Computational Algebraic Geometry for Feynman Integral Calculation

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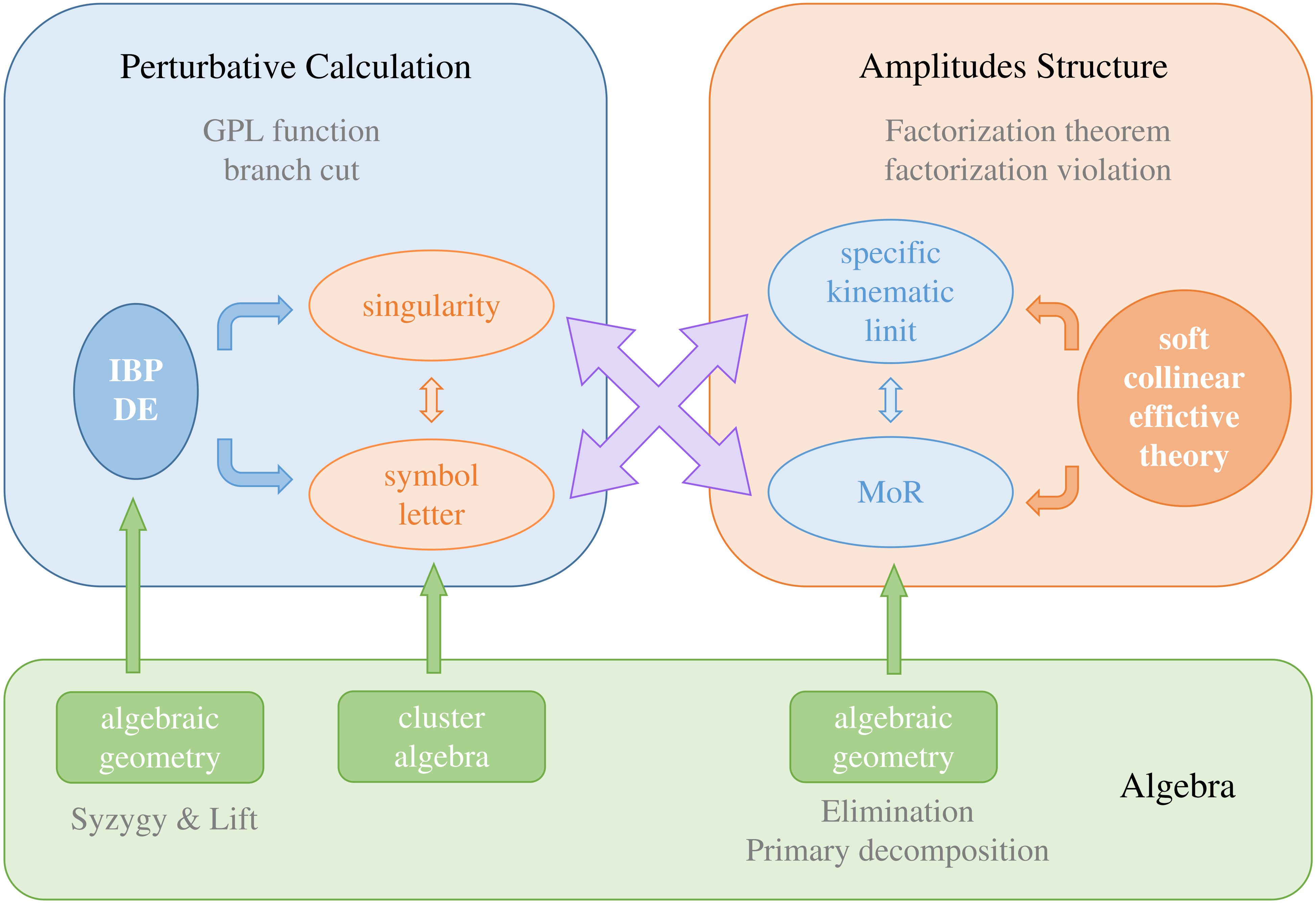
MathemAmplitudes
25/09/2025



Perturbative Calculation
GPL function
branch cut
IBP
DE
singularity
symbol
letter
Amplitudes Structure
Factorization theorem
factorization violation
specific
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MoR
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Syzygy & Lift
cluster
algebra
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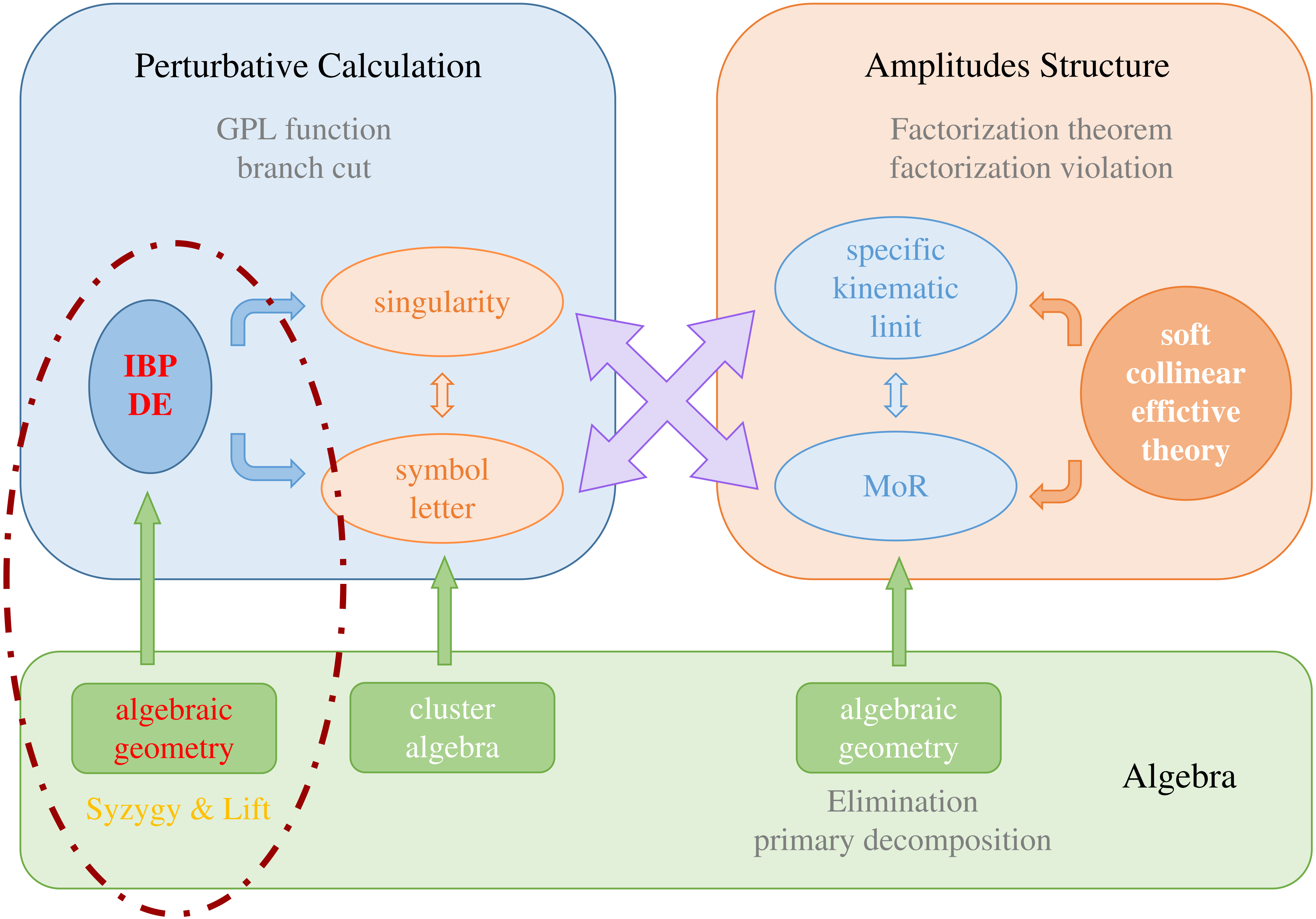
Elimination
Primary decomposition
Algebra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Algebraic Geometry & IBP

