Periods from mirror symmetry

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MathemAmplitudes: Co-homology and Combinatorics of GKZ systems, Euler-Mellin-Feynman Integrals and Scattering Amplitudes

Motivation

Mirror Symmetry

Type IIA strings with branes ⟷ **Type IIB strings with branes**

$Y_{\vec{y}}$ *(fixed Kähler)*

A-branes (half-dim cycles)

Periods (Feynman like integrals)

X *(fixed complex structure)*

$H_0(X,\mathbb{Q})\,,\ H_2(X,\mathbb{Q})\,,\ H_4(X,\mathbb{Q})$

$\vec{J} \in H^2(X,\mathbb{Q})$ (Kähler cone)

Hosono conjecture

- Choose a basis $\left\{ Q_j \right\}$ of $H^*(X, \mathbb{Q})$
- Then, it exists a cohomology valued hypergeometric function $w\left(\vec{y}, \vec{J}/2\pi i\right)$ such that if you set

$$w\left(\vec{y}, \vec{J}/2\pi i\right) =: \sum_j w_j(\vec{y})\, Q_j$$

then $\longrightarrow$ $$mir \;:\; K^c(X) \longmapsto H_3(Y_y, \mathbb{Q})$$

such that

$$w_j(\vec{y}) = \int_{mir(B_j)} \Omega(Y_{\vec{y}})$$

Holomorphic n-form $\left(CY_n\right)$

Brane configuration with charge Q_j

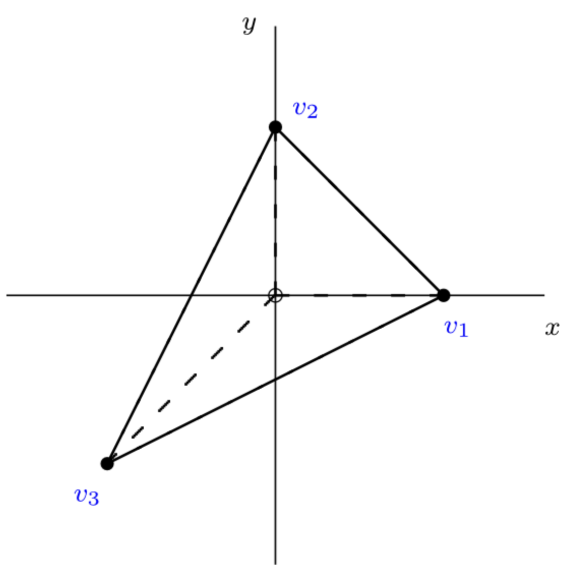
The local CY manifold

The toric orbifold

$$\mathbb{C}^3/\mathbb{Z}_3 = \left\{(x, y, z) \in \mathbb{C}^3 | (x, y, z) \sim (\omega x, \omega y, \omega z),\ \omega^3 = 1\right\}.$$

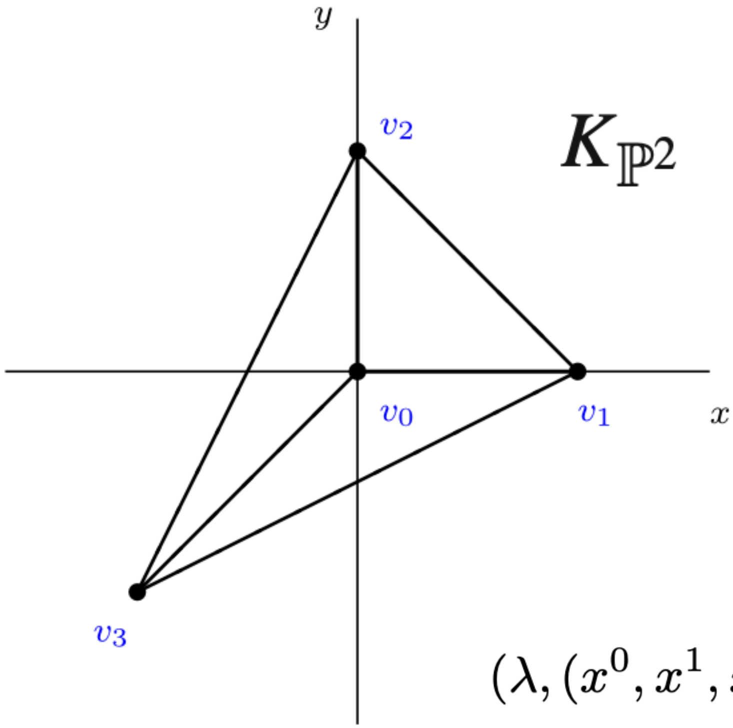
It is a non compact toric variety[3] with fan Δ generated by the three vectors

