Symmetry Invariant Twisted Cohomology

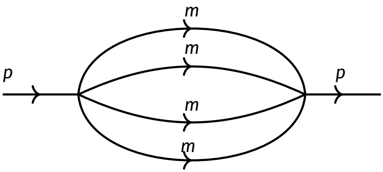
MathemAmplitudes 2025

Cathrin Semper
Based on work in progress
with Claude Duhr, Sara Maggio, Franziska Porkert and Sven Stawinski
csemper@uni-bonn.de

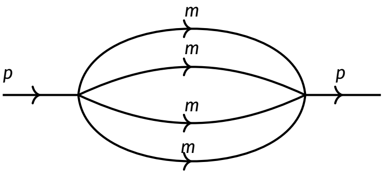
Motivation

- [Mizera, 2018] [Mizera, Mastrolia, 2019] identified twisted cohomology as appropriate mathematical framework for Feynman integrals (in dimensional regularization) [Background e.g. in Eric's talk]
- On the other hand: In common IBP approaches e.g. using [Laporta, 2000][see Gaia's, Rourou's, Tiziano's talks] symmetries play a crucial role to reduce number of master integrals
- [Gasparotto, Weinzierl, Xu, 2023][Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi et al, 2019] describe how symmetries can be understood for spanning differential forms of a cohomology integrated over the Feynman contour
- Effects of the symmetry on the periods and intersection pairings is so far less understood

Motivation



Motivation

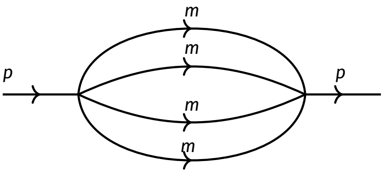


IBP-reduction

(or remember from [Wojciech's talk])

$\longrightarrow$ 3 master integrals

Motivation



IBP-reduction

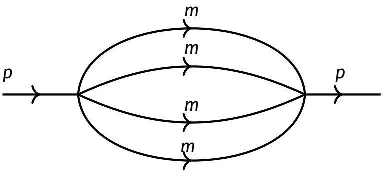
(or remember from [Wojciech's talk])

$\longrightarrow$ 3 master integrals

Twisted cohomology

$\longrightarrow \dim(H^5) = 11$

Motivation



IBP-reduction

(or remember from [Wojciech's talk])

$\longrightarrow$ 3 master integrals

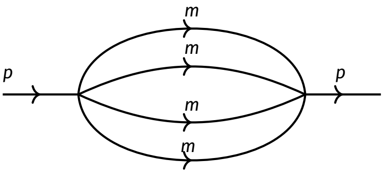
exclude symmetries

$\longrightarrow$ 11 master integrals

Twisted cohomology

$\longrightarrow \dim(H^5) = 11$

Motivation



IBP-reduction

(or remember from [Wojciech's talk])

$\longrightarrow$ 3 master integrals

exclude symmetries

$\longrightarrow$ 11 master integrals

Twisted cohomology

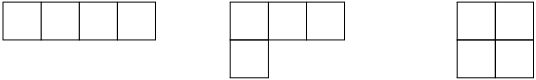
$\longrightarrow \dim(H^5) = 11$

- **We can find a corresponding 3 dim. intersection matrix of some smaller cohomology**

? **How is it related to the 11. dim one?**

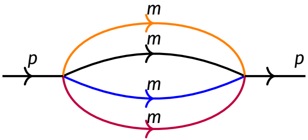
? **How does the period matrix transform?**

Outline

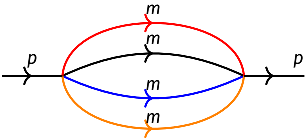


- Symmetry
  - in loop-momentum space
  - in Baikov representation
  - for the vector of master integrals
- Symmetry-reduced basis of a cohomology group
- Action of the symmetry on period and intersection pairing
- Some nice consequences