NeatIBP : a package generating small-size integration-by-parts relations for Feynman integrals

Comput.Phys.Commun. 316 (2025) 109798
Comput.Phys.Commun. 295 (2024) 108999

collaborate with Zihao Wu, Janko Boehm, Johann Usovitsch, Yingxuan Xu, Yang Zhang

Differential Equations for Energy Correlators in Any Angle

arXiv: 2506.02061

collaborate with Jianyu Gong, Jingwen Lin, Kai Yan and Yang Zhang

Singularity-Free Feynman Integral Bases

arXiv: 2508.04394

collaborate with Stefano De Angelis, David A. Kosower, Zihao Wu and Yang Zhang

Traditional IBP

$$0=\int\frac{\mathrm{d}^Dl_1}{i\pi^{D/2}}\cdots\frac{\mathrm{d}^Dl_L}{i\pi^{D/2}}\frac{\partial}{\partial l_k^\mu}\frac{v^\mu}{D_1^{\alpha_1}\cdots D_n^{\alpha_n}}=\int\frac{\mathrm{d}^Dl_1}{i\pi^{D/2}}\cdots\frac{\mathrm{d}^Dl_L}{i\pi^{D/2}}\frac{\frac{\partial v^\mu}{\partial l_k^\mu}-v^\mu\sum_{i=1}^n\frac{\partial D_i}{\partial l_k^\mu}\frac{\alpha_k}{D_i}}{D_1^{\alpha_1}\cdots D_n^{\alpha_n}}$$

target integrals

```
{G[1, 1, 1, 1, 1, 1, 1, 1, 0, -4, 0], G[1, 1, 1, 1, 1, 1, 1, 1, 0, -3, -1],
 G[1, 1, 1, 1, 1, 1, 1, 1, 0, -2, -2], G[1, 1, 1, 1, 1, 1, 1, 1, 0, -1, -3],
 G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, -4], G[1, 1, -2, 1, 1, 1, 1, 1, 0, 0, 0], ...
```

redundant integrals G[1, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0], G[1, 1, 2, 1, 1, 1, 1, 1, 0, -1, 0] ...

master integrals

```
G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, -1], G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0],
G[1, 1, 1, 0, 1, 1, 1, 1, 0, -1, 0], G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, -2],
G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, -1], G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0],...
```

redundant IBPs Time consuming! Memory consuming!

NeatIBP in Baikov representation

$$I[\alpha_1, \cdots, \alpha_n] = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

$\big\downarrow$ Baikov representation

$$I[\alpha_1, \cdots, \alpha_n] = C \int \mathrm{d}z_1 \cdots \mathrm{d}z_n P(z)^{\alpha} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

IBP operator $\quad O_{IBP} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} (a_i \cdot \,) \qquad a_i \in Q(\vec{s}_{ij})[z_1, \cdots, z_n]$

IBP relation $\quad \left(O_{IBP} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} (a_i \cdot \,) \right) I[\alpha_1, \cdots, \alpha_n] = 0$

Syzygy IBP in Baikov represetation

$$0=\int \mathrm{d}z_1\cdots\mathrm{d}z_n\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\left(a_i(z)P^{\alpha}\frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}\right)$$

$$\boxed{\alpha=\frac{D-L-E-1}{2}}$$

$$=\int \mathrm{d}z_1\cdots\mathrm{d}z_n\left(\sum_{i=1}^{n}\frac{\partial a_i}{\partial z_i}P^{\alpha}+\sum_{i=1}^{n}\alpha a_i\frac{\partial P}{\partial z_i}P^{\alpha-1}-P^{\alpha}\sum_{i=1}^{n}\alpha_i\frac{a_i}{z_i}\right)\frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}$$

$$\left(\textstyle\sum_{i=1}^{n}a_i(z)\frac{\partial P}{\partial z_i}\right)+b(z)P=0 \qquad a_i(z)=b_i(z)z_i \quad \text{for } i\in\{j|\alpha_j>0\}$$

Relate integrals in the same dimention

Avoid incearsing the degree of propagators

$$0=\int \mathrm{d}z_1\cdots\mathrm{d}z_n\left(\sum_{i=1}^{n}\left(\frac{\partial a_i}{\partial z_i}-\alpha_i b_i\right)+\alpha b\right)P^{\alpha}\frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}$$

Syzygy equation

$$O_{IBP} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} (a_i \cdot \ ) \qquad a_i \in Q(\vec{s}_{ij})[z_1, \cdots, z_n]$$

① $$\left(\sum_{i=1}^{n} a_i(z) \frac{\partial P}{\partial z_i}\right) + b(z) P = 0$$