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$



The local CY manifold

The crepant resolution



$K_{\mathbb{P}^2}$

It is the total space of the canonical bundle of $\mathbb{P}^2$

Add the vector $v_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$X = (\mathbb{C}^4 - \mathbb{C} \times (0,0,0))/\mathbb{C}^*$$

$$(\lambda, (x^0, x^1, x^2, x^3)) \longmapsto (\lambda^{-3} x^0, \lambda x^1, \lambda x^2, \lambda x^3)$$

Some structure

Four invariant divisors

$$D_j\,, \qquad j = 0, 1, 2, 3$$

$$D_1 \sim D_2 \sim D_3$$ Three relations

$$D_1 + D_2 + D_3 + D_0 \sim 0$$ Calabi Yau condition!

Six invariant curves

$$C_j \equiv C_{0j} \qquad C_{ij} \qquad i, j = 1, 2, 3$$

Five relations $$C_{ij} \sim C_{12}, \qquad C_j \sim -3C_{12}$$

$$C \equiv C_{12}$$

Chow Ring

	D_0	D_1	D_2	D_3
C	-3	1	1	1

Take T=D_1 as a generator of the Kähler cone

$$A^*(X) = \mathbb{Z}[X, T, C]/R$$

	X	T	C
X	X	T	C
T	T	C	0
C	C	0	0

Compact

$$A^c_*(X) = \mathbb{Z}[D^c_0, C^c_1, p^c]/R^c$$

	D^c_0	C^c_1	p^c
D^c_0	D^c_0	C^c_1	p^c
C^c_1	C^c_1	p^c	0
p^c	p^c	0	0

Intersection pairing

$$A^*(X) \otimes A^c_*(X) \longrightarrow A^c_*(X)$$

	D_0^c	C_1^c	p^c
X	D_0^c	C_1^c	p^c
T	C_1^c	p^c	0
C	p^c	0	0

And K theory

$$\mathrm{ch} \ : \ K(X) \to A^*(X) \otimes \mathbb{Q} \simeq H^*(X, \mathbb{Q})$$

$$\mathrm{ch}^{\mathrm{c}} \ : \ K^{\mathrm{c}}(X) \to A^{\mathrm{c}}_*(X) \otimes \mathbb{Q} \ \simeq H^{\mathrm{c}}_*(X, \mathbb{Q})$$

Chern classes

$$c(\mathcal{F}) := c_0(\mathcal{F}) + c_1(\mathcal{F}) + \cdots + c_n(\mathcal{F})$$

Some formulas

$$c(\mathcal{O}_X(D)) = X + D$$

$$\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_X(D_i) \Longrightarrow c(\mathcal{F}) = \prod_{i=1}^{r}(X + D_i)$$

$$\mathrm{ch}(\mathcal{F}) := \sum_{i=1}^{r} \mathrm{e}^{D_i} = r\,X + c_1(\mathcal{F}) + \frac{1}{2}(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}))$$

$$\mathrm{ch}(\mathcal{O}_X(D)) = X + D + \tfrac{1}{2}D^2$$

$$\mathrm{td}(\mathcal{F}) := \prod_{i=1}^{r} \frac{D_i}{1 - \mathrm{e}^{-D_i}} = X + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}(c_1(\mathcal{F})^2 + c_2(\mathcal{F}))$$

$$c(T_X) = X + \sum[C_{ij}] \Rightarrow \mathrm{td}(X) = \mathrm{td}(T_X) = X + \frac{1}{12}\sum[C_{ij}]$$

