Refined Genealogical Constraints

Maria Polackova

Minimal cuts and Genealogical Constraints

arXiv:2406.05943

In collaboration with Hofie Hannesdottir, Luke Lippstreu, and Andrew McLeod

Refining Genealogical Constraints

Upcoming paper

In collaboration with Giulio Crisanti, Luke Lippstreu, and Andrew McLeod

Bootstrapping Feynman integrals:

- Compute the symbol since it preserves all analytical properties of the Feynman integral up to transcendental constants

$$S(I) = \sum c_{i_1,i_2,\ldots,i_n} \lambda_{i_1} \otimes \cdots \otimes \lambda_{i_n}$$

$$\lambda = \sum_i P_i Q_i^{\nu_i}$$

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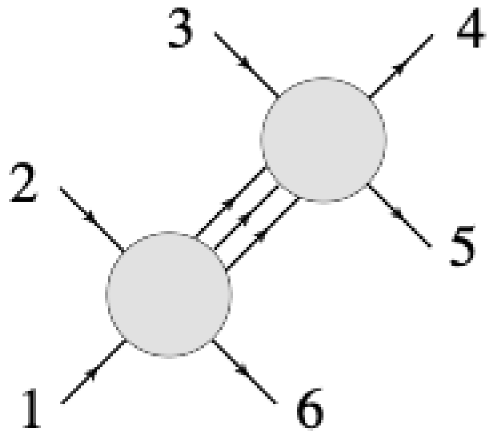
- For this we want to know all singularities using algorithms like PLD – Mizera, Fevola, Telen (2023) or Sofia – Correia, Giroux, Mizera (2025)
- Then we can use Landau Bootstrap – Hannesdottir, McLeod, Schwartz, Vergu (2024) to find the Symbol

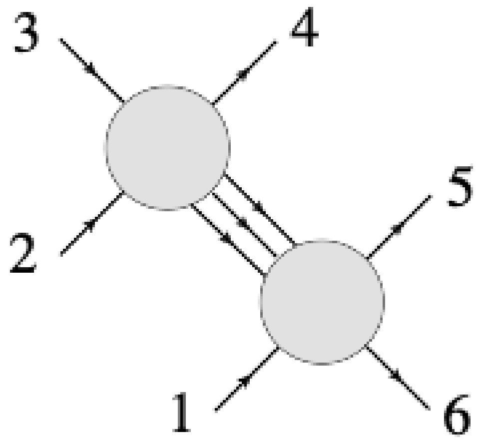
Constraints on the letter's sequences

- Steinmann relations:

$$\mathrm{Disc}_{s_{\mathcal{J}}}\mathrm{Disc}_{s_{\mathcal{L}}} I = 0 \text{ for } \mathcal{J} \not\subset \mathcal{L} \wedge \mathcal{L} \not\subset \mathcal{J} \wedge \mathcal{J} \cap \mathcal{L} \neq 0$$



vs.



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To place hierarchical constraints, we need to figure out how letters constrain the Feynman integral space

Landau equations

Landau (1959)

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$$I_F(p_i, m_i) \propto \int_0^\infty \frac{d\,\alpha_1 \ldots d\,\alpha_E}{\mathcal{G}^{\frac{D}{2}}}.$$

$$\mathcal{G} = U + F = (\alpha_i - r_1)(\alpha_i - r_2) \ldots (\alpha_i - r_n)$$

$\alpha_i = 0$ or $\frac{\partial \mathcal{G}}{\partial \alpha_i} = 0$ for $i = 1, \ldots, E$

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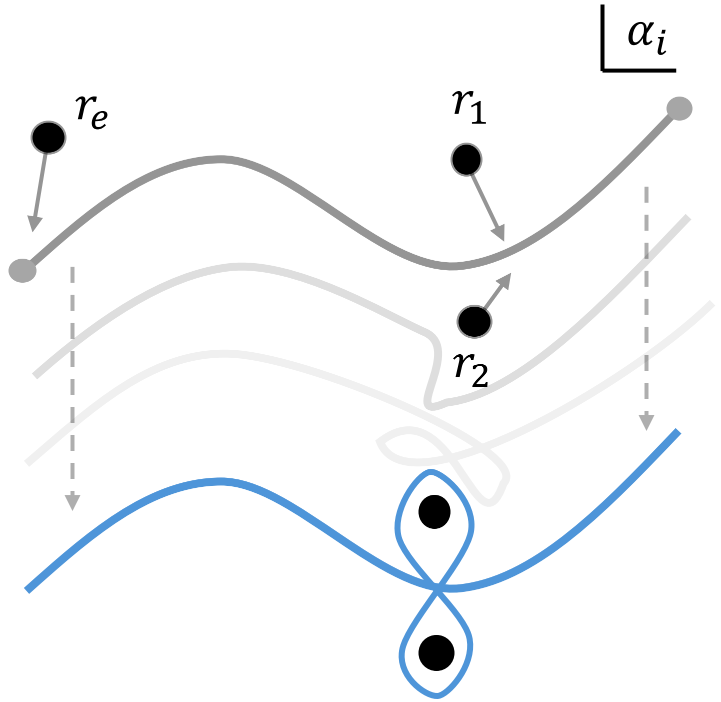

