Thimble decomposition and Wall Crossing Structure for Physical Integrals

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DER FORSCHUNG | DER LEHRE | DER BILDUNG

Based on

[hep-th: 2506.03252] with S.L. Cacciatori and A. Massidda

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Motivations

Analysis of complex **integrals**
describing generic physical systems

$$I = \int_\Gamma F(x_1, x_2, \ldots; \mu_1, \mu_2, \ldots) dx_1^2\, dx_2^2 \ldots$$

Direct computation

Resolution of systems of differential equations

Geometric interpretation in order to simplify the integral.

Periods

[Kontsevich, Zagier 2001]

Values of integrals of algebraically defined differential forms over certain chains in algebraic varieties

$$I = \int_\Gamma F(x_1, x_2, \ldots, x_n; \underbrace{\mu_1, \mu_2, \ldots}_{\text{Moduli of the integration contours}})dx_1\, dx_2 \ldots dx_n$$

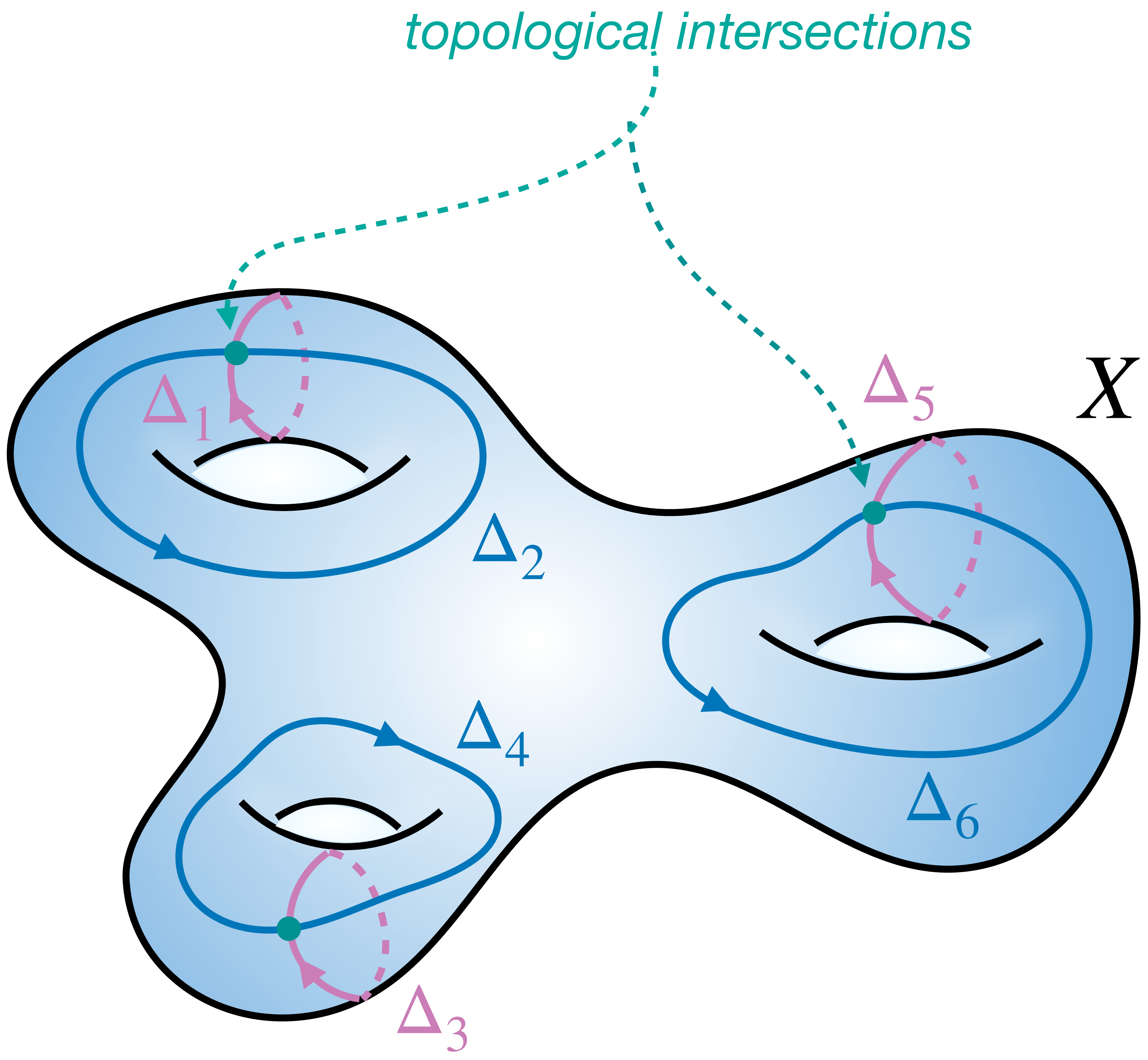
PERIOD = n- chain in the complex algebraic variety **X** $\otimes$ n- form in the complex algebraic variety **X**

≀ ≀ ≀

Non-degenarete pairing = **HOMOLOGY** $H_n(X)$ $\otimes$ **COHOMOLOGY** $H^n(X)$

The advantages

- The homology and cohomology rings are finitely generated
- There is a non-degenerate **internal product** in homology given by topological intersection among cycles and its dual in cohomology



$$I = \int_\Gamma \omega(\bar{x};\bar{\mu})$$ = Element of a finite dimensional **vector space**

[Mastrolia, Mizera - '18]

with double projection:

- with respect to a basis of cycles: $H_n(X) = \langle \Delta_1, \ldots, \Delta_N \rangle$

Master Integrals

Intersection numbers in homology

$$I = (\Delta, \Delta_1) \cdot \int_{\Delta_1} \omega + (\Delta, \Delta_2) \cdot \int_{\Delta_2} \omega + \ldots + (\Delta, \Delta_N) \cdot \int_{\Delta_N} \omega$$

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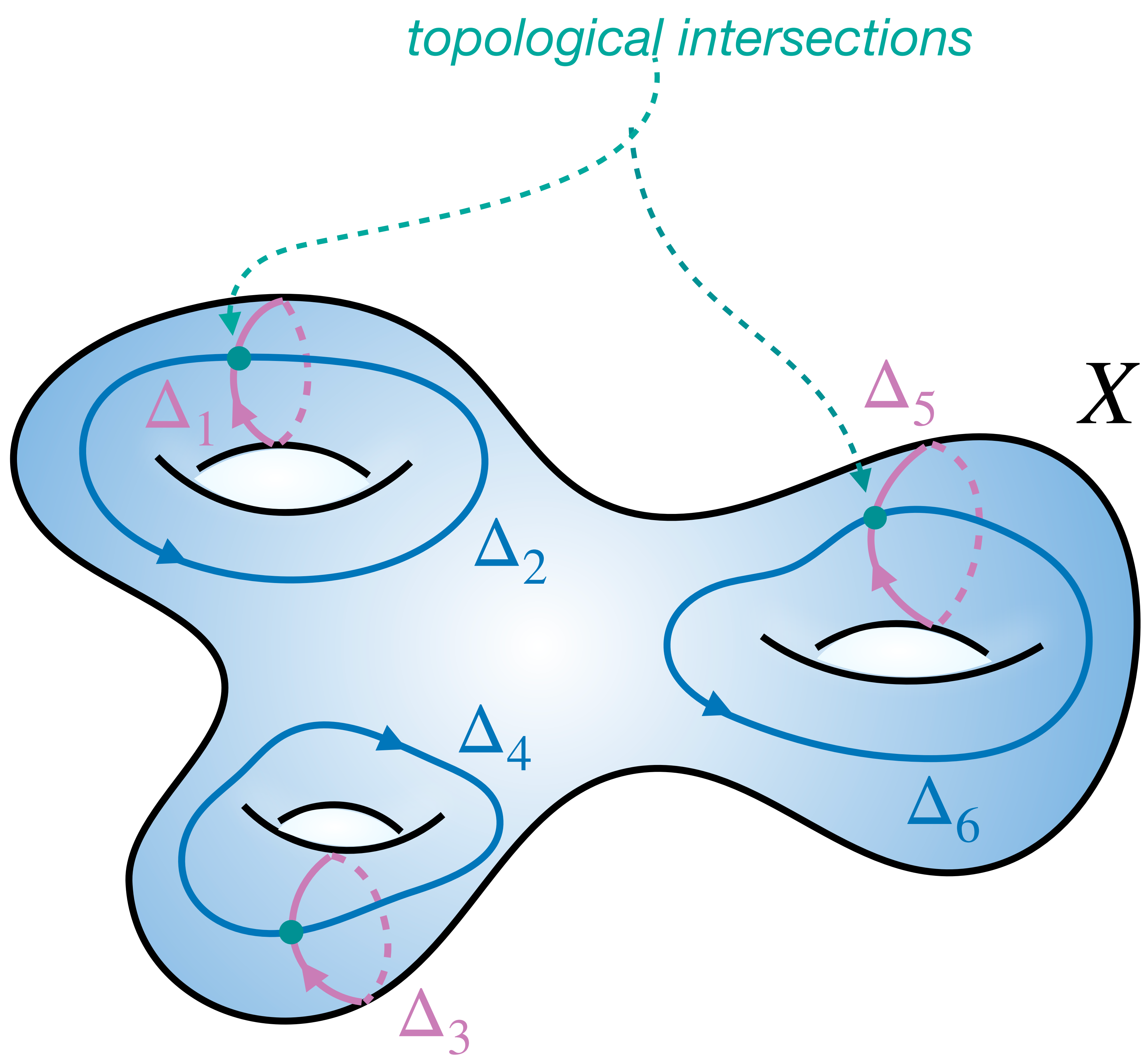

with double projection:

- with respect to a basis of forms: $H^n(X) = \langle \omega_1\,, \omega_2\,, \ldots, \omega_N \rangle$

Master Integrals

Intersection numbers in cohomology

$$I = (\omega, \omega_1)\cdot \int_\Delta \omega_1 + (\omega, \omega_2)\cdot \int_\Delta \omega_2 + \ldots + (\omega, \omega_N)\cdot \int_\Delta \omega_1$$

The problems

Feynman integral in
Baikov representation

$$I = \int_\Gamma \mathscr{B}(x_i, \mu_j)^{-\gamma}\, \omega$$

In dimensional regularization
$\gamma(D) \neq \mathbb{Z}$

Rational form

Baikov polynomial

- Multivalued integral with a potentially complicate monodromy
- Special values of the parameters at which the manifold X becomes singular
- …

Main Task

To indentify the **right** homology/cohomology to define the pairing

The exponential period map

[Kontsevich, Soibelman - 2024]

- X $\longrightarrow$ n-dim complex algebraic variety
- $f : X \mapsto \mathbb{C}$ $\longrightarrow$ Complex valued function
- μ $\longrightarrow$ Holomorphic volume form over X
- Γ $\longrightarrow$ Open integration chain on $X \backslash D_0$

$$\int_\Gamma e^{-f}\mu \; : \; H_\bullet^{Betti,\, global}\big((X, D_0), f\big) \otimes H^\bullet_{dR,\, global}\big((X, D_0), f\big) \longmapsto \mathbb{C}$$

Feynman integrals

Path integral in QFTs

CFT correlators

Non-perturbative computations in String Theory

...

