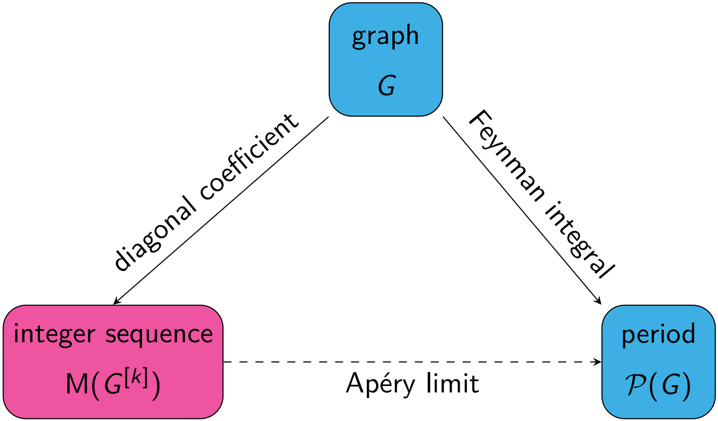
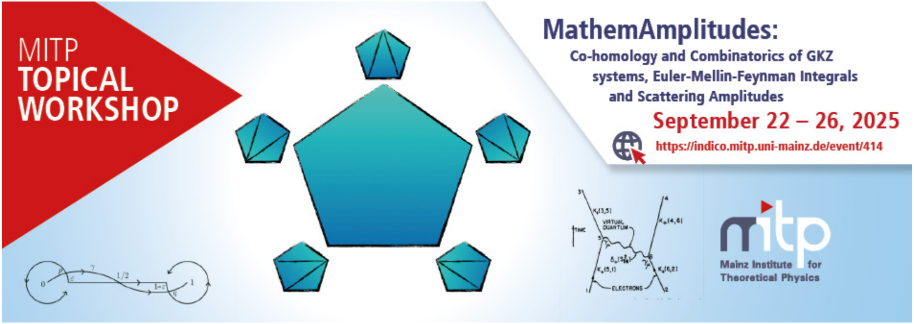
Combinatorial Feynman rules



based on 2304.05299 with **Karen Yeats** (Waterloo)
and work in progress with **Francis Brown** (Oxford)

Erik Panzer (Oxford) 26 September 2025
MITP Mainz



MathemAmplitudes: Co-homology and **Combinatorics of** GKZ systems, **Euler-Mellin-Feynman Integrals** and Scattering Amplitudes

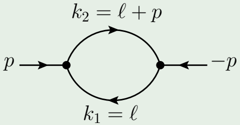
Analytic structure of Euler-Mellin integrals in regulators

[Berkesch–Forsgård–Nilsson–Passare '14] [Speer '69 for $\mathcal{I}_G$]

$$\mathcal{I}(s,z) = \int_{\mathbb{R}_+^N} \frac{x_1^{s_1}\cdots x_N^{s_N}}{f_1(x,z)^{s_{N+1}}\cdots f_M(x,z)^{s_{N+M}}} \frac{\mathrm{d}x_1\cdots\mathrm{d}x_N}{x_1\cdots x_N}$$

1. meromorphic function of $s \in \mathbb{C}^{N+M}$
2. interior of Newton polytopes ⇒ domain of convergence
3. facets of Newton polytopes ⇒ location of (simple) poles
4. restriction of f_i ⇒ residue as $N-1$ fold Euler-Mellin integral

Example



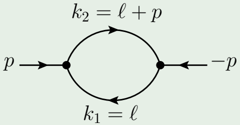
$$\mathcal{I}(s_1, s_2, D, p^2) = \int_{\mathbb{R}_+^2} \frac{x_1^{s_1} x_2^{s_2}}{\mathcal{G}(x, p^2)^{D/2}} \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2}$$

$$\mathcal{G}(x, p^2) = x_1 + x_2 + p^2 x_1 x_2$$

$$\mathrm{New}\,(\mathcal{G}) = \mathrm{conv}\left\{ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \right\}$$

$$= \{s_1 \leq 1\} \cap \{s_2 \leq 1\} \cap \{s_1 + s_2 \geq 1\}$$

Example



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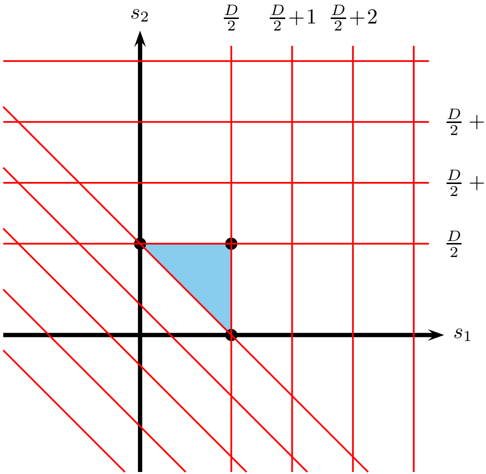


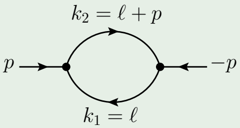
$$\mathcal{G}(x, p^2) = x_1 + x_2 + p^2 x_1 x_2$$

$$\mathrm{New}\,(\mathcal{G}) = \mathrm{conv}\left\{ \binom{0}{1}, \binom{1}{1}, \binom{1}{0} \right\}$$

