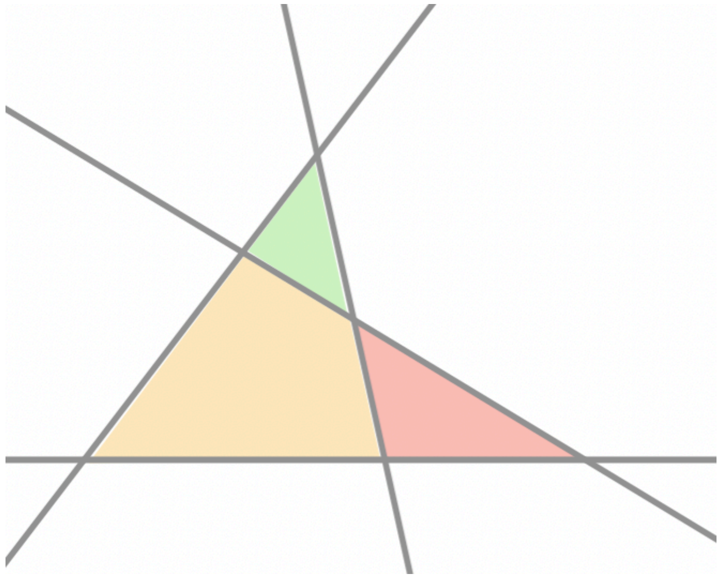
Eric Pichon-Pharabod

Max Planck Institute for Mathematics in the Sciences

Canonical form of differential systems of displaced hyperplane arrangements from positive geometry



Canonical forms

[Henn]

$$\int_\Gamma \ell_1^{\varepsilon_1} \cdots \ell_r^{\varepsilon_r} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,$$

One wants to find a span of integrals that is closed under differentiation $\mathscr{I} = (\mathscr{I}_1, \ldots, \mathscr{I}_s)$,
So that we have a differential system

$\mathrm{d}\mathscr{I} = A\mathscr{I}$ with A some matrix with rational coefficients in parameters $c_1, \ldots, c_r$.

We say the differential system is **in ε-factorised form** if $A = \varepsilon_1 A_1 + \cdots \varepsilon_r A_r$ where

$$A_j \in M_s\left(\mathbb{Q}(c_1, \ldots, c_r)\langle \mathrm{d}c_1, \ldots, \mathrm{d}c_r \rangle\right)$$

We say the differential system is **canonical form** if furthermore the entries of A_j are composed of dlogs

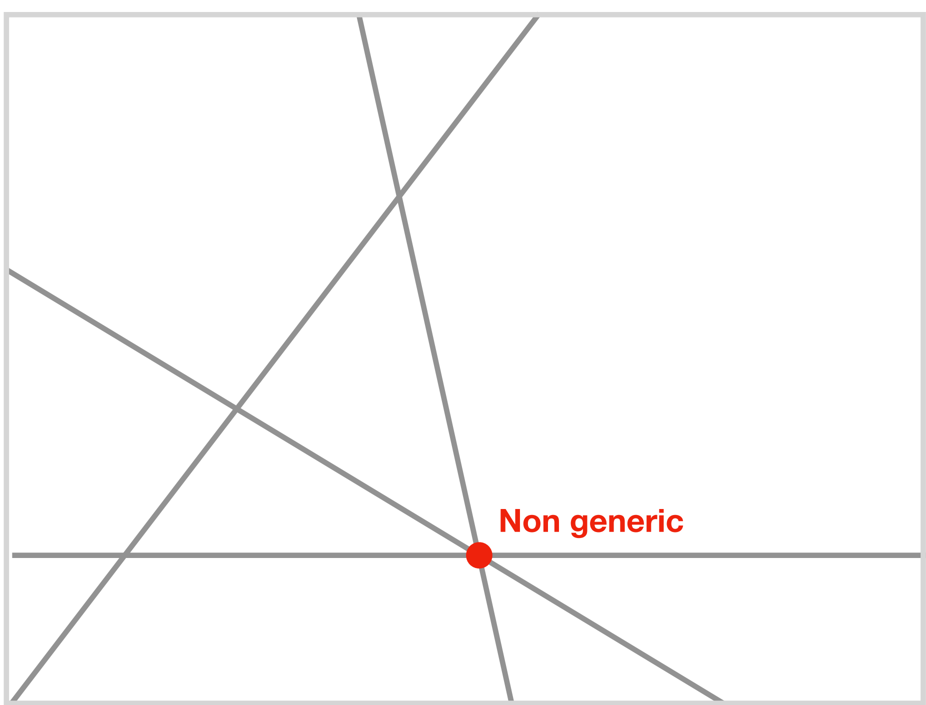
$$A_j \in M_s\left(\mathbb{Q}\langle \mathrm{dlog}\, f, f \in \mathbb{Q}(c_1, \ldots, c_r) \rangle\right)$$

Hyperplane arrangements

A hyperplane arrangement is a collection of hyperplanes $H_i = V(\ell_i)$ where $\ell_i = a_{i1}x_1 + \ldots + a_{in}x_n + c_i$

Displacement parameter

It is **generic** when the intersection of k hyperplanes has codimension k.



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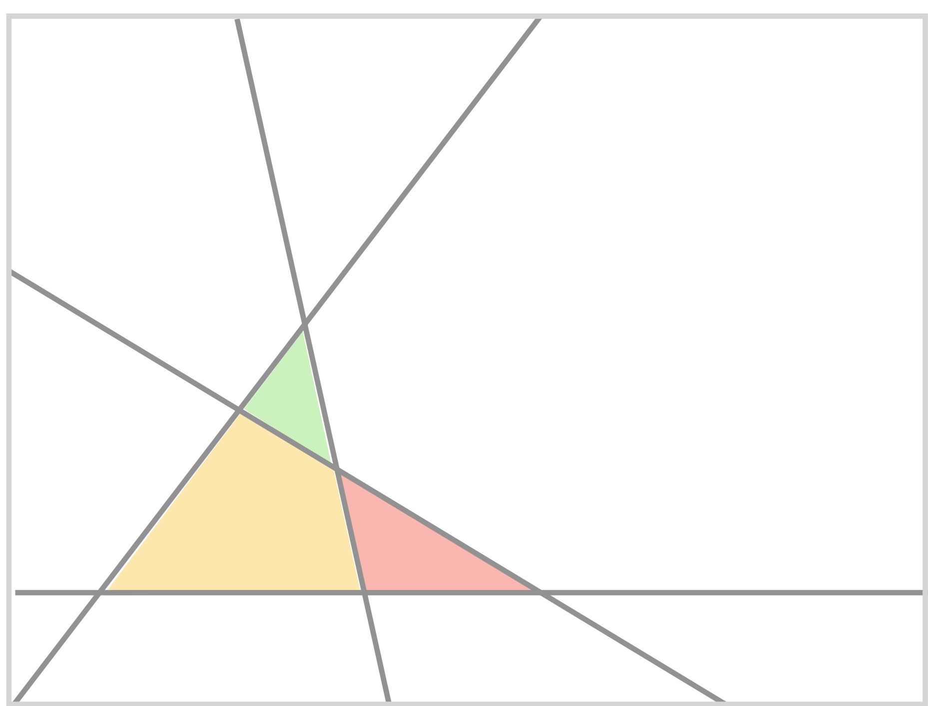
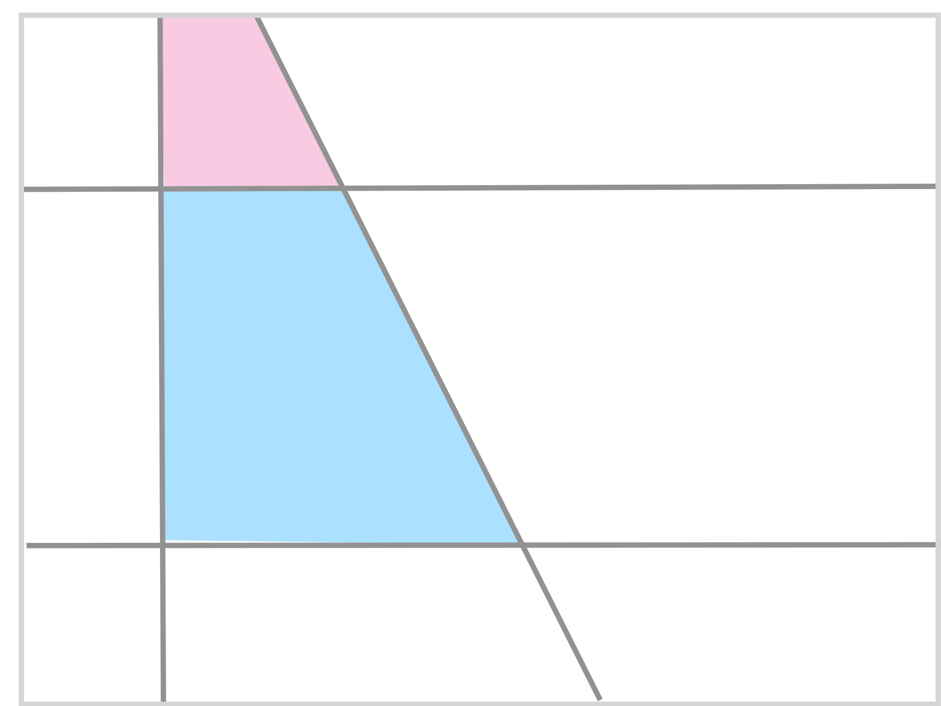




Non generic

∞

Up to moving displacement parameters, we can assume nonempty intersections are always generic.

Bounded regions are bounded connected components of the complement hyperplane arrangement.

Positive geometry

See also
[Arkani-Hamed et al. 2017]

[Brown Dupont 2025] defines for all hyperplane arrangements
a linear isomorphism called the **canonical map**

Space of
bounded regions
(relative homology)

$$\mathrm{can} : H_n(\mathbb{P}^n, \mathscr{A}) \to \Omega^n_{\log}(\mathbb{P}^n \backslash \mathscr{A})$$

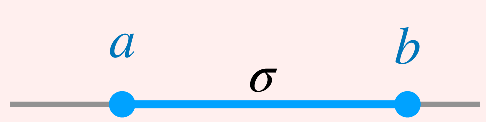
Space of
integrands

It sends a point to 1 and is compatible with residue and boundary maps:

$$\mathrm{can} \circ \partial_{H_i} = \mathrm{Res}_{H_i} \circ \mathrm{can}.$$

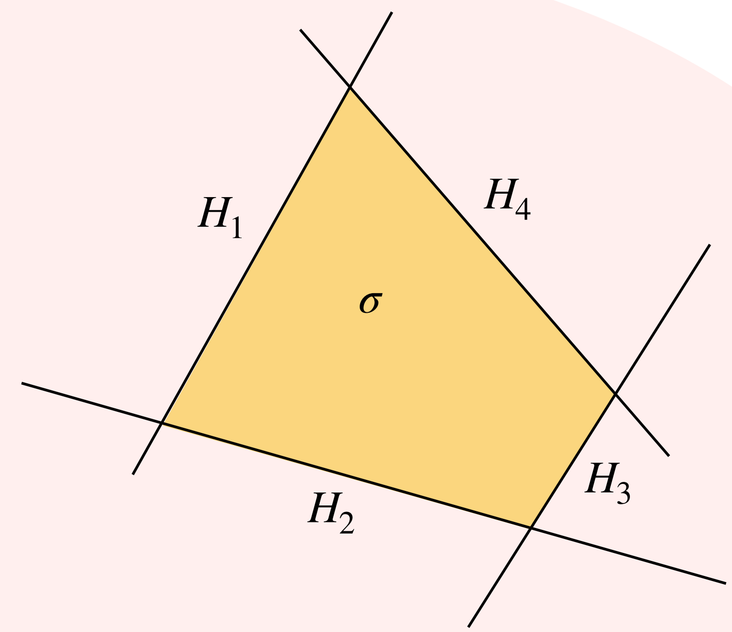
To simplify notations we will denote $\mathrm{can}(\sigma)$ by ϖ_σ.

Examples



$$\varpi_\sigma = \frac{(b-a)\mathrm{d}x}{(x-a)(b-x)} = \frac{\mathrm{d}x}{(x-a)} - \frac{\mathrm{d}x}{(x-b)}$$

$$\mathrm{Res}_a\, \varpi_\sigma = 1 \qquad \mathrm{Res}_b\, \varpi_\sigma = -1$$



$$\varpi_\sigma = \mathrm{dlog}\,\ell_1 \wedge \mathrm{dlog}\,\ell_2 + \mathrm{dlog}\,\ell_2 \wedge \mathrm{dlog}\,\ell_3 + \mathrm{dlog}\,\ell_3 \wedge \mathrm{dlog}\,\ell_4 + \mathrm{dlog}\,\ell_4 \wedge \mathrm{dlog}\,\ell_1$$

$$\mathrm{Res}_{H_1}\, \varpi_\sigma = \mathrm{dlog}\,\ell_2 - \mathrm{dlog}\,\ell_4$$

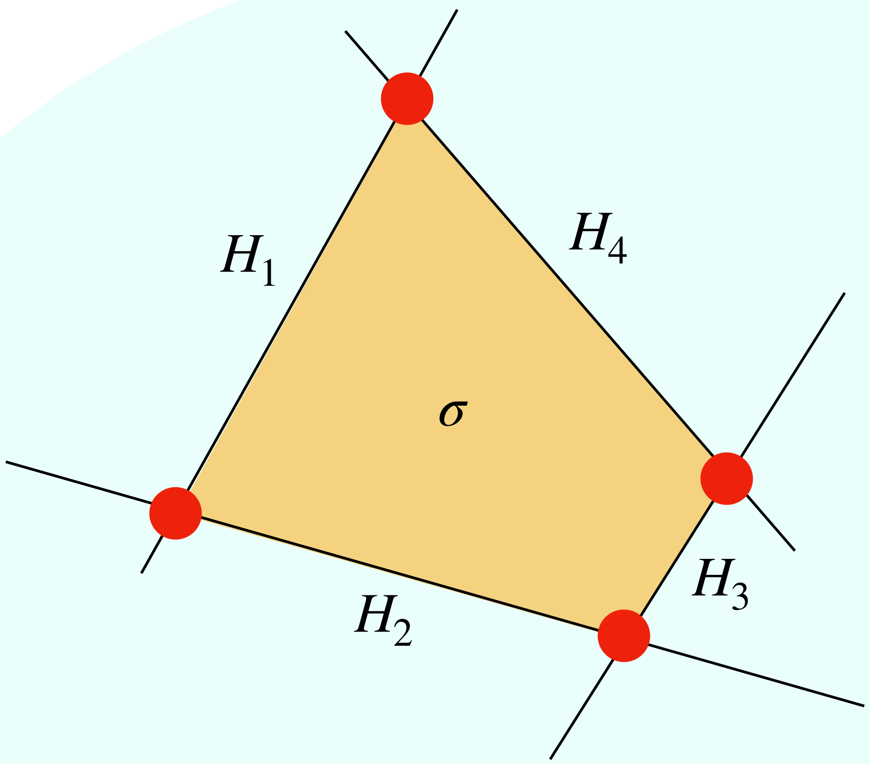
Canonical forms of bounded regions

The canonical form of a bounded region decomposes as a sum over its corners.

