From Critical Points to Syzygies for Feynman Integrals

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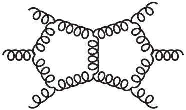
in collaboration with Qian Song

[arXiv:2509.17681]

Motivation

- Cutting-edge Feynman diagrams for colliders have gigantic integrand

 $\sim$ 100MB.

- Practical calculations reduce to basis of "master integrals":

$$\mathcal{I}_i = \sum_{j \in \text{masters}} c_{ij} \mathcal{I}_j .$$

Important question: how do we explicitly compute the c_{ij}?

Many talks focusing on this: [Gaia, Seva, Giacomo, Stefano, Rourou, Tiziano, Catherine]

Studying Feynman Integral Reduction

- In appropriate representation (e.g. Baikov), total derivatives vanish

$$\int \mathrm{d}\omega = 0.$$

[Tkachov, Chetyrkin '81]

- Feynman integrals live in (appropriate) cohomology groups:

$$\mathcal{I}_i \in H_\Gamma, \qquad H_\Gamma = \Omega^N/\mathrm{im}(\mathrm{d}).$$

- Two major strategies for integral reduction.

Indirect: "Integration by parts"

Explicitly construct $\mathrm{im}(\mathrm{d})$.

"Mod out" with linear algebra.

[Laporta '00]

Direct: "Intersection theory"

$$\mathcal{I}_k = \int \phi_k \quad \longrightarrow \quad \langle \phi_i \phi_j \rangle.$$

[Mastrolia, Mizera '18]

A Natural Question

Can the two approaches teach us about each other?

$$\text{``}\langle \phi_a \phi_b \rangle\text{''} \qquad \longleftrightarrow \qquad \text{``im}(\mathrm{d})\text{''?}$$

This talk: if we construct $\mathrm{im}(\mathrm{d})$ from syzygies, then yes!

Setup: Baikov Representation and Syzygies

Feynman Integrals in the Baikov Representation

$$\int_{\mathcal{C}} \mathrm{d}^N \vec{z} \left[B(\vec{s}, \vec{z})^{\gamma} \frac{\mathcal{N}(\vec{s}, \vec{z})}{\prod_{e \in \mathrm{props}(\Gamma)} z_e} \right].$$

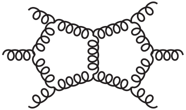
- Set of propagators in described by graph Γ, e.g. $\Gamma =$  .
- Baikov variables split into "propagators" (edges) and ISPs:

$$\{z_1, \ldots, z_N\} = \{z_e : e \in \mathrm{props}(\Gamma)\} \sqcup \{z_i : i \in \mathrm{ISPs}(\Gamma)\}.$$

- Function of kinematics $\vec{s}$, and regulator ϵ through $\gamma = \gamma_0 + \gamma_1 \epsilon$.
- Complexity in "Baikov polynomial", $\deg_z(B[\vec{s}, \vec{z}]) = 2 \times (\#\text{ loops})$.

Surface Terms in the Baikov Representation

- Physical integrands have restricted denominator. Live in Ω^N subspace.



$$= \int \mathrm{d}^{11} z \left[\frac{B(\vec{z})^\gamma \mathcal{N}(z_1, \ldots, z_{11})}{z_1 \cdots z_9} \right].$$

- Want corresponding subspace of $\mathrm{im}(\mathrm{d}) \Rightarrow$ “Surface terms”.

$$\mathrm{Surface}(\Gamma) = \left\{ \mathcal{S} \in R \;\; : \;\; \frac{B^\gamma \mathcal{S}}{\prod_{e \in \mathrm{props}(\Gamma)} z_e} = \partial_k \left[B^{\gamma - \Delta} \frac{a_k}{\prod_{e \in \mathrm{props}(\Gamma)} z_e^{\beta_e}} \right] \right\}.$$

[Ita '15]

