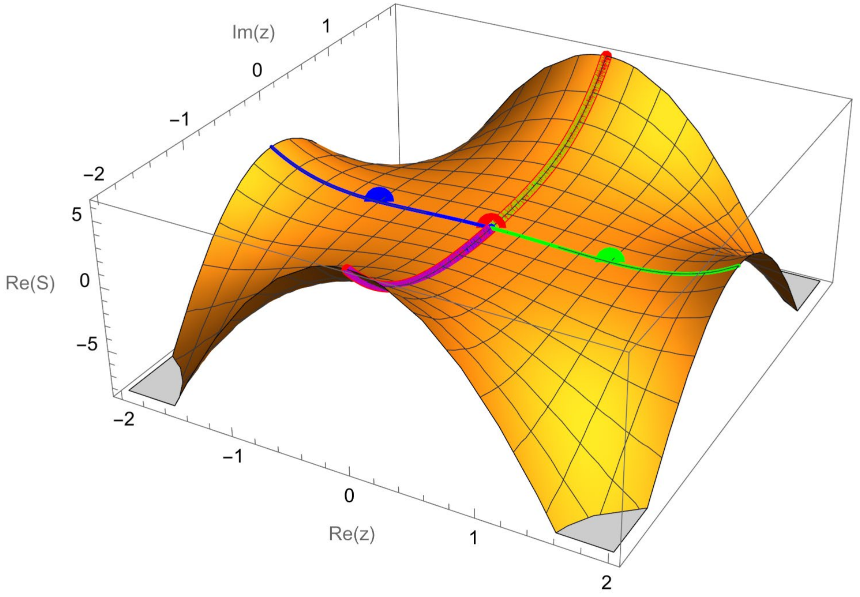
Pure imaginary part, twice that of lateral Borel resummation around diagrams

Work in progress : OPE cancellation of renormalons via thimbles

Conclusions

- Renormalon appears as saddle point of the effective action. Same footing as instantons!





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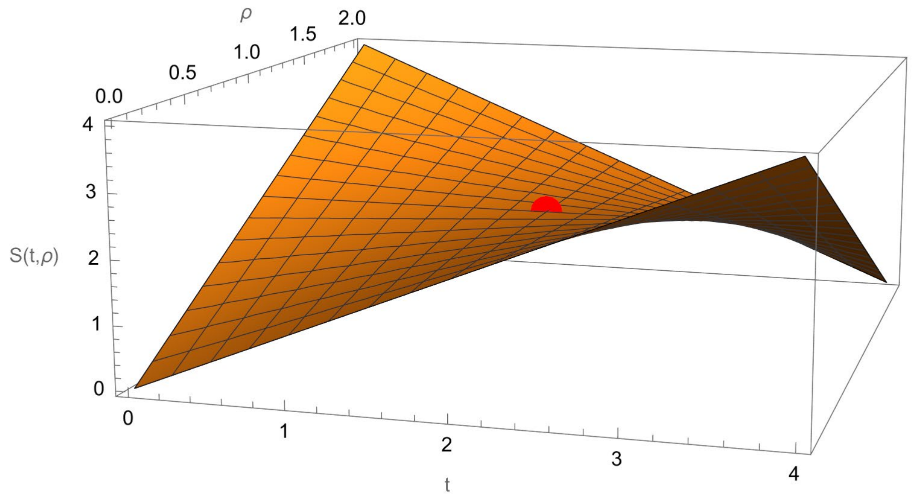