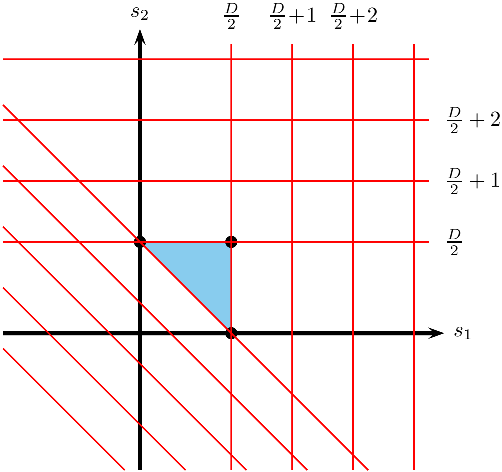
$$= \{s_1 \leq 1\} \cap \{s_2 \leq 1\} \cap \{s_1 + s_2 \geq 1\}$$

$$\mathrm{Poles}(\mathcal{I}) = \left\{ s_1 \in \frac{D}{2} + \mathbb{Z}_{\geq 0} \right\} \cup \left\{ s_2 \in \frac{D}{2} + \mathbb{Z}_{\geq 0} \right\} \cup \left\{ s_1 + s_2 \in \frac{D}{2} - \mathbb{Z}_{\geq 0} \right\}$$

Example



$$\mathcal{I}(s_1, s_2, D, p^2) = \frac{\Gamma(\frac{D}{2} - s_1)\Gamma(\frac{D}{2} - s_2)\Gamma(s_1 + s_2 - \frac{D}{2})}{(p^2)^{s_1+s_2-\frac{D}{2}}\Gamma(\frac{D}{2})}$$



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Analytic continuation: (assume $F(x)$ smooth at $x = 0$)

$$\underbrace{\int_0^\infty x^s \frac{\mathrm{d}x}{x} F(x)}_{\text{converges for } \mathrm{Re}(s) > 0} = \frac{1}{s(s+1)\cdots(s+k)} \underbrace{\int_0^\infty x^{s+k+1} \frac{\mathrm{d}x}{x} \left(-\frac{\partial}{\partial x}\right)^{k+1} F(x)}_{\text{converges for } \mathrm{Re}(s) > -k-1}$$

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Residues

$$\operatorname*{Res}_{s=-k} \left(\int_0^\infty x^s \frac{\mathrm{d}x}{x} F(x) \right) = \frac{1}{k!} \left(\frac{\partial}{\partial x} \right)^k_{x=0} F(x)$$

➡ $\mathrm{Res}^m(N\text{-fold Euler-Mellin } \mathcal{I}) = N - m$ fold Euler-Mellin integral

Question:

How to exploit this recursive structure?

Diagonal Mellin transform

$$\mathcal{I}(s) = \int_{\mathbb{R}_+^N} \frac{x_1^s \cdots x_N^s}{F(x)^s} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_N}{x_1 \cdots x_N}$$

➡ holomorphic on $s \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ (assume $(1, \ldots, 1) \in \mathrm{New}\,(F)^\circ$)

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Tropical limit

$$\mathcal{I}^{\mathrm{tr}} := \lim_{s \to 0} \left(s^N \mathcal{I}(s) \right) = \int_{\mathbb{R}_+^N} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_N}{F^{\mathrm{tr}}(x)}$$

Diagonal coefficients (deg $F = N$)

$$\lim_{s \to 0} \left(s^N \mathcal{I}(s-k) \right) = \mathcal{I}^{\mathrm{tr}} \cdot \left[x_1^k \cdots x_N^k \right] \left(F^k \right)$$

- $F = \sum_k c_k x^k \mapsto F^{\mathrm{tr}} = \max_{c_k \neq 0} x^k$

Diagonal Mellin transform

$$\mathcal{I}(s) = \int_{\mathbb{R}_+^N} \frac{x_1^s \cdots x_N^s}{F(x)^s} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_N}{x_1 \cdots x_N}$$

holomorphic on $s \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ (assume $(1, \ldots, 1) \in \mathrm{New}\,(F)^\circ$)

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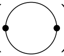
$$\lim_{s \to 0} \left(s^N \mathcal{I}(s-k) \right) = \mathcal{I}^{\mathrm{tr}} \cdot \left[x_1^k \cdots x_N^k \right] \left(F^k \right)$$

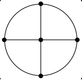
- $F = \sum_k c_k x^k \mapsto F^{\mathrm{tr}} = \max_{c_k \neq 0} x^k$
- $\lim_{\delta \to 0} \left(\delta^N \mathcal{I}(s_1 \delta, s_2 \delta, \ldots) \right) =$ canonical function $\in \mathbb{Q}(s_1, s_2, \ldots)$
- with kinematics: $z \mapsto z^{1/\delta}$

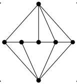
- $D = 4 - 2\varepsilon$
- log-divergent: $E = 2h_1$
- no subdivergences

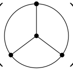
Feynman periods

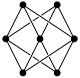
$$\mathcal{I}_G(D; \{\vec{p}_i \cdot \vec{p}_j, m_e^2\}) = \frac{\mathcal{P}(G)}{\varepsilon \cdot h_1} + \mathcal{O}\left(\varepsilon^0\right)$$

$\mathcal{P}\Big($  $\Big) = 1$

$\mathcal{P}\Big($  $\Big) = 20\zeta(5)$

$\mathcal{P}\Big($  $\Big) = \frac{1063}{9}\zeta(9) + 8\zeta(3)^3$

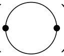
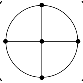
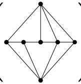
$\mathcal{P}\Big($  $\Big) = 6\zeta(3)$

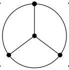
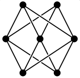
$\mathcal{P}\Big($  $\Big) = \frac{288}{5}\Big(58\zeta(8) - 24\zeta(3,5) - 45\zeta(3)\zeta(5)\Big)$

