

Annihilators and D-modules for Feynman integrals

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in collaboration with

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Motivation

- Feynman integrals are complicated functions whose computation is difficult.
- We would like to understand and utilize underlying mathematical structures.
- $\mathcal{D}$ -module approach to Feynman integrals [1-3].
- Can we find an algorithm to construct differential operators annihilating Feynman integrals?
- Can we build an annihilating ideal?
- Does there exist a generalization of the GKZ [4,5] construction for Feynman integrals?

Feynman integrals

Feynman parametrization:

$$\pi(\mathbf{s}) = \int_{\Gamma} \frac{U(\alpha)^{\kappa}}{F(\alpha, \mathbf{s})^{\eta}} \prod_{i=1}^n \alpha_i^{\nu_i-1} \mu \equiv \int_{\Gamma} \Omega,$$

- $\alpha = (\alpha_1, \dots, \alpha_n)$: integration variables,
- $\mathbf{s} = (\mathbf{s}_1, \dots, \mathbf{s}_m)$: external variables,
- $\mu \equiv \sum_{i=1}^n (-1)^{i-1} \alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_n$,
- $\Gamma = \{(\alpha_1, \dots, \alpha_n) | \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0\}$,
- U : homogeneous polynomial of degree ℓ ,
- F : homogeneous polynomial of $\ell + 1$ and also homogeneous in $\mathbf{s}$ of degree 1,
- $\kappa = \sum_{i=1}^n \nu_i - (\ell + 1)d/2$ and $\eta = \sum_{i=1}^n \nu_i - \ell d/2$,
- d : space-time dimension.

assumption: $\nu_i = 1$ for $i = 1, \dots, n$

$\mathcal{D}$ -modules

- m-th rational Weyl algebra:

$$\mathcal{R}_m = \mathbb{C}(x_1, \dots, x_m) \langle \partial_1, \dots, \partial_m \rangle \text{ with } [\partial_i, c(s)] = \frac{\partial c(s)}{\partial s_i},$$

- Any $L \in \mathcal{R}_m$ can be written in a normally ordered form as

$$L = \sum c_q(s) \partial^q,$$

multi-index notation, $\partial^q \equiv \partial_{s_1}^{q_1} \dots \partial_{s_m}^{q_m}$, $q = \{q_1, \dots, q_m\}$.

- Holonomic rank

$$r(\mathcal{I}) = \dim_{\mathbb{C}(x)} \mathbb{C}(x)[\xi] / (\mathbb{C}(x)[\xi] \cdot \text{char}(\mathcal{I})) = \dim_{\mathbb{C}(x)} (\mathcal{R} / \mathcal{R}\mathcal{I}) .$$

Macaulay matrix

- Macaulay matrix

$$M = (\partial^I \mathbf{A}_p) = 0, \quad |I| = 0, \dots, l,$$

where l is a chosen maximal degree of the derivatives, *i.e.*

$$\partial^I = \partial_{s_1}^{i_1} \cdots \partial_{s_m}^{i_m} \text{ with } i_1 + \dots + i_m = |I|.$$

- Standard monomials

$$\text{Std} = \{\partial^k\},$$

- Holonomic rank

$$r(\mathcal{I}) = |\text{Std}|.$$

Differential operators

- Goal:

$$\mathcal{I} = \{ \mathbf{A}_p \mid \mathbf{A}_p \pi(s) = 0 \},$$

- Shape:

$$\mathbf{A}_p \equiv \mathbf{D}_p + \mathbf{D}_{p-1},$$

- Ansatz w.r.t. external variables:

$$\mathbf{D}_p = \sum_{l, |l|=p} c_l \partial_s^l \implies \mathbf{D}_p \pi(s) = \int_{\Gamma} \frac{N_p}{F(\alpha, s)^p} \Omega,$$

where c_l are coefficients we want to fix.