Generalized Exponential Integral

- Rescaling the function: $f \mapsto \gamma f$ with $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$

Generalized exponential integral:

$$\int_\Gamma e^{-\gamma f}\mu \ : \ H_\bullet^{Betti,\,global}\big((X,D_0),\gamma f\big) \otimes H^\bullet_{dR,\,global}\big((X,D_0),\gamma f\big) \longmapsto \mathbb{C}$$

Wall Crossing Structure:

To study how the structure of the resulting integral depends on γ .

By varying the parameter γ, the homology and the cohomology groups involved in the pairing can change, and for special values of γ they can even reduce their dimensionality leading to a reduced number of MIs.

De Rham Cohomology

$$\left(X, D_0, f \right)$$

The integrand n-form μ is represented by a class $[\,\mu\,]$ in the degree n Twisted de Rham group

$$[\mu] \in H^n_{dR} \left(X, D_0, f\right)$$

Global Twisted de Rham

$$H^\bullet_{dR}(X, D_0, f) \cong \mathbb{H}^\bullet(X, \Omega^\bullet_{X,D_0}, \nabla_f)$$

Graded abelian group of equivalence classes of closed forms on $X \setminus D_0$ with respect to the differential

$$\nabla_f = d \; + df \wedge$$

where

$$(\Omega^\bullet_{X,D_0}, \nabla_f) : \; \Omega^0_{X,D_0} \xrightarrow{\nabla_f} \Omega^1_{X,D_0} \xrightarrow{\nabla_f} \dots \xrightarrow{\nabla_f} \Omega^n_{X,D_0}$$

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Generalized version:

$$H^\bullet_{dR,\gamma}(X, D_0, f) \cong \mathbb{H}^\bullet(X, \Omega^\bullet_{X,D_0}, \nabla_{\gamma f})$$

with respect to the differential

$$\nabla_f = d + \gamma\, df \wedge$$

Global Betti Homology

Graded abelian group which captures the topology of chains on $X \setminus D_0$ relative to the level sets of f at infinity:

$$H_\bullet^{Betti,global}\left((X,D_0),f,\mathbb{Z}\right) \equiv H_\bullet\left((X,D_0),f^{-1}(\infty),\mathbb{Z}\right)$$

- Let us fix $D_0 = \varnothing$ for simplicity.

  The integration cycle Γ is a **non-compact** n-dim cycle in the complex variety X with boundaries on the subset $\left\{\mathbf{z} \in X \,|\, f(\mathbf{z}) = \infty\right\} \subset X$. The cycle Γ is represented by the class $[\Gamma] \in H_n^{Betti,global}(X,f,\mathbb{Z})$

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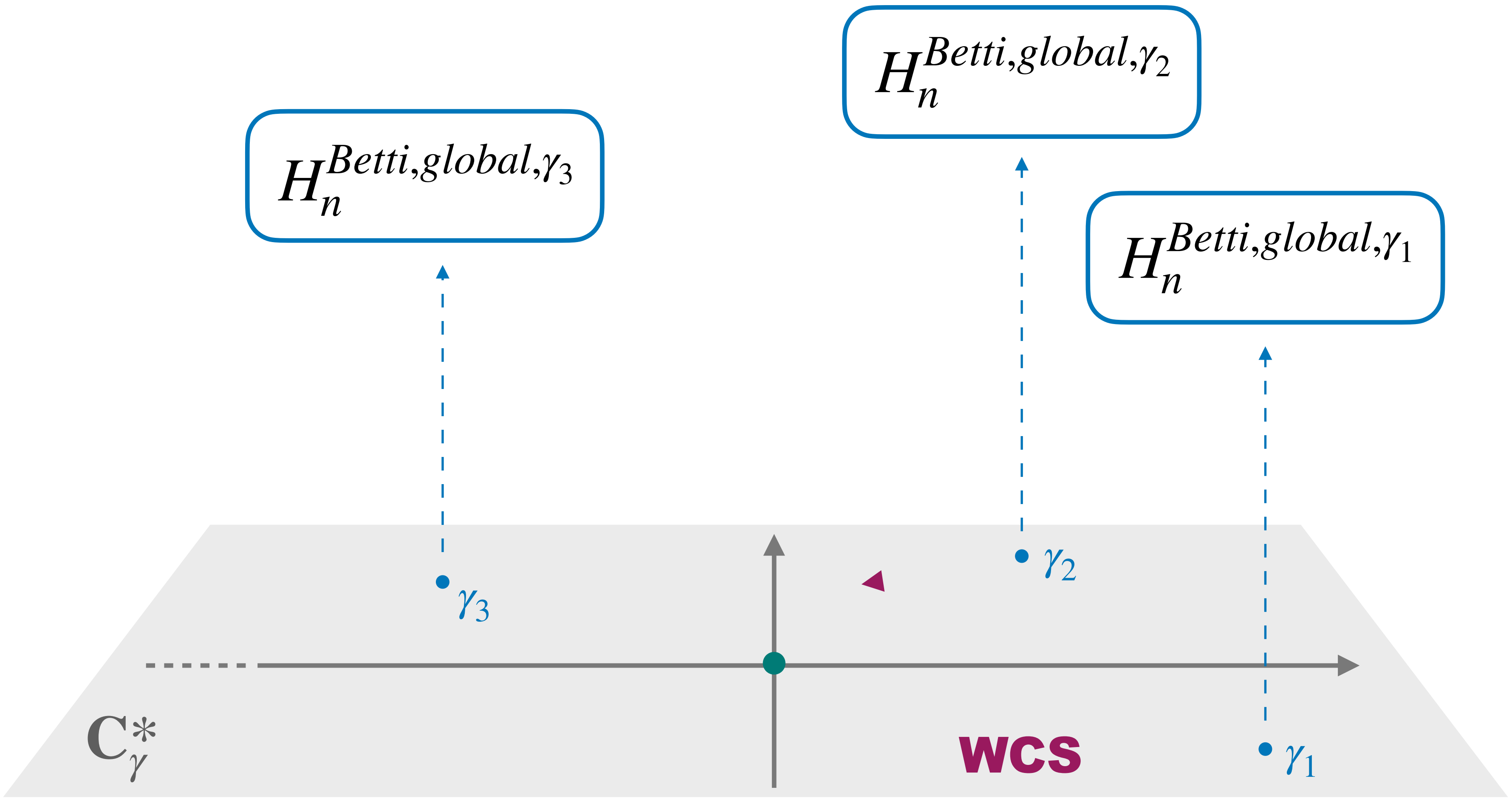
→ **Generalized version:**

$$H_\bullet^{Betti,global,\gamma}\left((X,D_0),f,\mathbb{Z}\right) \equiv H_\bullet\left((X,D_0),(\gamma f)^{-1}(\infty),\mathbb{Z}\right)$$



Master Integrals decomposition

Given the **exponential integral**, with fixed integrand and fixed $\gamma \in \mathbb{C}^*$

$$I = \int_\Gamma e^{-\gamma f}\mu$$

- Define a **basis** of integration contours $\{\Gamma_i\}_{i=1}^{N=dimH_n}$ for the Betti Homology group $H_n^{Betti,global,\gamma}(X, f, \mathbb{Z})$
- Define a non-degenerate internal product on the group $H_n^{Betti,global,\gamma}(X, f, \mathbb{Z})$:

$$\left(\Gamma_i, \Gamma_j\right) = c_{ij}$$

Intersection numbers

such that the integration contour Γ can be written in terms of the following **linear** combination:

$$\Gamma = \left(\Gamma, \Gamma_1\right)\Gamma_1 + (\Gamma, \Gamma_2)\Gamma_2 + \ldots + (\Gamma, \Gamma_N)\Gamma_N$$

leading to the following **MIs decomposition** for I

$$I = \left(\Gamma, \Gamma_1\right)\int_{\Gamma_1} e^{-\gamma f}\mu + (\Gamma, \Gamma_2)\int_{\Gamma_2} e^{-\gamma f}\mu + \ldots + (\Gamma, \Gamma_N)\int_{\Gamma_N} e^{-\gamma f}\mu$$

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Local Betti Homology

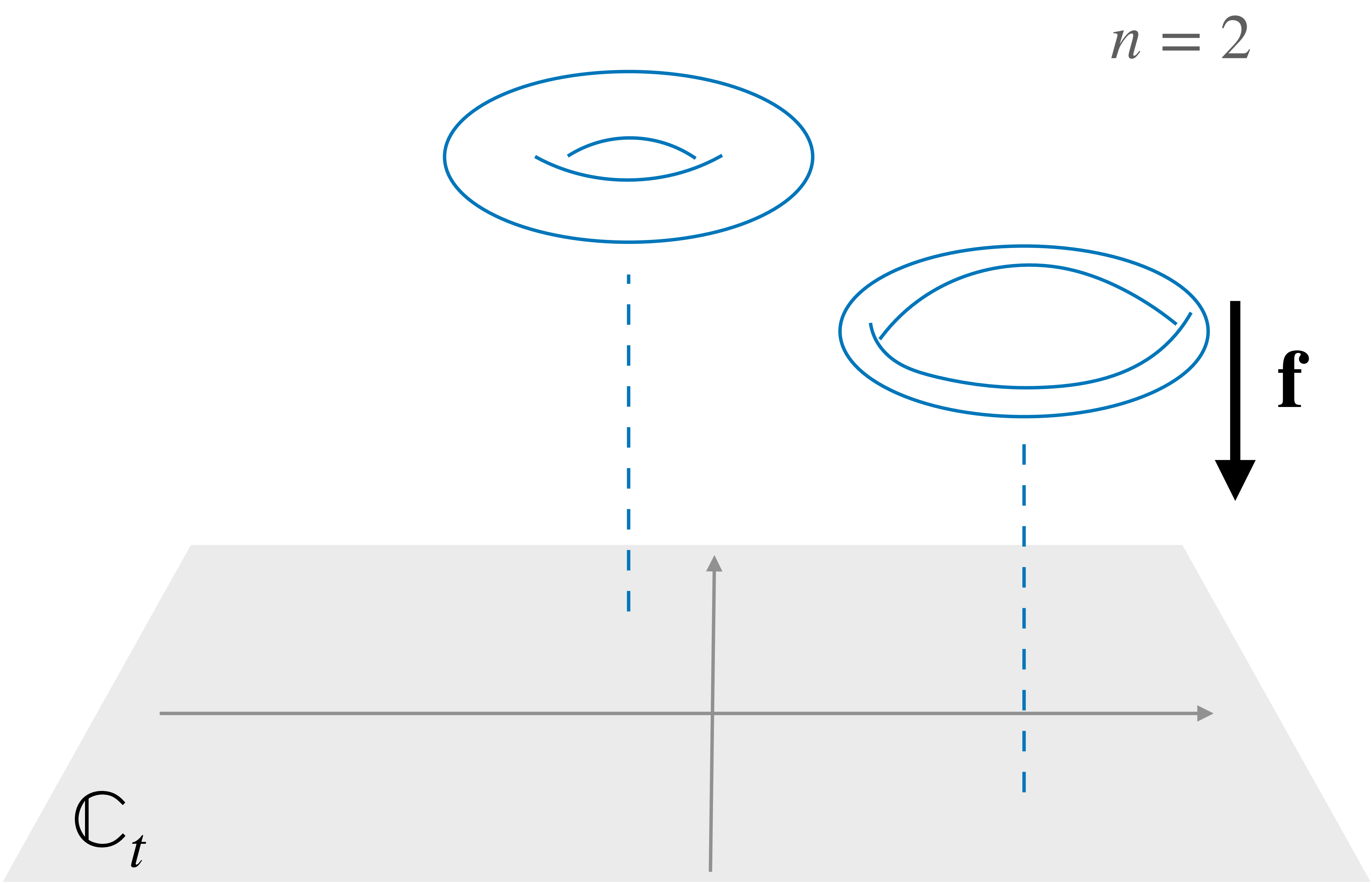
- We can reconstruct the **global** Betti Homology group using **local data**.