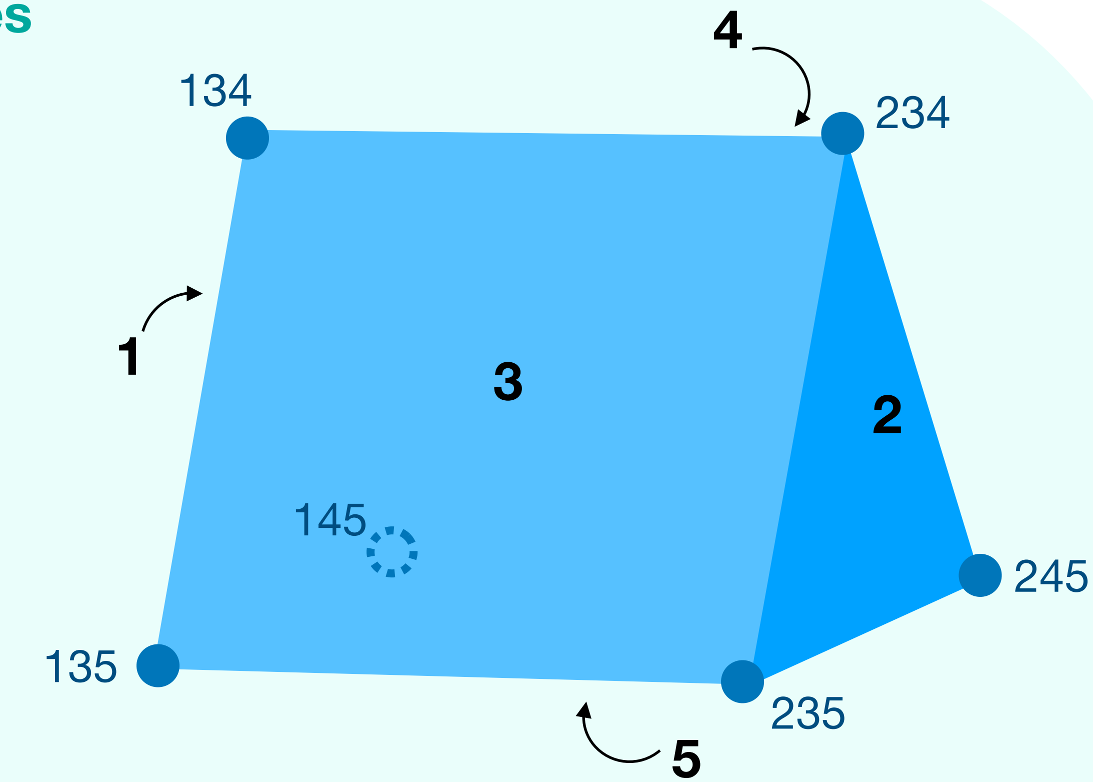
[Brown Dupont 2025] Define $\omega_i = \mathrm{dlog}\, \ell_i$ and $\omega_I = \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_{|I|}}$ for $I = \{i_1 < i_2 < \cdots < i_{|I|}\} \subset \{1, \ldots, r\}$. Then

$$\varpi_\sigma = \sum_I \partial_I(\sigma)\omega_I$$

Examples



$$\varpi_\sigma = \omega_{12} + \omega_{23} + \omega_{34} - \omega_{14}$$



$$\varpi_T = \omega_{134} + \omega_{145} - \omega_{135} - \omega_{234} - \omega_{245} + \omega_{235}$$

Twisted cohomology (in broad strokes)

Cohomology is the space of integrands up to integration by part relations:

$$\int_\Gamma \mathrm{d}\eta = 0 \qquad \text{so we say } \omega_1 \equiv \omega_2 \text{ if } \omega_1 - \omega_2 = \mathrm{d}\eta \text{ for some } \eta$$

Twisted cohomology is the same but you multiply the integrands by a function

$$\int_\Gamma \mathrm{d}\left(f^\varepsilon \omega\right) = \int_\Gamma f^\varepsilon \left(\mathrm{d}\omega + \varepsilon \frac{\mathrm{d}f}{f} \wedge \omega\right) = \int_\Gamma f^\varepsilon \mathrm{d}_\varepsilon \omega = 0 \qquad \text{where} \qquad \mathrm{d}_\varepsilon \square := \mathrm{d}\square + \varepsilon \frac{\mathrm{d}f}{f} \wedge \square$$

We say $\omega_1 \equiv \omega_2$ if $\omega_1 - \omega_2 = \mathrm{d}_\varepsilon \eta$ for some η.

In our case we have several twists

$$\int_\Gamma \ell_1^{\varepsilon_1} \cdots \ell_r^{\varepsilon_r} \omega$$

$$\mathrm{d}_\varepsilon \omega := \mathrm{d}\,\omega + \varepsilon_1 \,\mathrm{dlog}\, \ell_1 \wedge \omega + \ldots + \varepsilon_r \,\mathrm{dlog}\, \ell_r \wedge \omega \qquad \mathrm{d}\left(\ell_1^{\varepsilon_1} \cdots \ell_r^{\varepsilon_r} \omega\right) = \ell_1^{\varepsilon_1} \cdots \ell_r^{\varepsilon_r} \mathrm{d}_\varepsilon \omega\,.$$

Gauss-Manin connection

Consider an integral of a rational form depending on a parameter t.

One can differentiate with respect to this parameter.
It turns out that differentiation really acts on the cohomology.

$$\partial_t \int_\Gamma \omega_t = \int_\Gamma \nabla_t \omega_t$$

This defines a $\mathbb{C}$-linear map ∇_t on the $\mathbb{C}(t)$-vector space $H^n_{\mathrm{DR}}(X_t)$ called the Gauss-Manin connection.

In the twisted setting, the same holds, up to taking care of the twist:

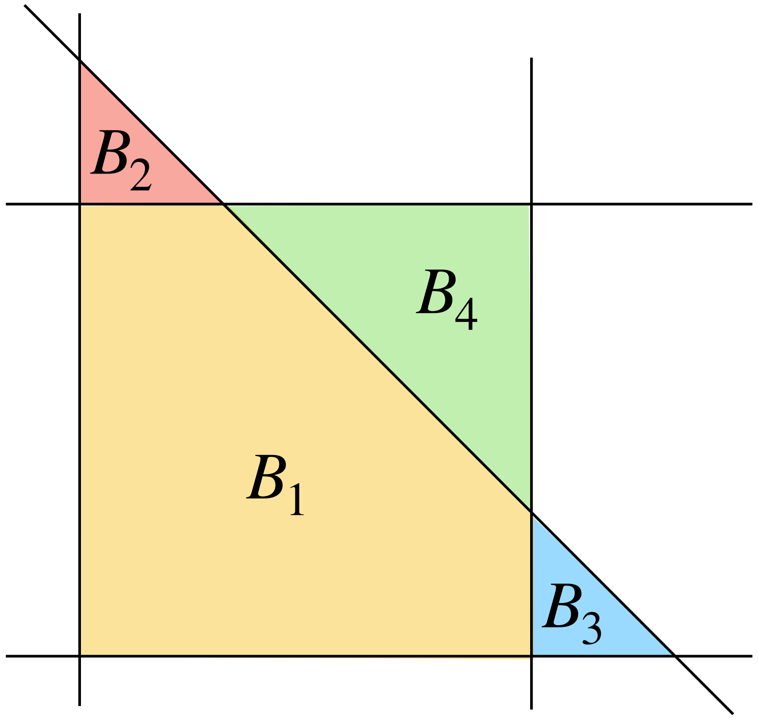
$$\partial_t \int_\Gamma f_t^\varepsilon \omega_t = \int_\Gamma f_t^\varepsilon \nabla_t^\varepsilon \omega_t \qquad \nabla_t^\varepsilon = \nabla_t - \varepsilon \frac{\partial_t f}{f}$$

Instead of a differential system of integrals $\mathrm{d}\mathcal{I} = A\mathcal{I}$,
we are looking for the Gauss-Manin connection on cohomology.

$$\nabla^\varepsilon \Omega = A\Omega$$

Canonical form from canonical forms

Theorem: Let $\mathscr{A}$ be an arrangement of r hyperplanes in n-dimensional space.
Let $B_1, \ldots, B_s$ be a basis of the space of bounded regions of $\mathscr{A}$.
Define $\Omega_1, \ldots, \Omega_s$ be the corresponding canonical forms.



Then $\langle \Omega_1, \ldots, \Omega_s \rangle$ is stable under the Gauss-Manin connection.
The differential system expressed in this basis is in canonical form.

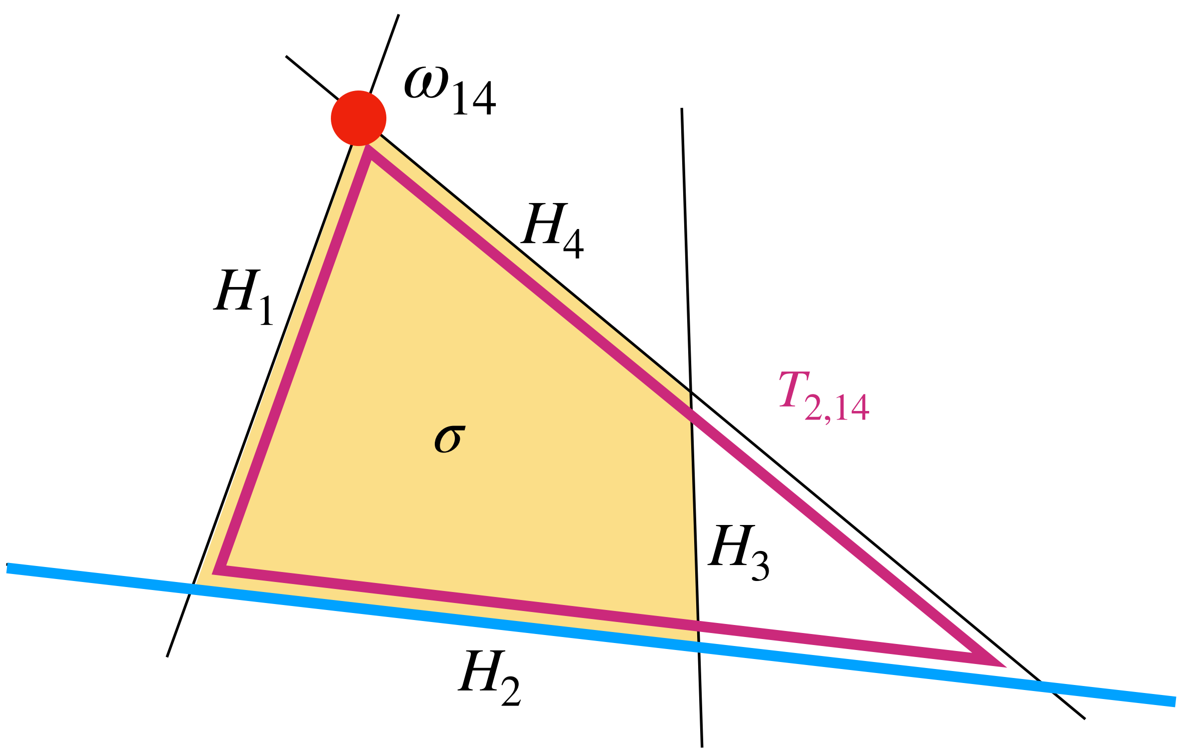
$$\nabla^{\varepsilon}\Omega = \sum_{j=1}^{r} \varepsilon_j M_j \Omega$$

The generic case

Twisted Gauss-Manin connection

$$\partial^{\varepsilon}_{c_i}\omega_i = -\frac{\varepsilon_i}{\ell_i}\omega_i + \frac{\mathrm{d}\ell_i}{\ell_i^2} = -\,\varepsilon_i\frac{1}{\ell_i}\omega_i + \sum_{j=1}^{r}\frac{\varepsilon_j}{\ell_i}\omega_j + \mathrm{d}_{\varepsilon}\omega_i$$

$$\partial_{c_i}\omega_j = 0 \text{ if } i \neq j$$



$$\varpi_\sigma = \omega_{12} + \omega_{23} + \omega_{34} - \omega_{14}$$

$$\partial^{\varepsilon}_{c_4}\varpi_\sigma = \partial^{\varepsilon}_{c_4}\omega_{12} + \partial^{\varepsilon}_{c_4}\omega_{23} + \partial^{\varepsilon}_{c_4}\omega_{34} - \partial^{\varepsilon}_{c_4}\omega_{14}$$

$$\partial^{\varepsilon}_{c_4}\omega_{14} = \frac{\varepsilon_1}{\ell_4}\omega_{11} + \frac{\varepsilon_2}{\ell_4}\omega_{12} + \frac{\varepsilon_3}{\ell_4}\omega_{13}$$