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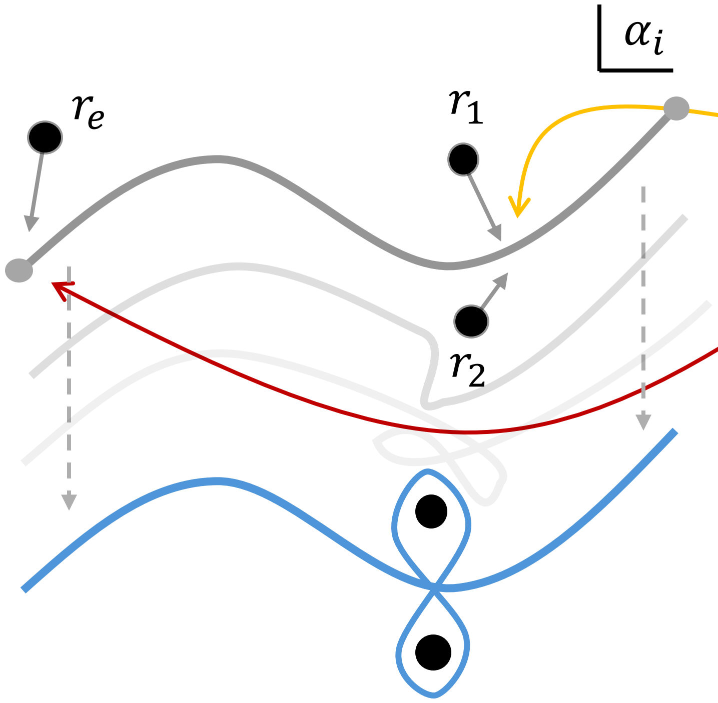

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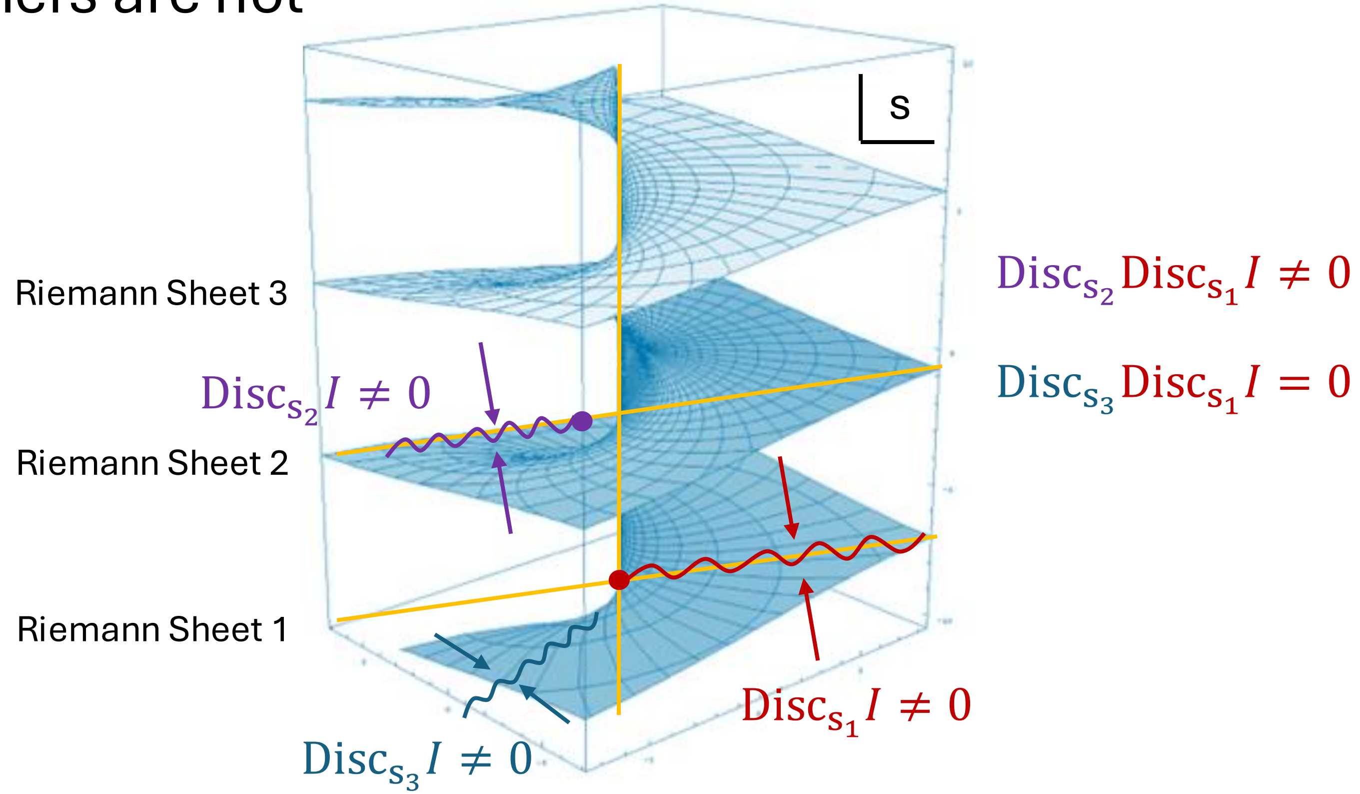
Discontinuities