- Action on $\pi(s)$:

$$\mathbf{A}_p \pi(s) = (\mathbf{D}_p + \mathbf{D}_{p-1}) \pi(s) = \int_{\Gamma} d\gamma = \int_{\partial\Gamma} \gamma = 0.$$

Griffiths-Dwork reduction

- Projective differential form [6-13]:

$$\begin{aligned} \gamma \equiv & \frac{1}{F^{k-1}} \sum_{i < j} (-1)^{i+j} (\alpha_i \lambda_j - \alpha_j \lambda_i) \frac{U(\alpha)^\kappa}{F(\alpha, \mathbf{s})^\eta} \\ & \times d\alpha_1 \wedge \dots \widehat{d\alpha_i} \wedge \dots \wedge \widehat{d\alpha_j} \wedge \dots \wedge d\alpha_n. \end{aligned}$$

- Total derivative

$$d\gamma = (\eta + k - 1) \frac{\sum_{i=1}^n \lambda_i \partial_i F}{F^k} \Omega - \frac{\sum_{i=1}^n \nabla_i \lambda_i}{F^{k-1}} \Omega,$$

with the *twisted covariant derivative* ∇_i , defined as,

$$\nabla_i \equiv \partial_i + \omega_i \wedge, \quad \text{with} \quad \omega_i \equiv \partial_i \log(U^\kappa),$$

Necessary conditions

A) Boundary condition:

$$\lambda_i \Big|_{\alpha_i=0} = 0,$$

B) Jacobian ideal membership condition:

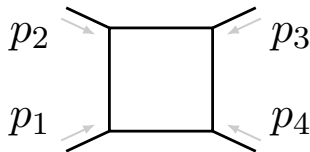
$$N_p = \sum_{i=1}^n \lambda_i \partial_i F,$$

implying that $N_p \in \langle \partial_1 F, \dots, \partial_n F \rangle$

C) Twisted covariant derivative condition:

$$N_{p-1} = \sum_{i=1}^n \nabla_i \lambda_i.$$

Example: Massless Box



Symanzik polynomials:

$$U = \sum_{i=1}^4 \alpha_i, \quad \text{and} \quad F = \alpha_1 \alpha_3 s + \alpha_2 \alpha_4 t,$$

with exponents $\kappa = 4 - d$ and $\eta = 4 - \frac{d}{2}$.

Example: Massless Box - First order operator

$$\mathbf{D}_1 \pi(s, t) = (c_1 \partial_s + c_2 \partial_t) \pi(s, t) = \int_{\Gamma} \frac{N_1}{F} \Omega,$$

$$\mathbf{D}_0 \pi(s, t) = \pi(s, t) = \int_{\Gamma} \Omega \implies N_0 = 1,$$

where

$$N_1 = \alpha_1 \alpha_3 c_1 \left(\frac{d}{2} - 4 \right) + \alpha_2 \alpha_4 c_2 \left(\frac{d}{2} - 4 \right).$$

- Conditions **A** and **B** $\rightarrow$ Syzygy equation:

$$\sum_{i=1}^4 q_i \alpha_i \partial_i F = q_0 N_1,$$

There are four generators of Syzygies $g_1, \dots, g_4$ and we choose the following combination:

$$q \equiv (q_1, \dots, q_0) = r_1 g_1 + r_2 g_2 + g_3, \text{ where } r_i \text{ are parameters.}$$

Example: Massless Box - First order operator

- $N_1 = \sum_{i=1}^4 \lambda_i \partial_i F$ where $\lambda_i = \frac{g_i}{q_0} \alpha_i$, explicitly:

$$\lambda_1 = \frac{\alpha_1(c_1(d-8)t + r_1)}{2st}, \quad \lambda_2 = \frac{\alpha_2(c_2(d-8)s + r_2)}{2st},$$
$$\lambda_3 = -\frac{\alpha_3 r_1}{2st}, \quad \lambda_4 = -\frac{\alpha_4 r_2}{2st}.$$

- Condition C gives:

$$\left\{ c_2 \rightarrow -\frac{4t}{(d-8)^2}, c_1 \rightarrow -\frac{4s}{(d-8)^2}, r_1 \rightarrow \frac{2st}{d-8}, r_2 \rightarrow \frac{2st}{d-8} \right\}.$$

- First order operator:

$$\mathbf{A}_1 = s\partial_s + t\partial_t + \left(4 - \frac{d}{2}\right),$$

where we took into account the prefactor $(\eta + k - 1)$ in the Griffiths-Dwork formula.