Some more formulas

If $i : V \hookrightarrow X$ is the embedding of V in X, we can define the local Chern character of S by

$$\mathrm{ch}^{\mathrm{c}}(S) = \mathrm{ch}(i_* S_V).$$

Grothendieck–Riemann–Roch theorem:

$$i_*(\mathrm{ch}(S_V)\mathrm{td}(V)) = \mathrm{ch}(i_* S_V)\mathrm{td}(X)$$

$$\mathrm{ch}^{\mathrm{c}}(S) = \mathrm{td}(X)^{-1}\, i_*(\mathrm{ch}(S_V)\mathrm{td}(V)).$$

$$\begin{aligned}
&\mathrm{ch}^{\mathrm{c}}(\mathcal{O}_{p^{\mathrm{c}}}) = p^{\mathrm{c}}, \quad \mathrm{ch}^{\mathrm{c}}(\mathcal{O}_{C^{\mathrm{c}}}(n)) = C^{\mathrm{c}} + (n+1)p^{\mathrm{c}},\\
&\mathrm{ch}^{\mathrm{c}}(\mathcal{O}_{D^{\mathrm{c}}}(C)) = i_*\Bigg(D^{\mathrm{c}} + \Big(C + \tfrac{1}{2}c_1(D^{\mathrm{c}})\Big)\\
&\qquad\qquad + \tfrac{1}{2}\Big(C^2 + c_1(D^{\mathrm{c}})\, C + \tfrac{1}{6}\big(c_1(D^{\mathrm{c}})^2 + c_2(D^{\mathrm{c}})\big)\Big)\Bigg)\\
&\qquad\qquad - \tfrac{1}{12}c_2(X)\, D^{\mathrm{c}}.
\end{aligned}$$

Duality, branes, charges

Duality is determined by the intersection product

$$(|) : K(X) \times K^c(X) \longrightarrow \mathbb{Z}, \quad (\boldsymbol{\phi}, \mathcal{B}) \longmapsto \int_X ch(\boldsymbol{\phi}) ch^c(\mathcal{B}) td(X)$$

Basis of branes

$$\mathcal{B}_0 := \mathcal{O}_{p^c}, \qquad \mathcal{B}_1 := \mathcal{O}_{C_1^c}(-T) \equiv \mathcal{O}_{C_1^c} \otimes \mathcal{O}_X(-D_1), \qquad \mathcal{B}_2 := \mathcal{O}_{D_0^c}(-2T) \equiv \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_X(-2D_1)$$

Dual generators defined by $(\boldsymbol{\phi}_i | \mathcal{B}_j) = \delta_{ij}$

Brane charges $\boldsymbol{Q}_i \equiv Q(\mathcal{B}_i) = ch(\boldsymbol{\phi}_i).$

$$td(X) = X - \frac{1}{2} C$$

$$\begin{aligned} ch^c(\mathcal{B}_0) &= p^c \\ ch^c(\mathcal{B}_1) &= C_1^c \\ ch^c(\mathcal{B}_2) &= D_0^c - \frac{1}{2} C_1^c + \frac{1}{2} p^c \end{aligned}$$

$$\begin{aligned} \mathbf{Q}_0 &= X \\ \mathbf{Q}_1 &= T + \frac{1}{2} C \\ \mathbf{Q}_2 &= C \end{aligned}$$

Mirror symmetry

From the toric data of X define the superpotential:

$$W(x_1, x_2; u, v; \boldsymbol{a}) = uv + a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1^{-1} x_2^{-1}$$

variables $(x_1, x_2; u, v) \in (\mathbb{C}^*)^2 \times \mathbb{C}^2$ and parameters $(a_0, \ldots, a_3) \in (\mathbb{C}^*)^4$.