$$M_1 = < f_1, f_2, \cdots >$$

② $$a_i(z) = b_i(z) z_i \ \text{ for } i \in \{j | \alpha_j > 0\}$$

$$M_2 = < g_1, g_2, \cdots >$$

Module Intersection $$\begin{pmatrix} a_i \\ b \end{pmatrix} \in M_1 \cap M_2$$

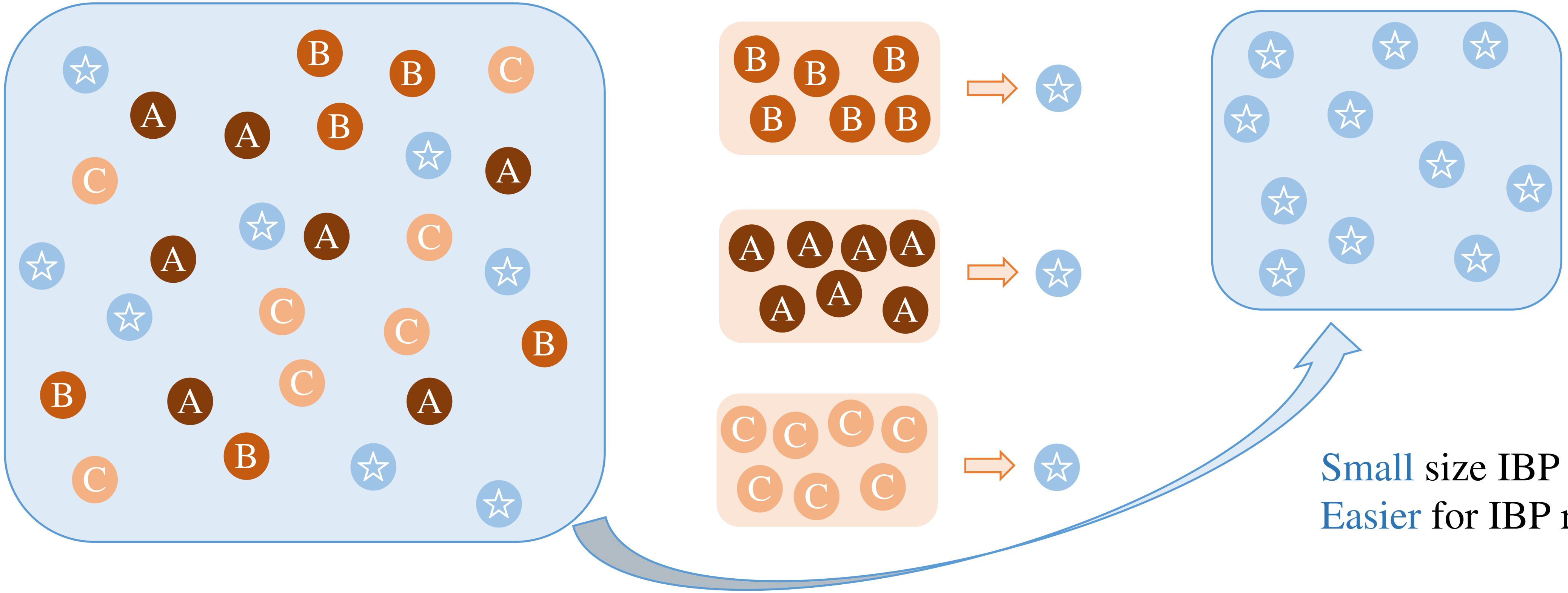
SINGULAR

The Magic of Syzygy

traditional IBP

syzygies make
the clever selections

NeatIBP

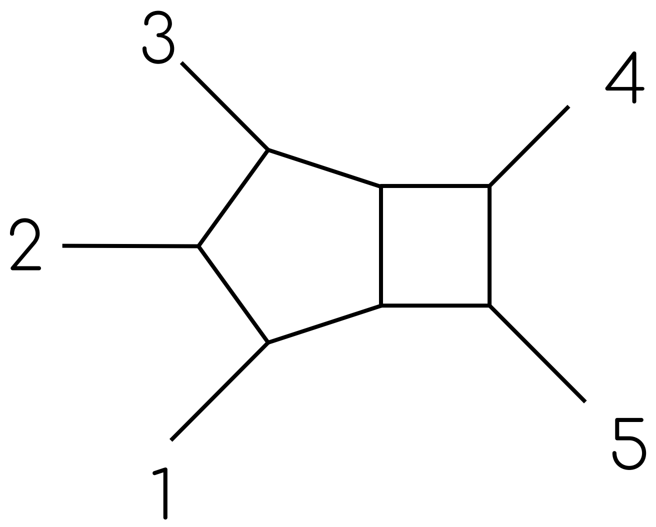


Small size IBP system
Easier for IBP reduction

expansive Gaussian elimination

Performance of NeatIBP

2L5P Example



Target integrals:
Max numerator degree: 5
Max denominator power: 1
Quantity: 2483

# of IBP (FIRE6): 11207942

# of IBP (NeatIBP): 14120

Time used: 27m at

CPU threads: 20
RAM: 128GB

Applications of NeatIBP:

https://github.com/yzhphy/NeatIBP

Studies that used NeatIBP	Reference	Date
Three-loop five-point pentagon-box-box Feynman diagram	*Phys.Rev.D* 112 (2025) 1, 016021	2025.7.23
Two-loop QCD helicity amplitudes for $g\ g \to g\ t\ \bar{t}$ at leading color	*JHEP* 03 (2025) 070	2025.3.11
Full-color double-virtual amplitudes for $q\ \bar{q} \to b\ \bar{b}\ H$	*JHEP* 03 (2025) 066	2025.3.11
Two-loop QCD corrections for $p\ p \to t\ \bar{t}\ j$	arXiv: 2411.10856	2024.11.16
Two-loop amplitudes for $W\ \gamma\ \gamma$ production at LHC	*JHEP* 12 (2025) 221	2024.12.30
NLO corrections to $J/\Psi\ c\ \bar{c}$ photoproduction	*Phys.Rev.D* 110 (2024) 9, 094047	2024.11.11
Two-loop five-point two-mass planar integrals	*JHEP* 10 (2024) 167	2024.10.23
Two-loop integrals for $t\ \bar{t}\ j$ production at hadron colliders in the leading color approximation	*JHEP* 07 (2024) 073	2024.7.9

Obtain more examples by using NeatIBP yourself

- People have a lot of research at the level of perturbative scattering amplitudes, and have a good understanding about their structures. However, the observables or cross section attract less attention.

- An interesting and simple class of observables are Energy Correlators.

Algebraic Geometry & IBP

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Motivation

- From the phenomenological point of view, energy correlators can be used as jet observables for verify the standard model or find new physics.

- From the computability, energy correlators is perhaps the simplest infared safe observable to calculate analytically.

$$Q(q) \to P_1(p_1) + P_2(p_2) + P_3(p_3) + P_4(p_4)$$

Z boson, off-shell photon or Higgs   q qbar g g   q qbar q qbar   g g g g

n-point energy correlator

$$E^n C(\theta_{ij}) = \int \prod_{i=1}^{n} d\Omega_{\vec{n}_i} \prod_{i\neq j} \delta(\vec{n}_i \cdot \vec{n}_j - \cos(\theta_{ij})) \frac{\int d^4 x e^{iqx} \langle 0|\mathcal{O}^{+}(x)\mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\mathcal{O}(0)|0\rangle}{(q^0)^n \int d^4 x e^{iqx} \langle 0|\mathcal{O}^{+}(x)\mathcal{O}(0)|0\rangle}$$

create the final state X

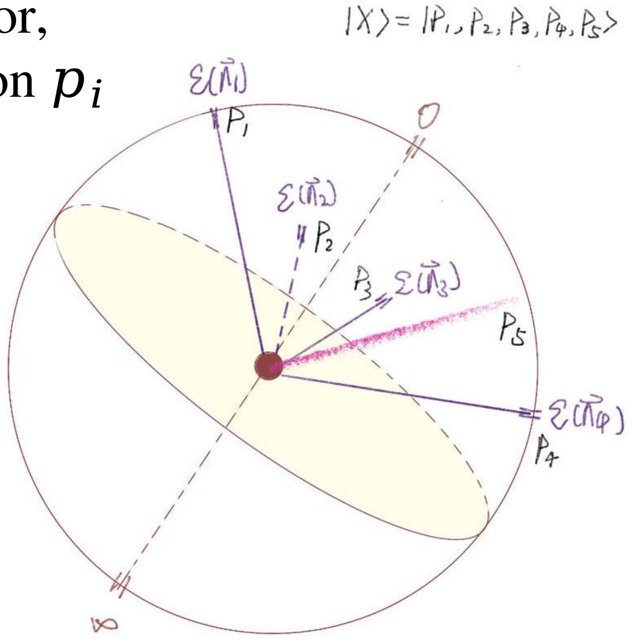
$$\Downarrow \quad \int d^4 x\, e^{iqx} \langle X|\mathcal{O}(x)|0\rangle \equiv (2\pi)^4 \delta^4(q - q_X) F_X$$

form factor, depends on p_i

$$\sim \sum_{(n_1,\cdots,n_n)\in X} \int d\Pi_X \left(\prod_{i=1}^{n} \delta^2(\vec{n}_i - \hat{p}_{n_i}) \frac{E_i}{q^0} \right) |F_X|^2$$

where $d\Pi_X$ is onshell phase-space of the finial state.

$$\frac{2\left(2q^4 (p_1+p_2+p_4+p_5)\cdot(p_2+p_3+p_4+p_5) + \cdots - q^4 (p_2+p_3+p_4+p_5)\cdot(p_1+p_2+p_3+p_4+p_5) + q^6\right)}{(p_3+p_4)^2 (p_1+p_5)^2 (p_1+p_4+p_5)^2 (p_1+p_2+p_4+p_5)^2 (p_3+p_4+p_5)^2 (p_2+p_3+p_4+p_5)^2}$$



Kai Yan and Xiaoyuan Zhang. Three-Point Energy Correlator in N=4 Supersymmetric Yang-Mills Theory. *Phys. Rev. Lett.*, 129(2):021602, 2022.