Loop-Momentum Space Symmetry



symmetry
=



$$\begin{aligned} D_1 &= k_1^2 - m^2 \\ D_2 &= k_2^2 - m^2 \\ D_3 &= (k_1 - k_3)^2 - m^2 \\ D_4 &= (k_2 - k_3 - p)^2 - m^2 \end{aligned} \xrightarrow[k_1 \to k_2 - k_3]{k_3 \to -k_1 + k_2 - p} \begin{aligned} D_1 &= (k_2 - k_3 - p)^2 - m^2 \\ D_2 &= k_2^2 - m^2 \\ D_3 &= (k_1 - k_3)^2 - m^2 \\ D_4 &= k_1^2 - m^2 \end{aligned}$$

Resulting Symmetry e.g.:

$$I_{2,1,1,1} = \int_\Gamma \mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3 \frac{\mathcal{N}}{D_1^2 D_2 D_3 D_4} I_{1,1,1,2}$$

$$I_{3,1,1,1} = \frac{2m^2 I_{3,1,1,1} - (\epsilon + 1)\left(I_{1,1,1,2} + I_{2,1,1,1}\right)}{2m^2}$$

Symmetries

In Loop-Momentum Space:

Generally: We are considering symmetries stemming from transforming integration variables interchanging propagators (graph symmetries)

[Z. Wu, Y. Zhang, 2025] [Z. Wu, J. Boehm, R. Ma, H. Xu and Y. Zhang, 2024] …

$$k_i \longrightarrow A_{ij} k_j + B_{ik} p_k \,, \qquad \det A = \pm 1$$

Symmetries

In Loop-Momentum Space:

Generally: We are considering symmetries stemming from transforming integration variables interchanging propagators (graph symmetries)

[Z. Wu, Y. Zhang, 2025] [Z. Wu, J. Boehm, R. Ma, H. Xu and Y. Zhang, 2024] …

$$k_i \longrightarrow A_{ij} k_j + B_{ik} p_k \,, \qquad \det A = \pm 1$$

In Baikov Representation:

- $z_i \longrightarrow \tilde{A}_{ij} z_j + \tilde{B}_{ik} p_k \qquad \det \tilde{A} \neq 0$

Toy Baikov Symmetry

$$\Psi_{F2} = x_1^{\alpha_1} x_2^{\alpha_2} (1 - x_1)^{\alpha_3} (1 - x_2)^{\alpha_4} (1 - x_1 y_1 - x_2 y_2)^{\alpha_5}$$

$$\downarrow y_1 = y_2 =: y$$

$$\Psi = x_1^{\alpha_1} x_2^{\alpha_2} (1 - x_1)^{\alpha_3} (1 - x_2)^{\alpha_4} (1 - x_1 y - x_2 y)^{\alpha_5}$$

Invariant under $x_1 \leftrightarrow x_2$

A basis: $\vec{\varphi} = \left\{ \frac{dx_1 \wedge dx_2}{(1-x_1)x_2}, \frac{dx_1 \wedge dx_2}{(1-x_2)x_1}, \frac{dx_1 \wedge dx_2}{(1-x_1)(1-x_2)}, \frac{dx_1 \wedge dx_2}{x_2 x_1} \right\}$

Symmetry: $\int_0^1 \int_0^1 \Psi \varphi_2 = \int_0^1 \int_0^1 \Psi \varphi_1$

Symmetry for Master Integrals

For a group G a representation R is map $G \longrightarrow GL(V)$, with V finite dim. vector space

Symmetry for Master Integrals

For a group G a representation R is map $G \longrightarrow GL(V)$, with V finite dim. vector space

Symmetry for Master Integrals

A symmetry for a vector of master integrals I acts like

$$RI = I$$

where R is a representation matrix of the graph symmetry group G

Symmetry for Master Integrals

For a group G a representation R is map $G \longrightarrow GL(V)$, with V finite dim. vector space