- $D = 4 - 2\varepsilon$
- log-divergent: $E = 2h_1$ ⇨
- no subdivergences

Feynman periods

$$\mathcal{I}_G(D; \{\vec{p}_i \cdot \vec{p}_j, m_e^2\}) = \frac{\mathcal{P}(G)}{\varepsilon \cdot h_1} + \mathcal{O}\left(\varepsilon^0\right)$$

$\mathcal{P}\Bigg($  $\Bigg) = 1$ $\mathcal{P}\Bigg($  $\Bigg) = 20\zeta(5)$ $\mathcal{P}\Bigg($  $\Bigg) = \frac{1063}{9}\zeta(9) + 8\zeta(3)^3$

$\mathcal{P}\Bigg($  $\Bigg) = 6\zeta(3)$ $\mathcal{P}\Bigg($  $\Bigg) = \frac{288}{5}\Big(58\zeta(8) - 24\zeta(3,5) - 45\zeta(3)\zeta(5)\Big)$

⇨ β function, critical exponents

⇨ > 1000 periods known [Broadhurst '85, &Kreimer '95, Schnetz '08]

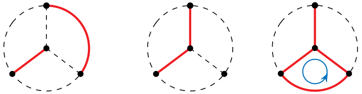
⇨ multiple polylogarithms at 2nd, 4th, 6th roots of unity

⇨ modular K3 surfaces [Brown&Schnetz '10]

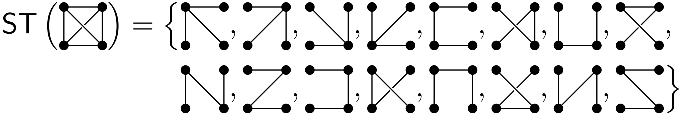
⇨ modular Calabi-Yau 3-folds [Logan '16]

Spanning trees (ST)

A spanning tree $T \subset G$ is a simply connected, spanning subgraph.



not connected   not spanning   has a loop



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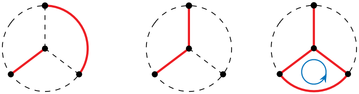
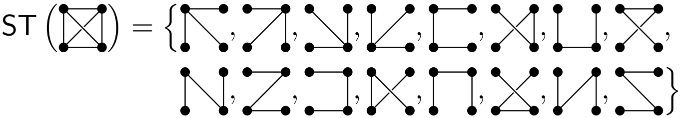




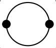
Symanzik polynomial

$$\mathcal{U}_G = \sum_{T \in \mathrm{ST}(G)} \prod_{e \notin T} x_e$$

Feynman period

$$\mathcal{P}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}_G^2 \big|_{x_1=1}}$$

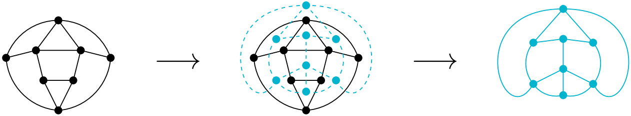
$\mathcal{U}$  $= x_1 + x_2$

$\mathcal{P}$ () $= \int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$

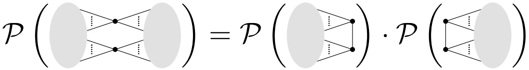
Identities

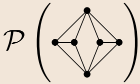
When is $\mathcal{P}(G_1) = \mathcal{P}(G_2)$?

1. **Planar duality:**



2. **Product:**





$= (6\zeta(3))^2$

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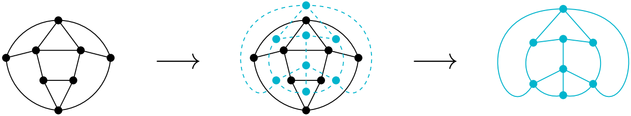

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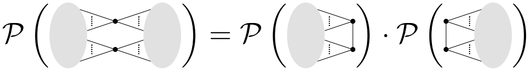
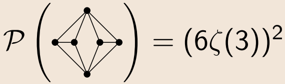
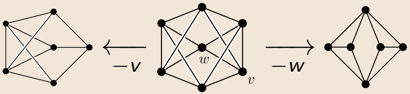
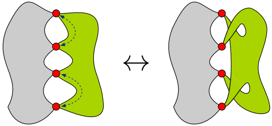




3. **Completion:** $\mathcal{P}(G \setminus v) = \mathcal{P}(G \setminus w)$



4. **Twist:**



Combinatorial Feynman rules

Goal:

Construct simpler graph invariants $G \mapsto F(G)$ with those symmetries.

$$\mathcal{P}(G_1) = \mathcal{P}(G_2) \;\Rightarrow\; F(G_1) = F(G_2)$$

1. point count $c_2(p^k)$ [Schnetz '09]
2. graph permanent [Crump&DeVos&Yeats '16]
3. Hepp bound $\mathcal{H}(G)$ [Panzer '19]
4. $\#\{$minimal 6-cuts$\}$ [Mercer&Panzer '21]
5. Martin sequence $\mathrm{M}(G^{[k]})$ [Panzer&Yeats '23]
6. Speyer's $g''(-1)$ [Panzer '25]

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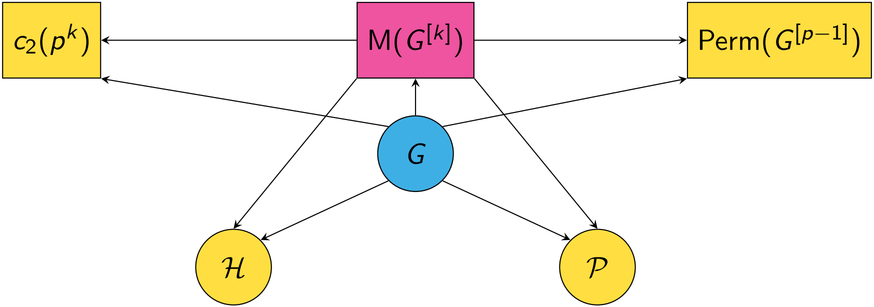
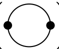