Setup

energy parameters: $x_i = \frac{2\,q\cdot p_i}{q^2}\,, \quad i = 1,\cdots,n$

angle parameters: $\zeta_{ij} = \frac{q^2\, p_i\cdot p_j}{2\,q\cdot p_i\, q\cdot p_j}\,, \qquad i,j = 1,\cdots,n$

$$\mathrm{E^nC}(\vec{\zeta}_{ij})\big|_{\mathrm{LO}} \sim \int d^4 p_{n+1}\delta^4(q - p_1 - \cdots - p_{n+1})\,\delta_+(p_{n+1}^2)$$
$$\times \int_0^1 dx_1\cdots dx_n\,(x_1\cdots x_n)^2\left[|F_{n+1}^{(0)}|^2(p_1,\cdots,p_{n+1}) + \mathrm{perm.}(1,\cdots,n+1)\right]$$

$\Downarrow$ integrate out p_{n+1}

$$\mathrm{E^nC}(\vec{\zeta}_{ij})\big|_{\mathrm{LO}} \sim \int_0^1 dx_1\cdots dx_n\,(x_1\cdots x_n)^2\delta(1 - Q_n)\,|F_{n+1}^{(0)}|^2_{\mathrm{sym.}}$$

where $Q_n = \sum_i x_i - \sum_{(ij)} x_i x_j \zeta_{ij}\,.$

Lift Differential Equations: workflow

$E^3\ C/S_3$ (permutation symmetry)

↓ Partial Fraction Decomposition

Classify simple finite integral families

↓ Syzygy Equations

Finite master integrals — IBP and iterative boundary IBP

↓ Lift Equations

Finite differential equations

↓ Cubically nilpotent

Canonical differetial equations

$\mathrm{E^3C}$ Divergent Region

- Propagators $\mathcal{D}_1 = \frac{s_{134}}{q^2} = -1 + x_2,\ \mathcal{D}_2 = \frac{s_{124}}{q^2} = -1 + x_3,\ \mathcal{D}_3 = \frac{s_{123}}{q^2} = -1 + x_1 + x_2 + x_3,$

$$\mathcal{D}_4 = \frac{s_{34}}{q^2 x_3} = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ \mathcal{D}_5 = \frac{s_{24}}{q^2 x_2} = -1 + x_1\zeta_{12} + x_3\zeta_{23}.$$

- Consider delta function measure

$$\delta(1 - x_1 - x_2 - x_3 + \zeta_{12}x_1x_2 + \zeta_{23}x_2x_3 + \zeta_{13}x_1x_3)$$

region 1 $\{\, x_1 \to 1 + (\zeta_{12} + \zeta_{13} - 2)cx_1,\ x_2 \to cx_2,\ x_3 \to cx_3 \,\}|_{c\to 0}\,,$

region 2 $\{\, x_1 \to cx_1,\ x_2 \to 1 + (\zeta_{12} + \zeta_{23} - 2)cx_2,\ x_3 \to cx_3 \,\}|_{c\to 0}\,,$

region 3 $\{\, x_1 \to cx_1,\ x_2 \to cx_2,\ x_3 \to 1 + (\zeta_{13} + \zeta_{23} - 2)cx_3 \,\}|_{c\to 0}\,.$

Potentially Divergent Region

- Power counting

$$\{\mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_3,\, \mathcal{D}_4,\, \mathcal{D}_5\} \xrightarrow[\text{counting}]{\text{power}} \begin{cases} \text{region 1} & \{0,\, 0,\, 1,\, 0,\, 0\} \\ \text{region 2} & \{1,\, 0,\, 1,\, 0,\, 0\}. \\ \text{region 3} & \{0,\, 1,\, 1,\, 0,\, 0\} \end{cases}$$

$$\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(\mathcal{D}_\delta) \xrightarrow[\text{counting}]{\text{power}} 2$$

Power Counting

$$D_1 = -1 + x_3,\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_3 = -1 + x_1\zeta_{12} + x_3\zeta_{23},$$
$$D_\delta = 1 - x_1 - x_2 - x_3 + x_1x_2\zeta_{12} + x_2x_3\zeta_{23} + x_1x_3\zeta_{13}$$

$$\mathrm{Int}[1,1,-1,1] = \int dx_1dx_2dx_3 \frac{D_3\,\delta(D_\delta)}{D_1D_2}$$

Finite integral
power counting of c is positive

```
{Int[-2, 0, 0, 1], Int[-2, 0, 1, 1], Int[-2, 1, -1, 1], Int[-2, 1, 0, 1],
 Int[-2, 1, 1, 1], Int[-2, 1, 2, 1], Int[-2, 2, -1, 1], Int[-2, 2, 0, 1],
 Int[-2, 2, 1, 1], Int[-1, -1, 0, 1], Int[-1, 0, -1, 1], Int[-1, 0, 0, 1],
 Int[-1, 1, -2, 1], Int[-1, 1, -1, 1], Int[-1, 1, 2, 1], Int[-1, 2, -2, 1],
 Int[-1, 2, -1, 1], Int[-1, 2, 0, 1], Int[-1, 2, 1, 1], Int[0, -1, -1, 1],
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 Int[0, 1, -2, 1], Int[0, 1, 2, 1], Int[0, 2, -3, 1], Int[0, 2, -2, 1],
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 Int[1, -1, -2, 1], Int[1, -1, -1, 1], Int[1, 0, -3, 1], Int[1, 0, -2, 1],
 Int[1, 1, -3, 1], Int[1, 1, -2, 1], Int[1, 2, -4, 1], Int[1, 2, -3, 1],
 Int[1, 2, -2, 1], Int[1, 2, -1, 1], Int[1, 2, 0, 1], Int[-1, 0, 1, 1],
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 Int[0, 1, 1, 1], Int[1, 0, 0, 1], Int[1, 1, -1, 1], Int[1, 1, 0, 1]}
```

region: $\{x_1 \to 0,\ x_2 \to 0,\ x_3 \to 1\}$

$$dx_1dx_2dx_3\delta(D_\delta) \xrightarrow[\text{counting}]{\text{power}} 2$$

$$\{D_1,\ D_2,\ D_3\} \xrightarrow[\text{counting}]{\text{power}} \{1,0,0\}$$

Syzygy for finite IBP

SINGULAR

$$\mathrm{Int}[n_1, n_2, n_3, 1] = \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \frac{\delta(D_\delta)}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_\delta}\Big|_{\mathrm{cut}(D_\delta)}$$

Integration by parts (IBP) $\longrightarrow$ Syzygy Equations

$$\mathcal{O}_{\mathrm{IBP}} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (a_i \cdot\ )$$

$$\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0,$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$

for divergent propagators and D_δ

Not increase the power of divergent propagators!

Lift for finite differential equation

Derivative of a certain parameter $\longrightarrow$ Lift Differential Equations

$$\mathcal{O}_{\partial\zeta_{**}} = \frac{\partial}{\partial\zeta_{**}} + \mathcal{O}_{\mathrm{IBP}} \equiv \boxed{\frac{\partial}{\partial\zeta_{**}}} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a_i$$

$$\frac{\partial}{\partial\zeta_{**}} D_j + \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0,$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$

Find the differential equations of a certain kinematic

May bring higher power of divergent propagators

When the Lift equation is satisfied, the integral will be finite!

Boundary IBP

$$D_1 = -1 + x_3,\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_3 = -1 + x_1\zeta_{12} + x_3\zeta_{23},$$
$$D_\delta = 1 - x_1 - x_2 - x_3 + x_1x_2\zeta_{12} + x_2x_3\zeta_{23} + x_1x_3\zeta_{13}$$

$$\mathcal{O}_{\mathrm{IBP}}\,\mathrm{Int}[n_1, n_2, n_3, 1] = \sum_{i=1}^{3}(\mathrm{BT}_{x_i=1} - \mathrm{BT}_{x_i=0}).$$

subfamily 1 $\quad D_1 = -1 + x_1\zeta_{12},\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_\delta = 1 - x_1 - x_2 + x_1x_2\zeta_{12}\,;$

subfamily 2 $\quad D_1 = -1 + x_3\zeta_{13},\ D_2 = -1 + x_1\zeta_{13},\ D_\delta = 1 - x_1 - x_3 + x_1x_3\zeta_{13}\,;$

subfamily 3 $\quad D_1 = -1 + x_3\zeta_{23},\ D_2 = -1 + x_2\zeta_{23},\ D_\delta = 1 - x_2 - x_3 + x_2x_3\zeta_{23}\,.$