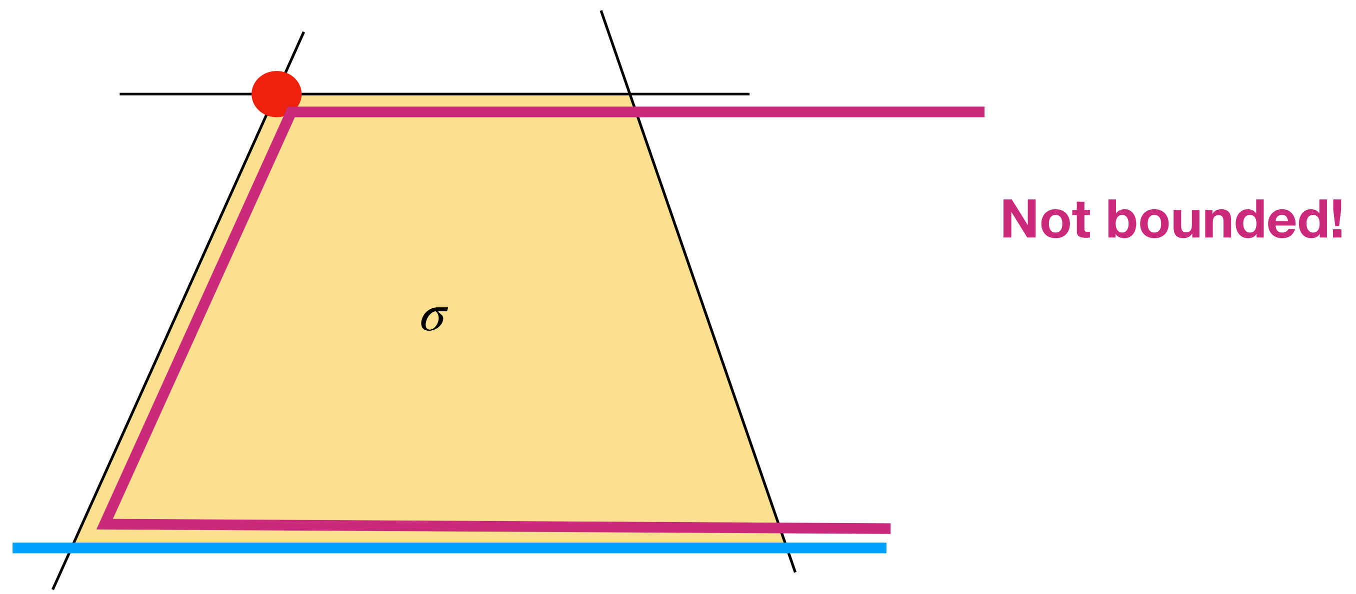
$$\frac{1}{\ell_4}\omega_{12} = \frac{\partial_{c_4}\alpha_{214}}{\alpha_{214}}\underbrace{\left(\omega_{12} + \omega_{24} - \omega_{14}\right)}_{\varpi_{T_{2,14}}}$$

$$\alpha_{214} = \det\begin{pmatrix} c_2 & c_1 & c_4 \\ a_{21} & a_{11} & a_{41} \\ a_{22} & a_{12} & a_{42}\end{pmatrix}$$

We get a sum over all pairs of corners and twisted hyperplanes (ω_I, H_i) of canonical forms of tetrahedron:

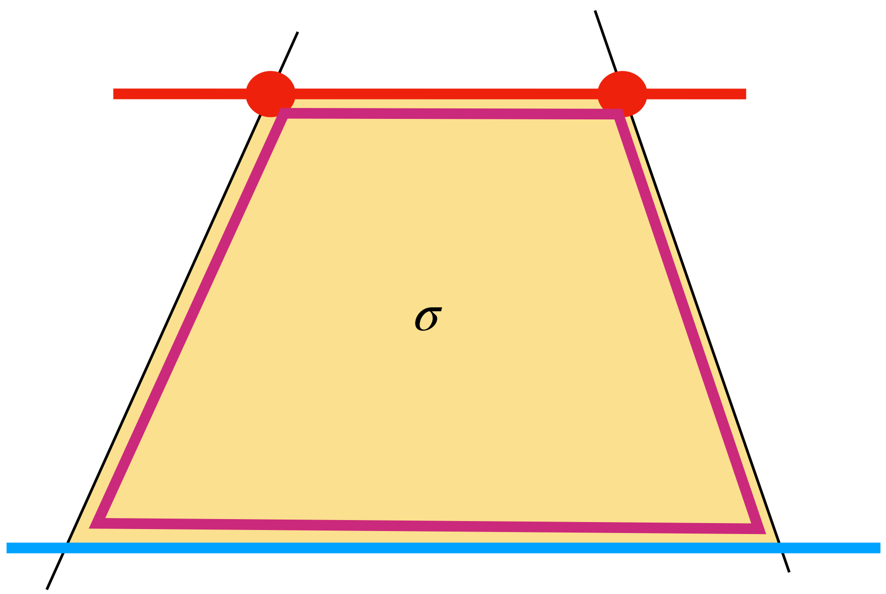
$$\partial^{\varepsilon}_{c_i}\varpi_\sigma = \sum_{I\ni i, |I|=r}\ \sum_{j=1}^{r} \partial_I(\sigma)\,\varepsilon_j\,\frac{\partial_{c_i}\alpha_{I,j}}{\alpha_{I,j}}\,\varpi_{T_{j,I}}$$

Prismatoids



Prismatoids

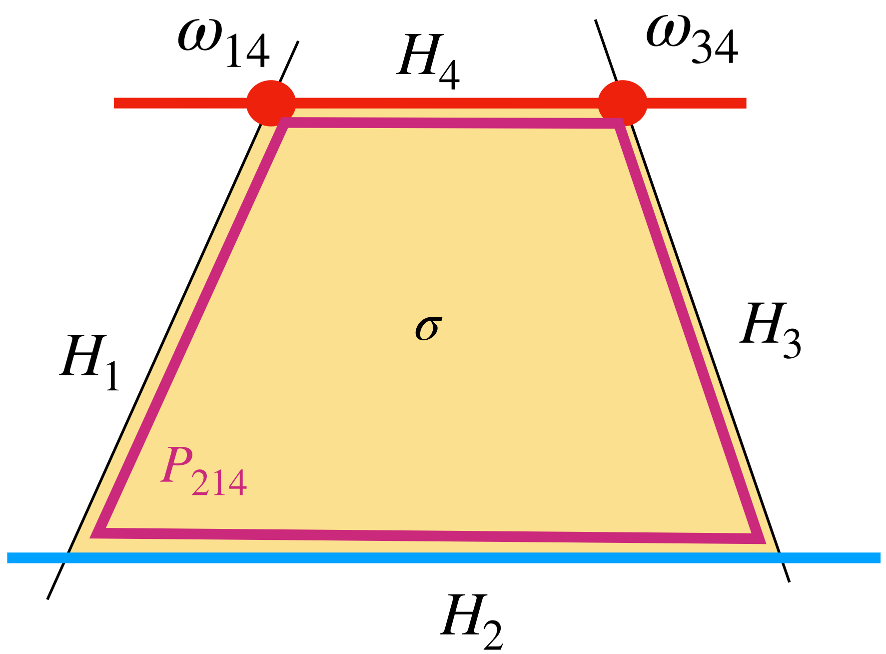
A **flat** of the arrangement is an intersection of hyperplanes.



We are interested in maximal flats parallel to a given twisted hyperplane.

A **prismatoid** is the bounded region σ, obtained from a hyperplane H,
a maximal flat F parallel to H and
a bounded region of the restricted hyperplane arrangement on F.

The non-generic case



$$\varpi_\sigma = \omega_{12} + \omega_{23} + \omega_{34} - \omega_{14}$$

$$\partial^{\varepsilon}_{c_4}(\omega_{14} - \omega_{34}) = \frac{\varepsilon_1}{\ell_4}\omega_{11} + \frac{\varepsilon_2}{\ell_4}\omega_{12} + \frac{\varepsilon_3}{\ell_4}\omega_{13} - \frac{\varepsilon_1}{\ell_4}\omega_{13} - \frac{\varepsilon_2}{\ell_4}\omega_{23} - \frac{\varepsilon_3}{\ell_4}\omega_{33}$$

$$\frac{1}{\ell_4}(\omega_{12} - \omega_{23}) = \frac{\partial_{c_4}\alpha_{214}}{\alpha_{214}}\underbrace{(\omega_{12} - \omega_{14} + \omega_{34} + \omega_{23})}_{\varpi_{P_{214}}}$$

We get a sum over all pairs of twisted hyperplanes and maximal parallel flats (H_i, P_j) of canonical forms of prismatoids:

$$\partial^{\varepsilon}_{c_i}\varpi_\sigma = \sum_{I \ni i, |I| = r} \sum_{j=1}^{r} \partial_I(\sigma)\, \varepsilon_j \frac{\partial_{c_i}\alpha_{I,j}}{\alpha_{I,j}} \varpi_{P_{j,I}}$$

Example — Toblerone

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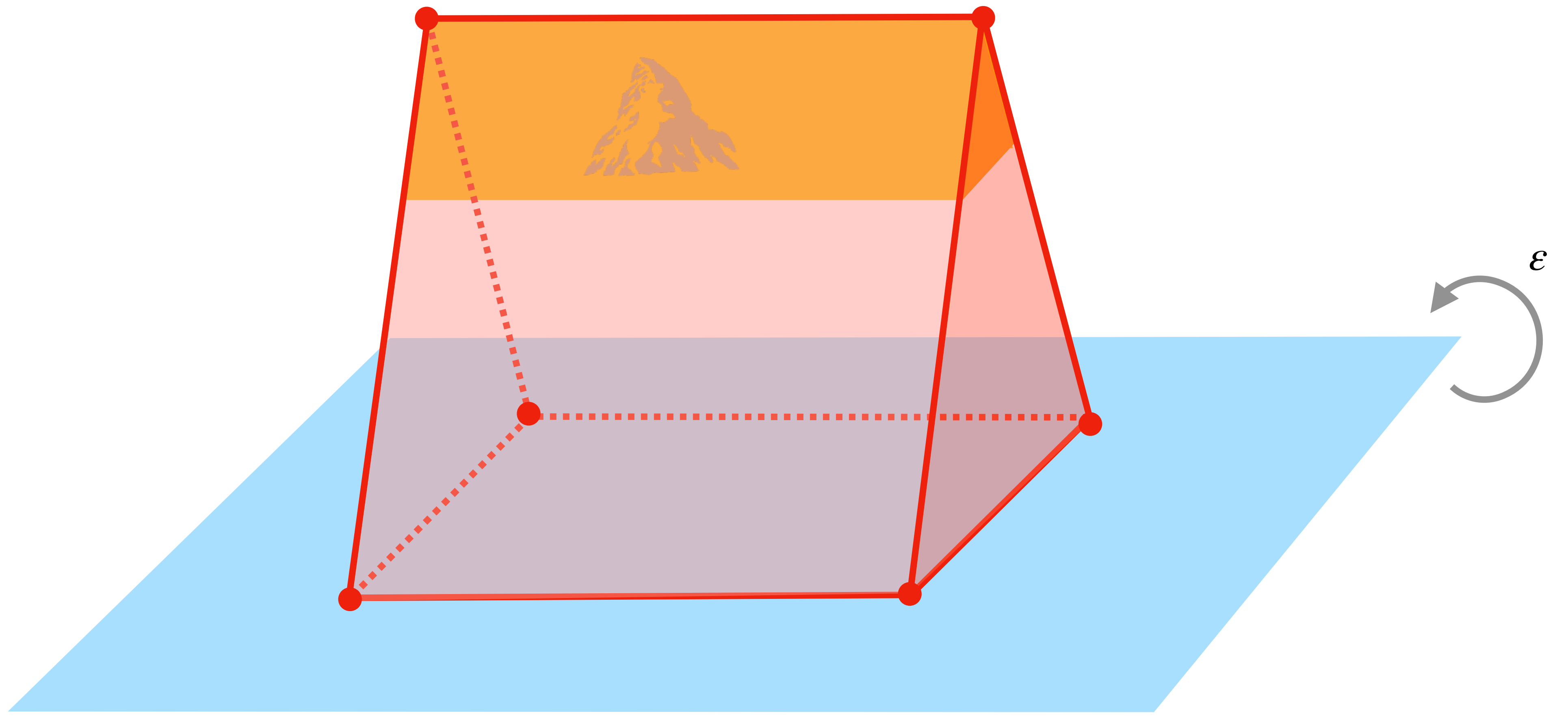

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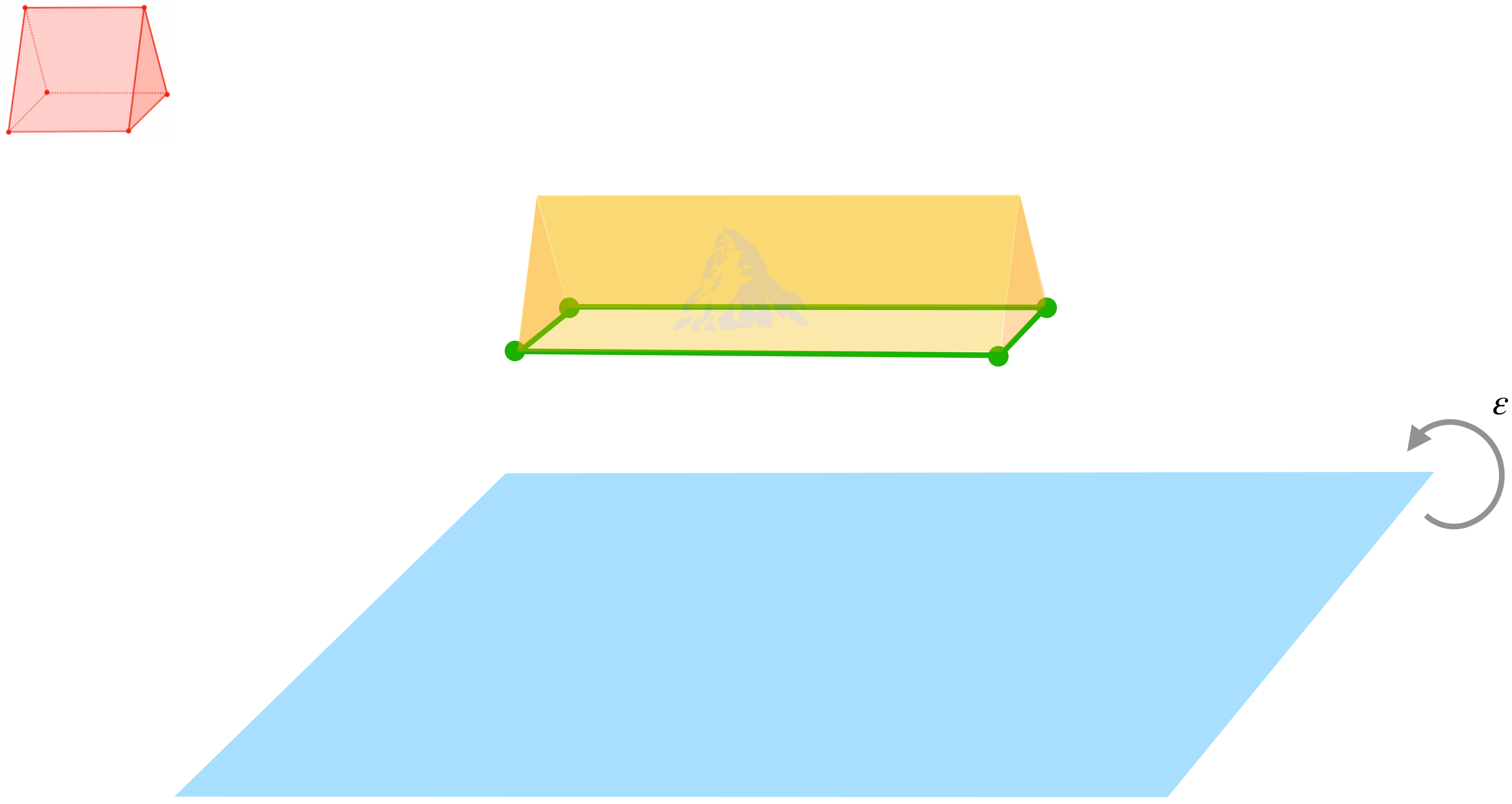