$$y = \boldsymbol{a}^{\ell}, \qquad \ell \in \mathbb{Z}^4, \qquad \ell_j = C \cdot D_j \qquad\qquad y = \frac{a_1 a_2 a_3}{a_0^3}$$

The mirror family Y_y in $(\mathbb{C}^*)^2 \times \mathbb{C}^2$

$$W(x_1, x_2; u, v; y) = uv + 1 + x_1 + x_2 + y x_1^{-1} x_2^{-1} = 0.$$

Homological mirror symmetry is an equivalence of triangulated categories

$$Mir : D^b\big(Coh(X_{\mathbf{t}})\big) \longrightarrow DFuk^o(Y_{\mathbf{y}}, \omega)$$

The bounded derived category of coherent sheaves $D^b\big(Coh(X_{\mathbf{t}})\big)$ depends on the choice of a complex structure on X, mirror to a (complexified) symplectic structure ω on Y used to define the derived Fukaya category $DFuk^o(Y_{\mathbf{y}}, \omega)$. The functor Mir induces a linear transformation

$$mir : K^{\mathrm{c}}(X_{\mathbf{t}}) \longrightarrow H_3(Y_{\mathbf{y}}, \mathbb{Q}),$$

which is symplectic with respect to the symplectic form on $K^{\mathrm{c}}(X_{\mathbf{t}})$ given by the Euler characteristic χ and the one on $H_3(Y_{\mathbf{y}}, \mathbb{Q})$ given by the intersection form.

Double fibration

$$x_1 x_2^2 + x_1^2 x_2 + y + z x_1 x_2 = 0 \qquad uv + z = 0$$

$$x_i \to y^{\frac{1}{3}} x_i \qquad z = -\frac{uv}{y^{\frac{1}{3}}} - \frac{1}{y^{\frac{1}{3}}} \qquad (X_0 : X_1 : X_2) = (1 : x_1 : x_2)$$

$$X_1^2 X_2 + X_2^2 X_1 - z X_1 X_2 X_0 + X_0^3 = 0.$$

$$\Omega_3 = \frac{1}{(2\pi i)^3} \mathrm{Res}_{f=0,g=0} \times \left(\frac{(X_0 dX_1 \wedge dX_2 - X_1 dX_0 \wedge dX_2 + X_2 dX_0 \wedge dX_1) \wedge du \wedge dv \wedge dz}{f(X_0, X_1, X_2) g(u, v)} \right)$$

$$f(X_0, X_1, X_2) = X_1 X_2^2 + X_1^2 X_2 - z X_0 X_1 X_2 + X_0^3,$$
$$g(u, v) = y^{\frac{1}{3}} z + uv + 1.$$

$$X_0 = X, \qquad X_1 = Y + \frac{U}{2} + \frac{Xz}{2}, \qquad X_2 = -Y + \frac{U}{2} + \frac{Xz}{2}$$

$$Y^2 = X^3 + \left(\frac{z}{2}\right)^2 X^2 + \frac{z}{2} X + \frac{1}{4}$$

Hosono mirror map

The rational cohomology ring of X is

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[J]/J^3,$$

where J is the Poincaré dual of the class of a line H in the base $\mathbb{P}^2$

Introduce the cohomology valued hgf

$$w(y; J) := \sum_{n=0}^{\infty} \frac{y^{n+\rho}}{\Gamma(1+n+\rho)^3 \Gamma(1-3(n+\rho))} \Bigg|_{\rho=\frac{J}{2\pi i}}$$

Expand as $\quad w(y; J) = w_0(y) Q_0 + w_1(y) Q_1 + w_2(y) Q_2$

Then, mir maps the brane generators in Lagrangian cycles

$\mathcal{B}_i \mapsto mir(\mathcal{B}_i) \equiv L_i$ such that $w_i(y) = \int_{mir(\mathcal{B}_i)} \Omega_3(Y_y)$ and

the monodromy of the hypergeometric series is integral and symplectic with respect to the symplectic form defined in $K^c(X)$

$$\chi(\mathcal{B}_i, \mathcal{B}_j) = \int_X ch(\mathcal{B}_i^*) ch(\mathcal{B}_j) td(X)$$