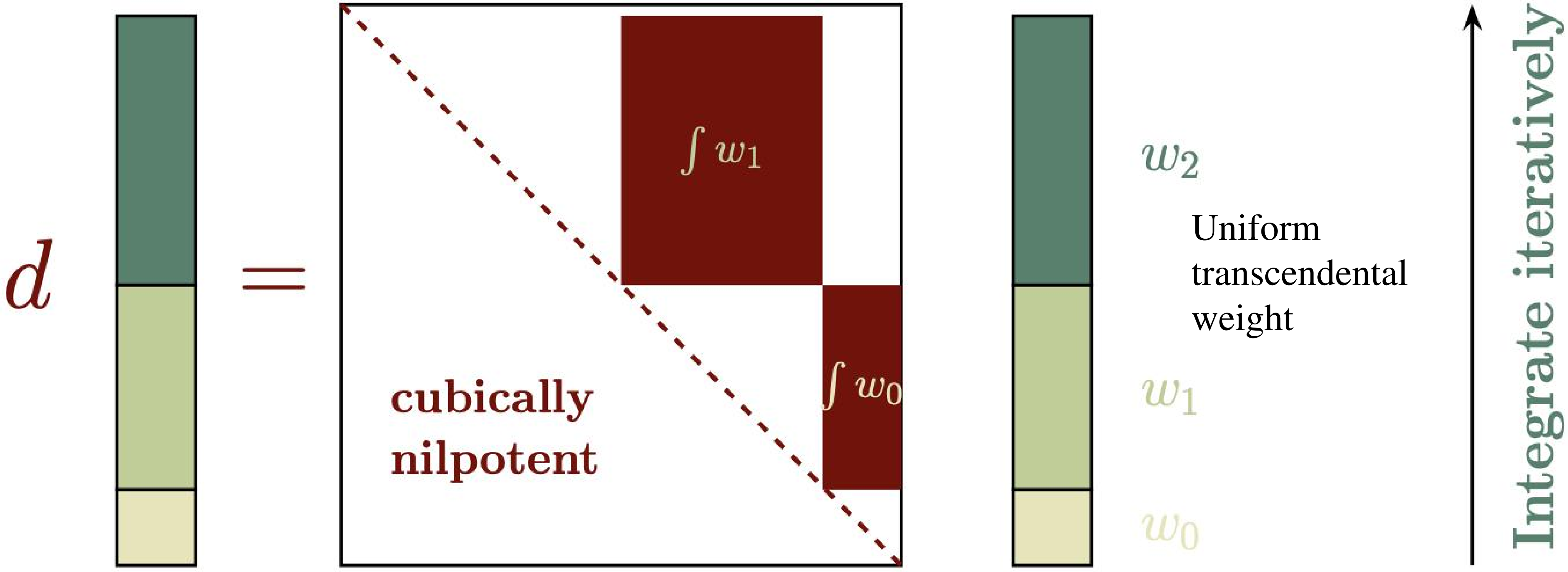
Master Integrals

$$\{\mathrm{Int}[1,1,-1,1],\ \mathrm{Int}[-1,1,1,1],\ \mathrm{Int}[1,1,0,1],\ \mathrm{Int}[0,1,1,1],\ \mathrm{Int}[1,0,0,1],$$
$$\mathrm{Int}_2[1,\{0,0,1\}],\ \mathrm{Int}_2[2,\{0,1,1\}],\ \mathrm{Int}_2[2,\{0,0,1\}],\ \mathrm{Int}_2[3,\{0,0,1\}],\ 1\}$$

Canonical Differential Equations

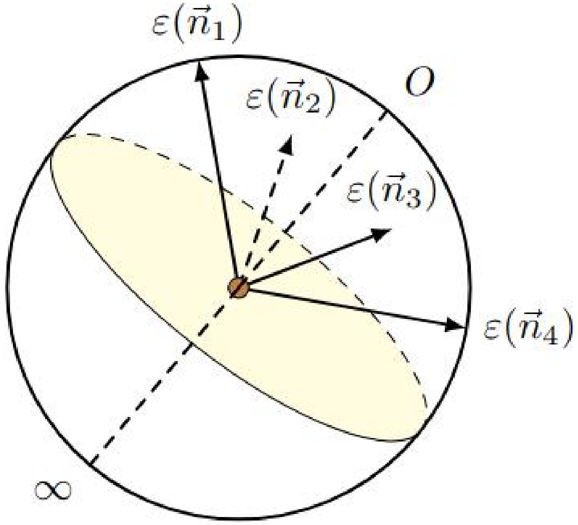
Differential Equations include boundary integrals

$$\frac{\partial}{\partial \zeta_{**}} \mathrm{Int}[n_1, n_2, n_3, 1] - \mathcal{O}_{\partial \zeta_{**}} \mathrm{Int}[n_1, n_2, n_3, 1] = \sum_{i=1}^{3} (\mathrm{BT}_{x_i=0}).$$



Uniform
transcendental
weight

Four Point Energy Correlator



one term in form factor

$$\frac{2\left(2q^4\left(p_1+p_2+p_4+p_5\right)\cdot\left(p_2+p_3+p_4+p_5\right)+\cdots-q^4\left(p_2+p_3+p_4+p_5\right)\cdot\left(p_1+p_2+p_3+p_4+p_5\right)+q^6\right)}{\left(p_3+p_4\right)^2\left(p_1+p_5\right)^2\left(p_1+p_4+p_5\right)^2\left(p_1+p_2+p_4+p_5\right)^2\left(p_3+p_4+p_5\right)^2\left(p_2+p_3+p_4+p_5\right)^2}$$

$$\frac{5x_1x_2x_3x_4\left(2\zeta_{12}x_2x_1+2\zeta_{13}x_3x_1+2\zeta_{14}x_4x_1+2\zeta_{23}x_2x_3+2\zeta_{34}x_3x_4-2x_1-x_2-2x_3-x_4+1\right)}{\left(\zeta_{12}x_2+\zeta_{13}x_3+\zeta_{14}x_4-1\right)\left(\zeta_{14}x_1+\zeta_{24}x_2+\zeta_{34}x_3-1\right)\left(\zeta_{13}x_1x_3+\zeta_{23}x_2x_3+\zeta_{34}x_4x_3+\zeta_{14}x_1x_4+\zeta_{24}x_2x_4-x_3-x_4\right)\cdots}$$

Dmitry Chicherin, Ian Moult, Emery Sokatchev, Kai Yan, and Yunyue Zhu. The Collinear Limit of the Four-Point Energy Correlator in $\mathcal{N}=4$ Super Yang-Mills Theory. 1 2024.

Propagators in E^{4}C

$$\delta(1 - x_1 - x_2 - x_3 - x_4 + x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_1x_4\zeta_{14} + x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}).$$

$$\frac{s_{45}}{x_4} = 1 - x_1\zeta_{14} - x_2\zeta_{24} - x_3\zeta_{34}\,, \quad \frac{s_{15}}{x_1} = 1 - x_2\zeta_{12} - x_3\zeta_{13} - x_4\zeta_{14}\,,$$

$$s_{2345} = 1 - x_1\,, \quad s_{1345} = 1 - x_2\,, \quad s_{1235} = 1 - x_4\,, \quad s_{1234} = 1 - x_1 - x_2 - x_3 - x_4\,,$$

$$\boxed{\begin{aligned} &s_{345} = 1 - x_1 - x_2 + x_1x_2\zeta_{12}\,, \quad s_{145} = 1 - x_2 - x_3 + x_2x_3\zeta_{23}\,, \quad s_{125} = 1 - x_3 - x_4 + x_3x_4\zeta_{34}\,,\\ &s_{123} = x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_2x_3\zeta_{23}\,, \quad s_{234} = x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}\,.\end{aligned}}$$