1. Consider the level sets associated with the function f : $\gamma f(\mathbf{z}) = t \in \mathbb{C}_t$

For each value of **t** *this equation defines a $(n-1)$-dim complex algebraic variety.*

Geometric point of view:

As t varies we have an entire family of algebraic varieties with **different sizes** for some of their internal $(n-1)$ -dimensional cycles.



Local Betti Homology

2. Define the set of critical loci of $f\,:\,\Sigma = \{\sigma_i \in X \,|\, df(\sigma_i) = 0\}$ and the corresponding set of critical values $S = \{f(\sigma_1) = t_1, f(\sigma_2) = t_2, \ldots, f(\sigma_N) = t_N\}$ (we are assuming non-degeneracy)

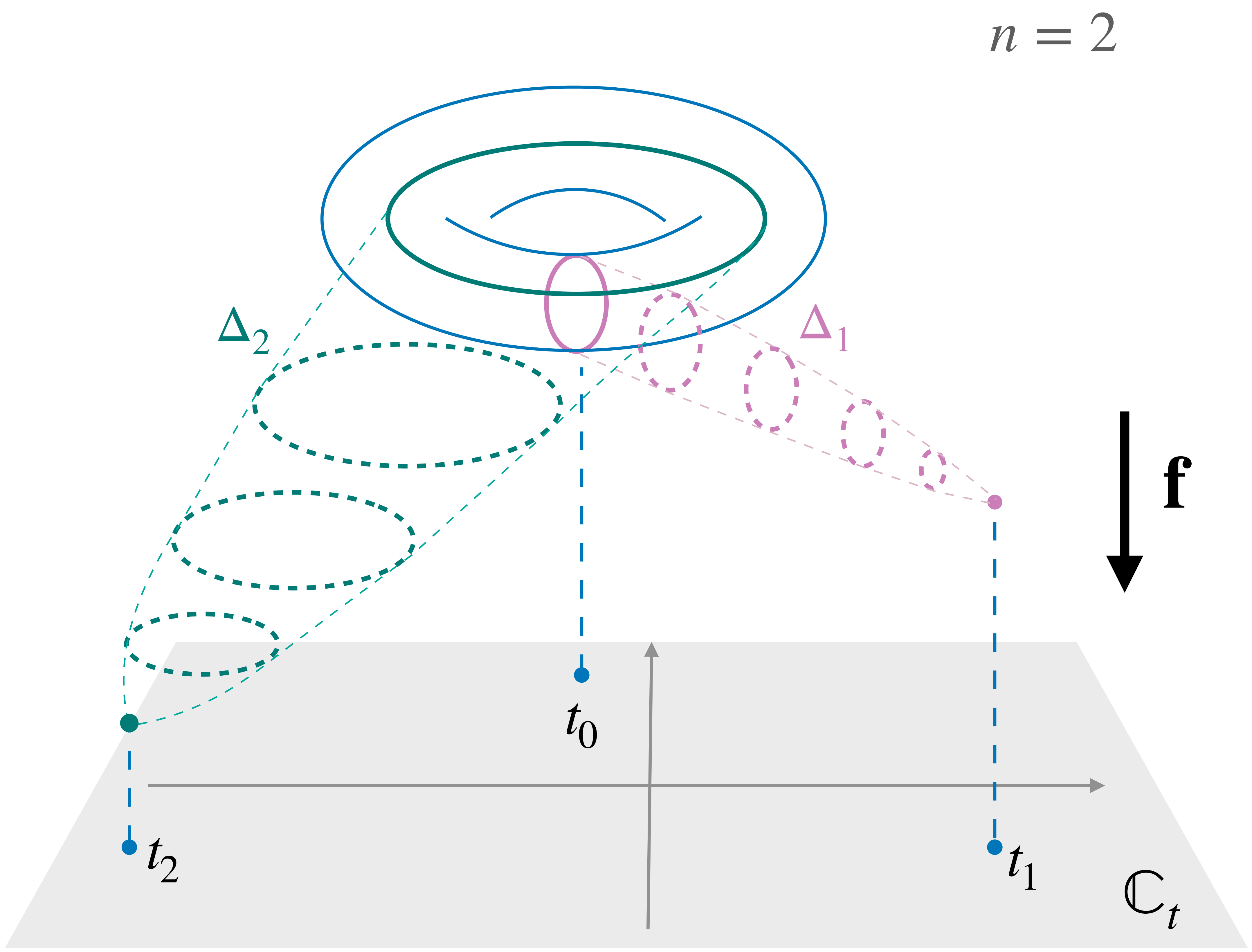
Geometric point of view:

In correspondence of these critical values the algebraic variety on the fiber is singular.

- Some of the internal $(n-1)$-cycles shrink to zero size.

↓

Vanishing cycles Δ_i

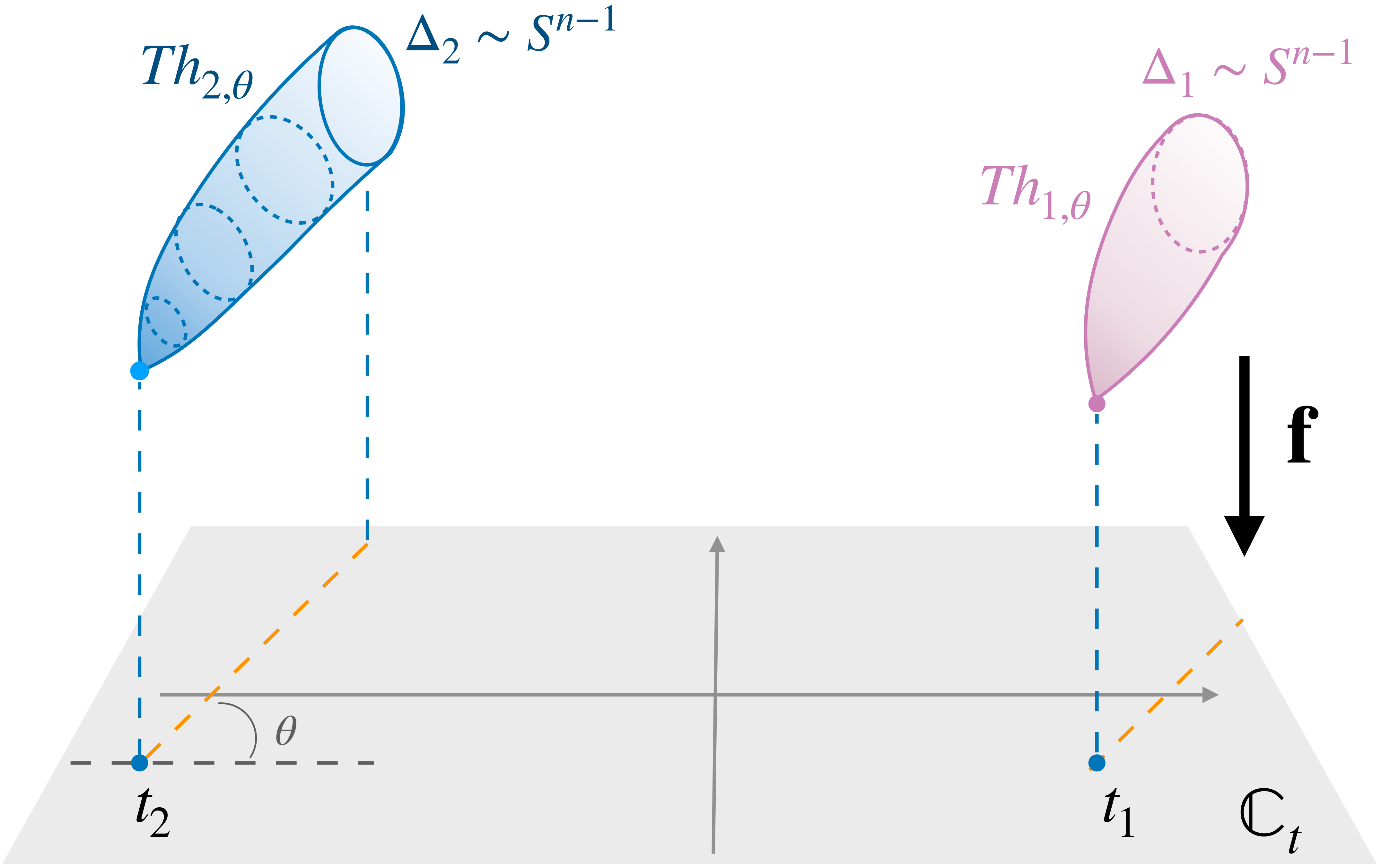


Local Betti Homology

- For each **critical point** t_i there is a **vanishing cycle** Δ_i.

(3) Consider the direction $\theta = arg(\gamma)$ in the t-plane and define the set of thimbles $\left\{Th_{i,\theta}\right\}_{i=1}^{N}$.

Thimble $Th_{i,\theta}$ **=** Trace of the vanishing cycle Δ_i along the direction $\theta = arg(\gamma)$ starting from the critical point t_i.



- **NOTE**: Thimbles are n-dimensional. They have the right dimension to be integration cycles!

The whole set of **thimbles** constructed along the direction $arg(\gamma)$ form a **basis** for the Betti homology

$$H_n^{Betti,global,\gamma}(X, f, \mathbb{Z}) \simeq \langle Th_{1,\theta}, Th_{2,\theta}, \ldots, Th_{N,\theta} \rangle$$

Master Integrals decomposition

Given the **exponential integral**, with fixed integrand and fixed $\gamma \in \mathbb{C}^*$

$$I = \int_\Gamma e^{-\gamma f}\mu$$

- Define a **basis** of integration contours $\{\Gamma_i\}_{i=1}^{N=dimH_n}$ for the Betti Homology group $H_n^{Betti,global,\gamma}(X, f, \mathbb{Z})$
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leading to the following **MIs decomposition** for I

$$I = \langle \Gamma, \Gamma_1 \rangle \int_{\Gamma_1} e^{-\gamma f}\mu + \langle \Gamma, \Gamma_2 \rangle \int_{\Gamma_2} e^{-\gamma f}\mu + \ldots + \langle \Gamma, \Gamma_N \rangle \int_{\Gamma_N} e^{-\gamma f}\mu$$

Internal product in Betti Homology

- For generic values of the angle $\theta = arg(\gamma)$, thimbles associated with different critical values never intersect

We can define an intersection pairing:

$$\langle \cdot \, , \cdot \rangle \quad : \quad H_n^{Betti,global,\gamma}(X, f, \mathbb{Z}) \times H_n^{Betti,global,e^{i\pi}\gamma}(X, f, \mathbb{Z}) \longmapsto \mathbb{Z}$$