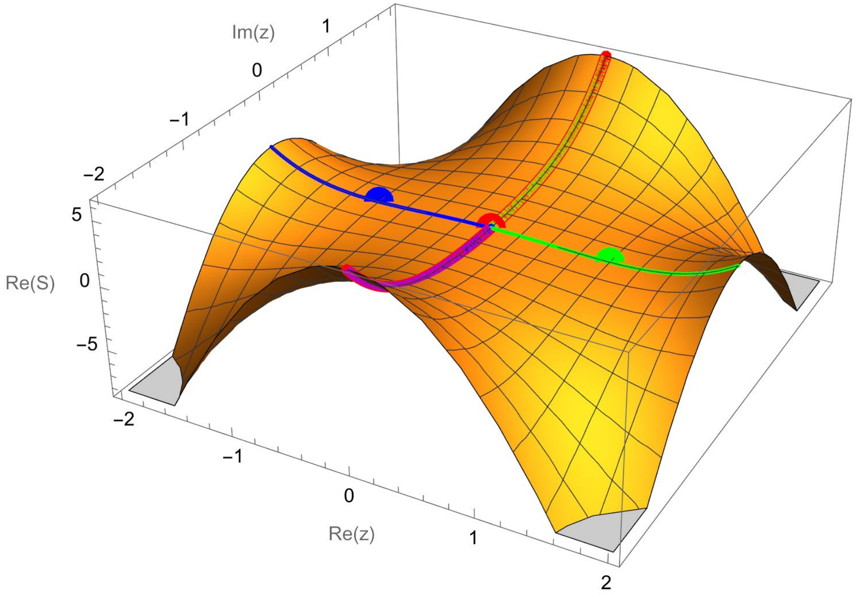

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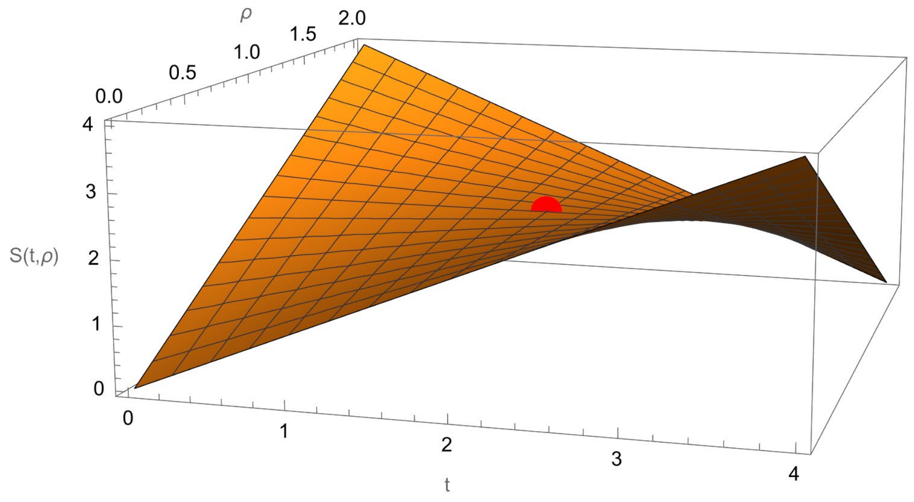

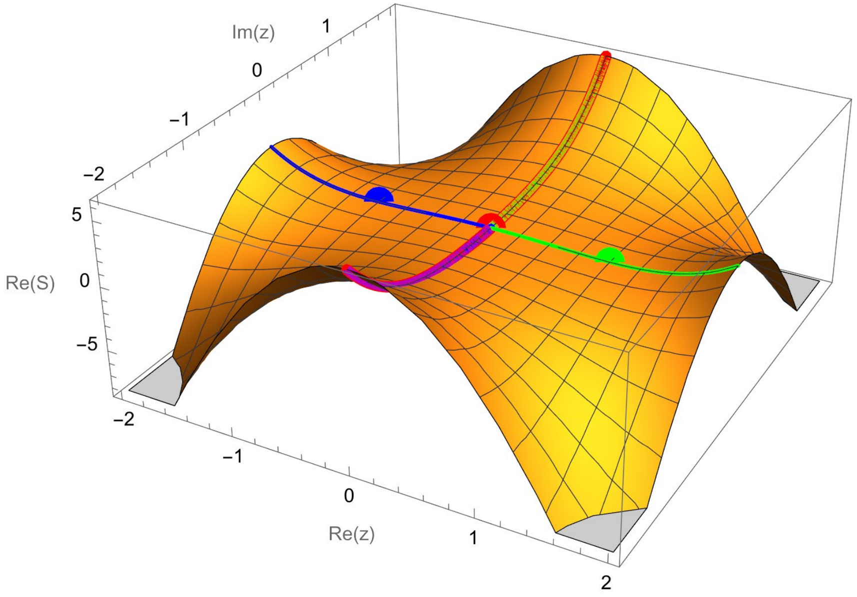

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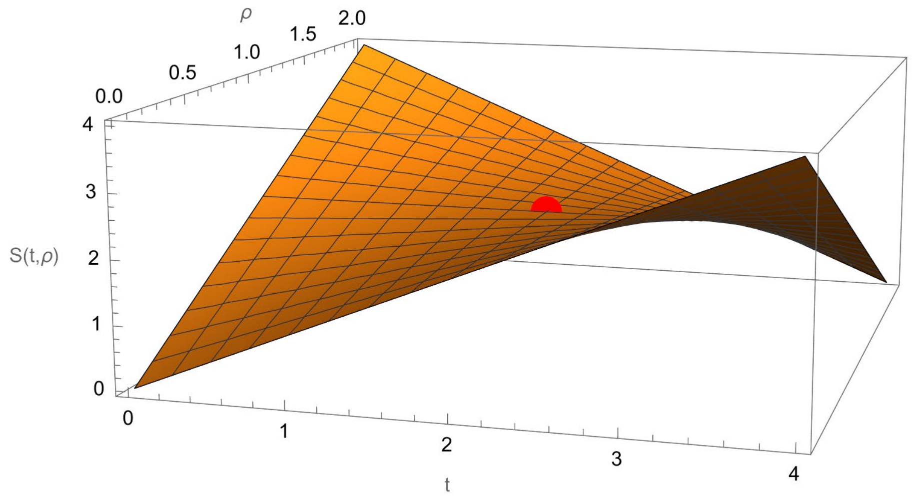
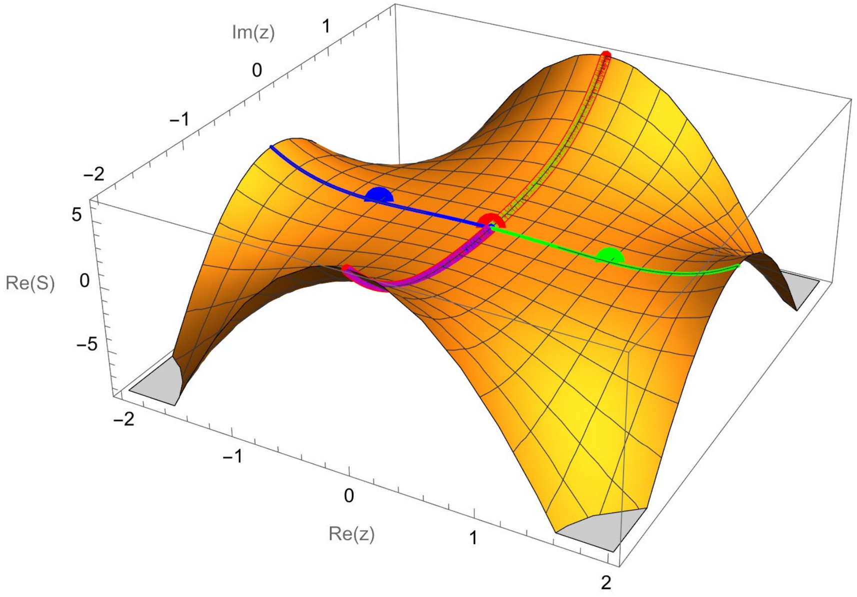
- Full trans-series of renormalons using thimble decomposition of the path integral?