Highly algebraic dependent

$\downarrow$ partial fraction decomposition

An integral countains three propagators at most

Partial Fraction Deconposition

$$\langle D_1, D_2, \cdots, D_n \rangle = \langle 1 \rangle$$

coprime polynomials
GCD=1

$$1 = a_1 D_1 + a_2 D_2 + \cdots + a_n D_n$$

$$\frac{1}{D_1 D_2 \cdots D_n} = \frac{a_1 D_1 + a_2 D_2 + a_n D_n}{D_1 D_2 \cdots D_n} = \sum_{i=1}^{n} \frac{a_i}{D_1 D_2 \cdots \hat{D}_i \cdots D_n}$$

Reduction at integrand level

Lift Differential Equations: Overview

4-point energy correlator

E^3C/S^3

↓ Partial Fraction Decomposition

Classify simple finite integral families

↓ Syzygy Equations

IBP and iterative boundary IBP

↓ Lift Equations

Finite differential equations

↓ Cubically nilpotent

Canonical differetial equations

+ **Combine Divergent Integrals into Finite Integrals**

+ **polynomial reduction integrand reduction**

Difficulties of Higher Points

- Increasing number of integral variables
- Polynomial Propagators
- Complex delta function

Four-Point Energy Correlators

$$\delta(1 - x_1 - x_2 - x_3 - x_4 + x_1 x_2 \zeta_{12} + x_1 x_3 \zeta_{13} + x_1 x_4 \zeta_{14} + x_2 x_3 \zeta_{23} + x_2 x_4 \zeta_{24} + x_3 x_4 \zeta_{34}).$$

$$D_1 = -1 + x_1 \zeta_{14} + x_2 \zeta_{24} + x_3 \zeta_{34}\,, \quad D_2 = -1 + x_2 \zeta_{12} + x_3 \zeta_{13} + x_4 \zeta_{14}\,,$$

$$D_3 = -1 + x_1\,, \quad D_4 = -1 + x_2\,, \quad D_5 = -1 + x_4\,, \quad D_6 = -1 + x_1 + x_2 + x_3 + x_4\,,$$

$$\boxed{\begin{aligned} &D_7 = -1 + x_1 + x_2 - x_1 x_2 \zeta_{12}\,, D_8 = -1 + x_2 + x_3 - x_2 x_3 \zeta_{23}\,, D_9 = -1 + x_3 + x_4 - x_3 x_4 \zeta_{34}\,, \\ &D_{10} = x_1 x_2 \zeta_{12} + x_1 x_3 \zeta_{13} + x_2 x_3 \zeta_{23}\,, D_{11} = x_2 x_3 \zeta_{23} + x_2 x_4 \zeta_{24} + x_3 x_4 \zeta_{34}\,. \end{aligned}}$$

$\{D_6, D_7\}$ $\{D_6, D_8\}$ $\{D_6, D_9\}$
$\{D_7, D_9\}$

$\{D_{10}, D_{11}\}$

elliptic

hyperelliptic g=2

- We are very happy to see the integrals in energy correlators are finit in dimension 4

- Amplitudes may divergent for d=4, this ask for the dimensional regulator $\varepsilon = d - 4$

- it is important to handle ε singularity in an appropriate way.

Algebraic Geometry & IBP

NeatIBP : a package generating small-size integration-by-parts relations for Feynman integrals

Comput.Phys.Commun. 316 (2025) 109798
Comput.Phys.Commun. 295 (2024) 108999

collaborate with Zihao Wu, Janko Boehm, Johann Usovitsch, Yingxuan Xu, Yang Zhang

Differential Equations for Energy Correlators in Any Angle

arXiv: 2506.02061

collaborate with Jianyu Gong, Jingwen Lin, Kai Yan and Yang Zhang

Singularity-Free Feynman Integral Bases

arXiv: 2508.04394

collaborate with Stefano De Angelis, David A. Kosower, Zihao Wu and Yang Zhang

Singularity-Free Bases

- no explicit singularities in ϵ appear in the coefficients of IBP reduction result
- no explicit singularities in ϵ are present in the definition of any basis integral

local ring $R \equiv \mathbb{F}[x]_{\mathbf{m}}$ is a polynomial ring $\mathbb{F}[x]$ *localized* on a maximal ideal $\mathbf{m}$

$$R = \left\{ \frac{f(x)}{g(x)}, \quad f(x), g(x) \in \mathbb{F}[x], g(x) \notin \mathbf{m} \right\}.$$

- We are working in the local ring $\mathbb{Q}[\epsilon]_{\langle \epsilon \rangle}$

Gaussian Elimination in a Local Ring

$p = 1$, IBP matrix $C(m, n)$, relevant sorted integrals I

$C = C^{[p]}$, $I = I^{[p]}$

Divide each row by its greatest common divisor

Find a candidate pivot

(a) $C_{uv}^{[p]}|_{\varepsilon\to 0} \neq 0$

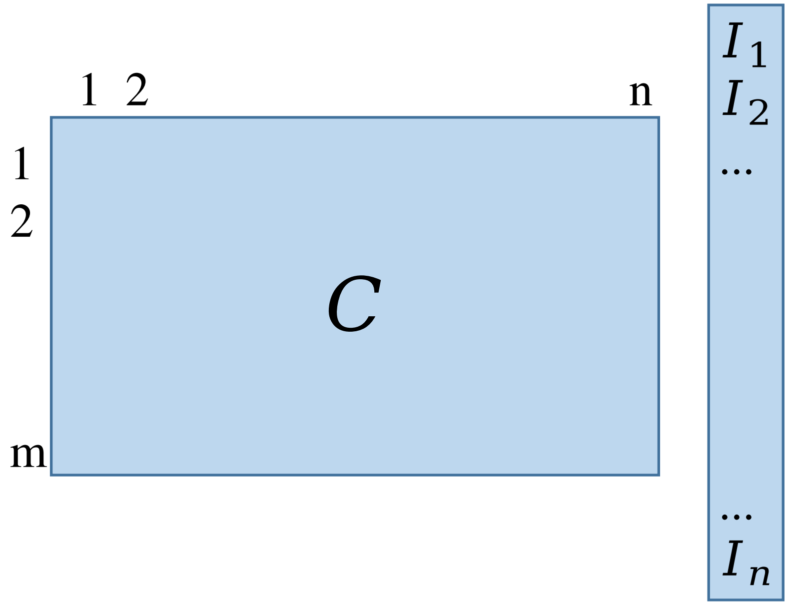
(b) for $I_v^{[p]}$, v is the smallest

(c) the u^th^ row is the sparsest

Pivot

Normalize & Reduce

if $p \leq m$, $p = p + 1$



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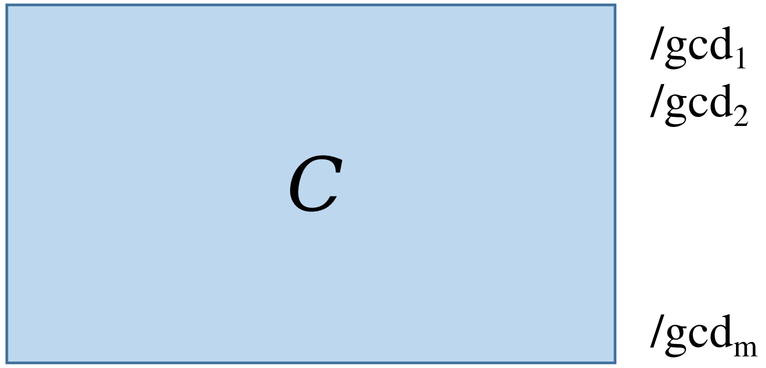