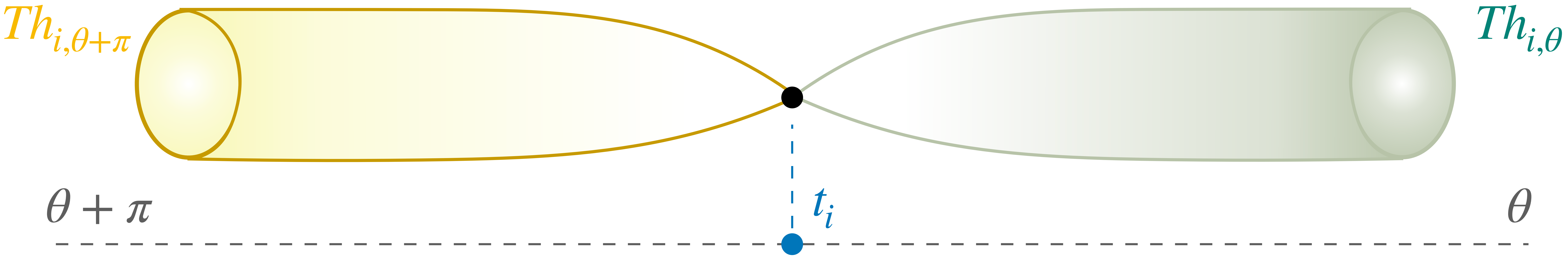
Betti homology

$$\{Th_{i,\theta}\}_{i=1}^{N}$$

Dual Betti homology

$$\{Th_{i,\theta+\pi}\}_{i=1}^{N}$$

With respect to these basis the intersection pairing is given by: $\langle Th_{i,\theta}, Th_{j,\theta+\pi} \rangle = \delta_{ij}$

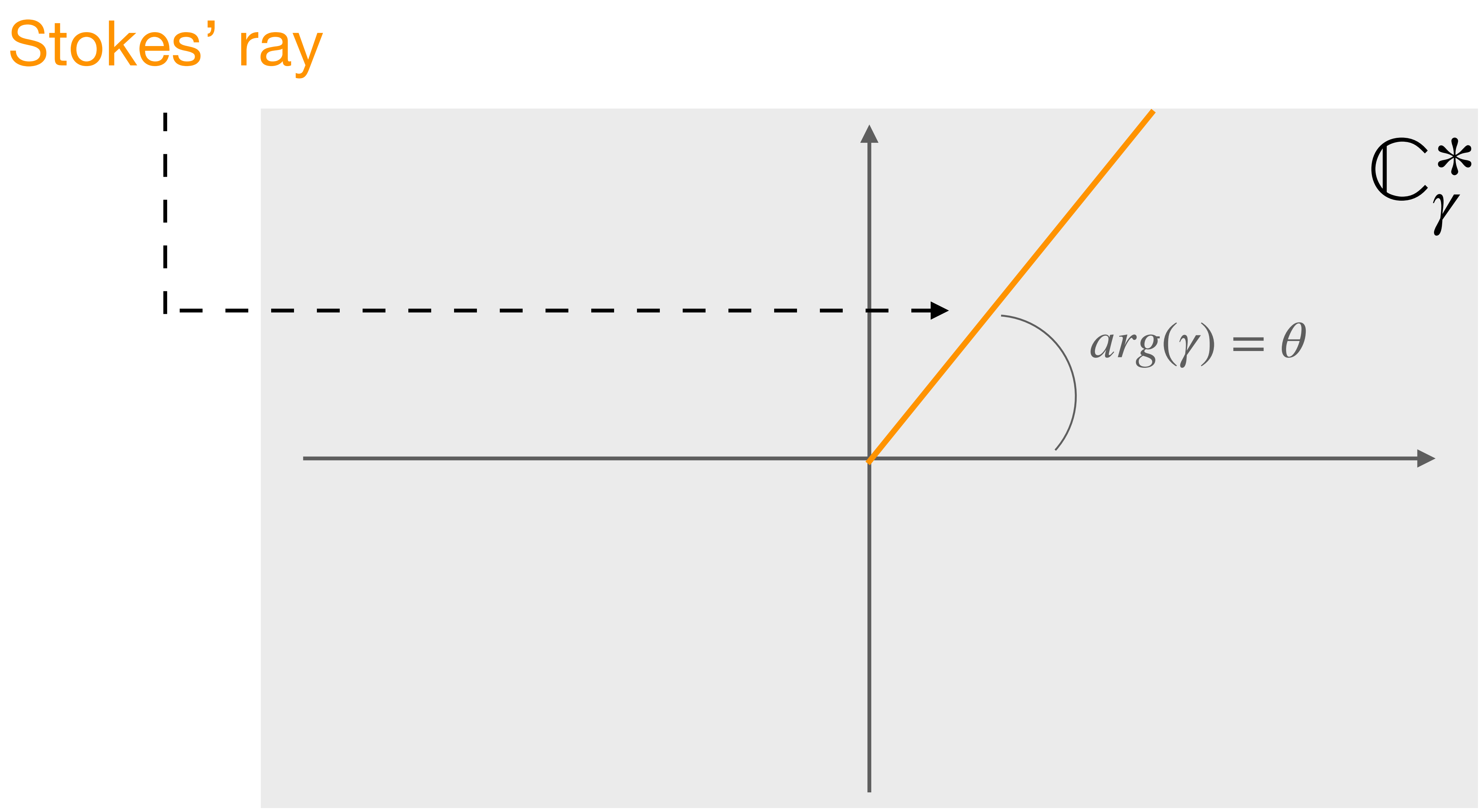


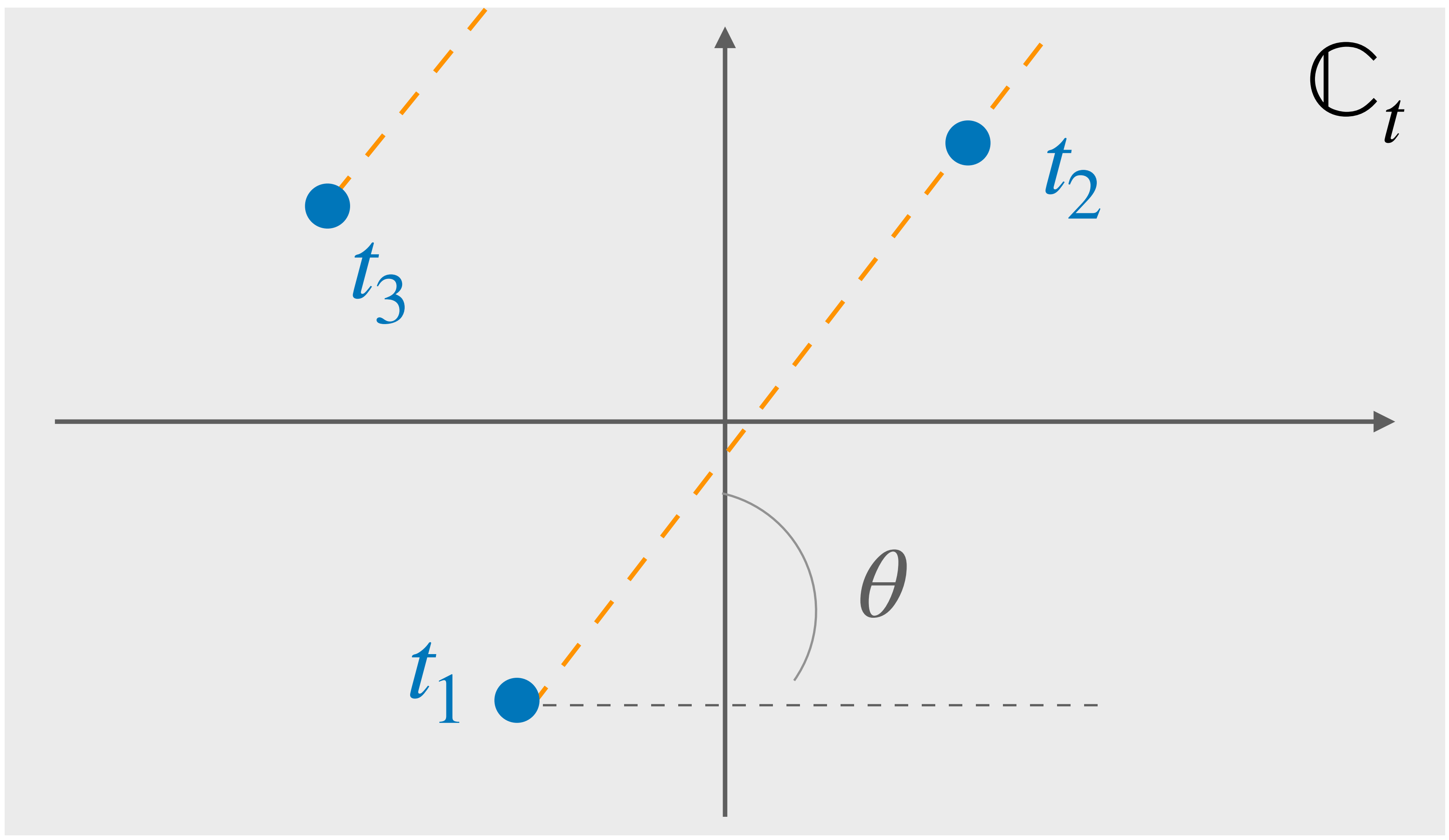
Thimbles degeneration

- Problems appear for special values of γ for which the lines with direction $\theta = \arg(\gamma)$ in the plane $\mathbb{C}_t$ intersect two or more critical points.

Degeneration of the corresponding thimbles

Appearence of **Stokes rays** in the plane $\mathbb{C}^*_\gamma$. In correspondence of this lines we have a **wrong number** of thimbles, then a corresponding **wrong computation** of the number of MIs.