Hepp bound `arXiv:1908.09820`

$$\mathcal{H}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{(\mathcal{U}^{\mathrm{tr}})^2|_{x_1=1}}$$

$$\mathcal{U}^{\mathrm{tr}} = \max_{T \in \mathsf{ST}} \prod_{e \notin T} x_e$$

$\mathcal{H}\Big($  $\Big)$

$$= \int_0^\infty \frac{\mathrm{d}x_2}{(\max\{1, x_2\})^2}$$

$$= \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2}$$

$$= 2$$

Hepp bound arXiv:1908.09820

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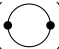
$\mathcal{H}\Big($  $\Big)$

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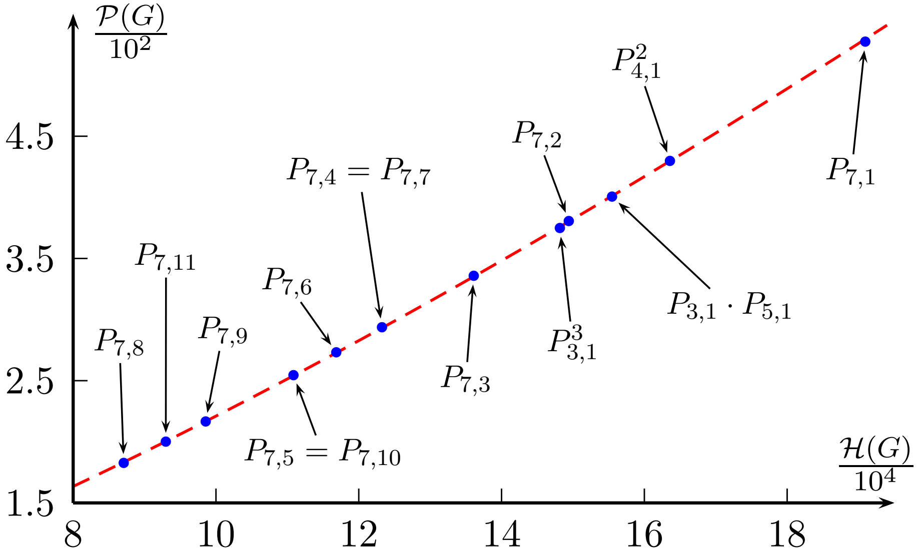
$$= \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2}$$

$$= 2$$

G	$\mathcal{P}(G \setminus v)$	$\mathcal{H}(G \setminus v)$
$P_{7,1}$	527.7	190952
$P_{4,1} \cdot P_{4,1}$	430.1	163592
$P_{3,1} \cdot P_{5,1}$	400.9	155484
$P_{7,2}$	380.9	149426
$P_{3,1} \cdot P_{3,1} \cdot P_{3,1}$	375.2	148176
$P_{7,3}$	336.1	136114
$\{P_{7,4}, P_{7,7}\}$	294.0	123260
$P_{7,6}$	273.5	116860
$\{P_{7,5}, P_{7,10}\}$	254.8	110864
$P_{7,9}$	216.9	98568
$P_{7,11}$	200.4	92984
$P_{7,8}$	183.0	87088

- all identities
- $\mathcal{H}(G) > \mathcal{P}(G) > \dfrac{\mathcal{H}(G)}{|\mathrm{ST}(G)|^2}$
- easy to compute



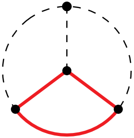
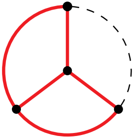
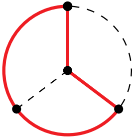
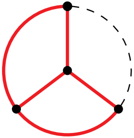
➡ efficient numerics (tropical Monte Carlo) for large Feynman integrals

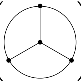
[Borinsky]

$$\omega(\gamma_\ell) = |\gamma| - 2\ell$$

Theorem

$$\mathcal{H}(G) = \sum_{\substack{\gamma_1\subset\gamma_2\subset\cdots\subset\gamma_\ell=G\\ \text{each }\gamma_i\text{ is 1PI}}} \frac{|\gamma_1|\cdot|\gamma_2\setminus\gamma_1|\cdots|G\setminus\gamma_{\ell-1}|}{\omega(\gamma_1)\cdots\omega(\gamma_{\ell-1})}$$

γ_1	$\subset$	γ_2	summand	#	$\sum$
	$\subset$		$\frac{3\cdot 2\cdot 1}{1\cdot 1} = 6$	12	72
	$\subset$		$\frac{4\cdot 1\cdot 1}{2\cdot 1} = 2$	6	12

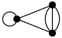
$\Rightarrow \mathcal{H}\Big($  $\Big) = 84$

Martin invariant [Bouchet&Ghier '96]

1.  $\subset G \Rightarrow \mathrm{M}(G) = 0$
2. $|V| = 3 \Rightarrow \mathrm{M}(G) = 1$
3. $\mathrm{M}($  $) = \mathrm{M}($  $) + \mathrm{M}($  $) + \mathrm{M}($  $)$

$\mathrm{M}\Big($  $\Big)$

$= 3 \cdot \mathrm{M}\Big($  $\Big)$

$= 3 \cdot \mathrm{M}\Big($  $\Big) + 6 \cdot \mathrm{M}\Big($  $\Big)$

$= 6$

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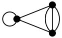
3. M() = M() + M() + M()