- If we know the solutions to Landau equations for each letter λ_i, we know its cut
- we compute discontinuities in a sequence and constraining the kinematic variety increasingly
- Some sequences are allowed, others are not



Discontinuities rules: Hierarchical constraints

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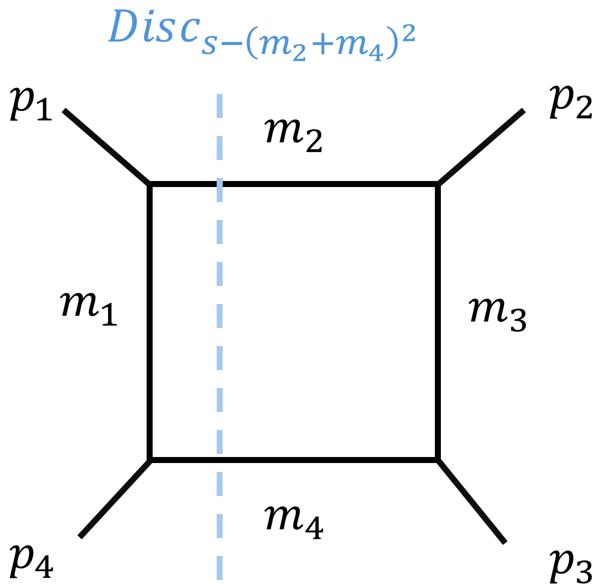

Cutkosky's rules:

Cutkosky (1960)

$$\mathrm{Disc}_{s-(m_2+m_4)^2} I(p_i) \propto \int_0^\infty \frac{\mathrm{d}^D k \; \boxed{\delta(p_2^2 - m_2^2)\delta(p_4^2 - m_4^2)}}{(p_1^2 - m_1^2)(p_3^2 - m_3^2)}$$

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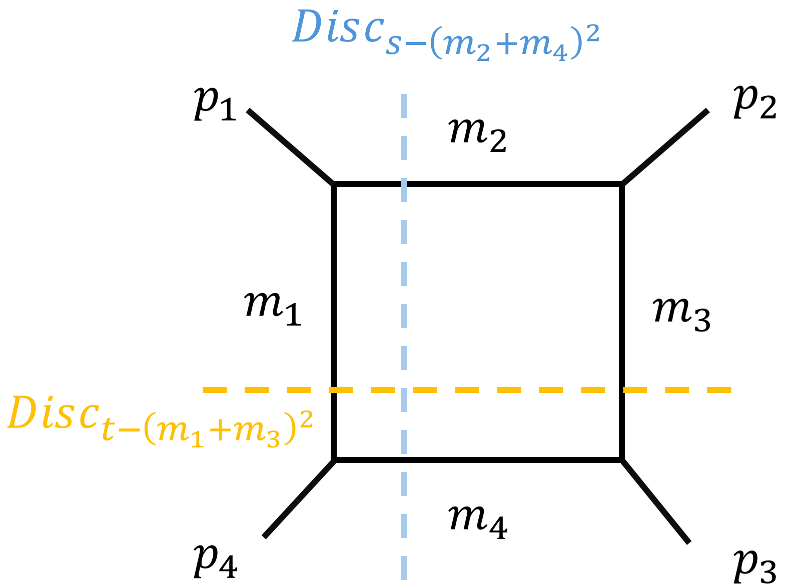

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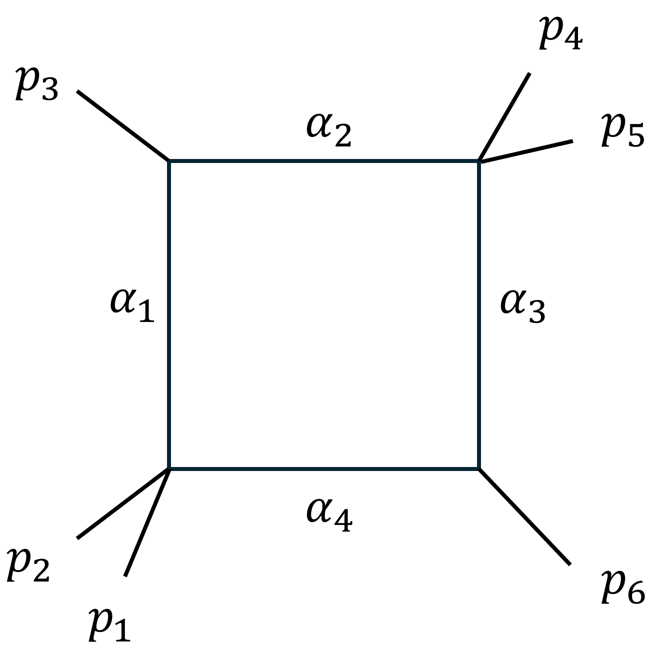
$$\mathrm{Disc}_{t-(m_1+m_3)^2} \mathrm{Disc}_{s-(m_2+m_4)^2} I(p_i) = 0$$