Symmetry for Master Integrals

A symmetry for a vector of master integrals I acts like

$$RI = I$$

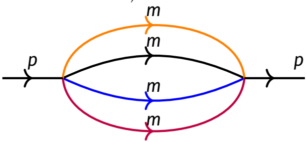
where R is a representation matrix of the graph symmetry group G

- Symmetry group of F_2 is S_2 (with $S_n = \{$permutations on the set {1,...,n}$\}$)
- Representation of S_2 acting on basis I is $R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Symmetry Groups

- Symmetry group of four-loop equal mass banana is S_4
- One of the representation matrices e.g. looks like

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2m^2+s & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & -2 & 0 \\ -s & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -s & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ m^2-\frac{s}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon+1}{2m^2} & 0 & 0 & \frac{-\epsilon-1}{2m^2} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Symmetry and Cohomology

Naively: Reduced master integrals because of symmetry $\longrightarrow$ reduced dimension of twisted cohomology group?

Symmetry and Cohomology

Naively: Reduced master integrals because of symmetry $\longrightarrow$ reduced dimension of twisted cohomology group?

$$\Psi = x_1^{\alpha_1} x_2^{\alpha_2} (1-x_1)^{\alpha_3} (1-x_2)^{\alpha_4} (1-x_1 y - x_2 y)^{\alpha_5}\ , \ \vec{\varphi} = \left\{ \frac{dx_1 \wedge dx_2}{(1-x_1)x_2}, \frac{dx_1 \wedge dx_2}{(1-x_2)x_1}, \frac{dx_1 \wedge dx_2}{(1-x_1)(1-x_2)}, \frac{dx_1 \wedge dx_2}{x_2 x_1} \right\}$$

$$C \overset{y_1 = y_2}{\sim} \begin{pmatrix} 1 & \frac{2}{7} & -\frac{3}{7} & -\frac{3}{7} \\ \frac{2}{7} & 1 & -\frac{3}{7} & -\frac{3}{7} \\ -\frac{3}{7} & -\frac{3}{7} & 1 & \frac{2}{7} \\ -\frac{3}{7} & -\frac{3}{7} & \frac{2}{7} & 1 \end{pmatrix}$$

Symmetry and Cohomology

Naively: Reduced master integrals because of symmetry $\longrightarrow$ reduced dimension of twisted cohomology group?

$$\Psi = x_1^{\alpha_1} x_2^{\alpha_2} (1-x_1)^{\alpha_3} (1-x_2)^{\alpha_4} (1-x_1 y - x_2 y)^{\alpha_5}\,,\ \vec{\varphi} = \left\{ \frac{dx_1 \wedge dx_2}{(1-x_1)x_2}, \frac{dx_1 \wedge dx_2}{(1-x_2)x_1}, \frac{dx_1 \wedge dx_2}{(1-x_1)(1-x_2)}, \frac{dx_1 \wedge dx_2}{x_2 x_1} \right\}$$

$$C \overset{y_1=y_2}{\sim} \begin{pmatrix} 1 & \frac{2}{7} & -\frac{3}{7} & -\frac{3}{7} \\ \frac{2}{7} & 1 & -\frac{3}{7} & -\frac{3}{7} \\ -\frac{3}{7} & -\frac{3}{7} & 1 & \frac{2}{7} \\ -\frac{3}{7} & -\frac{3}{7} & \frac{2}{7} & 1 \end{pmatrix}$$

$\det C \neq 0$!