M ()

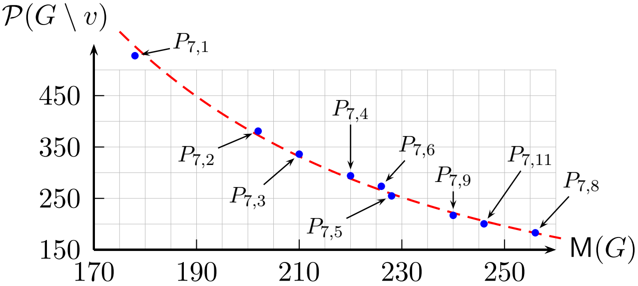
= 3 · M ()

= 3 · M () + 6 · M ()

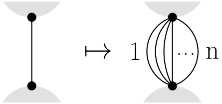
= 6

- easy to compute
- $\mathrm{M}(G) = O(-2)$ symmetry factor

Graph G	$\mathcal{P}(G \setminus v)$	$\mathrm{M}(G)$
$P_{7,1}$	527.7	178
$P_{7,2}$	380.9	202
$P_{7,3}$	336.1	210
$\{P_{7,4}, P_{7,7}\}$	294.0	220
$P_{7,6}$	273.5	226
$\{P_{7,5}, P_{7,10}\}$	254.8	228
$P_{7,9}$	216.9	240
$P_{7,11}$	200.4	246
$P_{7,8}$	183.0	256



$G^{[n]}$:



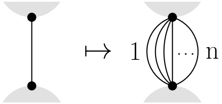
Martin sequence

$$A_n(G) := \frac{1}{(n!)^{2V-6}} \mathrm{M}\left(G^{[n]}\right)$$

- integral: $A_n \in \mathbb{Z}$ for all n
- P-recursive: $\exists p_i \in \mathbb{Z}[n]$ s.t.

$$\sum_{i=0}^{r} A_{n+i} p_i(n) = 0$$

$G^{[n]}$:



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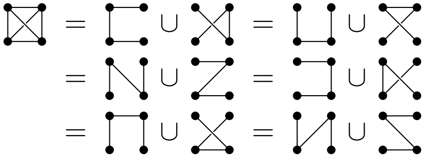
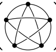
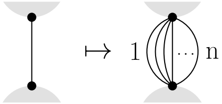
Theorem (Panzer&Yeats)

$$\mathrm{M}(G) = \# \left\{ \begin{array}{c} \text{partition edges of } G \setminus v \\ \text{into } k \text{ spanning trees} \end{array} \right\}$$



- $A_1 = \mathrm{M}\Big($  $\Big) = 6$

$G^{[n]}$:



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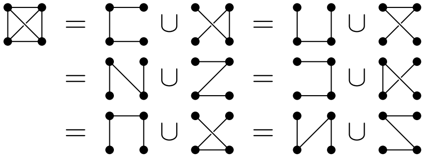
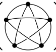
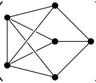
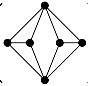


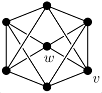
$A_1 = \mathrm{M}\left(\right.$  $\left.\right) = 6$

Corollary (diagonal coefficient)

$$A_r(G) \cdot \binom{2r}{r} = [x_1^r \cdots x_N^r] \, \mathcal{U}_{G \setminus v}^{2r}$$

- RHS independent ov v

$\#\mathrm{ST}$ () $= 135 \neq 128 = \#\mathrm{ST}$ ()

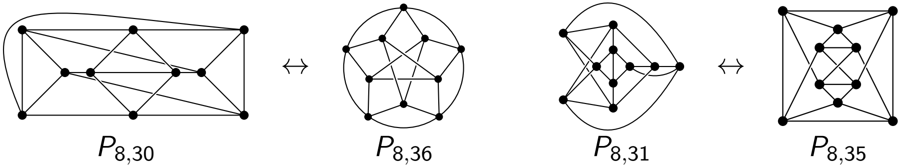
M (

) $= 36$

Further identities of Feynman integrals?

- $\mathcal{H}(P_{8,30}) = \frac{1724488}{3} = \mathcal{H}(P_{8,36})$
- $\mathcal{P}(P_{8,30}) \approx 505.5 \approx \mathcal{P}(P_{8,36})$ [Borinsky]
- $A_r(P_{8,30}) = A_r(P_{8,36})$ for $r = 1, \ldots, 8$

Conjecture: $\mathcal{P}(P_{8,30}) = \mathcal{P}(P_{8,36})$ and $\mathcal{P}(P_{8,31}) = \mathcal{P}(P_{8,35})$



Apéry limits

$$\sum_k A_{n+k} \cdot p_k(n) = 0 \qquad \mapsto \qquad \lim_{n\to\infty} \frac{B_n}{A_n}$$

Recurrence

$$n^4 \cdot u_n = (65n^4 - 130n^3 + 105n^2 - 40n + 6)u_{n-1} - 4(n-1)^2(4n-3)(4n-5)u_{n-2}$$

Particular solutions:

1. $(A_n)_{n\geq 0} = (1, 6, 126, 3948, 149310, \ldots)$
2. $(B_n)_{n\geq 0} = (0, 1, \frac{173}{8}, \frac{73219}{108}, \frac{4922855}{192}, \ldots)$

Recurrence AESZ#28

$$n^4 \cdot u_n = (65n^4 - 130n^3 + 105n^2 - 40n + 6)u_{n-1} - 4(n-1)^2(4n-3)(4n-5)u_{n-2}$$

Particular solutions:

1. $(A_n)_{n\geq 0} = (1, 6, 126, 3948, 149310, \ldots)$ $\quad A_n = \sum_{i,j=0}^{n} \binom{n}{i}^2 \binom{n}{j}^2 \binom{i+j}{n}^2$
2. $(B_n)_{n\geq 0} = (0, 1, \frac{173}{8}, \frac{73219}{108}, \frac{4922855}{192}, \ldots)$

Miracle 1

$A_n \in \mathbb{Z}$ for all n

Miracle 2

$$\lim_{n\to\infty} \frac{B_n}{A_n} = \frac{1}{7}\zeta(3)$$

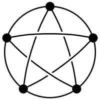
Applications:

- rational approximation
- mirror symmetry
- connects combinatorics with periods

  ⇨ *"A positive integer sequence counts something."*

Martin sequence

$$A_n := \frac{1}{(n!)^{2V-6}} \mathrm{M}\left(G^{[n]}\right)$$

$G =$ 

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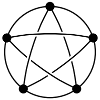
$G =$ 

$$A_n = \sum_{i,j=0}^{n} \binom{n}{i}^2 \binom{n}{j}^2 \binom{i+j}{n}^2 = (1, 6, 126, \ldots)$$

Conjecture 1

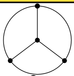
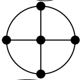
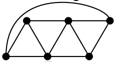
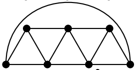
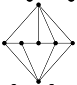
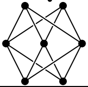
$\mathcal{P}(G \setminus v)$ determined by $(A_n)_{n\geq 1}$.

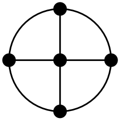
Conjecture 2

$\mathcal{P}(G \setminus v)$ is an Apéry limit of $(A_n)_{n\geq 1}$.

$\mathcal{P}(G \setminus v) = \mathcal{P}\Big($  $\Big) = 6\zeta(3) = \lim\limits_{n\to\infty} \dfrac{42 B_n}{A_n}$

$$\sum_{i=0}^{r} A_{n+i} p_i(n) = 0$$

graph G	deg p_i	r	#limits	Feynman period $\mathcal{P}(G)$
	4	2	1	$6\zeta(3)$
	9	3	2	$20\zeta(5)$
	44	6	3	$\frac{441}{8}\zeta(7)$
	15	3	2	$168\zeta(9)$
	136	10	5	$\frac{1063}{9}\zeta(9) + 8\zeta(3)^3$
	29	4	3	$\frac{288}{5}(58\zeta(8) - 24\zeta(3,5) - 45\zeta(3)\zeta(5))$



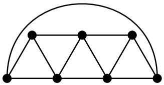
$(r = 3,\ \deg = 9)$

$$\begin{aligned}
0 = &-A_n \cdot n^6(220n^3 - 1089n^2 + 1804n - 1000)\\
&+A_{n-1} \cdot \big(109120n^9 - 867504n^8 + 2944040n^7 - 5614916n^6 + 6680560n^5\\
&\qquad - 5192600n^4 + 2658484n^3 - 869726n^2 + 165672n - 14040\big)\\
&+A_{n-2} \cdot \big(-436480n^9 + 4779456n^8 - 22975392n^7 + 63561032n^6\\
&\qquad - 111388080n^5 + 128093460n^4 - 96583656n^3 + 46031076n^2\\
&\qquad - 12588784n + 1506960\big)\\
&+A_{n-3} \cdot 4(4n-7)(4n-9)(220n^3 - 429n^2 + 286n - 65)(n-2)^4
\end{aligned}$$

Apéry limits

$$\mathbb{Q} \cdot \zeta(3) + \mathbb{Q} \cdot \zeta(5)$$

similar to [Zudilin]



$(r = 3,\ \deg = 15)$

$$
\begin{aligned}
0 = &-4A_n \cdot \Big((4n+3)(4n+5)(n+1)^8(25308n^5 + 298775n^4 + 1412460n^3 + 3342450n^2 + 3959240n + 1878054)\Big)\\
&+4A_{n+1} \cdot \begin{pmatrix} 2542239216n^{15} & +68146134540n^{14} & +849559488776n^{13} & +6534425534877n^{12} \\ +34679822409004n^{11} & +134532598771509n^{10} & +394103528221152n^{9} & +887818139797959n^{8} \\ +1550815354095216n^{7} & +2100644109315855n^{6} & +2188654694832820n^{5} & +1722670844465451n^{4} \\ +991612492036488n^{3} & +394125208734390n^{2} & +96726402422744n & +11050937494611 \end{pmatrix}\\
&-2A_{n+2} \cdot \begin{pmatrix} 373242384n^{15} & +11871181380n^{14} & +175529005034n^{13} & +1600239292249n^{12} \\ +10057451334808n^{11} & +46149899390390n^{10} & +159686953534620n^{9} & +424196553668790n^{8} \\ +872048772460200n^{7} & +1387115829582870n^{6} & +1692983204143564n^{5} & +1556787478598090n^{4} \\ +1043920355787024n^{3} & +481865853026354n^{2} & +136900787609638n & +18045933253725 \end{pmatrix}\\
&+A_{n+3} \cdot \Big((n+3)^{10}(25308n^5 + 172235n^4 + 470440n^3 + 644640n^2 + 443160n + 122271)\Big)
\end{aligned}
$$

Apéry limits

$$\mathbb{Q} \cdot \zeta(5) + \mathbb{Q} \cdot \zeta(9)$$

backups

Diagonals and Mellin transforms

$$A_r \propto \left(\prod_{e=2}^{N} \frac{1}{2\mathrm{i}\pi} \oint \frac{\mathrm{d}x_e}{x_e} \right) \left(\frac{\mathcal{U}_{G\setminus v}^2}{x_1 \cdots x_N} \right)^r$$