Hierarchical principle: 2-mass easy box



$\xrightarrow{\text{Cuts}}$

#	Singularity	Onshell lines
1	s_{123}	$\alpha_2,\ \alpha_4$
2	s_{345}	$\alpha_1,\ \alpha_3$
3	s_{12}	$\alpha_1,\ \alpha_4$
4	s_{45}	$\alpha_2,\ \alpha_3$
5	$s_{12}-s_{123}$	$\alpha_1,\ \alpha_2,\ \alpha_4$
6	$s_{12}-s_{345}$	$\alpha_1,\ \alpha_3,\ \alpha_4$
7	$s_{123}-s_{45}$	$\alpha_2,\ \alpha_3,\ \alpha_4$
8	$s_{345}-s_{45}$	$\alpha_1,\ \alpha_2,\ \alpha_3$
9	$s_{12}s_{45}-s_{123}s_{345}$	$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$
10	$s_{12}-s_{123}-s_{345}+s_{45}$	$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$

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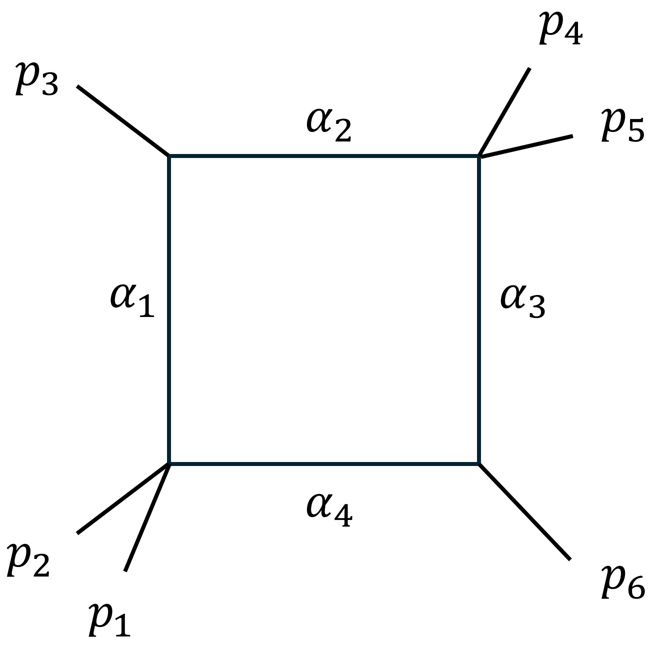
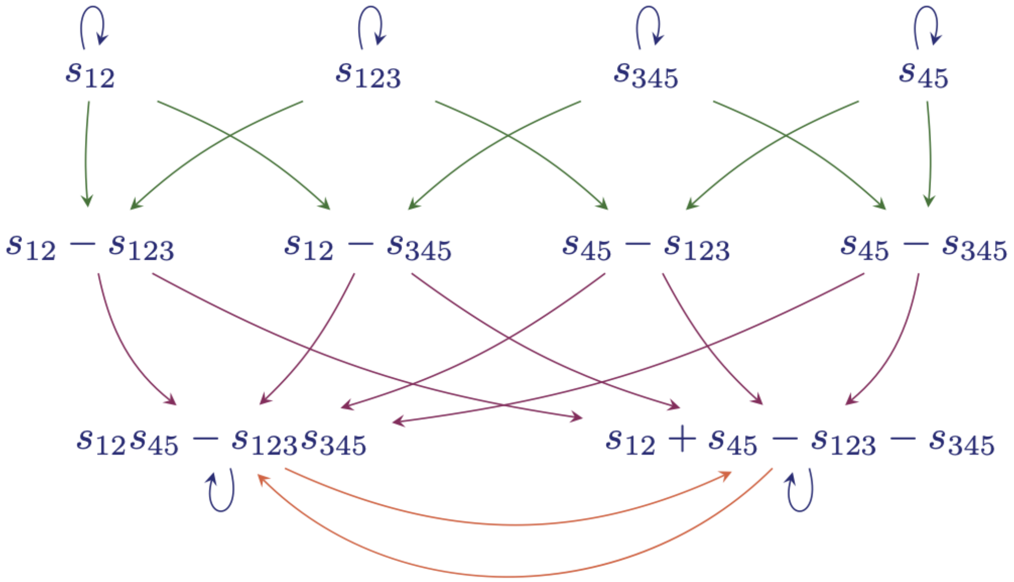


$\xrightarrow[\text{Hierarchical principle}]{\text{Cuts}}$

Allowed discontinuities by
genealogical constraints



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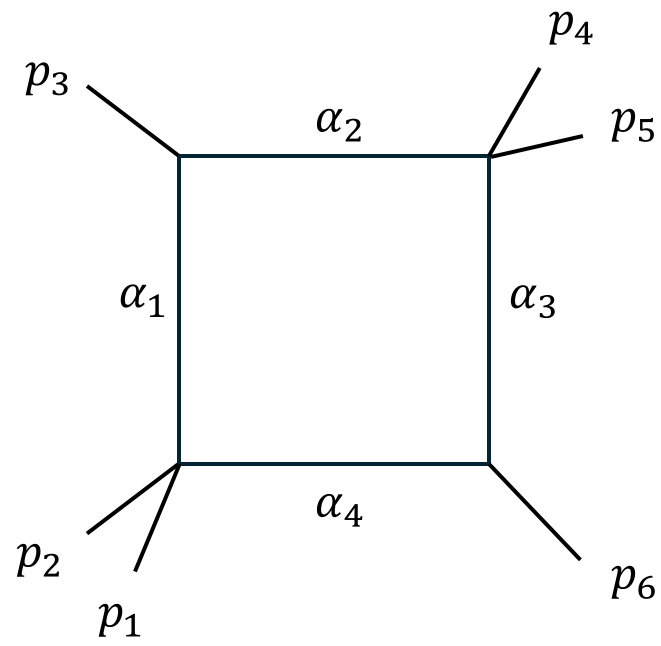