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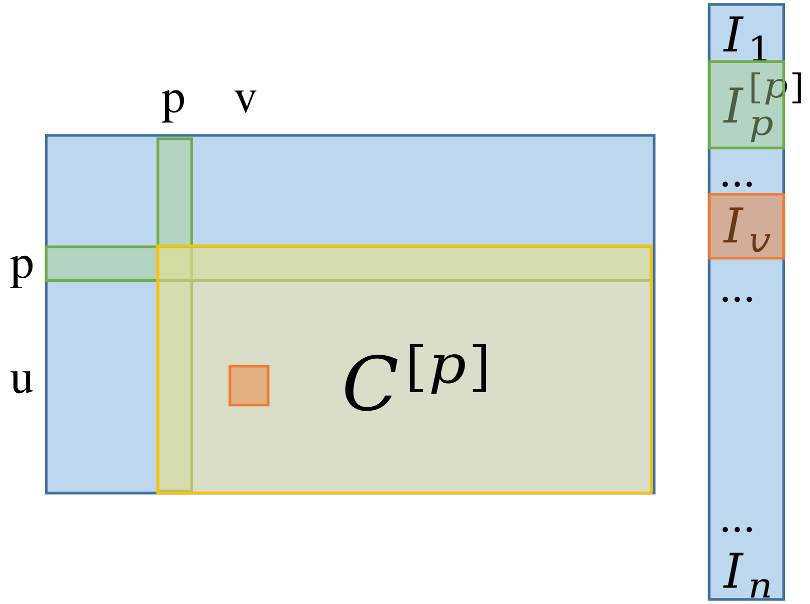

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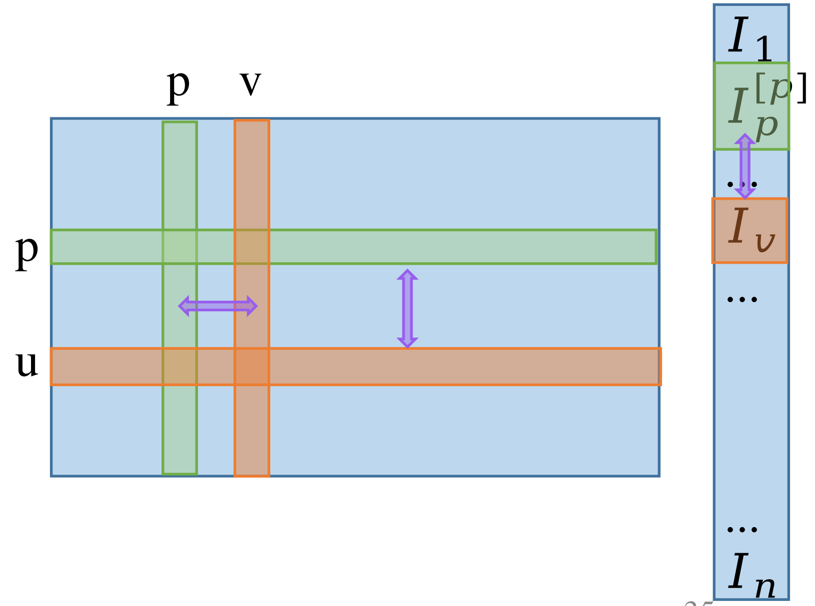

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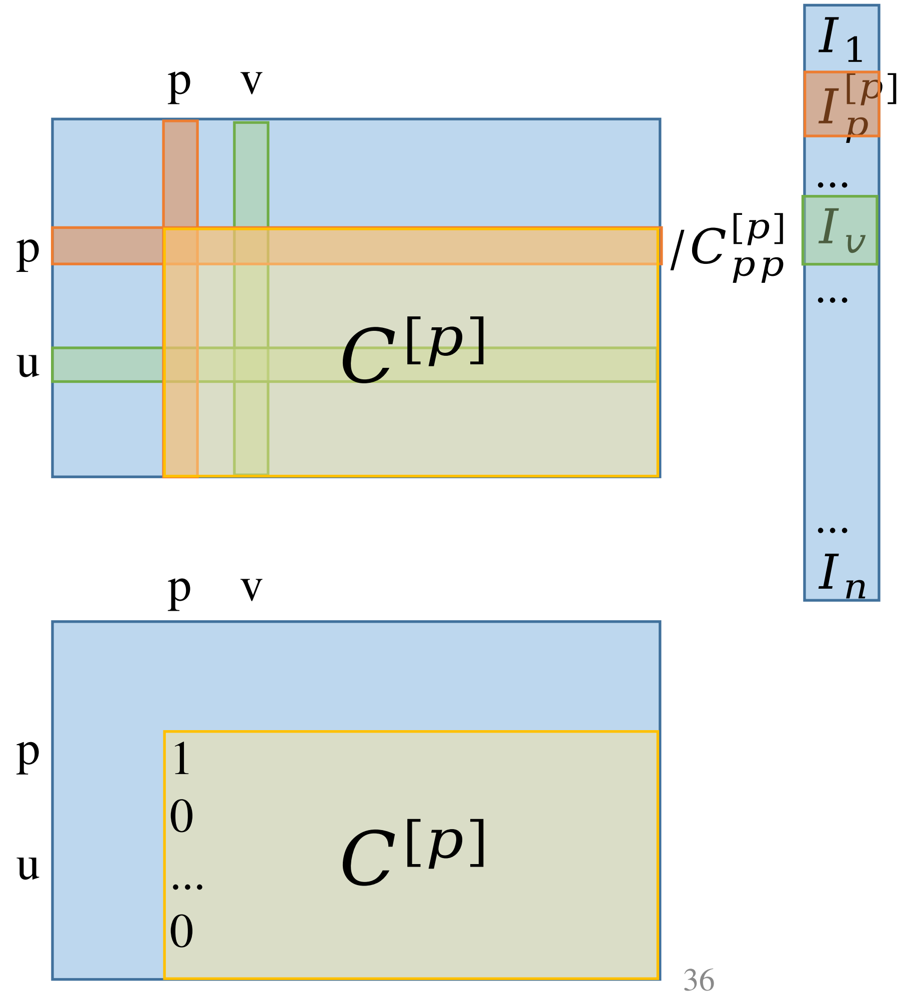
if $p \leq m$



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if $p \leq m$, $p = p + 1$

ε - singularity-free bases

Summary

NeatIBP : **syzygies make IBP neat and quick**

Energy Correlators : **Phase space integrals with polynomial propagators**
Elliptic and hyperelliptic integrals even at tree level

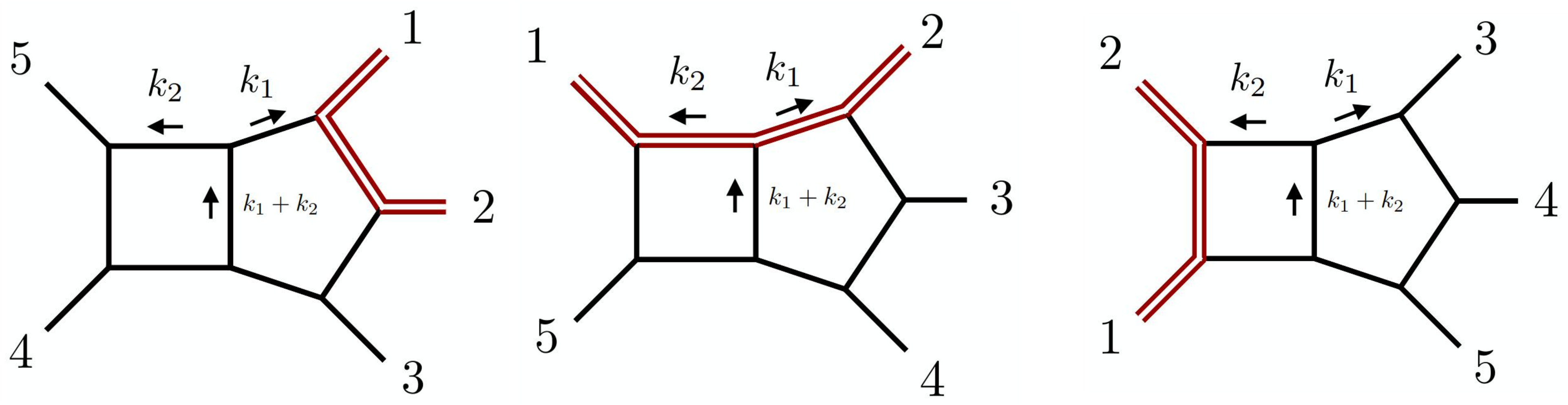
Singularity-Free Feynman Integral Bases
Gaussian Elimination in a Local Ring