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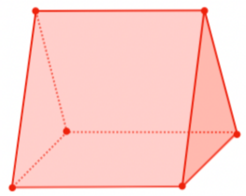

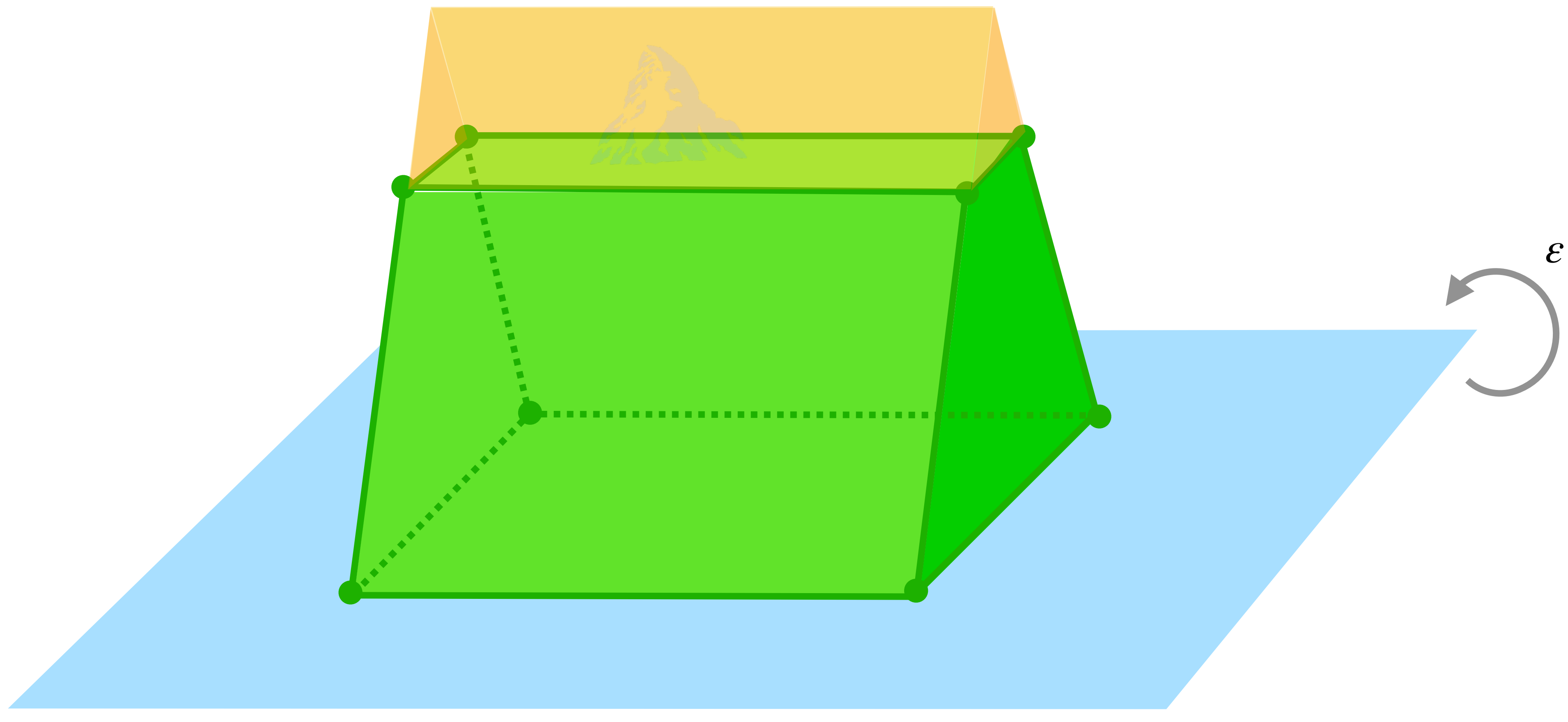

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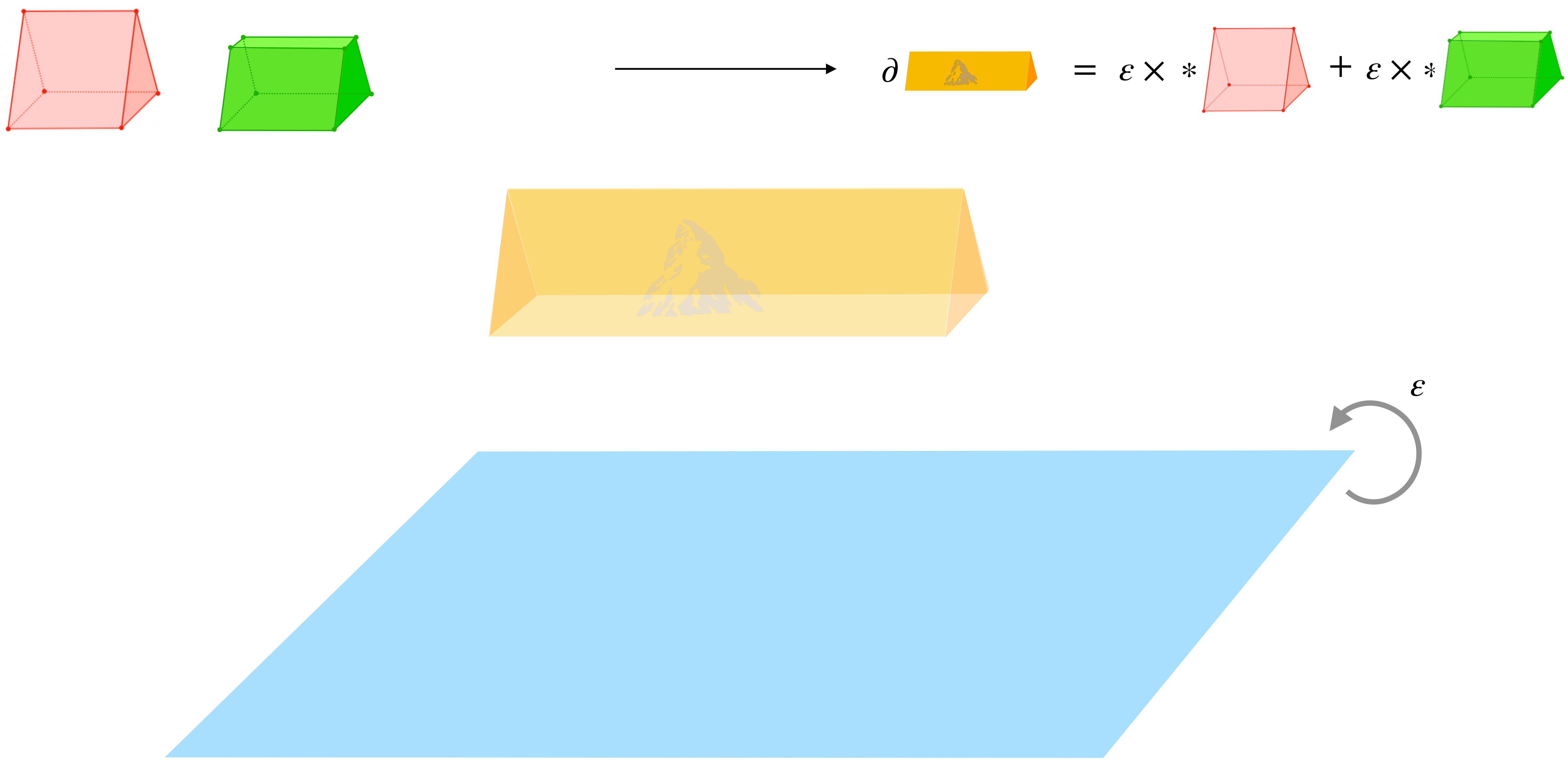
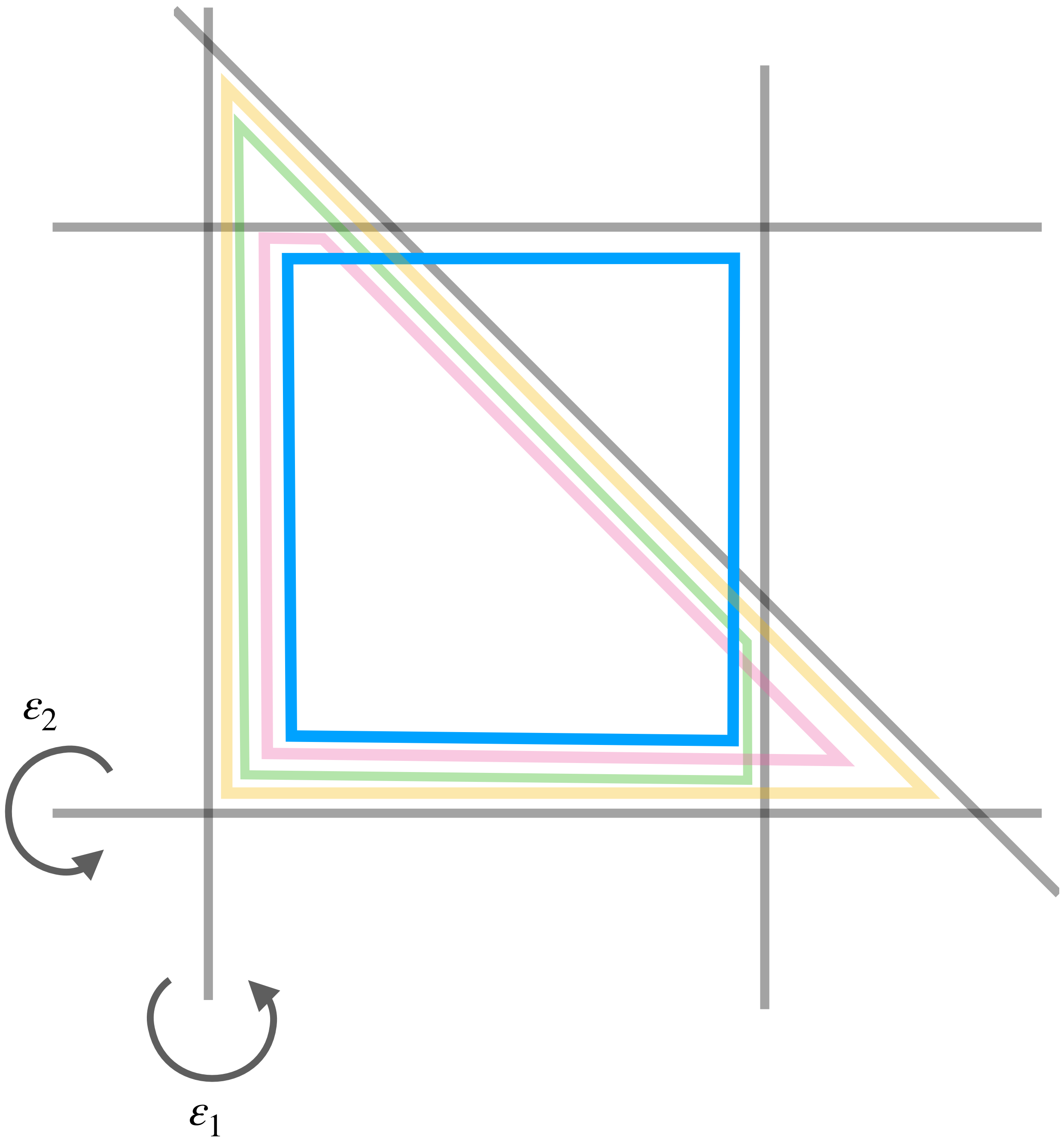
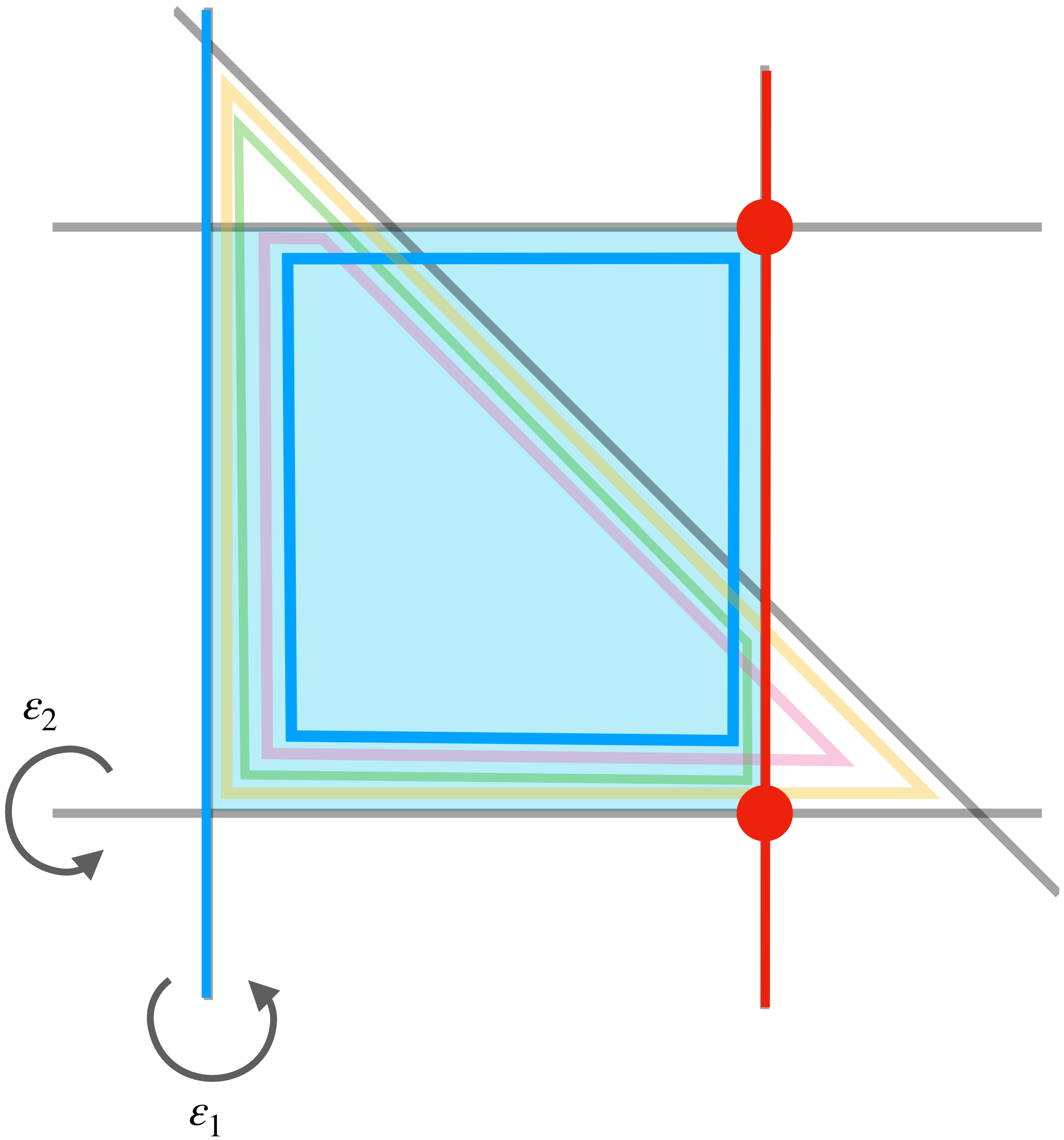