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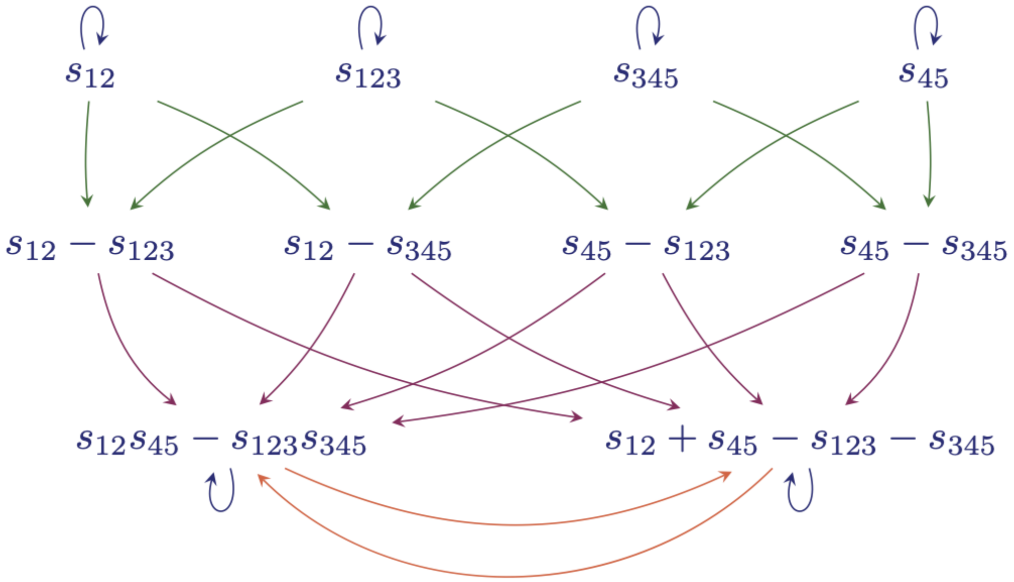

All 64 hierarchical constraints of the type: $\ldots \mathrm{Disc}_{\lambda'} \ldots \mathrm{Disc}_{\lambda} \ldots I(p_i) = 0$

Goal

- We need to find solutions to Landau equations for each letter -> find its cut
- Impose hierarchical principle so that we are sensitive to which integration contours are disallowed

Ruth's talk on Monday

Imposing hierarchy: Counting critical points

From the Picard-Lefschetz theory: the $\mathcal{G}$ polynomial is characterized by the number of master integrals = number of independent integration contours (in α)

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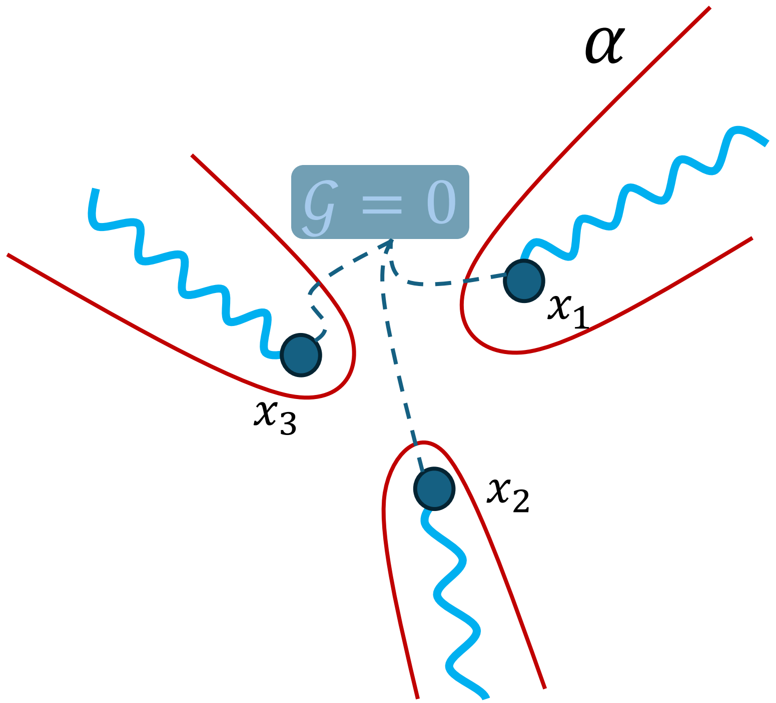

- $\mathcal{G} = (\alpha - x_1)(\alpha - x_2)(\alpha - x_3) \longrightarrow$ we have 2 independent integration contours