Mellin transform

$$I_G(s) = \left(\prod_{e=2}^{N} \int_0^\infty x_e^{s-1} \mathrm{d}x_e \right) \frac{1}{\mathcal{U}_G^{2s}|_{x_1=1}}$$

- meromorphic function of $s \in \mathbb{C}$
- uniquely determined by the sequence of *Mellin moments*

$$I_G(1) = \mathcal{P}(G) = \int_0^\infty \frac{\mathrm{d}x}{\mathcal{U}_G^2}, \quad I_G(2) = \int_0^\infty \frac{x_1 \cdots x_N}{\mathcal{U}_G^2} \frac{\mathrm{d}x}{\mathcal{U}_G^2}, \quad I_G(3), \quad \ldots$$

Mellin transform

$$I_G(s) = \left(\prod_{e=2}^{N} \int_0^\infty x_e^{s-1} \mathrm{d}x_e \right) \frac{1}{\mathcal{U}_G^{2s}|_{x_1=1}}$$

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$$\begin{tikzcd}[column sep=small]
& \boxed{\displaystyle\int_\sigma \frac{\mathrm{d}x_2 \cdots \mathrm{d}x_N}{x_1 \cdots x_N - t\mathcal{U}_{G\setminus v}^2}} \arrow[dl, "\sigma = (S^1)^{N-1}"', "|t| \to 0"] \arrow[dr, "\sigma = [0,\infty)^{N-1}", "|t| \to \infty"'] & \\
\boxed{\displaystyle(2\mathrm{i}\pi)^{N-1} \sum_{r \geq 0} A_r \binom{2r}{r} t^r} & & \boxed{\displaystyle-\sum_{s=1}^{\infty} I_{G\setminus v}(s) t^{-s}}
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=small]
& \boxed{\displaystyle\int_\sigma \frac{\mathrm{d}x_2\cdots\mathrm{d}x_N}{x_1\cdots x_N - t\mathcal{U}^2_{G\setminus v}}} \arrow[dl, "\sigma=(S^1)^{N-1}"', "|t|\to 0"] \arrow[dr, "\sigma=[0,\infty)^{N-1}", "|t|\to\infty"'] & \\
\boxed{\displaystyle(2\mathrm{i}\pi)^{N-1}\sum_{r\geq 0} A_r\binom{2r}{r}t^r} & & \boxed{\displaystyle-\sum_{s=1}^{\infty} I_{G\setminus v}(s)t^{-s}}
\end{tikzcd}
$$

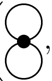
Claim (Brown & Panzer)

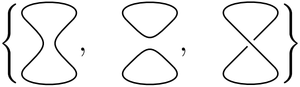
$$\sum_{i=0}^{r} p_i(n)A_{n+i}=0 \qquad \Rightarrow \qquad \sum_{i=0}^{r} p_{r-i}(-n)I_G(n+i)=0$$

$$\lambda_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$

Circuit partition polynomial

$$J(G,N) = \sum_{i_1,\ldots,i_E=1}^{N} \prod_v \lambda_{i_v^{(1)} i_v^{(2)} i_v^{(3)} i_v^{(4)}}$$
$$= \sum_P N^{\#\text{circuits}(P)}$$

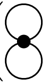
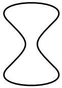
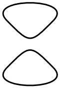
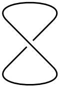
$J\Big($  $, N\Big) = \sum_{i,j=1}^{N} \lambda_{iijj} = N(N+2)$

$P \in$ 

$$\lambda_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$

Circuit partition polynomial

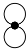
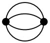
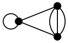
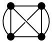
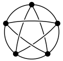
$$J(G,N) = \sum_{i_1,\ldots,i_E=1}^{N} \prod_v \lambda_{i_v^{(1)} i_v^{(2)} i_v^{(3)} i_v^{(4)}}$$
$$= \sum_P N^{\#\text{circuits}(P)}$$

$J\Big($  $, N\Big) = \sum_{i,j=1}^{N} \lambda_{iijj} = N(N+2)$ $\qquad P \in \Big\{$ , ,  $\Big\}$

Martin polynomial

$$J(G,N) = N \cdot \mathfrak{m}(G, N+2)$$

[Martin '77]
[Las Vergnas '83]

G							
$\mathfrak{m}(G,x)$	x	$3x$	x^2	x^2+6x	$3x^2$	$5x^2+12x$	$15x^2+36x$

$$J'(G,0) = \mathfrak{m}(G,2) = \#\{\text{Eulerian circuits}\}$$

$$\lambda_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$

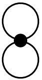
Circuit partition polynomial

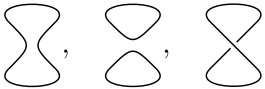
$$J(G,N) = \sum_{i_1,\dots,i_E=1}^{N} \prod_v \lambda_{i_v^{(1)} i_v^{(2)} i_v^{(3)} i_v^{(4)}} = \sum_P N^{\#\text{circuits}(P)}$$

$O(N)$ vector model

$$\mathscr{L} = \tfrac{1}{2}\sum_{i=1}^N \left(\|\nabla\phi_i\|^2 + m^2\phi_i^2\right) + \tfrac{g}{4!}\|\phi\|^4$$

$$\|\phi\|^4 = (\phi_1^2 + \dots + \phi_N^2)^2 = \frac{1}{3}\sum_{i,j,k,l=1}^{N} \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l$$