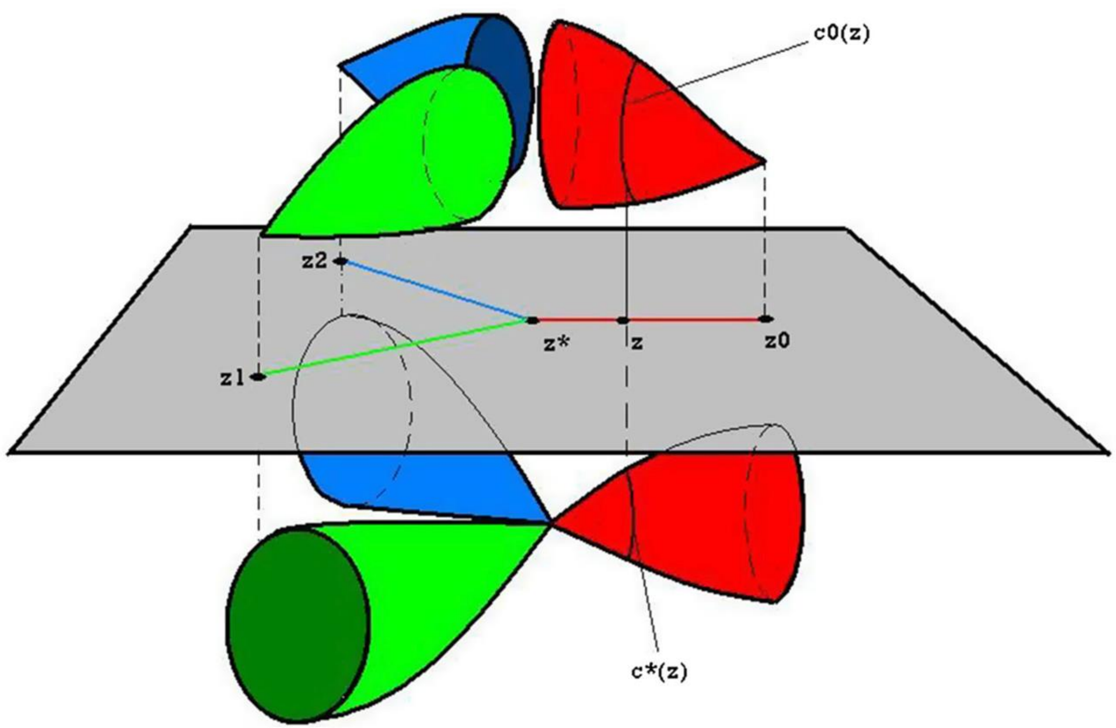
The Lagrangian cycles

$$Y^2 = X^3 + \left(\frac{z}{2}\right)^2 X^2 + \frac{z}{2} X + \frac{1}{4}.$$

In particular the critical values of the elliptic fibration are

$$z_0 = 3, \qquad z_1 = 3\omega, \qquad z_2 = 3\omega^2,$$

where $\omega := -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, whereas the singular point for the $\mathbb{C}^*$ fibration is $z_* = -\frac{1}{y^{\frac{1}{3}}}$.



The Lagrangian cycles

The cycles are Lagrangian 3-spheres

$$
\begin{aligned}
C_i &\simeq \left\{(z,w)\in\mathbb{C}^2 | z = Rte^{i\phi},\, w = R\sqrt{1-t^2}e^{i\phi},\, \phi\in[0,2\pi],\, \psi\in[0,2\pi],\, t\in[0,1]\right\} \\
&= \left\{(z,w)\in\mathbb{C}^2 \,||z|^2+|w|^2 = R^2\right\} \simeq S^3.
\end{aligned}
$$