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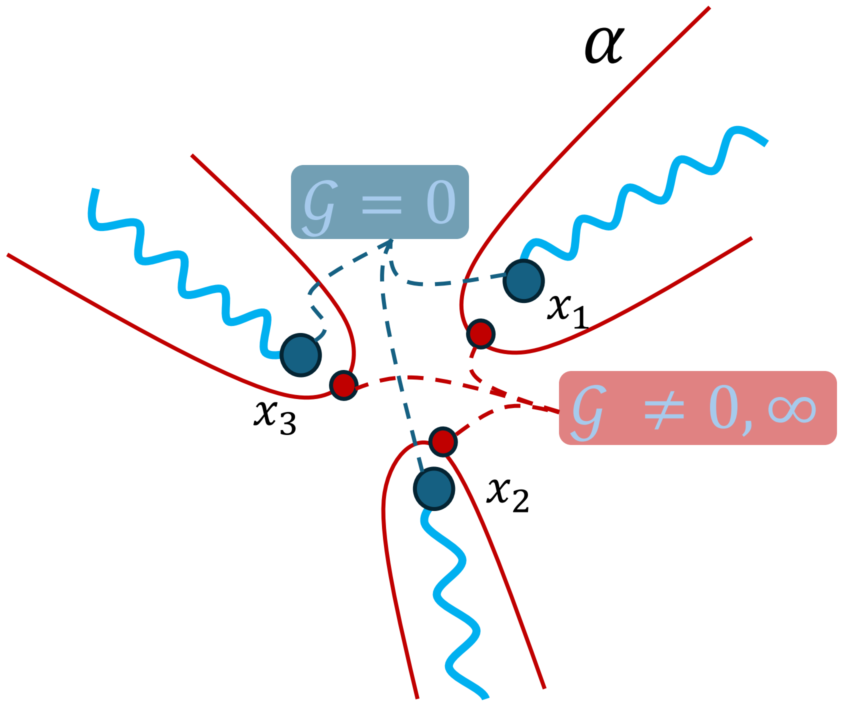


- $\mathcal{G} = (\alpha - x_1)(\alpha - x_2)(\alpha - x_3) \longrightarrow$ we have 2 independent integration contours

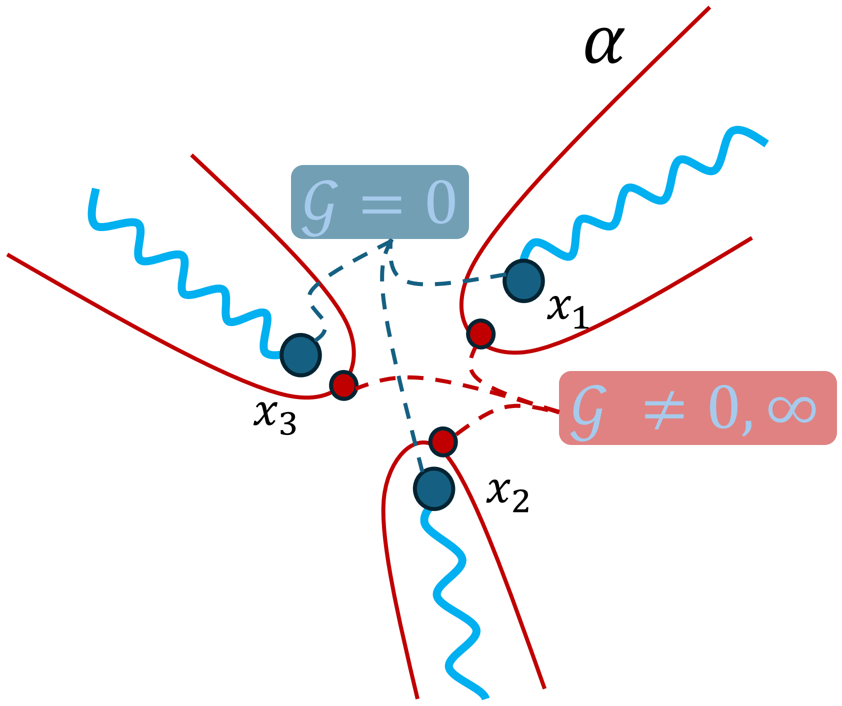
The integration contours can be chosen such that they cross critical points $\longrightarrow$ Lefschetz thimble

- Critical points can be found when satisfying:

$$\partial_\alpha \mathcal{G} = 0 \text{ with } \mathcal{G} \neq 0, \infty$$

Imposing hierarchy: Counting critical points

By counting the critical points, we can track how many integration contours vanished when imposing $\mathrm{Disc}_{\lambda} I$