Example — two-site



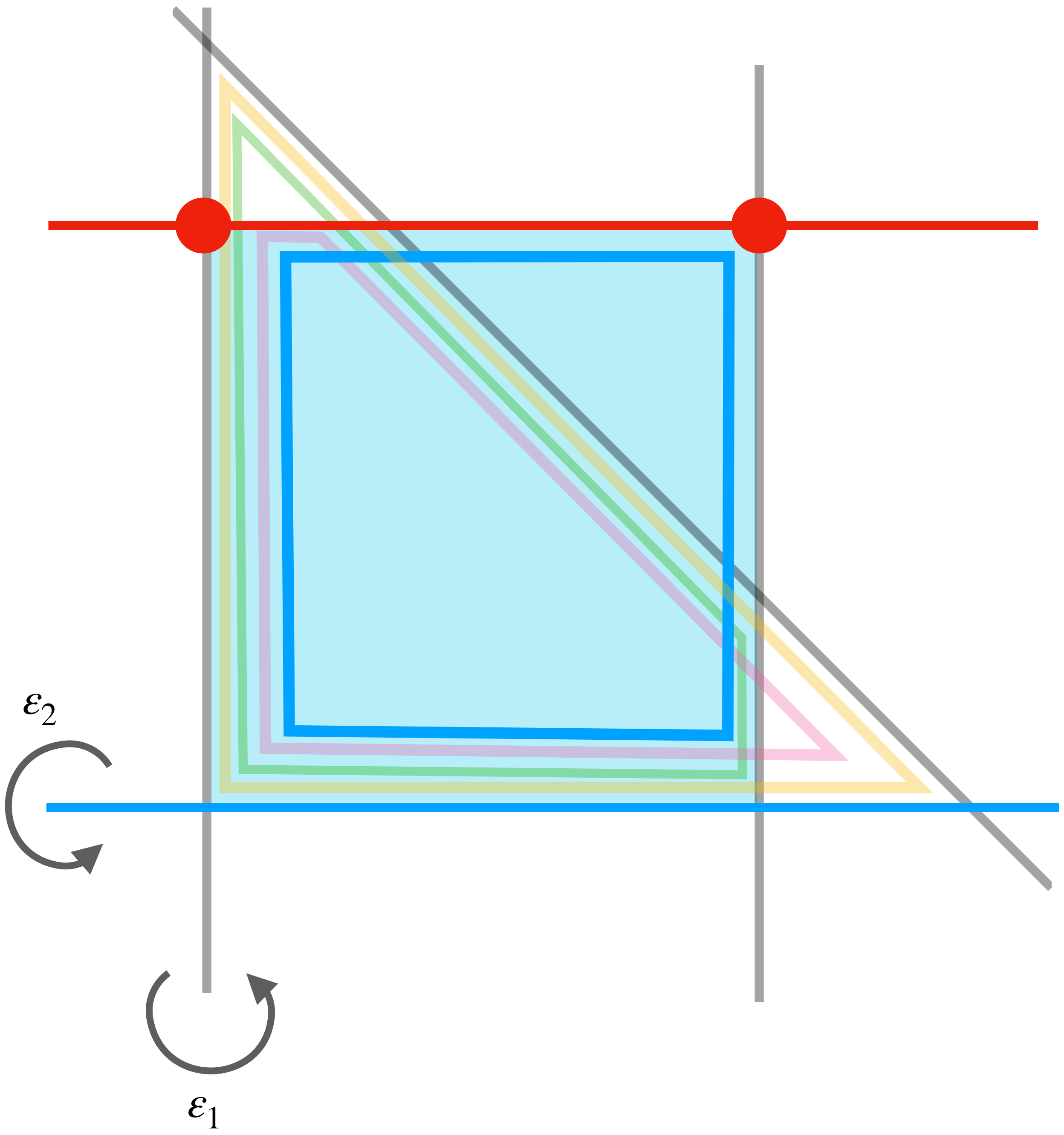
d $\square$ =

Example — two-site



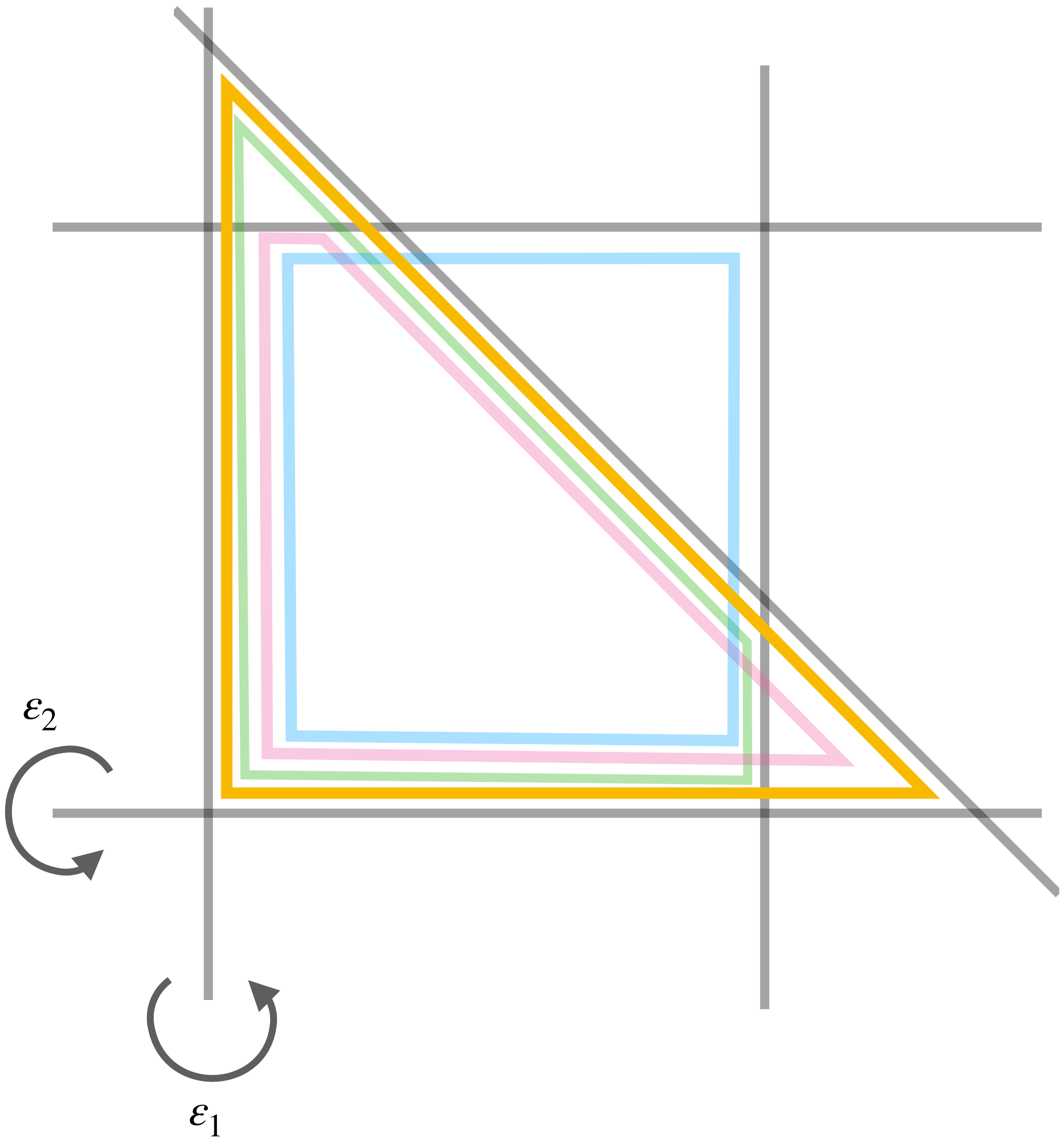
$$\mathrm{d}\,\square = \varepsilon_1 * \square$$