- Total derivatives specified by polynomial $\mathcal{S}$ in

$$R = \mathbb{C}(p_i \cdot p_j, m_k^2, \epsilon)[z_1, \ldots, z_N].$$

Baikov Representation Syzygies and Surface Terms

- Organize surface terms by constructing them from "syzygy equation"

$$a_0 B + \sum_{e \in \text{props}(\Gamma)} \tilde{a}_e z_e B + \sum_{i \in \text{ISPs}(\Gamma)} a_i \partial_i B + \sum_{e \in \text{props}(\Gamma)} \bar{a}_e z_e \partial_e B = 0.$$

[Gluza, Kajda, Kosower '09; Ita '15; Zhang, Larsen '15]

- Set of solutions $\vec{a}$ form so-called "syzygy module", $\text{Syz}(\Gamma)$.

$$\lambda_i \in R, \quad \vec{a}_i \in \text{Syz}(\Gamma) \qquad \Rightarrow \qquad \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 \in \text{Syz}(\Gamma).$$

- Each element $\vec{a} \in \text{Syz}(\Gamma)$ corresponds to an $\mathcal{S} \in \text{Surface}(\Gamma)$:

$$S_\Gamma(\vec{a}) = a_0 + \sum_{e \in \text{props}(\Gamma)} \tilde{a}_e z_e - \frac{1}{\gamma} \left[\sum_{i \in \text{ISPs}(\Gamma)} \partial_i a_i + \sum_{e \in \text{props}(\Gamma)} (z_e \partial_e \bar{a}_e) \right].$$

Syzygies and Critical Points

An Intriguing Connection

- As $\epsilon \to \infty$, intersection numbers simplify: (NB, max-cut)

$$\langle \phi_a \phi_b \rangle = \sum_{\vec{z}_i : \mathrm{dlog}(B) = 0} \left. \frac{\hat{\phi}_a \hat{\phi}_b}{\det(\Phi)} \right|_{\vec{z}_i} + \mathcal{O}(\epsilon^{-1}).$$

[Mizera, Pokraka '19],

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- New observation: surface terms also simplify!

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- Max-cut a_0 is relevant term. Is piece of the syzygy.

"Calculus becomes algebra" in limit.

Critical Points

- Solution set of $\mathrm{d}\log(B) = 0$ specifies algebraic variety, $U^{\Gamma}_{\mathrm{crit}[\log(B)]}$,

$$\partial_i B = 0, \quad i \in \mathrm{ISPs}(\Gamma), \qquad z_e = 0, \quad e \in \mathrm{props}(\Gamma), \quad B \neq 0.$$

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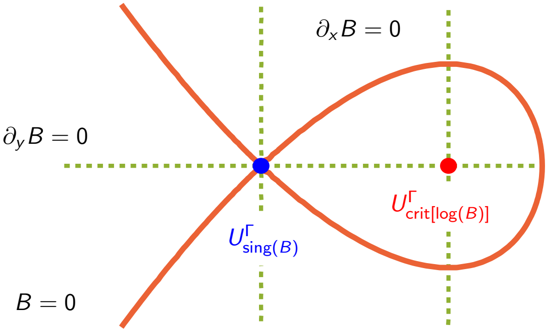
- As a toy example (implicitly on cut): $\qquad B = y^2 - x^2(1-x)$,



The Syzygy on Critical Points

- Let's consider syzygy equation on $U^{\Gamma}_{\text{crit}[\log(B)]}$

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- Simple vanishing condition for the a_0 term!

$$\Rightarrow a_0\Big|_{U^{\Gamma}_{\text{crit}[\log(B)]}} = 0.$$

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- Simple vanishing condition for the a_0 term!

$$\Rightarrow a_0 \big|_{U^{\Gamma}_{\text{crit}[\log(B)]}} = 0.$$

Questions:

- This is a necessary condition. But is it sufficient?

- Can we use this to construct syzygies?

$\Rightarrow$ We turn to the theory of ideals.