Thanks

Application of NeatIBP:

Two-loop-five-point amplitudes for $p\ p \to t\ \bar{t}\ j$ at leading color

Simon Badger, Matteo Becchetti, Nicolò Giraudo, Simone Zoia, JHEP 07 (2024) 073



(diagrams from JHEP 07 (2024) 073)

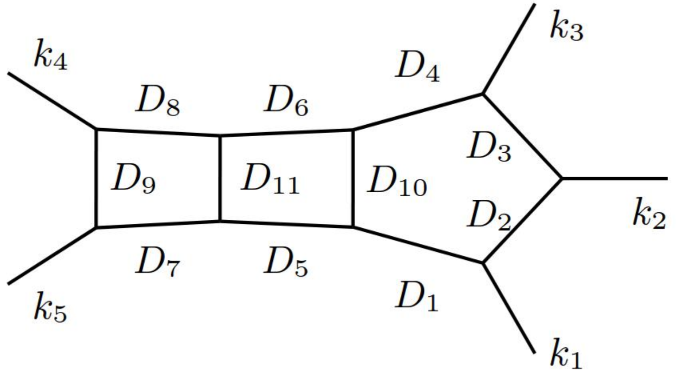
"The latter package (NeatIBP), in particular, allows us to obtain optimized systems of IBP relations through the solution of syzygy equations, this way making their solution substantially simpler

Application of NeatIBP:
Three-loop-five-point diagrams

Yuanche Liu, Antonela Matijašić, Julian Miczajka, Yingxuan Xu, Yongqun Xu, Yang Zhang, arXiv:2411.18697

Analytic evaluation of the master integrals for the Pentagon-box-box diagram via differential equations

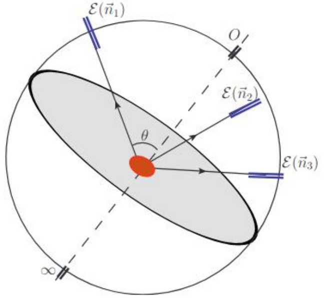
Target integrals: integrals needed for DE of master integrals

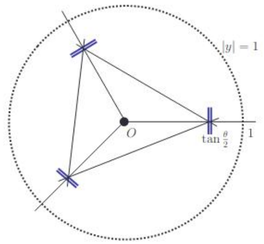


(diagram from arXiv:2411.18697)

"On a laptop, NeatIBP finds about 85,000 IBP identities in full kinematics dependence within several hours These relations are sufficient to derive the differential equation. As a comparison, standard IBP tools would generate three orders of magnitude more IBP relations, which make the reduction computation impossible."

Analytic result





$$\zeta_{ij} = \frac{|z_i - z_j|^2}{(1+|z_i|^2)(1+|z_j|^2)}, \quad z_1 = 0,\ \bar{z}_2 = z_2$$

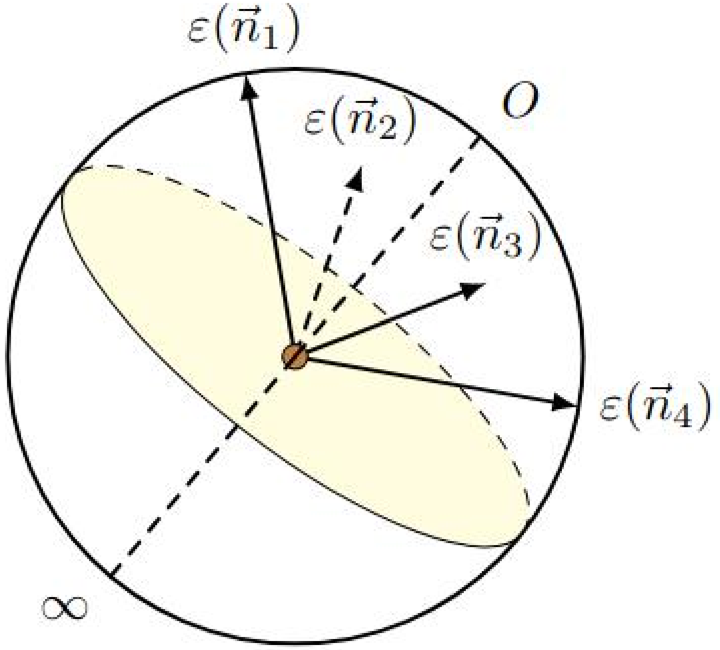
letters

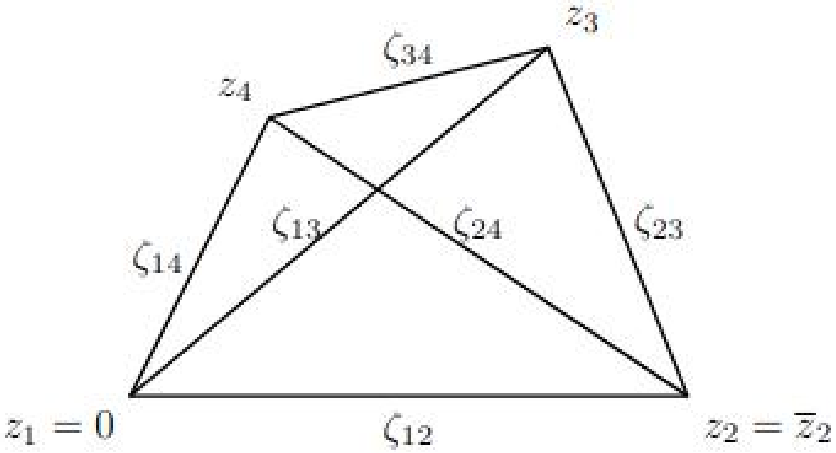
$$z_2,\ z_2 - z_3,\ z_3,\ z_3 - z_2,\ \bar{z}_3,\ \bar{z}_3 - z_2,\ \bar{z}_3 - z_3,$$
$$z_2^2 + 1,\ z_2 z_3 + 1,\ z_2 \bar{z}_3 + 1,\ z_3 \bar{z}_3 + 1,$$

$$\begin{aligned}
\mathrm{UT}_2 = \ & 2\mathrm{Li}_2\left(\frac{z_2\left(z_3\bar{z}_3+1\right)}{z_2-\bar{z}_3}\right) - 2\mathrm{Li}_2\left(\frac{z_3\left(z_2\bar{z}_3+1\right)}{z_3-\bar{z}_3}\right) + 2\mathrm{Li}_2\left(-\frac{\left(z_2^2+1\right)z_3}{z_2-z_3}\right) \\
& - 2\mathrm{Li}_2\left(\frac{z_2}{z_2-\bar{z}_3}\right) + 2\mathrm{Li}_2\left(\frac{z_3}{z_3-\bar{z}_3}\right) - 2\mathrm{Li}_2\left(-\frac{z_3}{z_2-z_3}\right) \\
& - 2\log\left(\frac{\left(z_2z_3+1\right)\bar{z}_3}{\bar{z}_3-z_3}\right)\log\left(z_2\bar{z}_3+1\right) + 2\log\left(\frac{\left(z_2z_3+1\right)\bar{z}_3}{\bar{z}_3-z_2}\right)\log\left(z_3\bar{z}_3+1\right) \\
& + 2\log\left(z_2^2+1\right)\log\left(\frac{z_2\left(z_2z_3+1\right)}{z_2-z_3}\right) - \log^2\left(z_2z_3+1\right)
\end{aligned}$$

Four point energy correlator

$$E^4 C(\vec{\zeta}_{ij})|_{LO} = \int_0^1 dx_1 \cdots dx_4 (x_1 \cdots x_4)^2 \delta(1 - Q_4) |F_5^{(0)}|^2_{sym}$$





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