Example — two-site



$$\mathrm{d}\,\square = \varepsilon_1 * \square + \varepsilon_2 * \square$$

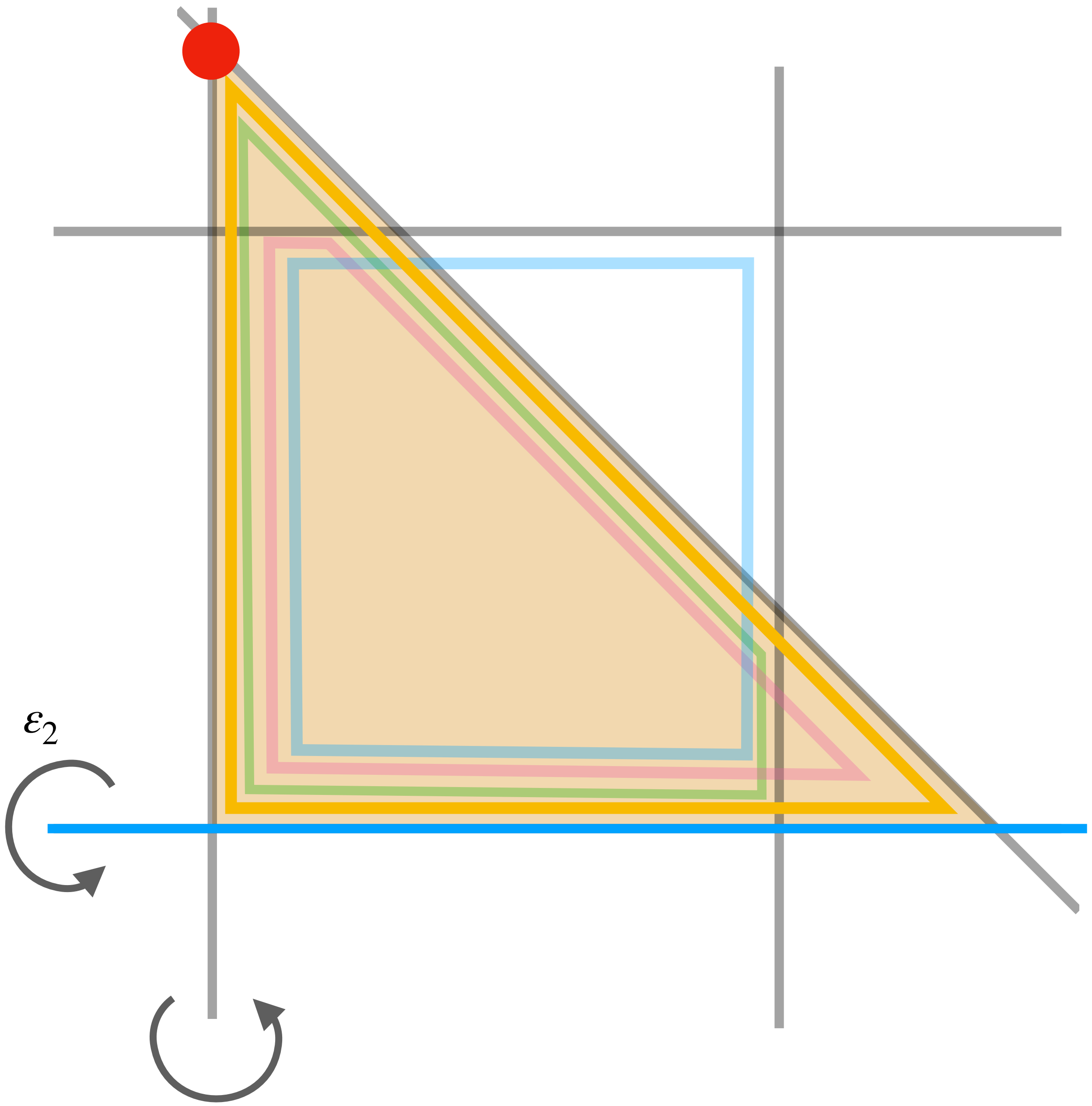
Example — two-site



$$d\,\square = \varepsilon_1 * \square + \varepsilon_2 * \square$$

$$d\,\triangle =$$

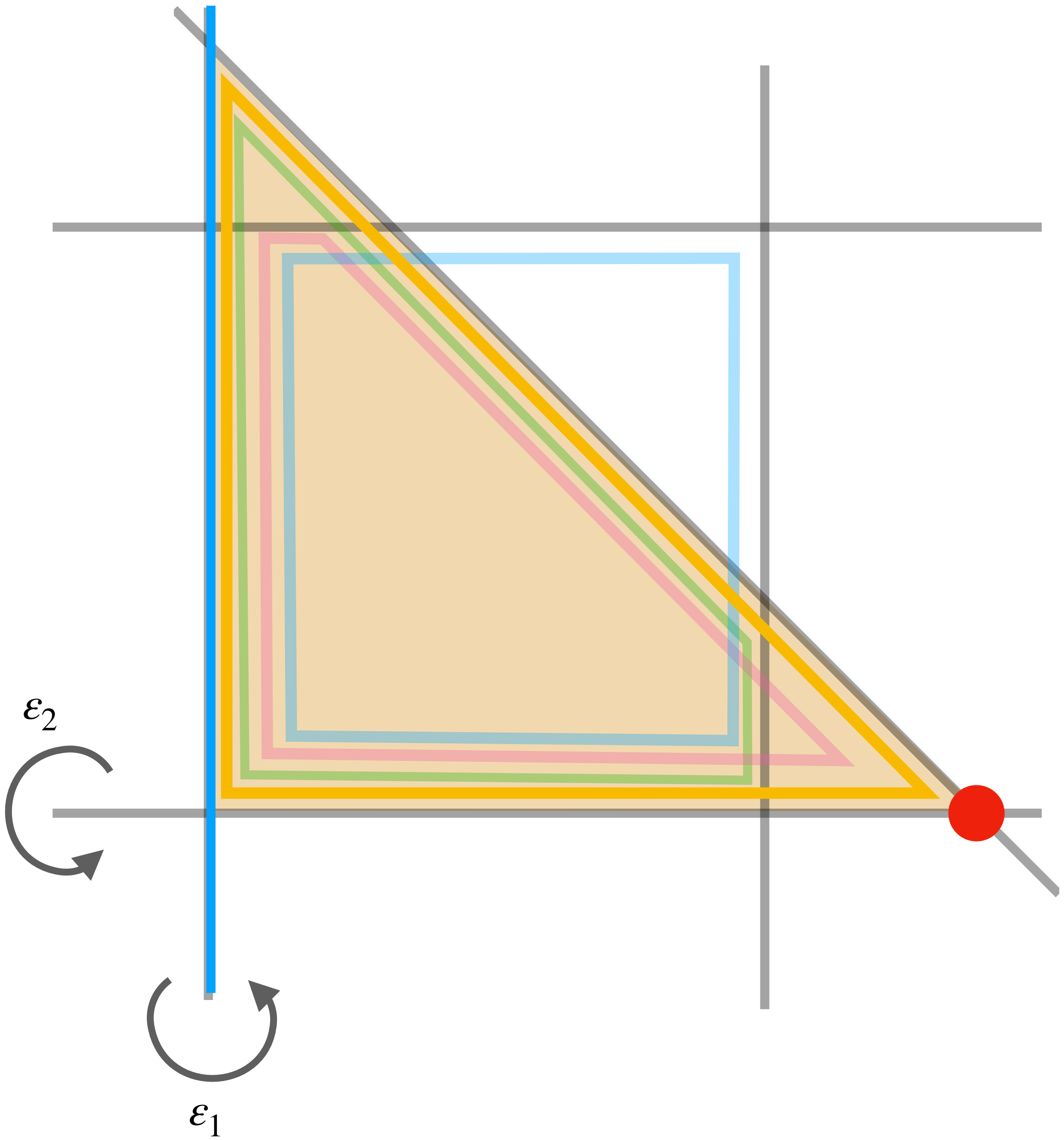
Example — two-site



$$d\,\square = \varepsilon_1 * \square + \varepsilon_2 * \square$$

$$d\,\triangle = \varepsilon_1 * \triangle$$

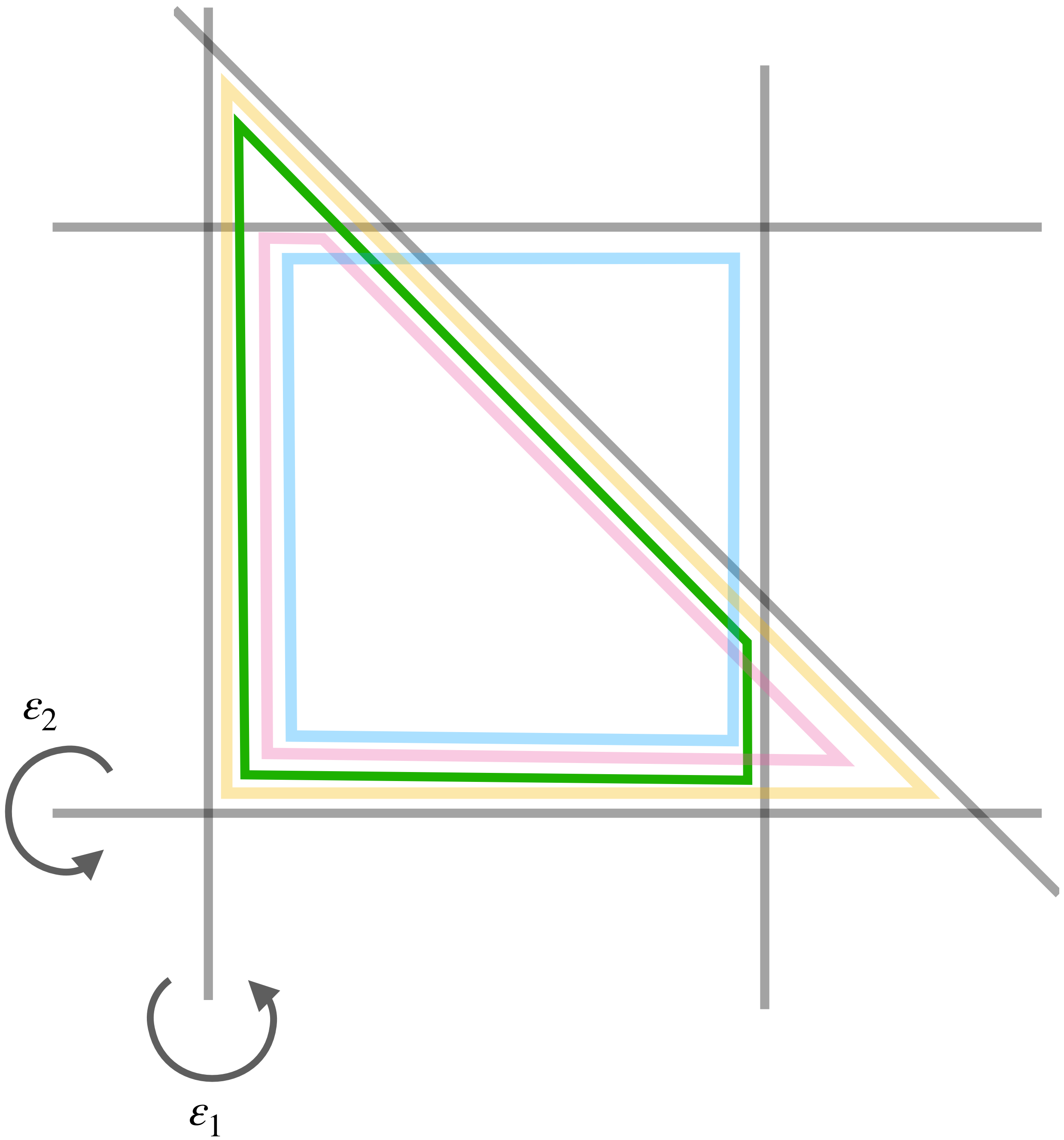
Example — two-site

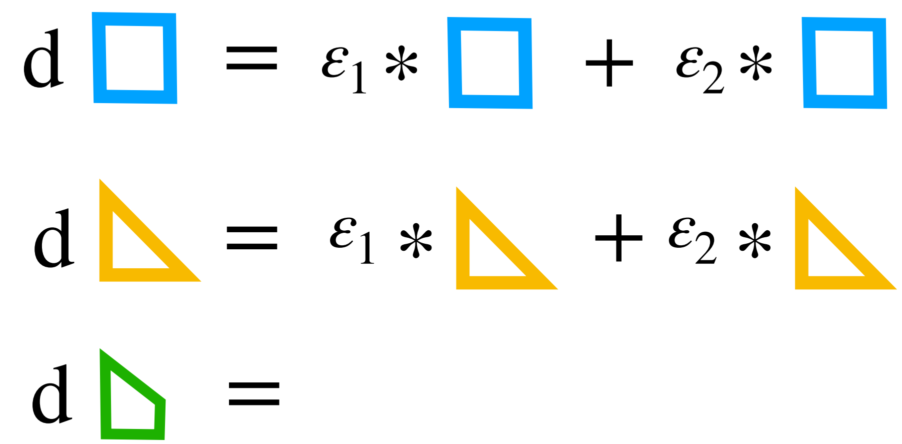


$$\mathrm{d}\,\square = \varepsilon_1 * \square + \varepsilon_2 * \square$$

$$\mathrm{d}\,\triangle = \varepsilon_1 * \triangle + \varepsilon_2 * \triangle$$

Example — two-site





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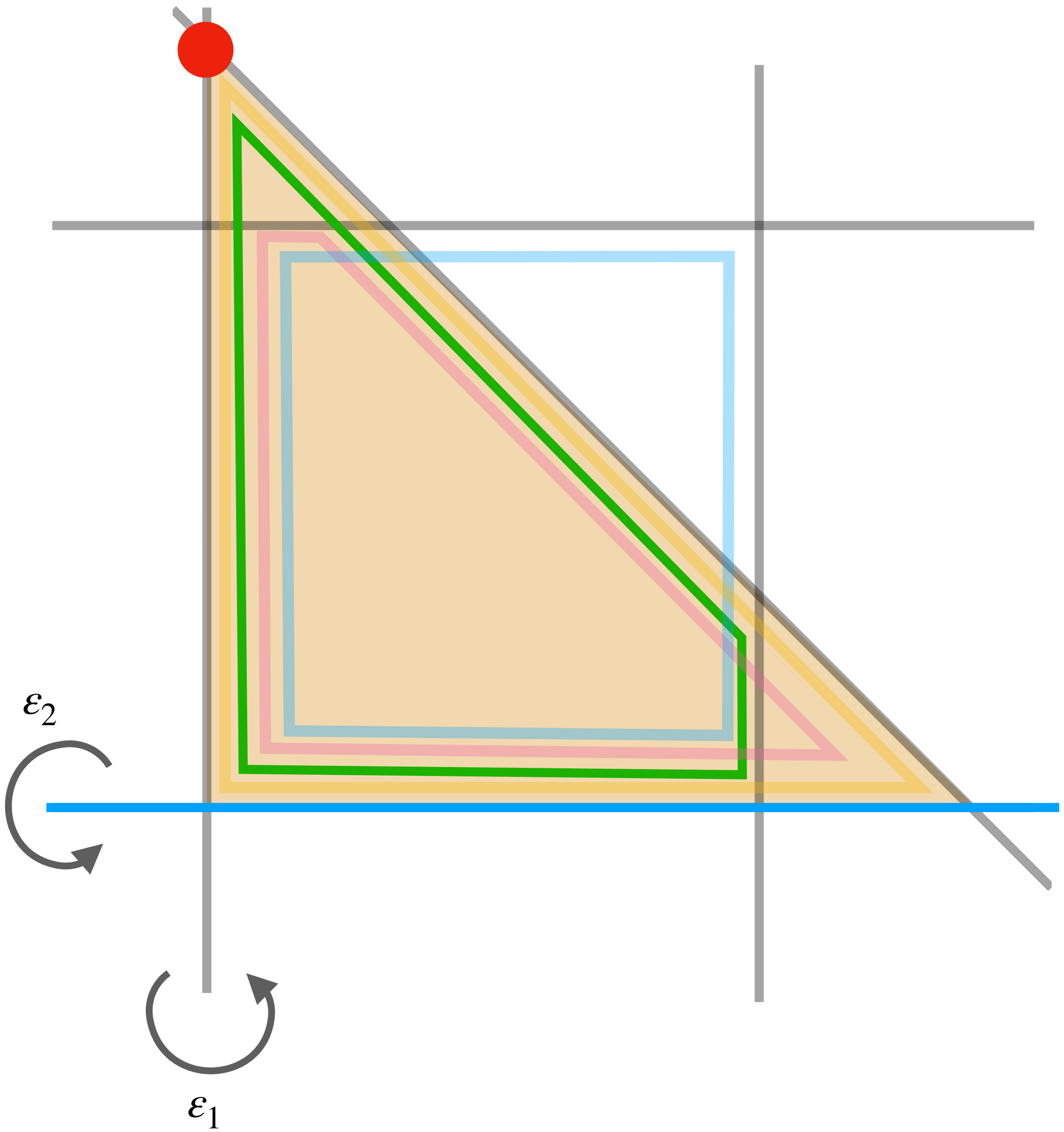