Ideal Basics

- Polynomial ideal is set of polynomial combinations of generators:

$$\langle p_1, \ldots p_n \rangle = \left\{ \sum_i a_i p_i \ : \ a_i \in \mathbb{C}[z_1, \ldots, z_N] \right\}.$$

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- Geometric operations on varieties correspond to operations on ideals:

intersection of ideals $\leftrightarrow$ union of varieties

ideal quotients/saturations $\leftrightarrow$ difference of varieties

$\ldots$

The Ideal of a_0 Terms

- The a_0 of a syzygy belongs to a set of terms, A_0^Γ.

$$
\begin{aligned}
A_0^\Gamma &= \left\{ a_0 \in R \;\;:\;\; a_0 B \in J_{\mathrm{syz}}^\Gamma \right\}, \\
J_{\mathrm{syz}}^\Gamma &= \langle \partial_i B \;:\; i \in \mathrm{ISPs}(\Gamma) \rangle + \langle z_e B, z_e \partial_e B \;:\; e \in \mathrm{props}(\Gamma) \rangle.
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- Geometrically, ideal quotient used to remove subvarieties.

$$V(J^\Gamma_{\mathrm{syz}} : \langle B \rangle^\mu) = \overline{V(J^\Gamma_{\mathrm{syz}}) \setminus V(\langle B \rangle)}.$$

- Saturation index μ describes multiplicity of $B = 0$ component of J^Γ_{syz}.

Simplifying the A_0^Γ Ideal

- Can "split" J_{syz}^Γ up, using B factors as

$$J_{\mathrm{syz}}^\Gamma = \underbrace{\left(\langle \partial_i B : i \in \mathrm{ISPs}(\Gamma)\rangle + \langle z_e : e \in \mathrm{props}(\Gamma)\rangle\right)}_{J_{\mathrm{crit}(B)}^\Gamma} \cap \underbrace{\left(J_{\mathrm{syz}}^\Gamma + \langle B^\mu \rangle\right)}_{J_{\mathrm{sing}}^{\subseteq\Gamma,\mu}}.$$

[BP, Song '25]

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$$\mu = 1 \qquad \Rightarrow \qquad A_0^\Gamma = J_{\text{crit}(B)}^\Gamma : \langle B^\mu \rangle.$$

- $\Rightarrow$ Vanishing condition is generally necessary, but not sufficient.

$$A_0^\Gamma \subseteq I(U_{\text{crit}[\log(B)]}).$$

Total Derivatives from Critical Points

A_0^Γ and Master Integral Counting

- The a_0 terms are intimately related to number of master integrals.

$$H_\Gamma = R/(\mathrm{Surface}[\Gamma] + \langle z_e : e \in \mathrm{props}[\Gamma]\rangle)\,.$$

- If $U^\Gamma_{\mathrm{crit}[\log(B)]}$ is set of points, they count number of master integrals:

$$\dim(U^\Gamma_{\mathrm{crit}[\log(B)]}) = 0 \quad \Rightarrow \quad \dim_{\mathbb{C}}(H_\Gamma) = \dim_{\mathbb{C}}(R/[J^\Gamma_{\mathrm{crit}(B)} : \langle B^\mu\rangle]).$$

[Lee, Pomeransky '13]

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[Lee, Pomeransky '13]

- If $\mu = 1$ max-cut surface terms in one-to-one correspondence with A_0^Γ!

$$A_0^\Gamma \simeq \mathrm{Surface}(\Gamma) + \langle z_e : e \in \mathrm{props}(\Gamma)\rangle.$$

[BP, Song '25]

Critical Syzygies

- Intuition: need only surface terms independent as $\epsilon \to \infty$, max-cut.

$$\text{Surface}(\Gamma)|_{\text{cut}} \quad \simeq \quad \lim_{\epsilon \to \infty} \text{Surface}(\Gamma)|_{\text{cut}}.$$

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- This gives us a natural equivalence relation on $\mathrm{Syz}(\Gamma)$.

$$\vec{a}_1 \; \sim \; \vec{a}_2 \quad \leftrightarrow \quad \lim_{\epsilon \to \infty} S_\Gamma(\vec{a}_1)|_{z_e=0} = \lim_{\epsilon \to \infty} S_\Gamma(\vec{a}_2)|_{z_e=0}.$$

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- Define "critical syzygies" as inequivalent syzygies under $\sim$.