the periods are

$$
I_i = -\frac{1}{8\pi^2}\int_{z_*}^{z_i} dz \int_{c_i} \frac{dX}{\sqrt{(X-X_0)(X-X_1)(X-X_2)}},
$$

$$
\begin{aligned}
J_k &:= \int_{c_k} \frac{dX}{\sqrt{(X-X_i)(X-X_j)(X-X_k)}} \\
&= 2\int_{X_i}^{X_k} \frac{dX}{\sqrt{(X-X_i)(X-X_j)(X-X_k)}} - 2\int_{X_k}^{X_j} \frac{dX}{\sqrt{(X-X_i)(X-X_j)(X-X_k)}} \\
&= \frac{2\pi}{\sqrt{X_j-X_i}}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{X_k-X_i}{X_j-X_i}\right) - \frac{2\pi}{\sqrt{X_i-X_j}}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{X_k-X_j}{X_i-X_j}\right).
\end{aligned}
$$

Asymptotic periods

$$I_k = -\frac{1}{8\pi^2}\int_{z_*}^{z_k} J_k dz = -\frac{1}{8\pi^2}\int_0^{z_k} J_k dz$$
$$+\frac{1}{4\pi}\int_0^{-y^{-\frac{1}{3}}} \frac{1}{\sqrt{X_j - X_i}}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{X_k - X_i}{X_j - X_i}\right) dz$$
$$-\frac{1}{4\pi}\int_0^{-y^{-\frac{1}{3}}} \frac{1}{\sqrt{X_i - X_j}}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{X_k - X_j}{X_i - X_j}\right) dz =: A_k + B_k(y).$$

Expansion for large y

$$I_k(y) = \frac{1}{3}(-1)^k + \omega^k \frac{\sqrt{3}}{8\pi^3}\Gamma(1/3)^3 y^{-\frac{1}{3}} + \omega^{2k}\frac{\sqrt{3}}{16\pi^3}\Gamma(2/3)^3 y^{-\frac{2}{3}} + o(y^{-1}).$$

Compare with $w_j(y)$ (defined for small y). We need analytic continuation.

Theory of cohomological hgf

Let X be a finite-dimensional toric variety with the Chow ring $C(X)$ (possibly with a compact support). Let $f : D \longrightarrow \mathbb{C}$ be a complex function, holomorphic in the domain $D \subseteq \mathbb{C}^d$, where $d = \dim(C^2(X))$. Let $\underline{\rho} \equiv \{\rho_j\} \in C(X)$ be any set of generators of $C^2(X)$.

Definition 1. *We define the I cohomologisation of f polarized by $\underline{\rho}$ as the function*

$$f_{\underline{\rho}} : D \longrightarrow C(X), \qquad f_{\underline{\rho}}(z_1, \ldots, z_d) := f(z_1 + \rho_1, \ldots, z_d + \rho_d).$$

suppose ρ ha nilpotence N. Then, we have

$$f_\rho(z) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(z)\rho^n, \qquad Res(f_\rho, z_0) := \sum_{n=0}^{N} \frac{1}{n!} Res(f^{(n)}, z_0)\rho^n$$

Proposition 1. *In the same hypotheses as above, we have*

$$Res(f_\rho, z_0) = Res(f, z_0).$$

Proposition 2. *Let $f : D \longrightarrow \mathbb{C}$ be holomorphic in the domain D, $w \in D$ and $\rho \in C^2(X)$. Then,*

$$f_\rho(w) = Res\Big(\frac{f(z)}{z - w - \rho}, w\Big).$$

Theory of cohomological hgf

Proposition 3. *Γ_ρ is a holomorphic function on $\mathbb{C}\backslash\mathbb{N}$, with simple poles in $z=-n$, $n=0,1,2,\ldots$, with residues*

$$Res(\Gamma_\rho,-n)=\frac{(-1)^n}{n!},\qquad n=0,1,2,\ldots.$$

Moreover, it satisfies the identities

$$\begin{aligned}(z+\rho)\Gamma_\rho(z)=&\Gamma_\rho(z+1),\\ \Gamma_\rho(z)\Gamma_{-\rho}(1-z)=&\frac{\pi}{\sin(\pi(z+\rho))},\\ (2\pi)^{\frac{n-1}{2}}\Gamma_{n\rho}(nz)=&n^{nz+n\rho-\frac{1}{2}}\prod_{j=0}^{n-1}\Gamma_\rho\Big(z+\frac{j}{n}\Big).\end{aligned}$$