Since

$$\int_\Gamma \Psi \varphi_1 = \int_\Gamma \Psi \varphi_2$$

but

$$\varphi_1 \neq \varphi_2$$

[Gasparotto, Weinzierl, Xu, 2023], [Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi et al, 2019]

Symmetry and Cohomology

Naively: Reduced master integrals because of symmetry $\longrightarrow$ reduced dimension of twisted cohomology group?

$$\Psi = x_1^{\alpha_1} x_2^{\alpha_2} (1-x_1)^{\alpha_3} (1-x_2)^{\alpha_4} (1-x_1 y - x_2 y)^{\alpha_5} \, , \; \vec{\varphi} = \left\{ \frac{dx_1 \wedge dx_2}{(1-x_1)x_2}, \frac{dx_1 \wedge dx_2}{(1-x_2)x_1}, \frac{dx_1 \wedge dx_2}{(1-x_1)(1-x_2)}, \frac{dx_1 \wedge dx_2}{x_2 x_1} \right\}$$

$$C \overset{y_1 = y_2}{\sim} \begin{pmatrix} 1 & \frac{2}{7} & -\frac{3}{7} & -\frac{3}{7} \\ \frac{2}{7} & 1 & -\frac{3}{7} & -\frac{3}{7} \\ -\frac{3}{7} & -\frac{3}{7} & 1 & \frac{2}{7} \\ -\frac{3}{7} & -\frac{3}{7} & \frac{2}{7} & 1 \end{pmatrix}$$

$\longrightarrow$ The dimension of the twisted cohomology group does not reduce

(The Euler characteristic (mentioned in [Saiei's and Simon's talks]) does not drop

det $C \neq 0$!

Since

$$\int_\Gamma \Psi \varphi_1 = \int_\Gamma \Psi \varphi_2$$

but

$$\varphi_1 \neq \varphi_2$$

[Gasparotto, Weinzierl, Xu, 2023], [Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi et al, 2019]

Symmetry and Cohomology

- For a subgroup G' of G that leaves Ψ invariant [Gasparotto, Weinzierl, Xu, 2023] teaches us how we can find the lower-dimensional basis of a smaller cohomology group $H^n_{G'}$:

Symmetry and Cohomology

- For a subgroup G' of G that leaves Ψ invariant [Gasparotto, Weinzierl, Xu, 2023] teaches us how we can find the lower-dimensional basis of a smaller cohomology group $H^n_{G'}$:

- E.g. Here $\int_0^1 \int_0^1 \Psi \varphi_1 = \int_0^1 \int_0^1 \Psi \varphi_2$

  $\longrightarrow$ Choose $\varphi_{\text{sym}} = \left(\frac{1}{2}(\varphi_1 + \varphi_2), \varphi_3, \varphi_4, \frac{1}{2}(\varphi_1 - \varphi_2)\right)^T$

  $\longrightarrow$ $\int_0^1 \int_0^1 \Psi \varphi_{\text{sym},4} = 0$ and $R\vec{\varphi}_{G,j} = \vec{\varphi}_{G,j}$ for $j \leq 3$ new masters

Symmetry and Cohomology

- For a subgroup G' of G that leaves Ψ invariant [Gasparotto, Weinzierl, Xu, 2023] teaches us how we can find the lower-dimensional basis of a smaller cohomology group $H^n_{G'}$:

- E.g. Here $\int_0^1 \int_0^1 \Psi\, \varphi_1 = \int_0^1 \int_0^1 \Psi\, \varphi_2$

  $\longrightarrow$ Choose $\varphi_{\text{sym}} = \left(\frac{1}{2}(\varphi_1 + \varphi_2), \varphi_3, \varphi_4, \frac{1}{2}(\varphi_1 - \varphi_2)\right)^T$

  $\longrightarrow$ $\int_0^1 \int_0^1 \Psi\, \varphi_{\text{sym},4} = 0$ and $R\vec{\varphi}_{G,j} = \vec{\varphi}_{G,j}$ for $j \leq 3$ new masters

- Can be generalized to more complicated groups by identifying the distinct non-zero orbits, building invariant integrands

Symmetry and Cohomology

- For a subgroup G' of G that leaves Ψ invariant [Gasparotto, Weinzierl, Xu, 2023] teaches us how we can find the lower-dimensional basis of a smaller cohomology group $H^n_{G'}$:

- E.g. Here $\int_0^1 \int_0^1 \Psi \varphi_1 = \int_0^1 \int_0^1 \Psi \varphi_2$