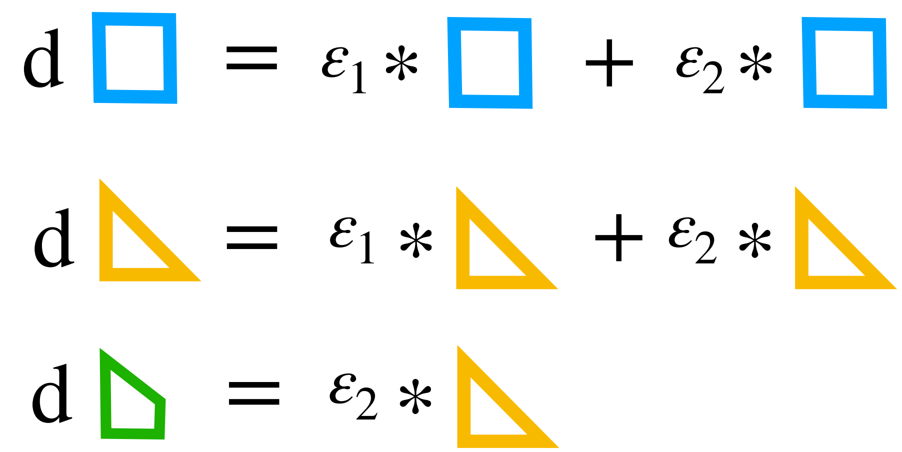

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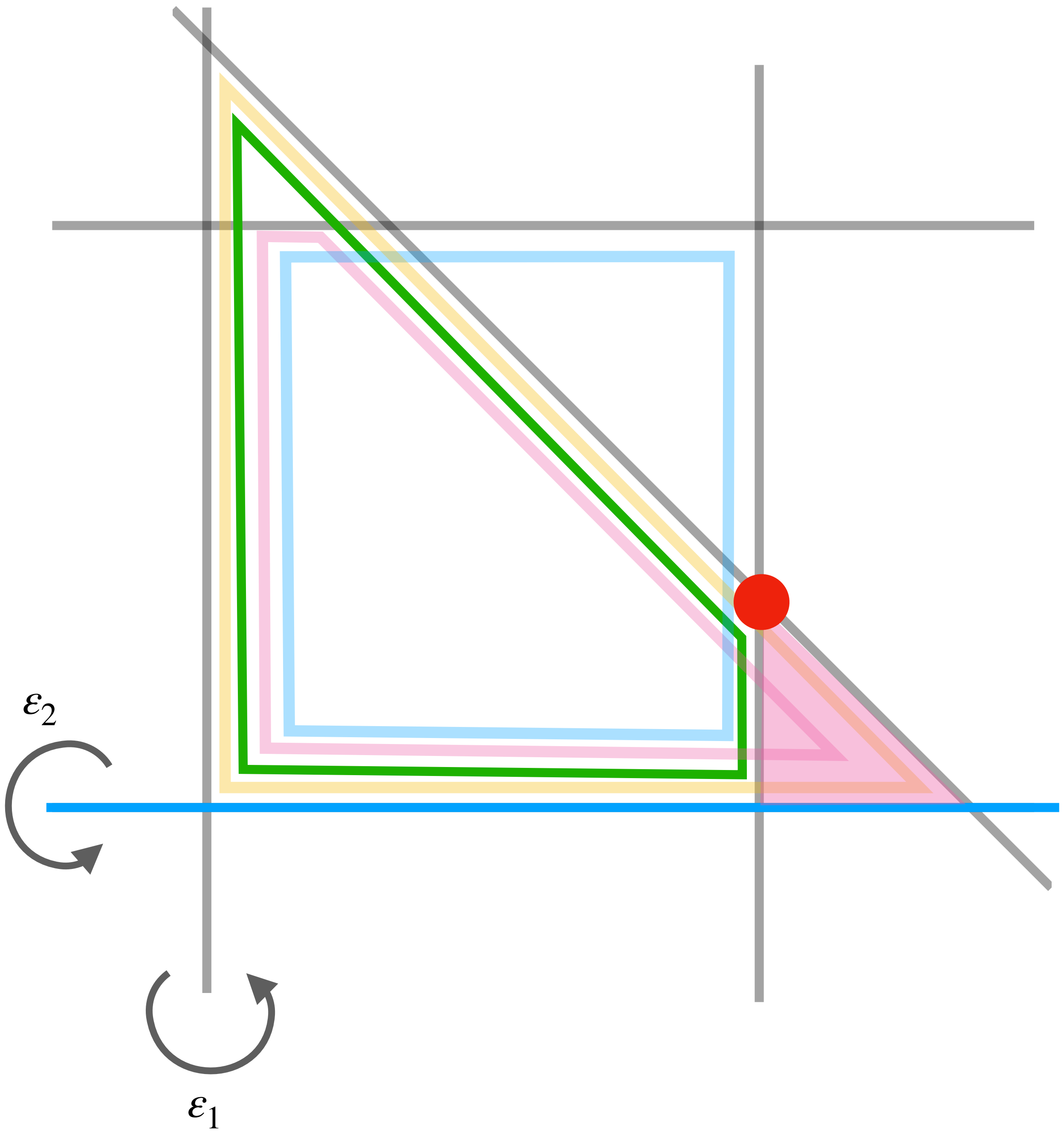
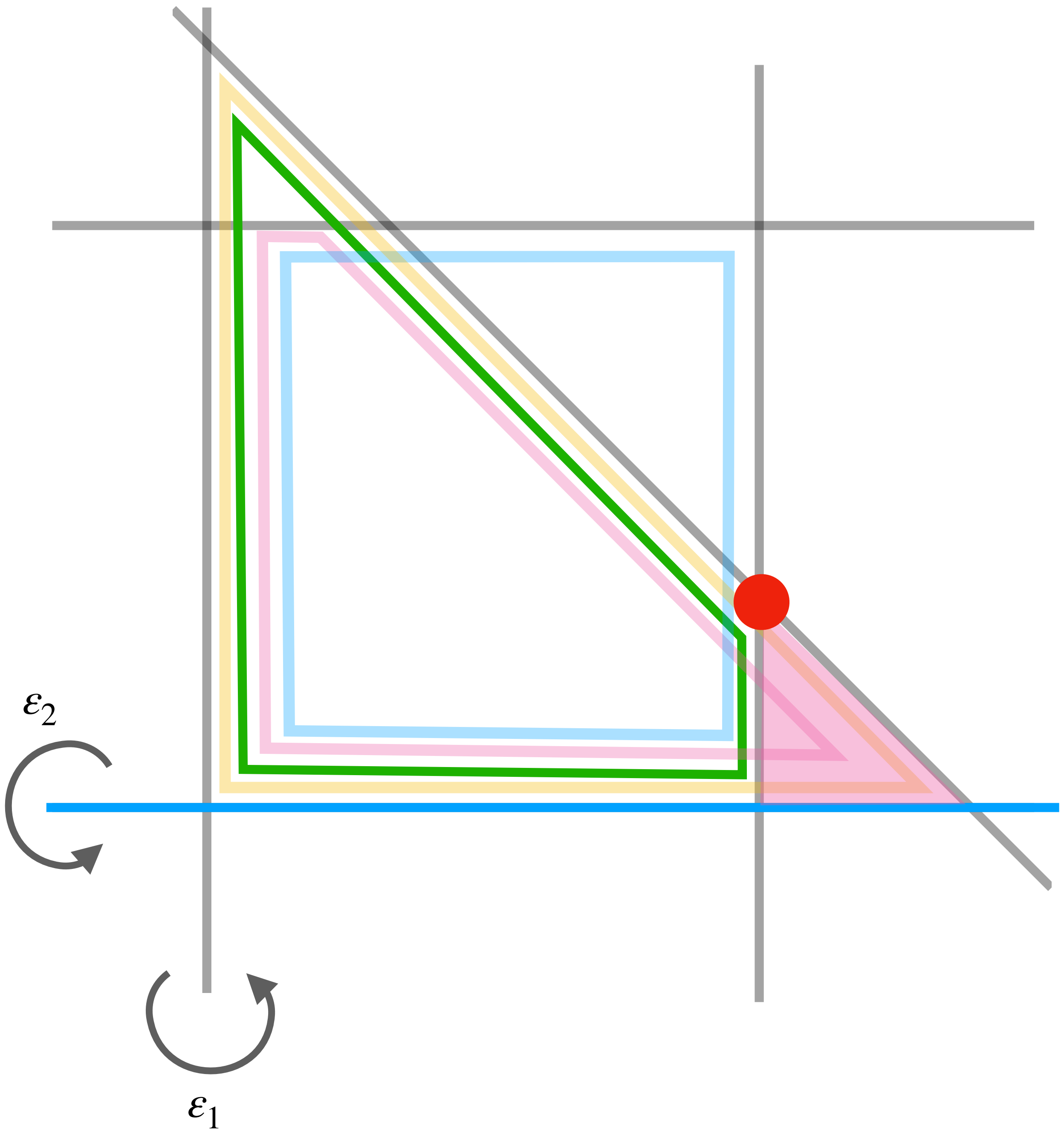


$$d\,\square = \varepsilon_1 * \square + \varepsilon_2 * \square$$

$$d\,\triangle = \varepsilon_1 * \triangle + \varepsilon_2 * \triangle$$

$$d\,\triangle = \varepsilon_2 * \triangle + \varepsilon_2 * \triangle$$

Example — two-site

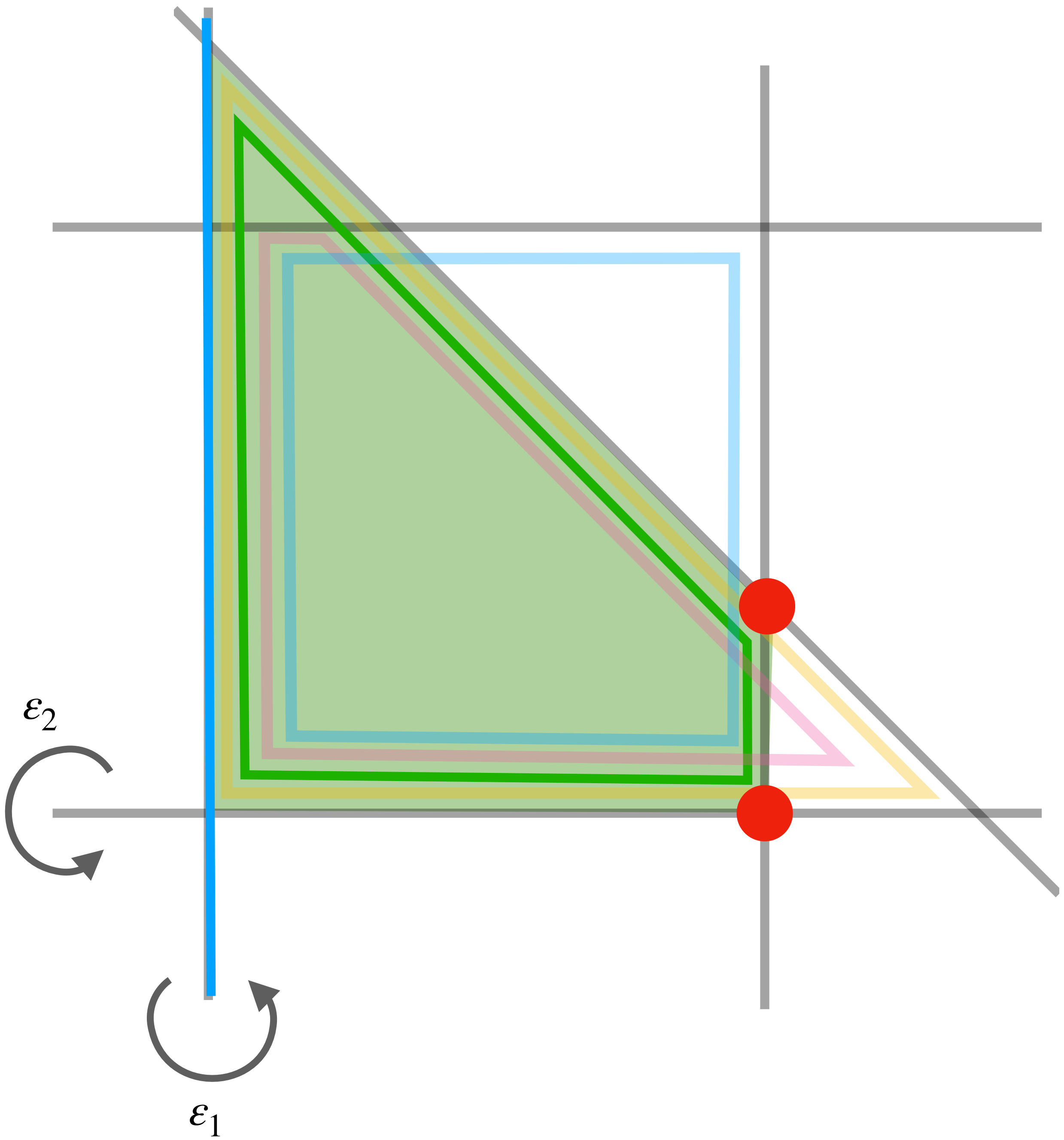


$$\mathrm{d}\,\square = \varepsilon_1 * \square + \varepsilon_2 * \square$$

$$\mathrm{d}\,\triangle = \varepsilon_1 * \triangle + \varepsilon_2 * \triangle$$

$$\mathrm{d}\,\triangle = \varepsilon_2 * \triangle + \varepsilon_2 * (\triangle - \triangle)$$