$$\mathrm{CSyz}(\Gamma) = \mathrm{Syz}(\Gamma)/\!\sim .$$

Critical Surface Terms

- Set of critical syzygies is "much smaller" than standard syzygies:

$$\mathrm{CSyz}(\Gamma) \simeq \underbrace{A_0^\Gamma / \langle z_e : e \in \mathrm{props}(\Gamma) \rangle}_{\text{single-element, on-shell}} .$$

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- Motivated to build surface terms from critical syzygies

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$$\mathrm{CSyzSurface}(\Gamma) = \{ S_\Gamma(\vec{a}) : \vec{a} \in \mathrm{CSyz}(\Gamma) \}.$$

- By construction, if $\dim(U^\Gamma_{\mathrm{crit}[\log(B)]}) = 0$, and the multiplicity, $\mu = 1$,

$$\mathrm{CSyzSurface}(\Gamma) \simeq \mathrm{Surface}(\Gamma) / (\mathrm{Surface}(\Gamma) \cap \langle z_e : e \in \mathrm{props}(\Gamma) \rangle).$$

$\Rightarrow$ CSyzSurface(Γ) is complete up to surface terms from pinches.

Concrete Studies

Building CSyz(Γ), preliminaries.

- Goal: explicit generating set of CSyz(Γ).

$$\mathrm{CSyz}(\Gamma) = \langle \vec{a}^{(1)}, \ldots \rangle, \qquad \vec{a}_0^{(i)} \in A_0^\Gamma.$$

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$$U_{\mathrm{sing}}^\Gamma = \emptyset \qquad \Rightarrow \qquad A_0^\Gamma = J_{\mathrm{crit}(B)}^\Gamma.$$

[See also: Zhang '16]

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- In general, finding a generating set is hard. $\Rightarrow$ Go computational.

One Loop

- One-loop Baikov polynomial is a quadric $\Rightarrow$ general analysis tractable.

$$B(\vec{z}) = \frac{1}{2}(\vec{z}_i, \vec{z}_e, 1) \left(\begin{array}{ccc} \mathcal{H}_\Gamma & X & \vec{\mathcal{B}}_i \\ X^T & O & \vec{\mathcal{B}}_e \\ \vec{\mathcal{B}}_i^{\ T} & \vec{\mathcal{B}}_e^{\ T} & \mathcal{B}_0 \end{array} \right) \left(\begin{array}{c} \vec{z}_i \\ \vec{z}_e \\ 1 \end{array} \right).$$

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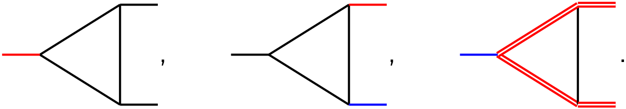
- If $\mathcal{H}_\Gamma$ is invertible, $B(\vec{z}) = 0$ is singular variety if

$$0 = B|_{U^\Gamma_{\mathrm{crit}(B)}} = \underbrace{\mathcal{B}_0 - \vec{\mathcal{B}}_i^{\,T} \mathcal{H}_\Gamma^{-1} \vec{\mathcal{B}}_i}_{\text{“Cayley determinant”}}.$$

- “Most cases” have Cayley $\neq 0$ (box, bubble, pentagon ...).

Singular Cases at One Loop

- Cayley$=0$ cases associated to IR divergent triangles



- Explicit calculation required, e.g. for massless triangle

$$S_\Gamma(\vec{a}) = -2s^2 - 8z_2(s + z_2 - [z_1 + z_3])\left(1 - \frac{1}{\gamma}\right) + \frac{s}{\gamma}(z_1 + z_3 + 2z_2).$$

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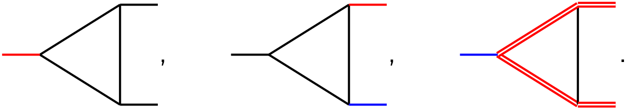

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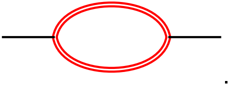
- $\mathcal{H}_\Gamma$ not invertible if Gram determinant is zero, e.g. $p^2 = 0$ bubble,



- In this case, $\dim(U^\Gamma_{\mathrm{crit}(\log(B))}) = 1 \Rightarrow$ critical syzygies insufficient.