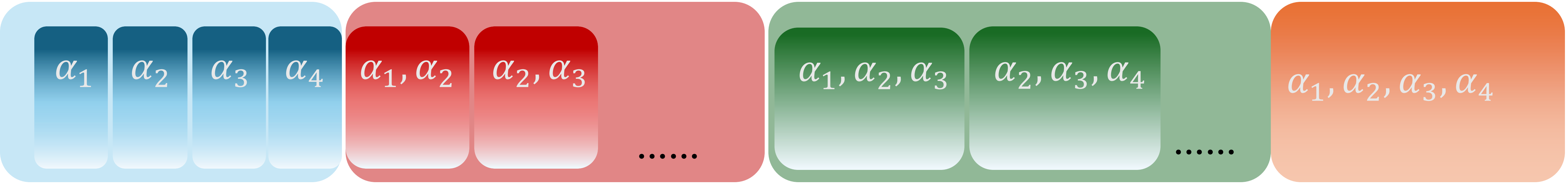
- When $\lambda = 0$, some of the $\alpha = x_i$ and the integration contour now go through $\mathcal{G} = 0$ point,
- Corresponding critical point vanishes and the overall number drops.
- This way we can detect whether the kinematic variety responds to $\lambda_i = 0$ and changes its topology
- In practice we count the number of critical points as number of irreducible monomials – and for this we use Gröbner basis (which turns out to be fast in this case!)

Imposing hierarchy: Counting critical points

- We must solve the Landau equations though and decide on which lines are on shell/which α's =0
- This becomes a problem very quickly with a more involved examples
- we use a sector-by-sector approach from Lee, Pomeransky (2013)

Sectors

- Sectors are sets of non-zero α_i's
- Some sectors do not contain any critical points in $\mathcal{G}$ – those which do are master integrals essentially.



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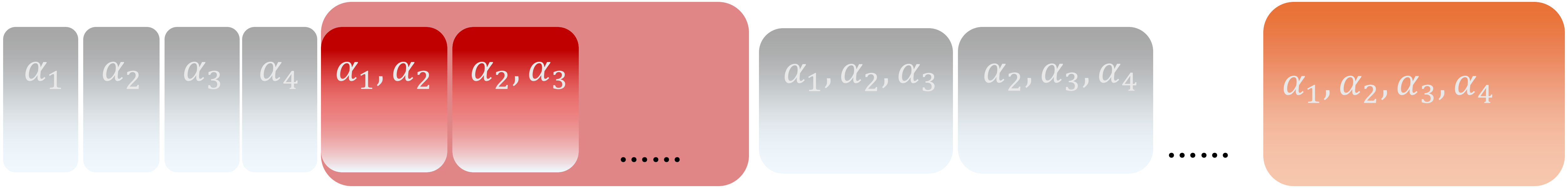
- Some sectors do not contain any critical points in $\mathcal{G}$ – those which do are master integrals essentially.
- As we constrain $\mathcal{G}$ more with $\lambda_i = 0$, more critical points drop in these selected sectors



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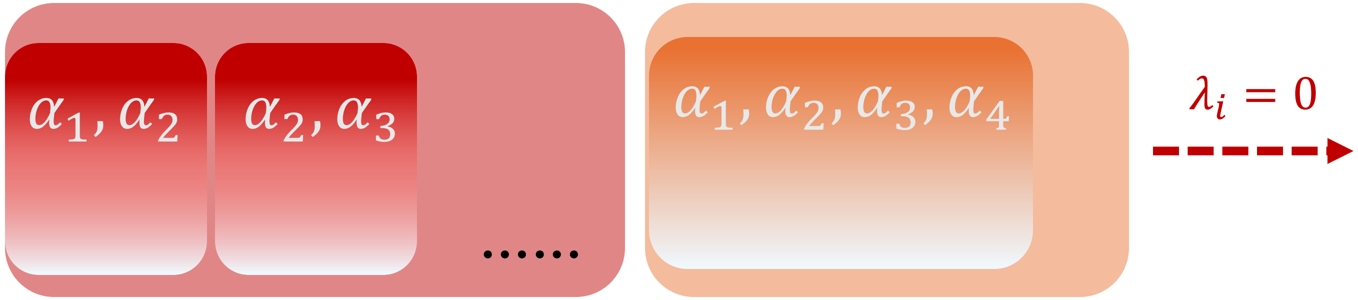

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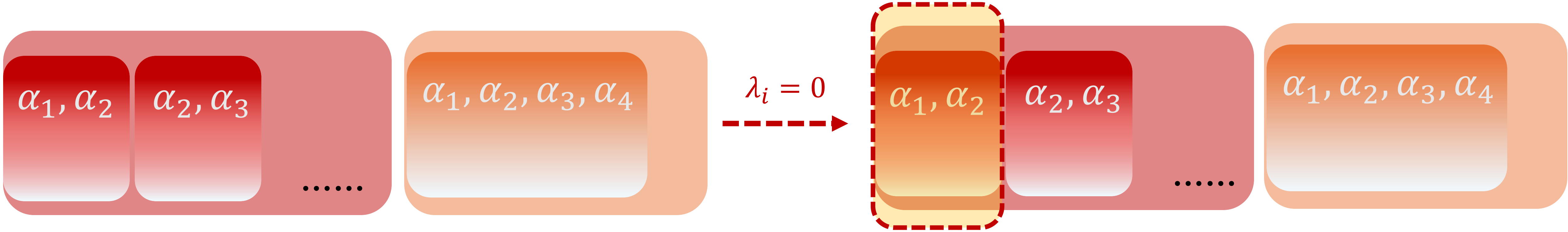