Example — two-site

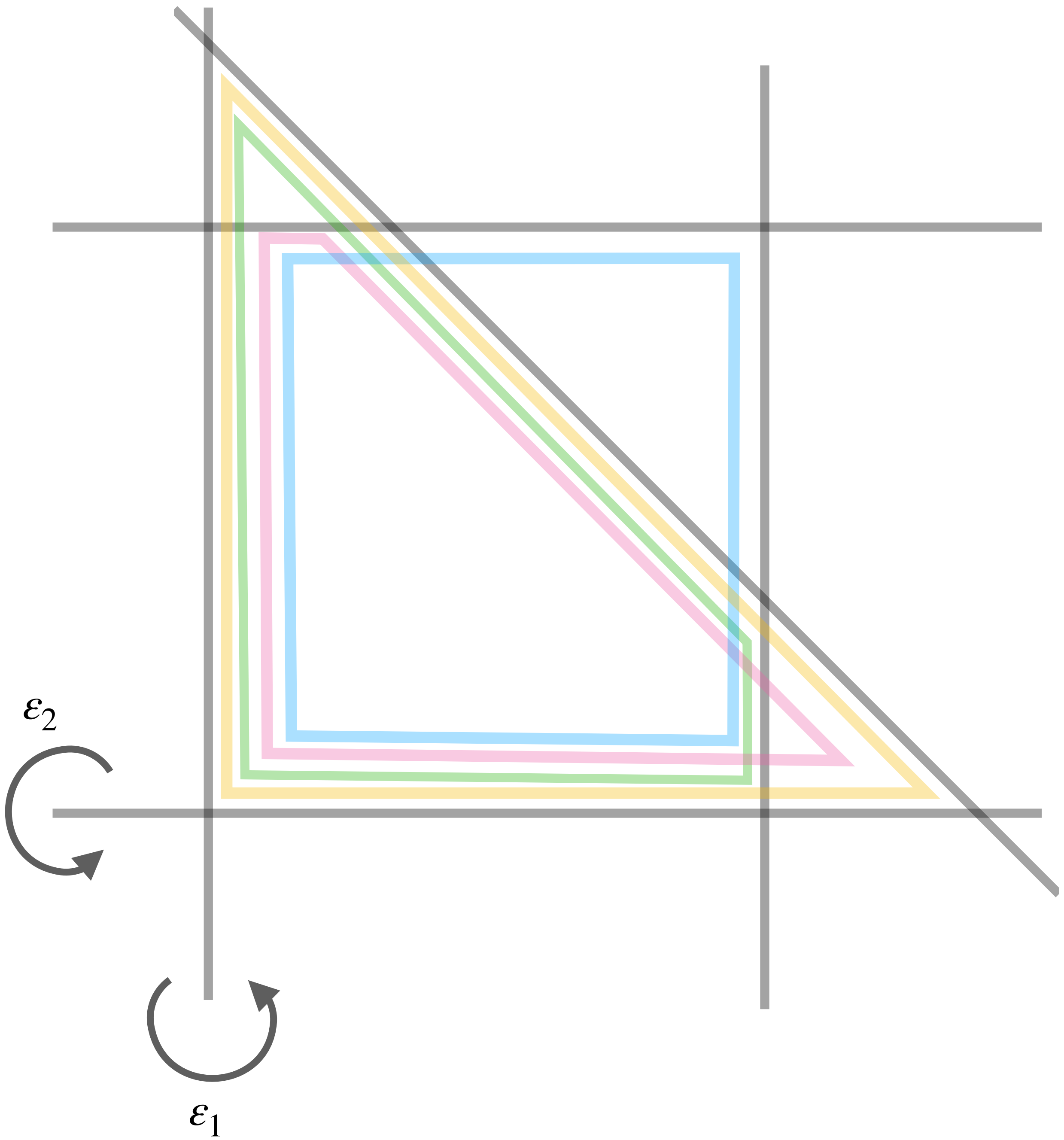


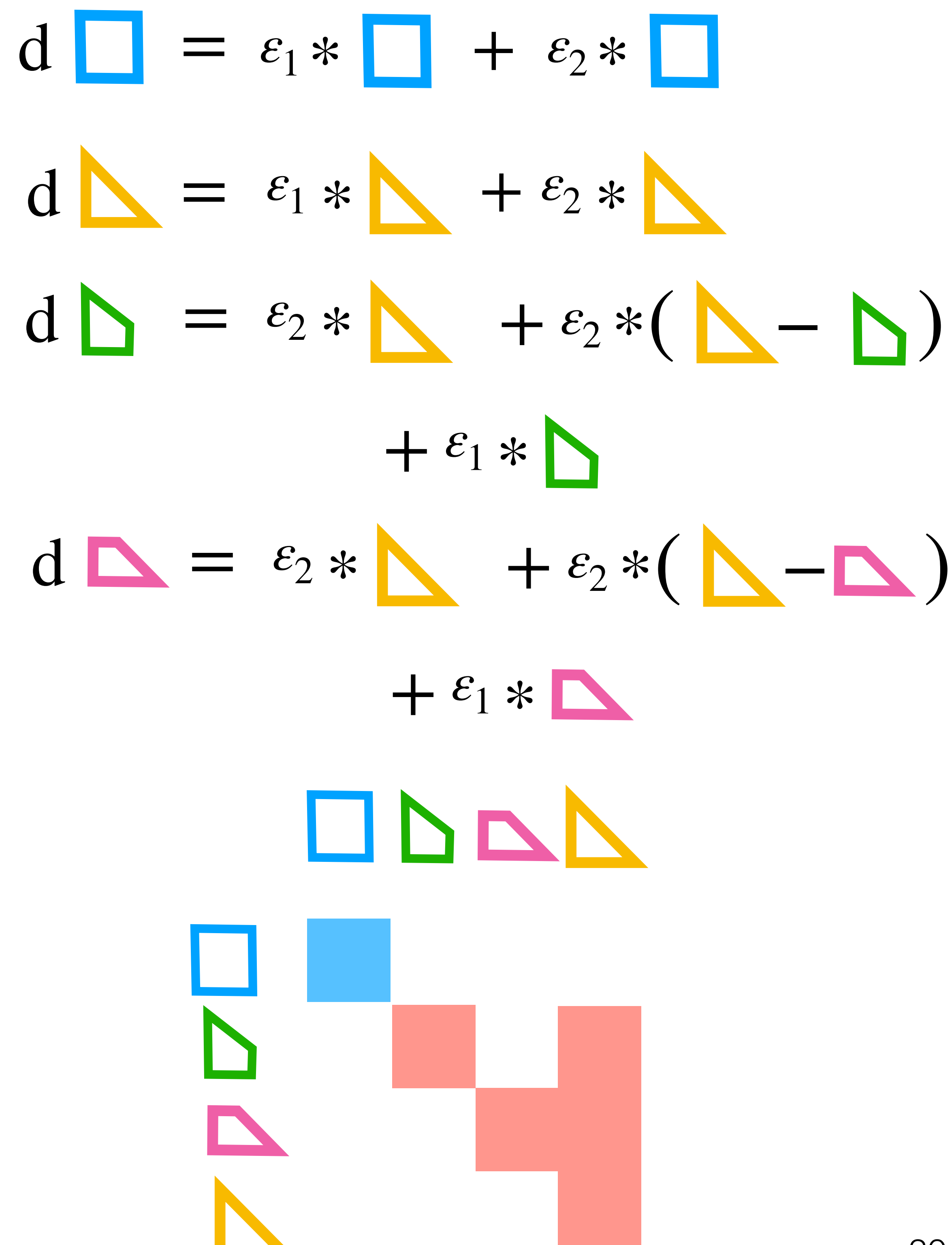
$$\mathrm{d}\,\square = \varepsilon_1 * \square + \varepsilon_2 * \square$$

$$\mathrm{d}\,\triangle = \varepsilon_1 * \triangle + \varepsilon_2 * \triangle$$

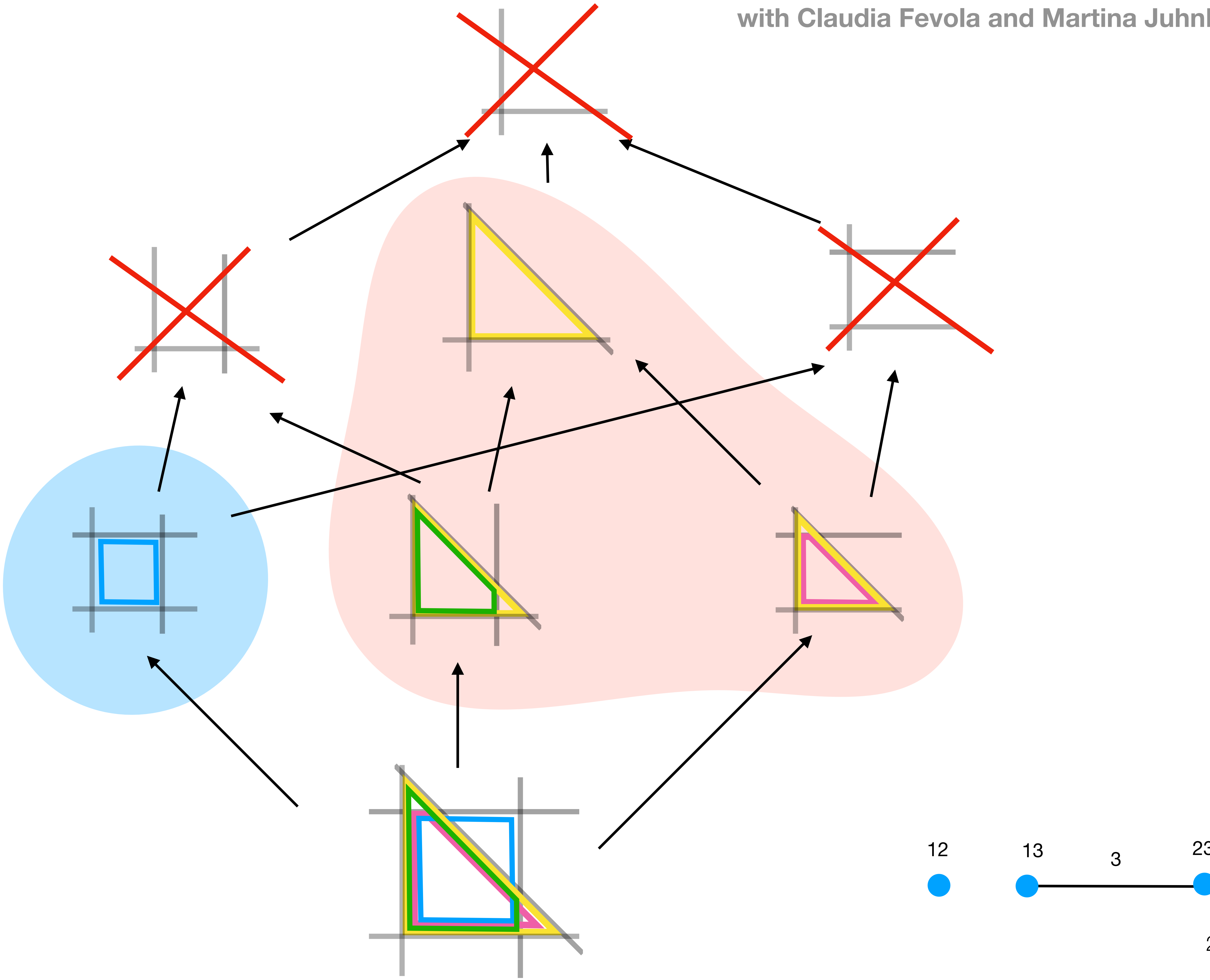
$$\mathrm{d}\,\triangle' = \varepsilon_2 * \triangle + \varepsilon_2 * (\triangle - \triangle') + \varepsilon_1 * \triangle'$$

Example — two-site



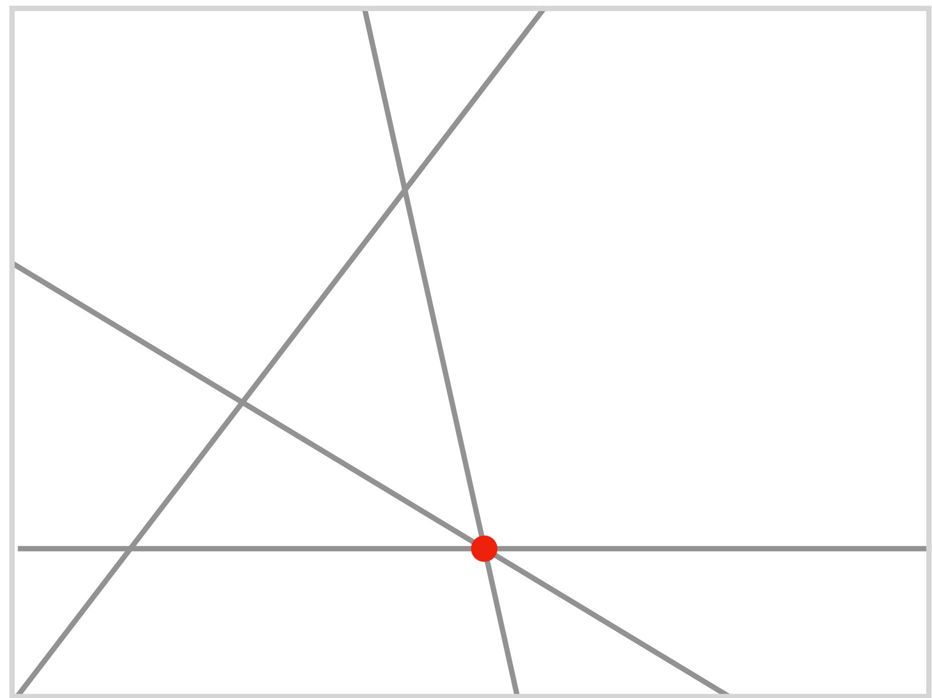


with Claudia Fevola and Martina Juhnke



Non-generic intersection

In physical examples (e.g. three-site), the displacement parameters are not independent.



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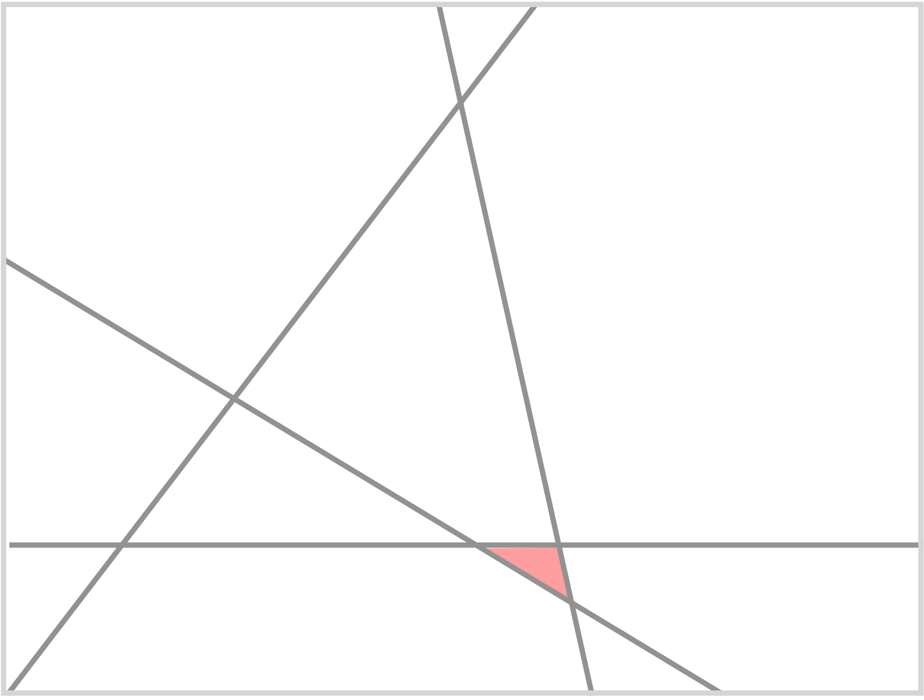

Degenerating the hyperplane arrangement amounts to:

- taking a quotient by bounded region(s) in the homology side
- adding relations between the integrands in the cohomology side

We can do both after computing the connection matrix.

Thank you!