  $\longrightarrow$ Choose $\varphi_{\text{sym}} = \left(\frac{1}{2}(\varphi_1 + \varphi_2), \varphi_3, \varphi_4, \frac{1}{2}(\varphi_1 - \varphi_2)\right)^T$

  $\longrightarrow$ $\int_0^1 \int_0^1 \Psi \varphi_{\text{sym},4} = 0$ and $R\vec{\varphi}_{G,j} = \vec{\varphi}_{G,j}$ for $j \leq 3$ new masters

- Can be generalized to more complicated groups by identifying the distinct non-zero orbits, building invariant integrands

? **What happens over other contours/the cycles**

? **How do the period matrix and intersection pairings transform?**

Symmetries

In Loop-Momentum Space:

Generally: We are considering symmetries stemming from transforming integration variables interchanging propagators (graph symmetries)

[Z. Wu, Y. Zhang, 2025] [Z. Wu, J. Boehm, R. Ma, H. Xu and Y. Zhang, 2024] ...

$$k_i \longrightarrow A_{ij} k_j + B_{ik} p_k \,, \qquad \det A = \pm 1$$

In Baikov Representation:

- $f \,:\, z_i \longrightarrow \tilde{A}_{ij} z_j + \tilde{B}_{ik} p_k \,, \qquad \det \tilde{A} \neq 0$

Symmetries

In Loop-Momentum Space:

Generally: We are considering symmetries stemming from transforming integration variables interchanging propagators (graph symmetries).
[Z. Wu, Y. Zhang, 2025] [Z. Wu, J. Boehm, R. Ma, H. Xu and Y. Zhang, 2024] …

$$k_i \longrightarrow A_{ij} k_j + B_{ik} p_k \,, \qquad \det A = \pm 1$$

In Baikov Representation:

- $f\,:\; z_i \longrightarrow \tilde{A}_{ij} z_j + \tilde{B}_{ik} p_k \,, \qquad \det \tilde{A} \neq 0$
- These transformations leave the Baikov polynomial invariant
  [Gasparotto, Weinzierl, Xu, 2023]

Symmetries

In Loop-Momentum Space:

Generally: We are considering symmetries stemming from transforming integration variables interchanging propagators (graph symmetries).
[Z. Wu, Y. Zhang, 2025] [Z. Wu, J. Boehm, R. Ma, H. Xu and Y. Zhang, 2024] …

$$k_i \longrightarrow A_{ij}k_j + B_{ik}p_k\,, \qquad \det A = \pm 1$$

In Baikov Representation:

- $f\,:\; z_i \longrightarrow \tilde{A}_{ij}z_j + \tilde{B}_{ik}p_k\,, \qquad \det \tilde{A} \neq 0$
- These transformations leave the Baikov polynomial invariant
  [Gasparotto, Weinzierl, Xu, 2023]

For the cohomology

We can show that such a transformation f

1. leaves the Baikov polynomial/twist invariant
2. leaves the intersection pairing invariant $RCR^T = C$
3. acts on the period paring as $\langle f^* \varphi | \gamma] = \langle \varphi | f_* \gamma]$

   such that the period pairing is invariant up to interchanging cycles $RP = PA^T$ with $\mathrm{d}A = 0$

Consequences for Intersection Pairing

$$\Psi = x_1^{\alpha_1} x_2^{\alpha_2} (1-x_1)^{\alpha_3} (1-x_2)^{\alpha_4} (1-x_1 y - x_2 y)^{\alpha_5}\,,\ \vec{\varphi} = \left\{ \frac{dx_1 \wedge dx_2}{(1-x_1)x_2}, \frac{dx_1 \wedge dx_2}{(1-x_2)x_1}, \frac{dx_1 \wedge dx_2}{(1-x_1)(1-x_2)}, \frac{dx_1 \wedge dx_2}{x_2 x_1} \right\}$$