Thank you for your time! Questions?

Backup

Two Loop Renormalon

$$\int_0^\infty dt \int_{R_0}^\infty \frac{dR}{R^{1+\Delta}}\ e^{-t/\alpha(\mu)-2\beta_0 t\ln\mu R+\frac{\beta_1 t}{\beta_0}\ln(\frac{1}{\alpha(\mu)}+\frac{\beta_1}{\beta_0})+\cdots}$$

Shift to

$$\frac{1}{\tilde{\alpha}(\mu)} = \frac{1}{\alpha(\mu)} + \frac{\beta_1}{\beta_0}$$

$$\int_0^\infty dt\ e^{-t/\tilde{\alpha}(\mu)} \int_{R_0}^\infty \frac{dR}{R^{1+\Delta}} (\mu R)^{-2\beta_0 t}\ \tilde{\alpha}(\mu)^{-\beta_1 t/\beta_0}\ e^{\beta_1 t/\beta_0}$$

$$= \mu^\Delta \int_0^\infty dt\ e^{-t/\tilde{\alpha}(\mu)} \int d\rho\ e^{-\rho(2\beta_0 t-\Delta)} \tilde{\alpha}(\mu)^{-\beta_1 t/\beta_0}\ e^{\beta_1 t/\beta_0}$$

Integration by parts

$$\mu^\Delta \int_0^\infty dt\ e^{-t/\tilde{\alpha}(\mu)} \int d\rho\ e^{-\rho(2\beta_0 t-\Delta)} \tilde{\alpha}(\mu)^{-2\beta_1/\beta_0^2}\ e^{\beta_1 t/\beta_0}$$

$$\propto \mu^\Delta \int_0^\infty dt\ e^{-t/\tilde{\alpha}(\mu)} \frac{1}{(2\beta_0 t-\Delta)^{1+\Delta\beta_1/\beta_0^2}} e^{-t\beta_1/\beta_0}$$

Form of two loop running included in renormalons from diagrams Mueller 1992, Grunberg 1993

Renormalons in SYM

Some literature claim that $\langle \text{Tr } G^2(0)\rangle$ vanishes in $\mathcal{N} = 1$ SYM Shifman : 1411.4004, Dunne, Shifman, Unsal : 1502.06680. Does the leading renormalon in the Adler function vanish?

Diagrammatic probe is hard since we rely on large N_f and naïve non-abelianization

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Path integral methods can answer this! Potential fermion zero-modes (as present in instanton-anti-instanton) can make the Borel transform / operator vev zero.

$$\int [da_n]\, e^{-S_0(a_n)} \quad \underbrace{\frac{d^4x_0\ d\rho}{\rho^5}}_{\text{Bosonic Zero Modes}} \times \underbrace{\Pi_{n=1}^{2Nk} d\theta}_{\text{Fermionic Zero Modes}}$$

$n_f = 2Nk$ zero modes in $SU(N)$ with $\mathcal{N}=k$ SUSY

$$\lambda^\alpha \sim \frac{\rho}{(\rho^2+(x+x_0)^2)^{3/2}}\xi^\alpha$$

Kills Bosonic operator expectation values

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$\langle (\bar{\psi}\psi)^2(0) \rangle$ gives the leading renormalon at $t = 3/\beta_0$. Interesting to check and prove this using diagrams!

OPE and Renormalons

In precision QCD, the OPE is a Λ_{QCD}/Q expansion

$$\sigma = \boxed{C_0(Q,\mu)}\mathcal{O}_0(\mu) + \boxed{C_1(Q,\mu)}\frac{\langle \mathcal{O}_2(\mu)\rangle}{Q} + \cdots$$

Feynman diagrams contribute
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NP operator matrix elements
Contain imaginary contributions that
cancel with the IR renormalons
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