Apply to

$$f(y,\rho)=\sum_{n=0}^{\infty}\frac{y^{n+\rho}}{\Gamma(1+n+\rho)^3\Gamma(1-3n-3\rho)},$$

$$\begin{aligned}f(y,\rho)=&1+\rho\sum_{n=0}^{\infty}Res\Big(\frac{y^{-s}\Gamma(-3s)}{\Gamma(1-s)^3}\frac{3\pi}{\sin(\pi(s+\rho))},s=-n\Big)\\ =&1+\frac{\rho}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{y^{-s}\Gamma(-3s)}{\Gamma(1-s)^3}\frac{3\pi}{\sin(\pi(s+\rho))}\,ds\\ =&1+\frac{\rho}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}y^{-s}\Gamma(-3s)\Gamma(s)^3\frac{3\sin^2(\pi s)}{\pi^2}\Big(1+\rho\pi\frac{\cos(\pi s)}{\sin(\pi s)}\Big)\,ds.\end{aligned}$$

Final result

$$f(y,\rho) = 1 - \rho\frac{3}{4\pi^2}\sum_{n=0}^{\infty} y^{-\frac{1}{3}-n}\Gamma\left(n+\frac{1}{3}\right)^3\left(1+\frac{\rho}{\sqrt{3}}\right) + \rho\frac{3}{4\pi^2}\sum_{n=0}^{\infty} y^{-\frac{2}{3}-n}\Gamma\left(n+\frac{2}{3}\right)^3\left(1-\frac{\rho}{\sqrt{3}}\right).$$

Comparison with the asymptotic expansion gives

$$(I_0, I_1, I_2) = (w_0, w_1, w_2)\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

$$w_0(y) = 1,$$

$$w_1(y) = \frac{1}{2\pi i}\ln(y) + \frac{3}{2\pi i}\sum_{m=1}\frac{(3m-1)!}{(m!)^3}(-y)^m,$$

$$w_2(y) = -\frac{1}{8\pi^2}(\ln(-y))^2 + \frac{1}{8} - \frac{3}{4\pi^2}\ln(-y)\sum_{m=1}\frac{(3m-1)!}{(m!)^3}(-y)^m$$

$$-\frac{9}{4\pi^2}\sum_{n=1}\frac{(3n-1)!}{(n!)^3}[\psi(3n) - \psi(n+1)](-y)^n.$$

Small y

Large y

$$w_1(y) = \frac{3}{2\pi i}\left[-\frac{y^{-\frac{1}{3}}}{4\pi^2}\sum_{n=0}^{\infty}\Gamma(n+1/3)^3\frac{(-1)^n}{(3n+1)!}\frac{1}{y^n} + \frac{y^{-\frac{2}{3}}}{4\pi^2}\sum_{n=0}^{\infty}\Gamma(n+2/3)^3\frac{(-1)^n}{(3n+2)!}\frac{1}{y^n}\right]$$

$$w_2(y) = \frac{1}{3} + \frac{\sqrt{3}}{4\pi}\left[-(1+i\sqrt{3})\frac{y^{-\frac{1}{3}}}{4\pi^2}\sum_{n=0}^{\infty}\Gamma(n+1/3)^3\frac{(-1)^n}{(3n+1)!}\frac{1}{y^n} + (-1+i\sqrt{3})\frac{y^{-\frac{2}{3}}}{4\pi^2}\sum_{n=0}^{\infty}\Gamma(n+2/3)^3\frac{(-1)^n}{(3n+2)!}\frac{1}{y^n}\right]$$

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