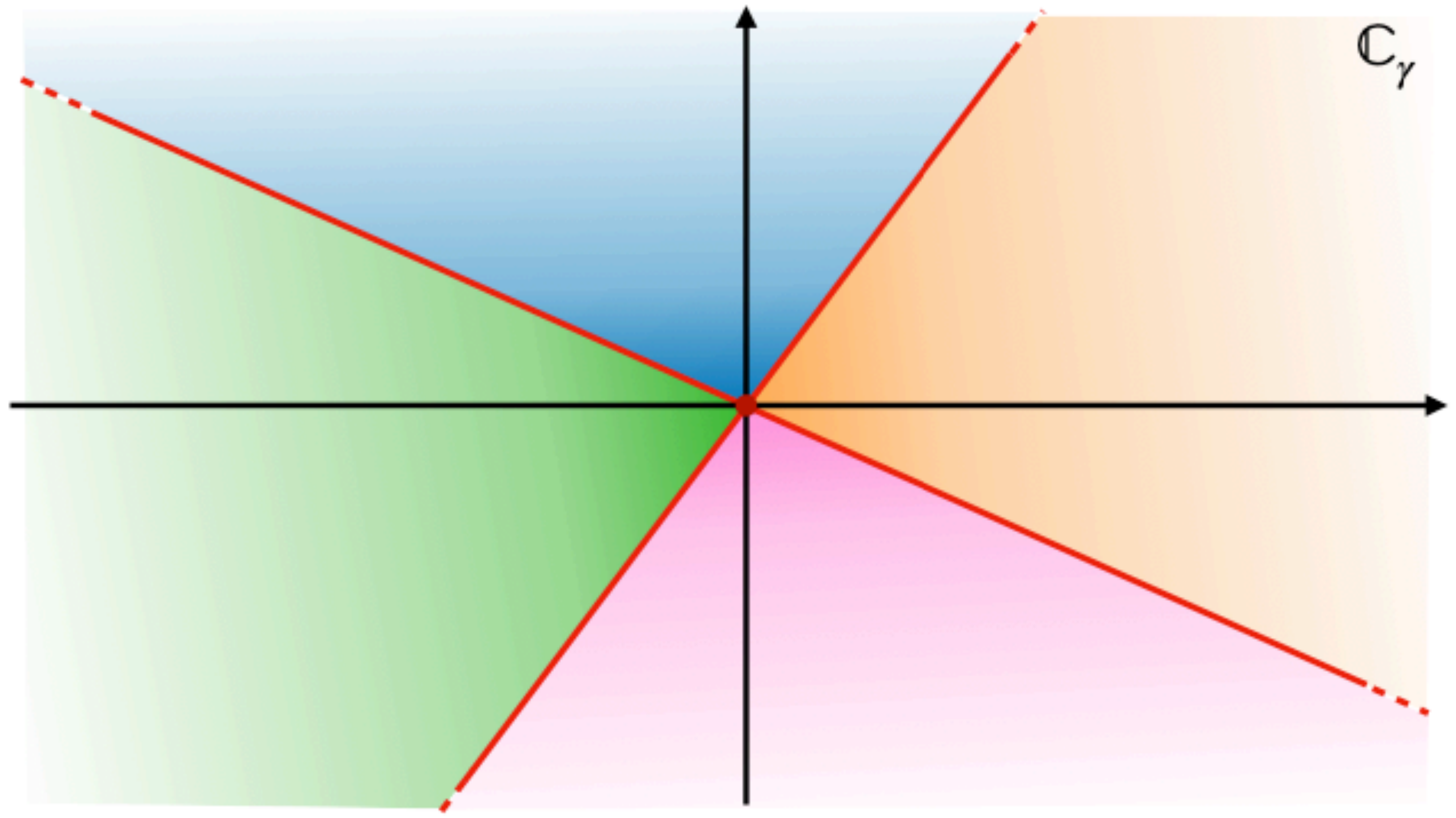




Wall Crossing Structure (WCS)

- Stokes'rays divide the plane $\mathbb{C}^*_\gamma$ in different sectors.

- Study how the basis of thimbles change when we cross these rays = **WCS**.



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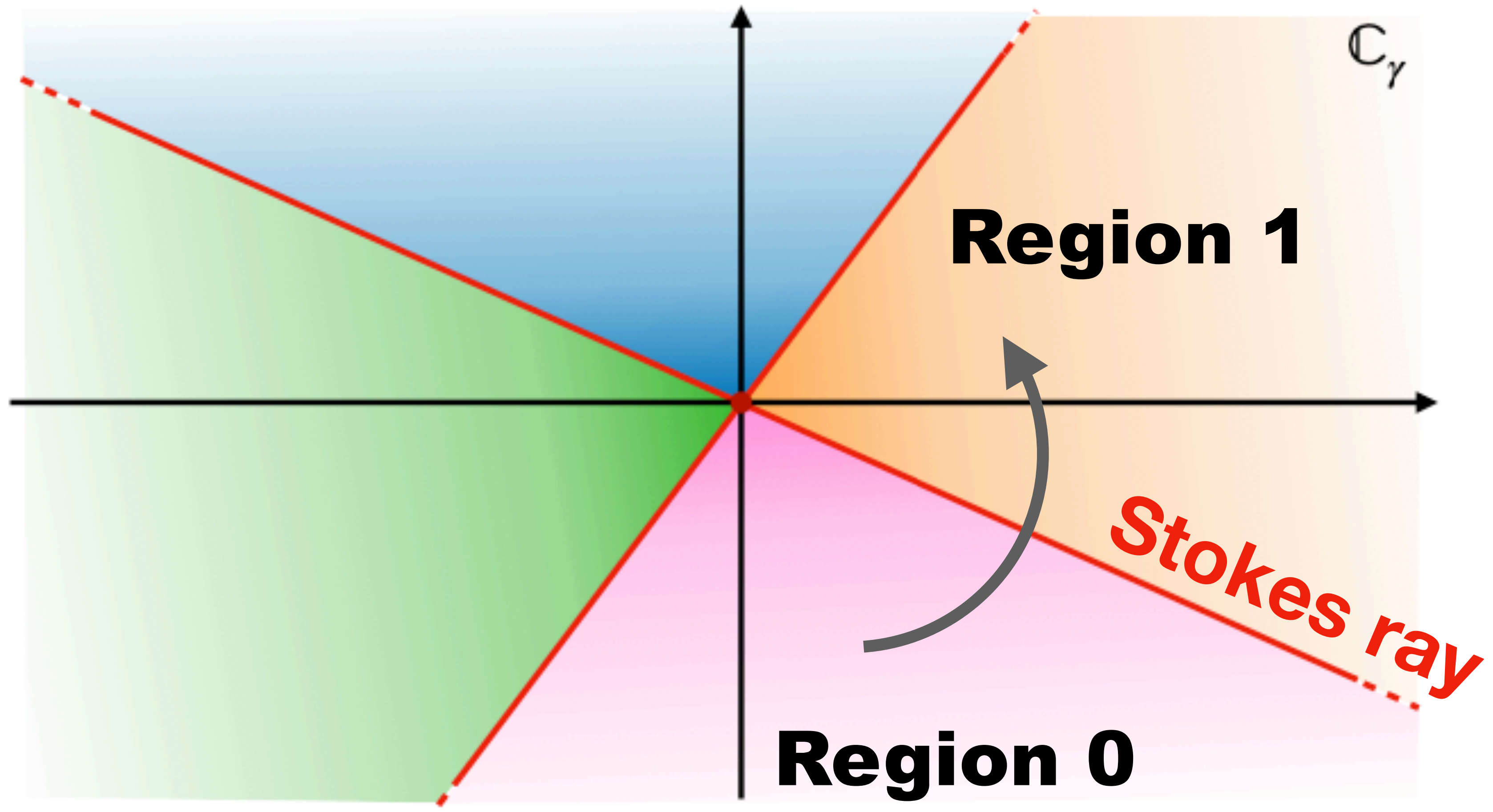


- To cross a Stokes ray corresponding to a Stokes line in the $\mathbb{C}_t$ plane connecting the critical values t_i and t_j imposes a discontinuity jump for the corresponding thimbles Th_i and Th_j described by matrix of the form:

$$\begin{pmatrix} \Gamma_i^{+(1)} \\ \Gamma_j^{+(1)} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_i^{+(0)} \\ \Gamma_j^{+(0)} \end{pmatrix}$$

Intersection numbers among the corresponding vanishing cycles

$$\Delta_{ij} = (\pm 1)\Delta_i \circ \Delta_j \,.$$

IMPORTANT!

Local monodromies around the critical points completely determine these numbers via Picard-Lefshetz theorem:

$$M_j(\Delta_i) = \Delta_i + (-1)^{n(n+1)/2}(\Delta_i \circ \Delta_j)\Delta_j$$

Holomorphic function

In the simple case in which the function $f(\mathbf{z}) : X \mapsto \mathbb{C}$ is **holomorphic** the previous analysis connects with the study of **Lefschetz thimbles** in (a complexified version of) **Morse theory**.

$$I = \int_\Gamma e^{-\gamma f} \mu$$

- $X = \mathbb{C}^n$
- $h = Re(\gamma f) \rightarrow$ **Morse function**
- $\Gamma_i \rightarrow$ **Lefschetz thimbles:**

Solutions of the gradient flow equations:

$$\frac{du^i}{d\tau} = g^{ij} \frac{\partial h}{\partial u^j}$$

- $Im(\gamma f) = const$
- Γ_i passes through the critical point σ_i
- h monotonically increases along Γ_i

Assure convergence of the integral along the paths Γ_i

Example: Pearcey's Integral

$$P(\gamma) = \int_{\Gamma} e^{-\gamma(z^4+bz^2+cz+d)} dz$$

Describing the Grand-canonical **partition function** of gauge Skyrme models for nuclear matter.

[Cacciatori, Canfora, Lagos, Muscolino,Vera - '21]

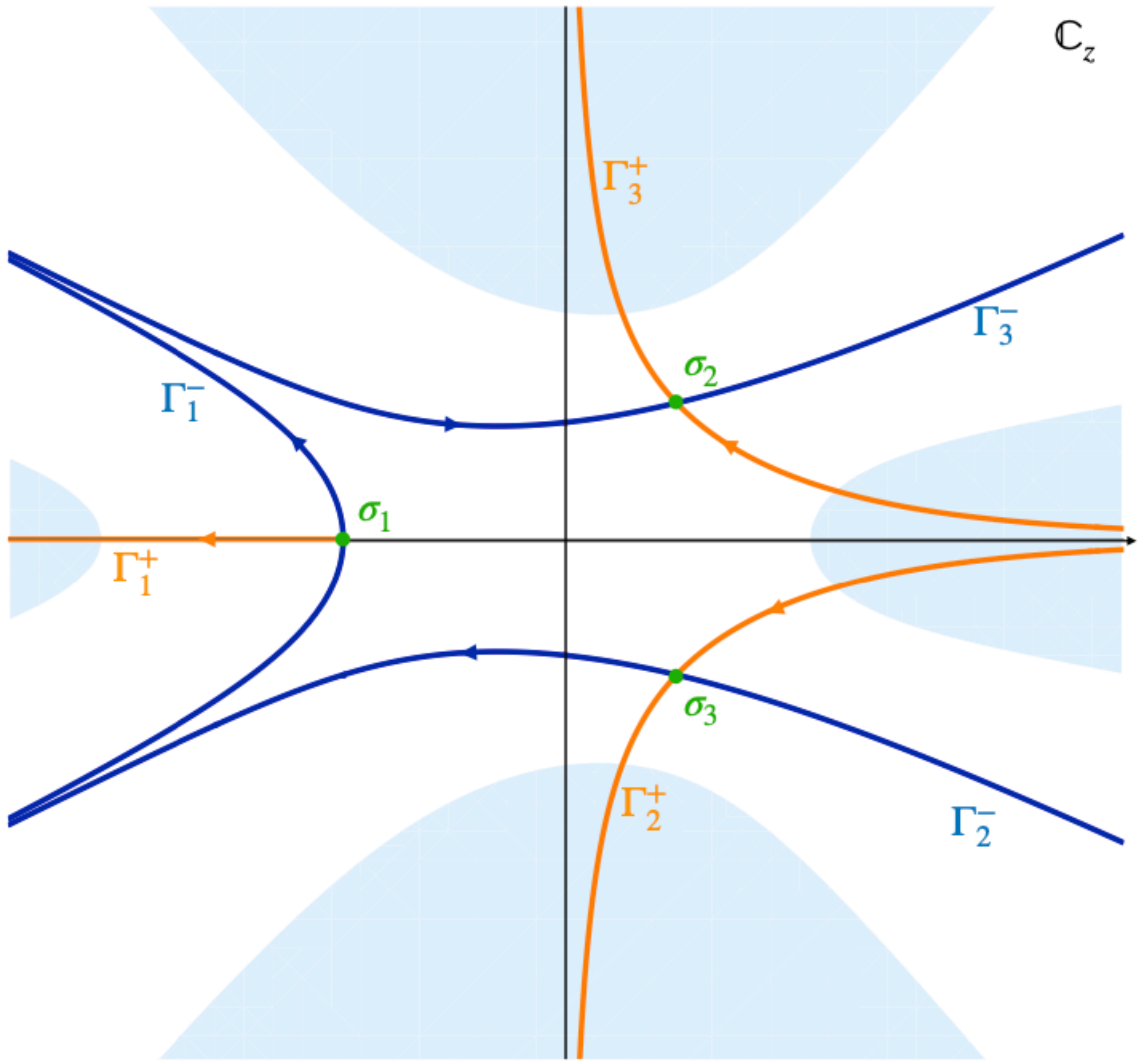
- The algebraic variety is $X = \mathbb{C}$.
- The critical points of f define the set: $\Sigma = \left\{ \mathbf{z} \in \mathbb{C} \,|\, f'(\mathbf{z}) = 4z^3 + 2bz + c = 0 \right\}$

According with the sign of the discriminant $\Delta = 8b^3 + 27c^2$ we have three different situations:

$$\Delta \equiv \begin{cases} > 0 & 1 \text{ real and } 2 \text{ complex conjugate solutions,} \\ < 0 & 3 \text{ real different solutions,} \\ = 0 & 3 \text{ real solutions with at least a multiple root.} \end{cases}$$

Three **different regions** in the parameter space (a, b) with different Betti homologies and different WCSs.