After the basis change to $I_{\text{sym}} = \left(\frac{1}{2}(I_1 + I_2), I_3, I_4, \frac{1}{2}(I_1 - I_2)\right)^T$

$$C_{\text{sym}} \sim \begin{pmatrix} \boxed{\begin{matrix} \frac{9}{14} & -\frac{3}{7} & -\frac{3}{7} \\ -\frac{3}{7} & 1 & \frac{2}{7} \\ -\frac{3}{7} & \frac{2}{7} & 1 \end{matrix}}^{\,C_G} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \frac{5}{14} \end{pmatrix}$$

The invariant differentials decouple.

Consequences for Intersection Pairing

$$\Psi = x_1^{\alpha_1} x_2^{\alpha_2} (1-x_1)^{\alpha_3} (1-x_2)^{\alpha_4} (1-x_1 y - x_2 y)^{\alpha_5}\ , \ \vec{\varphi} = \left\{ \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{(1-x_1)x_2}, \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{(1-x_2)x_1}, \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{(1-x_1)(1-x_2)}, \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{x_2 x_1} \right\}$$

After the basis change to $I_{\text{sym}} = \left(\frac{1}{2}(I_1 + I_2), I_3, I_4, \frac{1}{2}(I_1 - I_2)\right)^T$

$$C_{\text{sym}} \sim \begin{pmatrix} \overset{C_G}{\boxed{\begin{matrix} \frac{9}{14} & -\frac{3}{7} & -\frac{3}{7} \\ -\frac{3}{7} & 1 & \frac{2}{7} \\ -\frac{3}{7} & \frac{2}{7} & 1 \end{matrix}}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \frac{5}{14} \end{pmatrix}$$

The invariant differentials decouple.

We can show that this holds more generally and is a direct consequence of Schur's lemma if the group action is decomposed in terms of irreducible representations

Irreducible Representations

- A representation is completely reducible if $R = R_1 \oplus \cdots \oplus R_k$

$$MR(g)M^{-1} = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_k \end{pmatrix}, \tag{1}$$

Irreducible Representations

- A representation is completely reducible if $R = R_1 \oplus \cdots \oplus R_k$

$$MR(g)M^{-1} = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_k \end{pmatrix}, \tag{1}$$

e.g. for a basis change to $I_{\text{sym}} = \left(\frac{1}{2}(I_1 + I_2), I_3, I_4, \frac{1}{2}(I_1 - I_2)\right)^T$ we get

$$R_{\text{sym}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad C_{\text{sym}} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}$$

R_1 R_2 C_{11} C_{12} C_{21} C_{22}

This corresponds exactly to the irreps $R = \mathrm{id}_3 \oplus (-1)$

We had $C = R(g)CR(g)^T, \quad \forall g \in G \overset{\text{Schur}}{\longrightarrow} C_{12} = C_{21} = 0$

Symmetric Bananas

- For S_n finite number of irreducibles related to characters, e.g.

S_4	id	$(12)(34)$	(12)	(123)	(1234)
$\ydiagram{4}$	1	1	1	1	1
$\ydiagram{1,1,1,1}$	1	1	-1	1	-1
$\ydiagram{2,2}$	2	2	0	-1	0
$\ydiagram{3,1}$	3	-1	1	0	-1
$\ydiagram{2,1,1}$	3	-1	-1	0	1

Symmetric Bananas

- For S_n finite number of irreducibles related to characters, e.g.

S_4	id	$(12)(34)$	(12)	(123)	(1234)
(4)	1	1	1	1	1
$(1,1,1,1)$	1	1	-1	1	-1
$(2,2)$	2	2	0	-1	0
$(3,1)$	3	-1	1	0	-1
$(2,1,1)$	3	-1	-1	0	1

- 3-loop equal-mass banana: We find a basis that transforms under

$$R = \begin{pmatrix} (3,1) & & & & & \\ & (3,1) & & & & \\ & & (2,2) & & & \\ & & & (4) & & \\ & & & & (4) & \\ & & & & & (4) \end{pmatrix} \quad \dim(H_G^n) = 3$$

Symmetric Bananas

- For S_n finite number of irreducibles related to characters, e.g.