$J\Big($  $, N\Big) = \sum_{i,j=1}^{N} \lambda_{iijj} = N(N+2)$

$P \in \Big\{$  $\Big\}$

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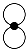
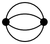
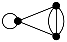
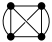
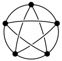
G							
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For an arbitrary Euler-Mellin integral $\mathcal{I}$:

- shifts $s_i \mapsto s_i \pm 1$ span vector space $\qquad D \to D \pm 2$

$$\mathfrak{M} := \sum_{\delta \in \mathbb{Z}^{N+M}} \mathbb{k} \cdot \mathcal{I}(s+\delta, z) \qquad \mathbb{k} = \mathbb{C}(s, z)$$

- $\mathfrak{M}$ is closed under $\partial / \partial z_i$
- $\#\{\text{master integrals}\} = \dim_{\mathbb{k}} \mathfrak{M} < \infty$ [Loeser–Sabbah '91] [Smirnov–Petukhov '11 for $\mathcal{I}_G$]

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[Smirnov–Petukhov '11 for $\mathcal{I}_G$]

Reduction problem:

- given: basis $\mathfrak{m} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_r)$ of $\mathfrak{M}$ and $\delta \in \mathbb{Z}^{N+M}$
- wanted: $\lambda_1, \ldots, \lambda_r \in \Bbbk$ such that

$$I(z, s+\delta) = \lambda_1 \cdot \mathfrak{m}_1 + \cdots + \lambda_r \cdot \mathfrak{m}_r$$

Theorem (Loeser–Sabbah, Bitoun–Bogner–Klausen–Panzer)

$$\dim_{\Bbbk} \mathfrak{M} = (-1)^N \chi\left((\mathbb{C}^*)^N \setminus \{f_1 \cdots f_M = 0\} \right)$$

Theorem (Sabbah)

$\exists$ basis $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of $\mathfrak{M}$ such that all $A_i \in \mathrm{GL}_r(\mathbb{k})$ have linear poles in s

$$\tau_i \mathfrak{m} = A_i \mathfrak{m}$$

Problem:

How to construct such a basis?

Regulators (D, s):

- difference equations (IBP)
- meromorphic (single-valued)
- simple poles on hyperplanes (linear)

Kinematics $z = (\{m_i^2\}, \{p_i \cdot p_j\})$:

- differential equations
- multi-valued
- Landau variety (non-linear)

Properties of $\mathcal{I}(s, z)$: [Berkesch–Forsgård–Nilsson–Passare '14]

1. converges in a non-empty, open cone of s [Speer '69 for $\mathcal{I}_G$]
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Theorem

Let $f_i(x,z) = \sum_{i,n} c_{i,n}(z) x^n$ completely nonvanishing (e.g. $c_{i,n} \in \mathbb{R}_{\geq 0}$).

1. *Set* $\mathrm{New}\,(f_i) = \mathrm{conv}\{n \in \mathbb{Z}^N \colon c_{i,n} \neq 0\}$. *Then $\mathcal{I}(s,z)$ converges in*

$$(\mathrm{Re}\, s_1, \ldots, \mathrm{Re}\, s_N) \in int\left(\sum_{i=1}^{M} (\mathrm{Re}\, s_{N+i}) \cdot \mathrm{New}\,(f_i)\right)$$

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2. *Set* $\mathbf{n}_\sigma \in \mathbb{Z}^{N+M}$ *so* $\sum_i s_{N+i} \cdot \operatorname{New}(f_i) = \bigcap_\sigma \{s \cdot \mathbf{n}_\sigma \leq 0\}$. *Then*
$$\operatorname{Poles}(\mathcal{I}(s)) \subseteq \bigcup_{\sigma} \bigcup_{k=0}^{\infty} \{s \colon s \cdot \mathbf{n}_\sigma = k\}$$

Graph polynomials and their Newton polytopes:

- Symanzik polynomials $\mathcal{G}(x,z) = \mathcal{U}(x) + \mathcal{F}(x,z)$:

$$\mathcal{U} = \sum_{\substack{\text{spanning}\\ \text{tree } T}} \prod_{e \notin T} x_e \qquad \mathcal{F} = \mathcal{U} \cdot \sum_{e} m_e^2 x_e - \sum_{\substack{\text{2-forests}\\ F = T_1 \sqcup T_2}} \prod_{e \notin F} x_e \sum_{\substack{i \in V(T_1)\\ j \in V(T_2)}} p_i \cdot p_j$$

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- New $(\mathcal{U})$ is a (graphical) matroid polytope;
  facets $\leftrightarrow$ subgraphs $\gamma\subset E(G)$ s.t. γ and G/γ biconnected

$$\partial_\gamma \mathrm{New}\,(\mathcal{U})=\mathrm{New}\,(\mathcal{U})\cap\{s\cdot\mathbf{n}_\gamma=h_1(\gamma)\},\qquad (\mathbf{n}_\gamma)_e=\begin{cases}1 & \text{if } e\in\gamma,\\ 0 & \text{else}\end{cases}$$

- (UV-)factorization: $\partial_\gamma \mathrm{New}\,(\mathcal{G}_G)=\mathrm{New}\,(\mathcal{U}_\gamma)\times\mathrm{New}\left(\mathcal{G}_{G/\gamma}\right)$

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Non-degenerate kinematics (Brown)

- facets of $\mathrm{New}\,(\mathcal{F}) \leftrightarrow$ certain "motic" subgraphs γ
- (IR-)factorization: $\partial_\gamma \mathrm{New}\,(\mathcal{G}_G) = \mathrm{New}\,(\mathcal{G}_\gamma) \times \mathrm{New}\left(\mathcal{U}_{G/\gamma}\right)$