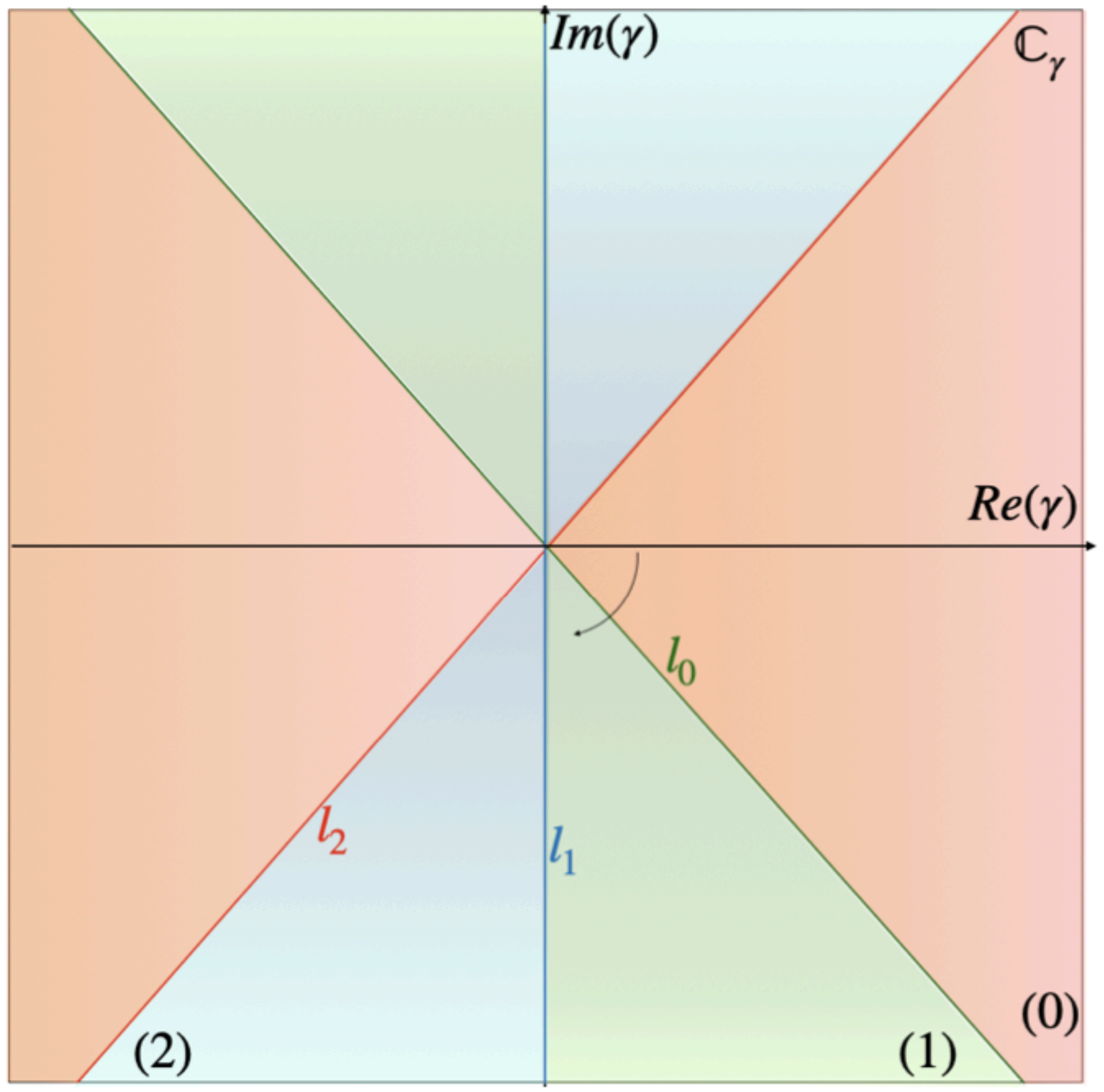
Region $\Delta > 0$

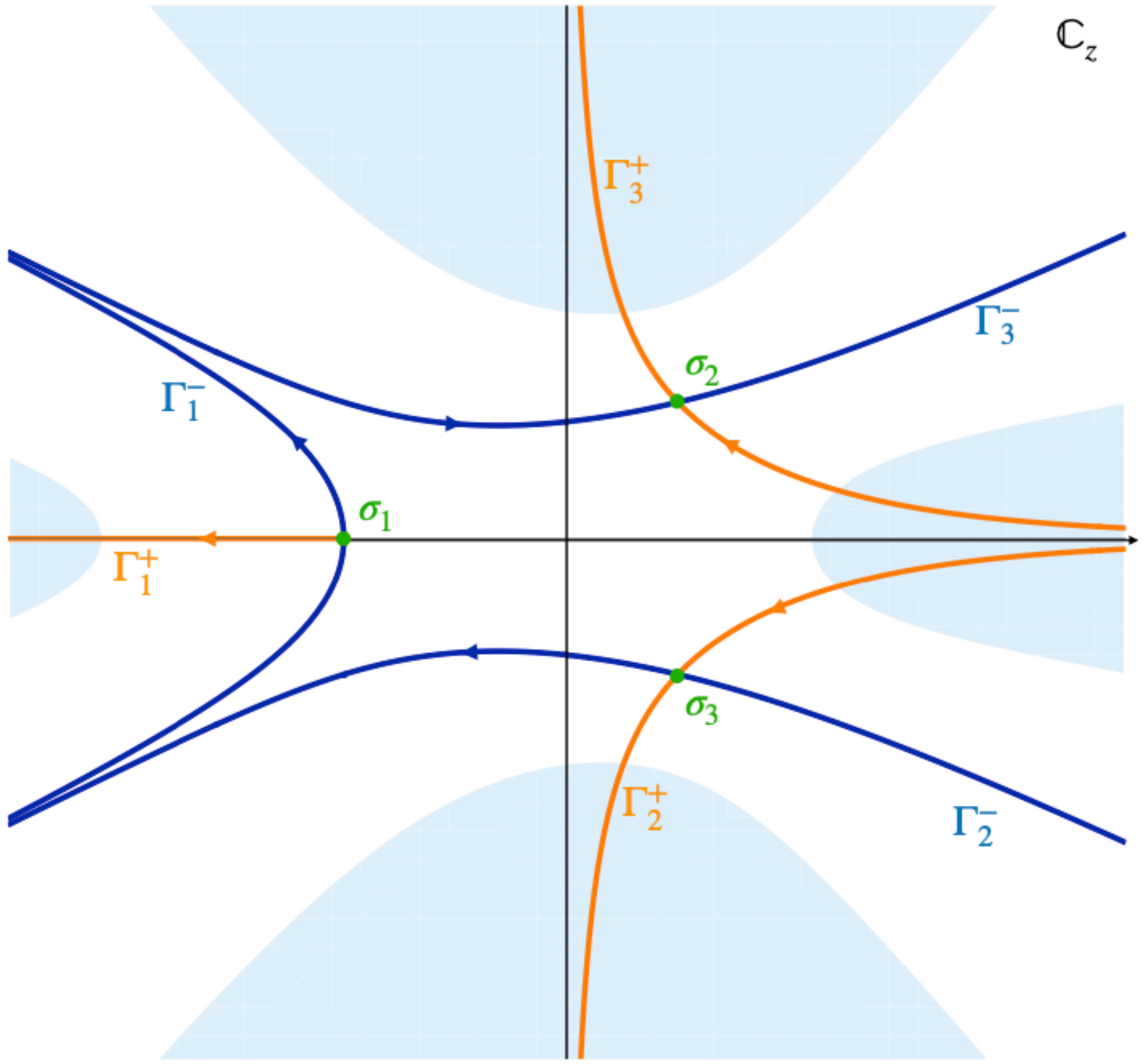


- 3 distinct critical points in $X = \mathbb{C}$
- 3 distinct thimbles $\left\{\Gamma_i^+\right\}_{i=1}^3$ for the Betti homology $H_n^{Betti,\gamma}(\mathbb{C}, f, \mathbb{Z})$ and 3 distinct thimbles for the dual homology

$$dimH_1^{Betti}(\mathbb{C},f,\mathbb{Z}) = 3$$

Region $\Delta > 0$





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$$dimH_1^{Betti}(\mathbb{C}, f, \mathbb{Z}) = 3$$

$$l_0 : Re(\gamma) = -\frac{11}{16}\sqrt{\frac{3}{2}}\mathrm{Im}(\gamma), \qquad \textit{where} \quad \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_1} = \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_2},$$