S_4	id	$(12)(34)$	(12)	(123)	(1234)
$\ydiagram{4}$	1	1	1	1	1
$\ydiagram{1,1,1,1}$	1	1	-1	1	-1
$\ydiagram{2,2}$	2	2	0	-1	0
$\ydiagram{3,1}$	3	-1	1	0	-1
$\ydiagram{2,1,1}$	3	-1	-1	0	1

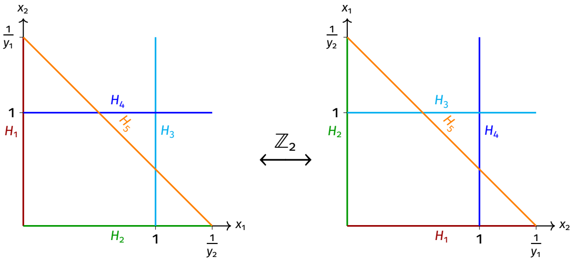
- 3-loop equal-mass banana: We find a basis that transforms under

$$R = \begin{pmatrix} \ydiagram{3,1} & & & \\ & \ydiagram{3,1} & & \\ & & \ydiagram{2,2} & \\ & & & \boxed{\begin{matrix} \ydiagram{4} & & \\ & \ydiagram{4} & \\ & & \ydiagram{4} \end{matrix}} \end{pmatrix} \quad \dim(H^n_G) = 3$$

- For a sub-symmetry, e.g. the $m_1,\ m_2 = m_3 = m_4$ configuration the number of masters follows from purely group theoretical arguments e.g. $\ydiagram{3,1} \longrightarrow \ydiagram{3} \oplus \ydiagram{2,1}$

More Block Structures

- Homology basis can be decomposed similarly $\longrightarrow$ same block structure as well as period matrix P and DEQ matrix Ω



All relevant information is captured in G-invariant (co-)homology groups H_G^n and $H_{n,G}$ with period pairing P_G

Some Consequences

- Observation: the group representation acting on canonical integrals (in the spirit of [Henn '13], [Primo, Tancredi '17], [Adams, Weinzierl '18] [Broedel, Duhr, Dulat, Penante, Tancredi '19]) is independent of the kinematics
  - → Decompose canonical intersection matrix into irreducibles by a constant rotation.
  - → Canonical intersection matrix is constant [Duhr, Porkert, Stawinski, CS '25]: It takes block structure for generic kinematic configurations even in the absence of symmetry. e.g. for 3-loop banana

$$\Delta = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & -\frac{1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{11}{12} \end{pmatrix}$$

Summary

- Generally $\dim(H^n) \geq \dim(\text{IBP master integrals})$
- For a cohomology H^n: Symmetry leaves twist and intersection pairings invariant

✓ **The G-invariant cohomology group H^n_G**

$\longrightarrow$ Decompose group action on differentials in terms of its irreducibles

$\longrightarrow$ Basis of H^n_G: Differentials $\vec{\varphi}_G$ that transform under the identity of G

$\longrightarrow$ The number of masters for sub-symmetries directly follows from the maximal symmetry

✓ **Blockstructure**

$\longrightarrow$ All pairings and the DEQ decompose into block-diagonal form with dimension of blocks given by the dimension of the irreps.

Outlook

- Can we understand the "canonical" symmetry better?
- [Erik's idea]: Can we find a topological interpretation? (Euler characteristic at the fixed point)
- Is there a twist associated to the G-invariant cohomology group?