Since there's a drop in number of critical points in α_1, α_2 sector, this is the cut for $\lambda_i = 0$

The algorithm

- Find a cut for $\lambda_i = 0$

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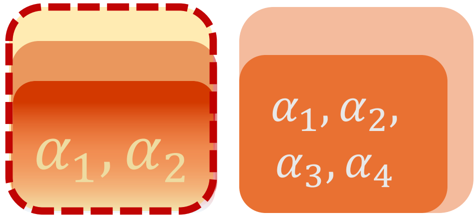
- Take sectors corresponding to the cut and its supersets



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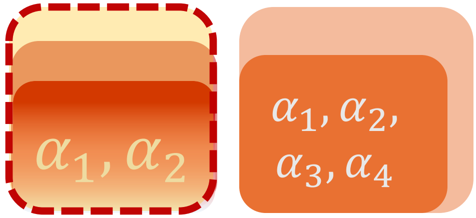
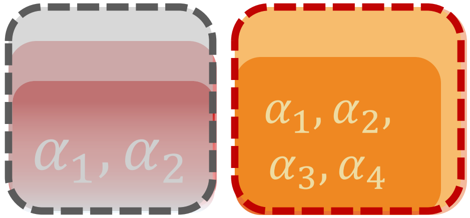


- Probe another $\lambda_j = 0$



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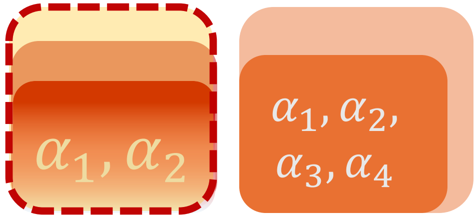
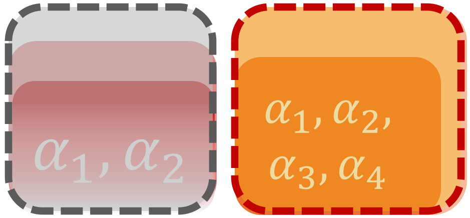
- Take sectors corresponding to the cut and its supersets 

- Probe another $\lambda_j = 0$ 

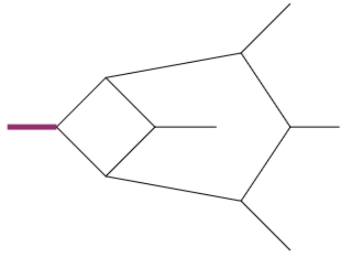
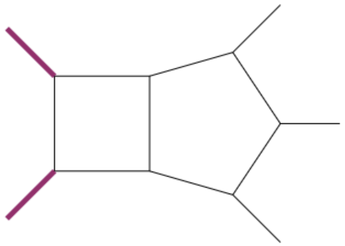
- If the number of critical points drops in the selected sectors, the letter λ_i can be followed by λ_j
- If there's no drop, the letter λ_i cannot be followed by λ_j

$$\ldots \mathrm{Disc}_{\lambda_j} \ldots \mathrm{Disc}_{\lambda_i} \ldots I = 0$$

Some examples

5pt 1-mass hexa-box families
Abreu, Ita, Page, Tschernow 2021

5pt 2-mass penta-box families
Abreu, Chicherin, Sotnikov, Zoia 2024

Family of Diagrams	Family	Alphabet^2	Predicted Disallowed Pairs	Completeness	Timing (s)
	zzz	$33^2 = 1089$	322	100%	200
	mzz	$26^2 = 676$	215	100%	190
	zmz	$29^2 = 841$	188	100%	165
	zzz	$34^2 = 1156$	376	100%	195
	zzm	$32^2 = 1024$	343	100%	236
	zmz	$35^2 = 1225$	391	100%	166
	mzz	$36^2 = 1296$	544	100%	148
	mzm	$33^2 = 1089$	498	100%	127
	mmz	$31^2 = 961$	254	100%	127

Advantages and caveats

- With this method we can find precisely which lines are on-shell for a given letter
- We can detect the non-repeating letters, i.e. $\ldots \mathrm{Disc}_{\lambda} \ldots \mathrm{Disc}_{\lambda} \ldots I = 0$
- Sometimes instead of finding critical point, we find a critical line
  - This is a rare case, and we have an empirical intuition on what predictions can be done
  - We should find critical points of the critical line as suggested in Lee, Pomeransky 2013

Algebraic letters

- Finding predictions for rational letters is straightforward
- For algebraic letters of the form $\lambda_A = \frac{a+\sqrt{b}}{a-\sqrt{b}}$:
  a) Find all rational letters which set the numerator or denominator of the algebraic letter to 0, i.e. $\left.\frac{a+\sqrt{b}}{a-\sqrt{b}}\right|_{s_{12}} = 0, \infty$
  b) Find all $b|_{\lambda_R} = 0$
- Then any λ_R can be followed by λ_A only if λ_R can be followed by b and all rational letters found in a)

What's next

- Genealogical constraints work also on non-polylogarithmic functions (without any changes)
- For instance, elliptic functions: one needs to identify branch points of elliptic functions appearing in the symbol and then run the genealogical constraints code on them
- Same for cosmological correlators

The goal after imposing the genealogical constraints is to find full adjacency constraints