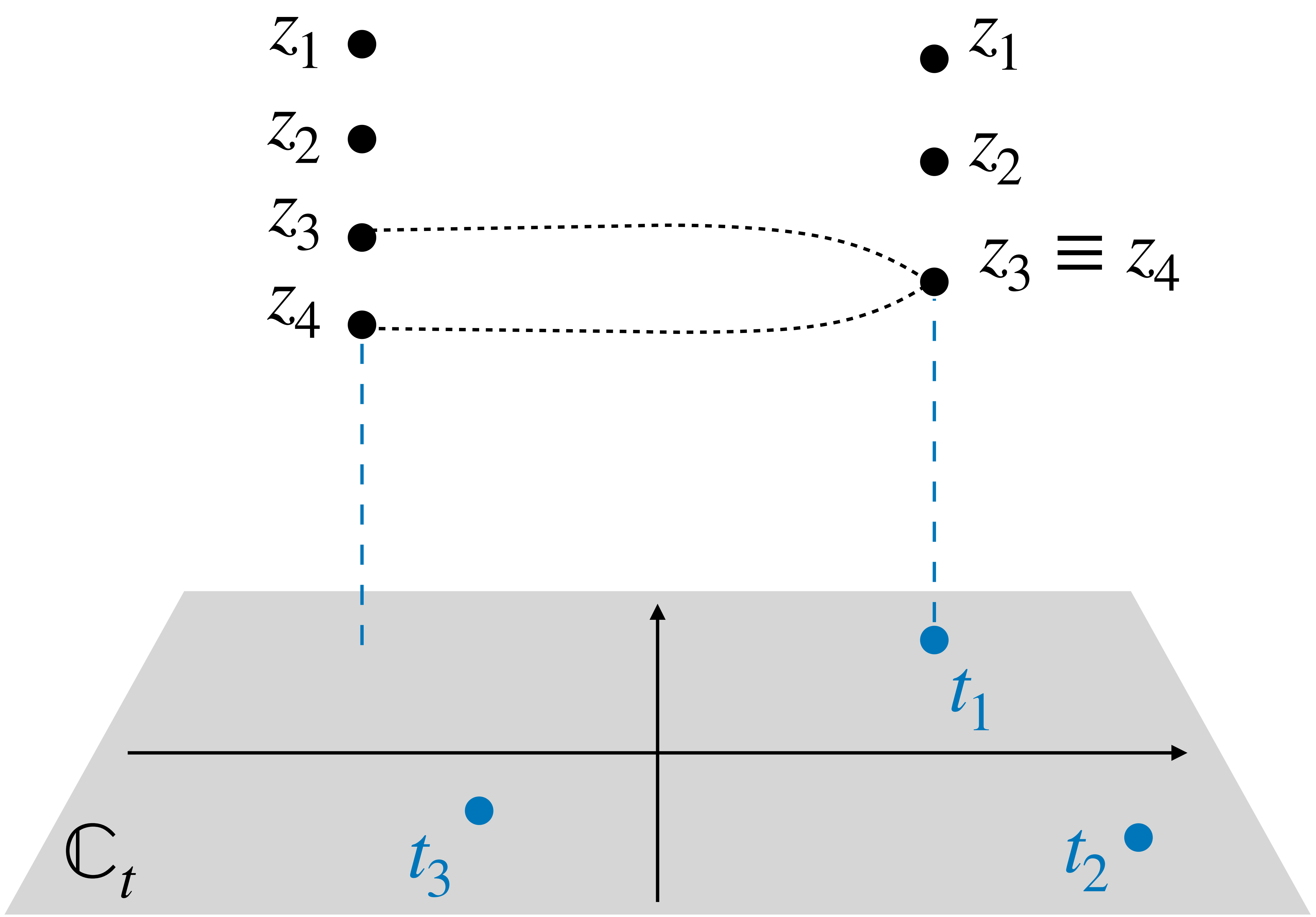
$$l_1 : Re(\gamma) = 0, \qquad \textit{where} \quad \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_2} = \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_3},$$

$$l_2 : Re(\gamma) = \frac{11}{16}\sqrt{\frac{3}{2}}\mathrm{Im}(\gamma), \qquad \textit{where} \quad \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_1} = \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_3}.$$

Stokes'rays separating the regions (0) , (1) and (2) in the plane $\mathbb{C}_\gamma$

Region $\Delta > 0$

- Fix $\gamma \notin l_i$
- **3 critical points** in Σ corresponding to the **3 critical values** $f(\sigma_1) = t_1, f(\sigma_2) = t_2$ and $f(\sigma_3) = t_3$
- Construct the **fibration** $f : \mathbb{C} \mapsto \mathbb{C}_t$, wich fibers are the four points $f^{-1}(t) = \left\{z_1(t), z_2(t), z_3(t), z_4(t)\right\}$



- Construct the vanishing cycles

$$\Delta_1 = \{z_3\} - \{z_4\} \quad \textit{and} \quad \Delta_2 = \Delta_3 = \{z_1\} - \{z_4\}$$

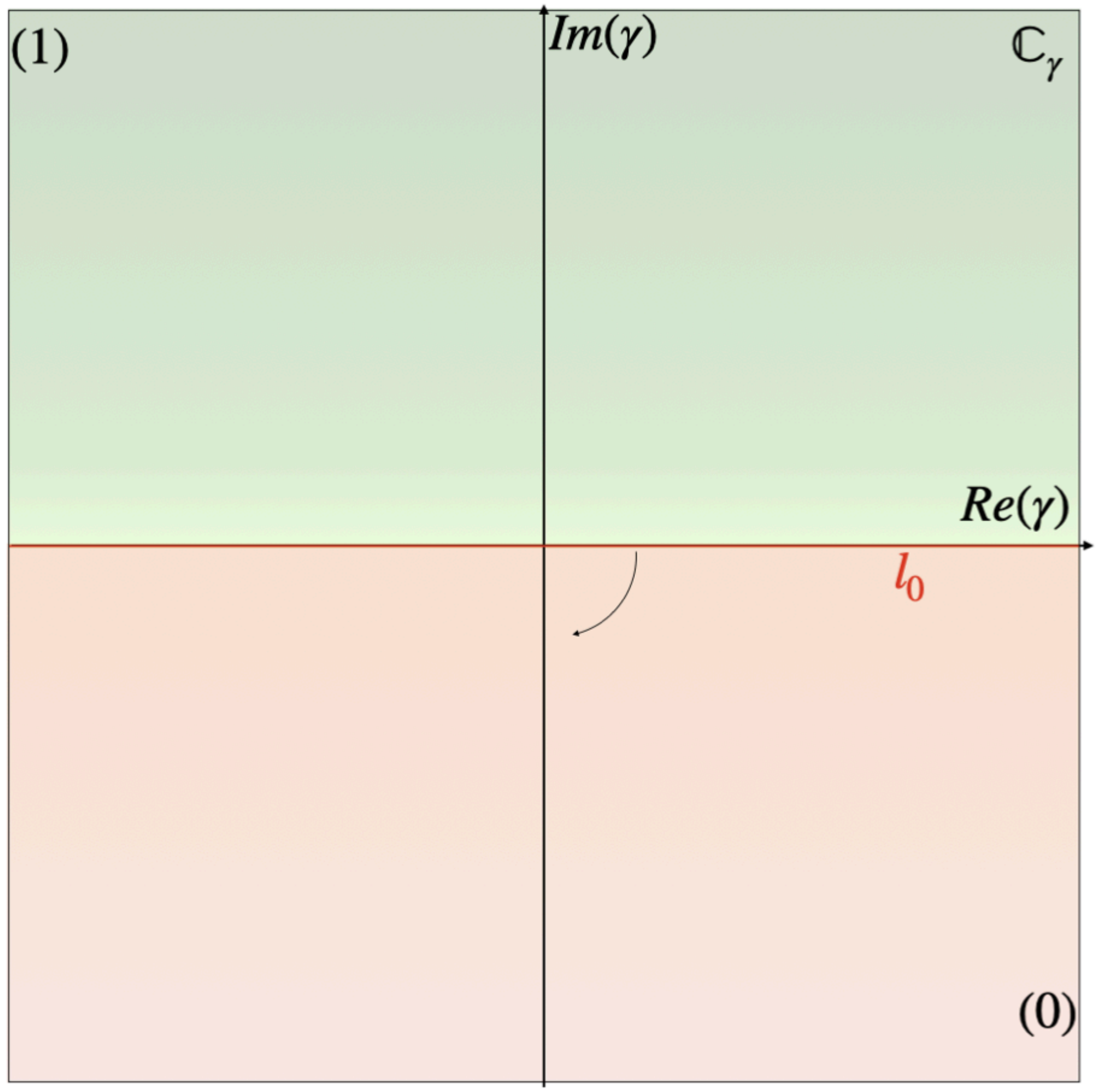
- Compute the monodromies around the critical values

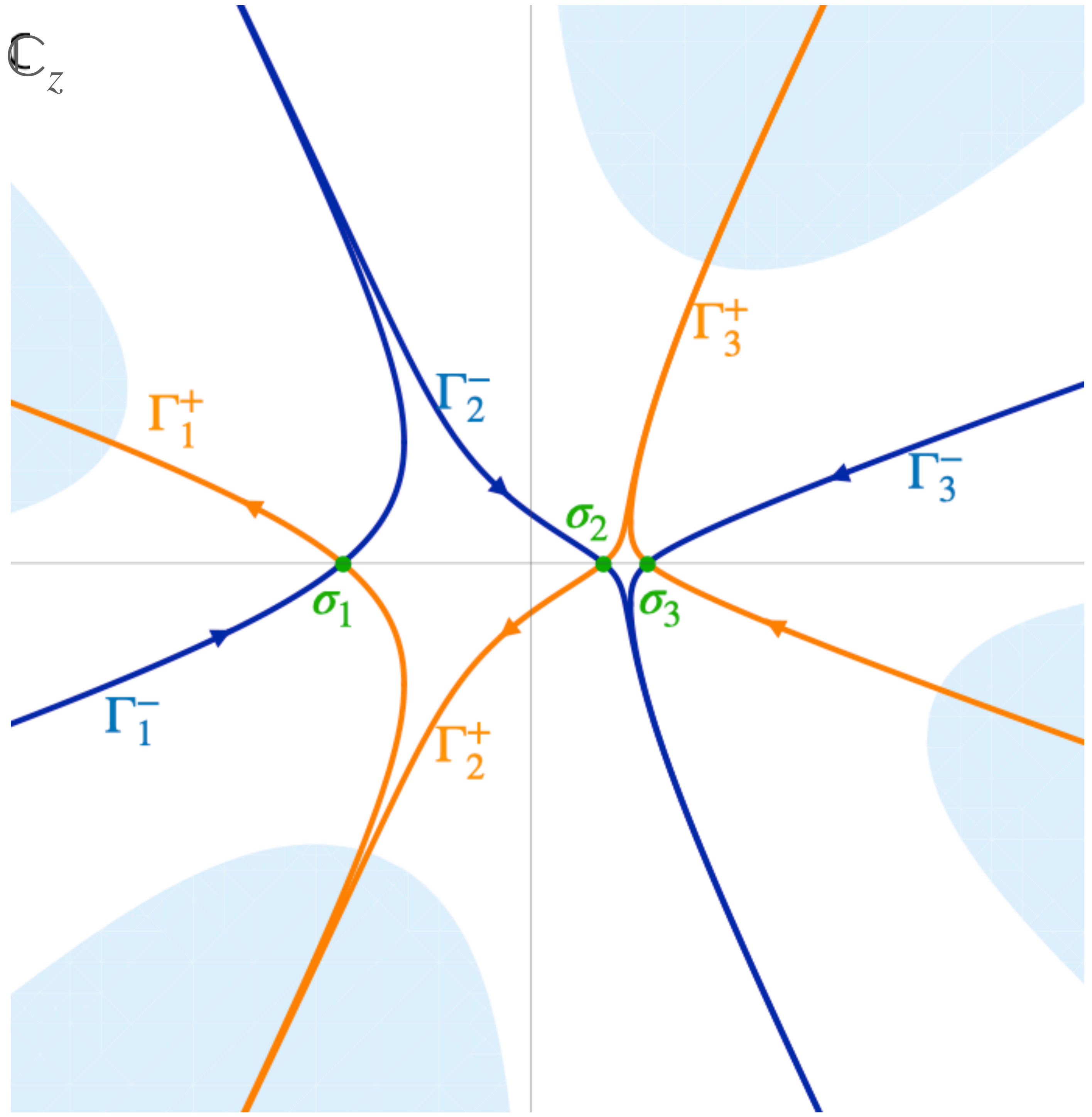
$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad M_2 = M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- Compute the jump matrices for the WCS

$$T^{(0)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \qquad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Region $\Delta < 0$