Symmetry and Cohomolgy

Can be generalized to more complicated groups [Gasparotto, Weinzierl, Xu, 2023]:

- Sort the basis such that $\vec{\varphi} = \{\varphi_1, \ldots, \varphi_{N_O}, \ldots \varphi_N\}$
- With N_O the number of non-zero orbits such that the first N_O elements are in distinct orbits
- The new set of master integrands is given by

$$o_j = \frac{1}{|G|} \sum_{g \in G} R(g)\, \varphi_j \,, \text{ with } g \in G \text{ and } 1 \leq j \leq N_O$$

$$\int_\Gamma \Psi \rightarrow \varphi_j = \int_\Gamma \Psi o_j$$

- What happens over other contours/the cycles?
- How does the period matrix/the intersection paring transform?

Irreducible Representations

- A representation is completely reducible if $R = R_1 \oplus \cdots \oplus R_k$
$$MR(g)M^{-1} = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_k \end{pmatrix}, \tag{2}$$
- Character + dimension of a representation $\longrightarrow$ decomposition in terms of irreducible representations
- Schur's lemma if R_1 and R_2 of different dimension
$R_1(g)A = AR_2(g)\,, \quad \forall G\,, \longrightarrow A = 0.$

Blockstructure

Schur's Lemma

- $R_1 : G \longrightarrow \mathrm{GL}(V),\ R_2 : G \longrightarrow \mathrm{GL}(W)$ irreducible with $V \not\simeq W$
- Matrix A of $\dim W \times \dim V$ with $R_1(g)A = AR_2(g)\,, \quad \forall G\,,$

$\longrightarrow A = 0.$

Blockstructure

Schur's Lemma

- $R_1 : G \longrightarrow \mathrm{GL}(V)$, $R_2 : G \longrightarrow \mathrm{GL}(W)$ irreducible with $V \not\simeq W$
- Matrix A of dim $W \times$ dim V with $R_1(g)A = AR_2(g)\,, \quad \forall G\,,$

$\longrightarrow A = 0.$

For a basis change to $I_{\mathrm{sym}} = \left(\frac{1}{2}(I_1 + I_2), I_3, I_4, \frac{1}{2}(I_1 - I_2)\right)^T$ we get

$$R_{\mathrm{sym}} = \begin{array}{cc} & R_1 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & R_2 \end{array} \qquad C_{\mathrm{sym}} = \begin{array}{ccc} & C_{11} & \\ & \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix} & \begin{array}{c} C_{12} \\ \\ \\ C_{22} \end{array} \\ C_{21} & & \end{array}$$

This corresponds exactly to the irreps $R = \mathrm{id}_3 \oplus (-1)$

Blockstructure

Schur's Lemma

- $R_1 : G \longrightarrow \mathrm{GL}(V)$, $R_2 : G \longrightarrow \mathrm{GL}(W)$ irreducible with $V \not\simeq W$
- Matrix A of dim $W \times$ dim V with $R_1(g)A = AR_2(g)$, $\quad \forall G$,

$\longrightarrow A = 0$.

For a basis change to $I_{\text{sym}} = \left(\frac{1}{2}(I_1 + I_2), I_3, I_4, \frac{1}{2}(I_1 - I_2)\right)^T$ we get

$$R_{\text{sym}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad C_{\text{sym}} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix}$$

R_1 R_2 C_{11} C_{12} C_{21} C_{22}

This corresponds exactly to the irreps $R = \mathrm{id}_3 \oplus (-1)$

We had $C = R(g)CR(g)^T$, $\quad \forall g \in G$

$\longrightarrow$ For symmetric group $R^{-1} = R^T$ such that $R(g)C = CR(g)$

$$R(g)C = CR(g) \longrightarrow \begin{pmatrix} C_{1,1}R_1 & C_{1,2}R_1 \\ C_{2,1}R_2 & C_{2,2}R_2 \end{pmatrix} = \begin{pmatrix} C_{1,1}R_1 & C_{1,2}R_2 \\ C_{2,1}R_1 & C_{2,2}R_2 \end{pmatrix} \xrightarrow{\text{Schur}} C_{12} = C_{21} = 0$$