Two-Loop

- At two loops, $B = 0$ is always singular on cut, i.e. $U^{\Gamma}_{\text{sing}} \neq \emptyset$.

- Four-dimensional locus is always (codim 3) sub-variety of U^{Γ}_{sing}.

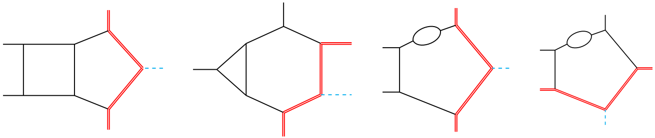
$$V(\langle \mu_{ij} \rangle) \subseteq U^{\Gamma}_{\text{sing}}, \qquad \mu_{ij} = G \begin{pmatrix} \ell_i & p_1 & p_2 & p_3 & p_4 \\ \ell_j & p_1 & p_2 & p_3 & p_4 \end{pmatrix}.$$

- $\Rightarrow$ Non-trivial critical syzygies are unavoidable. Hard problem!

- Here, we construct them computationally. [See paper for algorithm].

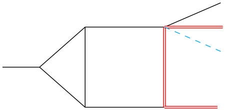
Leading-Color $pp \to t\bar{t}H$ Light Quark Loop

- Built critical surface terms* for all (sub-)topologies of



*Implemented in `Caravel` and checked against `FIRE`.

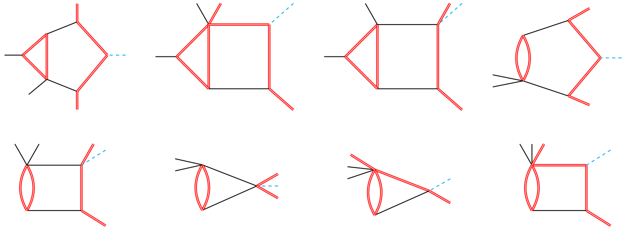
- $\dim(U^{\Gamma}_{\text{crit}[\log(B)]}) = 0$ everywhere. $\mu = 1$ almost everywhere.
- Single case where $\mu > 1$. (Interestingly: only 1 surface term missing).



- Insufficient surface terms from syzygies, needs "higher seeding".

Leading-Color $pp \to t\bar{t}H$ Heavy Quark Loop Pentabox

- A number of cases where $U^{\Gamma}_{\text{crit}[\log(B)]}$ is not zero-dimensional.



- Interestingly: $\dim(U^{\Gamma}_{\text{crit}[\log(B)]}) = 1$, maybe simple?

- Require analyzing syzygies with $a_0 = 0$, "sub-critical syzygies".

Conclusions

Summary:

- Large-ϵ limit surface terms vanish on critical points of the twist.
- Syzygy equation gives "lift" from large to finite ϵ.
- Critical syzygy formalism singles out minimal set of total derivatives.

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Outlook:

- Can we construct analytic CSyz(Γ) generators from geometry?
- Natural step: "sub-critical" syzygies, when $\dim(U^{\Gamma}_{\mathrm{crit}[\log(B)]}) > 0$.

Geometrical significance of J^{Γ}_{syz}

- $V(J^{\Gamma}_{\text{syz}})$ splits into subvarieties where $B = 0$, $B \neq 0$.

$$U^{\Gamma}_{\text{syz}} = \left[\bigcup_{\Gamma_k \subseteq \Gamma} U^{\Gamma_k}_{\text{sing}} \right] \cup U^{\Gamma}_{\text{crit}[\log(B)]}.$$

- First set of varieties correspond to singular loci of $B = 0$ on cuts.

$$B = 0, \qquad \partial_i B = 0 \;:\; i \in \text{ISPs}(\Gamma_k), \qquad z_e = 0 \;:\; e \in \text{props}(\Gamma_k).$$

- Second variety corresponds to critical locus of $\log(B)$ on max cut.

$$\partial_i \log(B) = 0 \;:\; i \in \text{ISPs}(\Gamma), \qquad z_e = 0 \;:\; e \in \text{props}(\Gamma).$$