- $dimH_1^{Betti}(\mathbb{C}, f, \mathbb{Z}) = 3$
- 3 vanishing cycles

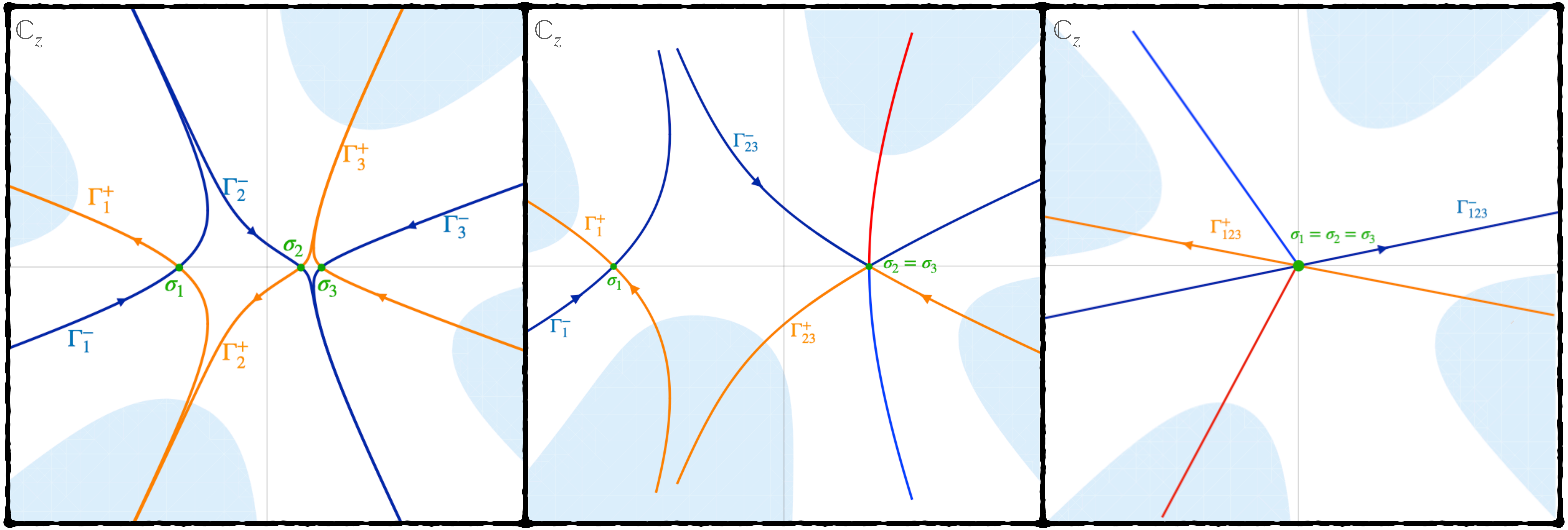
$$\Delta_1 = \{z_1\} - \{z_2\}\,,$$
$$\Delta_2 = \{z_3\} - \{z_4\},$$
$$\Delta_3 = \{z_1\} - \{z_4\}$$

- Monodromies

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \; \textit{and } M_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

- 1 Stokes'ray: study the corresponding WCS.

Region $\Delta = 0$



$H_1(X, D_N, \mathbb{Z}) = span\{\Gamma_1^+, \Gamma_{23}^+\} \cong \mathbb{Z}^2$ $\qquad$ $H_1(X, D_N, \mathbb{Z}) = span\{\Gamma_{123}^+\} \cong \mathbb{Z}$

$H_1(X, D_N, \mathbb{Z})^\vee = span\{\Gamma_1^-, \Gamma_{23}^-\} \cong \mathbb{Z}^2$ $\qquad$ $H_1(X, D_N, \mathbb{Z})^\vee = span\{\Gamma_{123}^-\} \cong \mathbb{Z}$

Multivalued function

Let us consider the case in which $f : X \mapsto \mathbb{C}$ is a multivalued function.

→ The main constructions underlying the analysis still work but we need some modifications!

MAIN POTENTIAL PROBLEM

Additional contribution to the 1-form α defining the **twist** in the de Rham cohomology

- Holomorphic case: $\alpha = df$ (**exact form**)
- Multivalued case: $\alpha = \alpha_{reg} + \alpha_{log} + \alpha_{\infty}$ (**closed form**, not necessary exact)

Geometric Manipulations:

- Choose a suitable compactification for $X \mapsto \overline{X}$ s.t.: $\overline{X} - X = D_h \cup D_v \cup D_{log}$

Multivalued function

Let us consider the case in which $f : X \mapsto \mathbb{C}$ is a multivalued function.

→ The main constructions underlying the analysis still work but we need some modifications!

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Additional contribution to the 1-form α defining the **twist** in the de Rham cohomology

- Holomorphic case: $\alpha = df$ (**exact form**)
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Geometric Manipulations:

- Choose a suitable compactification for $X \mapsto \overline{X}$ s.t.: $\overline{X} - X = D_h \cup D_v \cup D_{log}$
- Study the behavior of $\bar{f} : \overline{X} \mapsto \mathbb{C}$

Horizontal divisor
Locus at infinity where $\bar{f}$ is finite.

Vertical divisor
Locus at infinity where $\bar{f}$ diverges.

Log divisor
Locus at which $d\bar{f}$ has log poles.

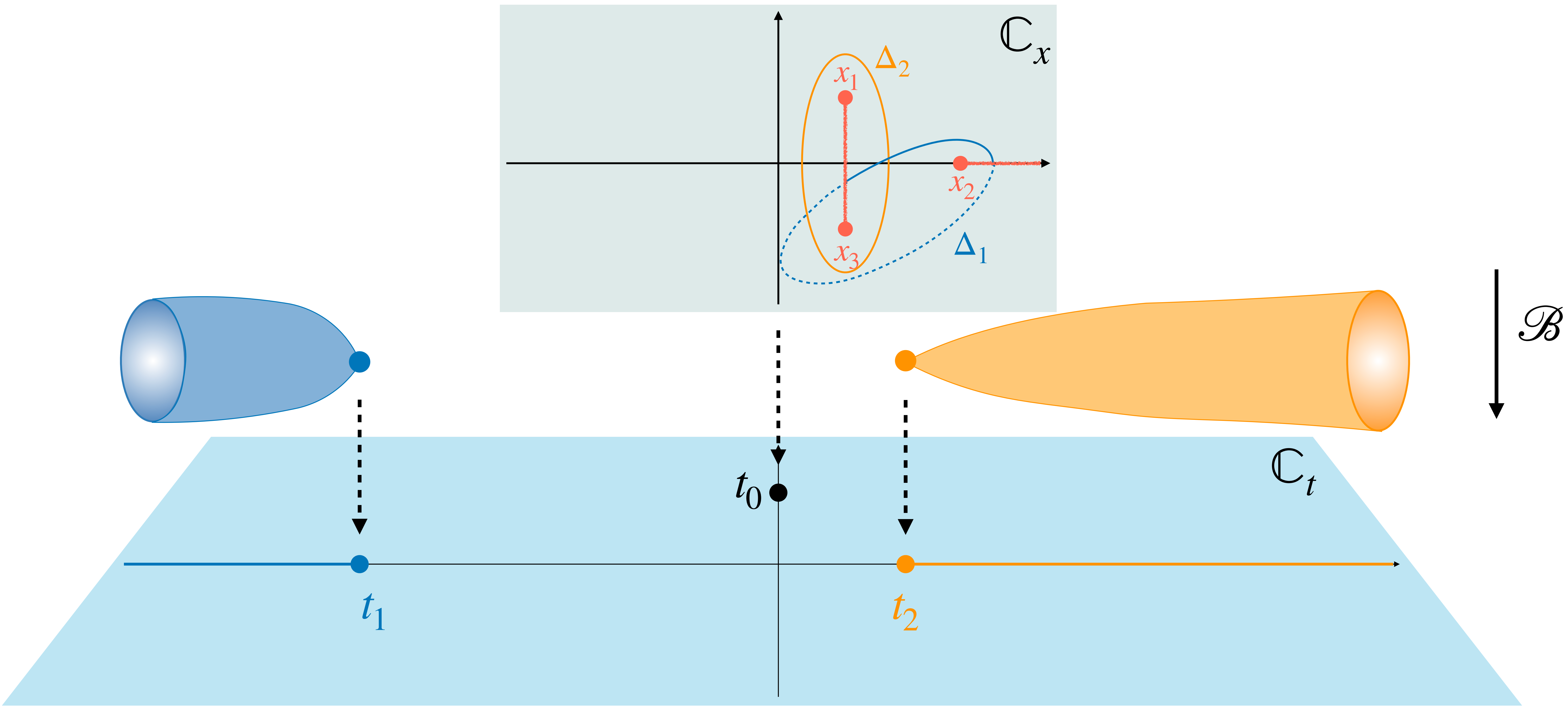
Logarithmic exponent

$$I = \int_\Gamma \frac{dx \wedge dy}{\left[y^2 + x(x-1)(x-\lambda)\right]^\gamma} = \int_\Gamma e^{-\gamma \log\left[y^2 + x(x-1)(x-\lambda)\right]} dx \wedge dy = \int_\Gamma e^{-\gamma \log \mathscr{B}(x,y;\lambda)} dx \wedge dy$$

- $X = \mathbb{C}^2 \backslash \{\mathscr{B} = 0\}$
- $\overline{X} = \mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$
- We study the closed form: $d\ \log \overline{\mathscr{B}} = \dfrac{2\eta y dy + [y^2 + x^2 + x\lambda(x-2\eta)]d\eta + [-3x^2 - \eta^2\lambda + 2x\eta(1+\lambda)]dx}{y^2\eta - x(x-\eta)(x-\eta\lambda)}$
- $D_h = D_v = \emptyset$
- $D_{log} = D_{\overline{\mathscr{B}}} \cup D_\infty$ with $D_{\overline{\mathscr{B}}} = \overline{\mathscr{E}}_\lambda = \{[x : y : \eta] \in \mathbb{P}^2 \,|\, \overline{\mathscr{B}} = 0\}$

$$D_\infty = \mathbb{P}^1 = \{[x : y : 0] \in \mathbb{P}^2\}$$

Logarithmic exponent



Rank (Betti homology) = 2 → Number of independent thimbles

Conclusions and future directions

- **Exponential integrals** provide a well defined pairing between twisted de Rham co-cycles and Betti cycles over complex manifolds that allow to accomodate in the same framework a wide range of physically relevant integrals.

- The **WCS** analysis allows for an analytic continuation of the MIs decomposition in the parameter γ to study Stokes' phenomena to assure a sharp counting of the co-homology dimension.

→ To test the analysis in Feynman integrals in different representations;
(WIP with S.L.Cacciatori, A.Massidda, P.Mastrolia and S.Noja)

→ To study in detail the dependence of the pairing on the kinematic parameters ($\sim$ complex structure moduli for vanishing cycles);
(WIP with S.L.Cacciatori, A.Massidda, P.Mastrolia and S.Noja)

→ To performe explicit computations for families of higher CYs (beyond elliptic curves): $K3, CY_3, CY_4, \ldots;$
(WIP with S.L.Cacciatori, A.Massidda, P.Mastrolia and S.Noja)

→ To use the same method to analyze conformal correlators (aka string amplitudes);
(Project in the definition stage)

→ …

Thank you for your attention!