Example: Massless Box

- Generators of $\mathcal{I}$:

$$\mathbf{A}_E = s\partial_s + t\partial_t + \left(4 - \frac{d}{2}\right),$$

$$\mathbf{A}_3 = p_{3,0} \partial_s^3 + p_{2,1} \partial_s^2 \partial_t + p_{1,2} \partial_s \partial_t^2 + p_{0,2} \partial_t^2.$$

- Holonomic rank:

$$r(\mathcal{I}) = 3.$$

- singular locus:

$$\text{sing}(\mathcal{I}) = st(s+t).$$

- Picard-Fuchs operator:

$$\mathbf{A}_{\text{PF}} = (\partial_s + r_1) \cdot (\partial_s + r_2) \cdot (\partial_s + r_3).$$

Example: Massless Box

- Differential system in the basis $\vec{f} = (l_{1,1,1,1}, \partial_s l_{1,1,1,1}, \partial_s^2 l_{1,1,1,1})^T$:

$$\partial_s \vec{f} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{d-8}{2s^2(s+t)} & \frac{(d-8)^2(-t)-(64-6d)s}{4s^2(s+t)} & \frac{(d-18)s+2(d-9)t}{2s(s+t)} \end{pmatrix} \vec{f}$$

- Change to the basis $\vec{g} = (l_{1,1,1,1}, l_{1,0,1,0}, l_{0,1,0,1})$:

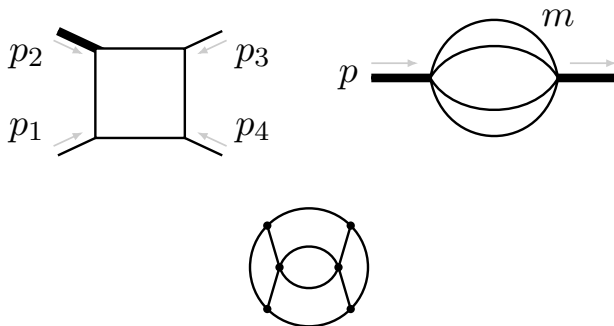
$$\vec{f} = B\vec{g},$$

- Differential system in new basis:

$$\partial_s \vec{g} = \begin{pmatrix} \frac{(d-6)t-2s}{2s(s+t)} & \frac{2(d-3)}{s^2(s+t)} & \frac{6-2d}{s^2t+st^2} \\ 0 & \frac{d-4}{2s} & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{g}$$

More applications

Hypergeometric ${}_2F_1$ and ${}_3F_2$



and more

Generators of the $\mathcal{D}$ -ideal

Conjecture

Under genericity conditions on U, F, κ, η and when p is sufficiently large, the holonomic rank r of the left ideal of the rational Weyl algebra R_m generated by $\mathbf{D}_i + \mathbf{D}_{i-1}, i \leq p$ is $r = |\chi(V)|$ where $\chi(V)$ is the Euler characteristic of the zero set V of $U(\alpha)F(\alpha, s)$ in the α -space $\{\alpha_1 \cdots \alpha_n \neq 0\}$ with generic s -parameters values.

How can we improve?

- What is the minimal linear combination of syzygy generators to get a non-trivial solution?
- Ansatz based approach

$$\lambda_i = \alpha_i \sum \beta_j \alpha^j \equiv \alpha_i p(\alpha),$$

where $\alpha^j = \alpha_1^{j_1} \dots \alpha_n^{j_n}$.

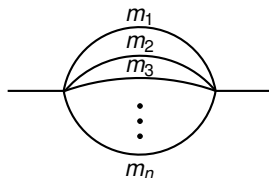
- $\deg(\lambda_i) = \deg(N_{p-1}) + 1 \Rightarrow \deg(p(\alpha)) = \deg(N_{p-1})$.
- Are the monomials of N_{p-1} sufficient? $\Rightarrow$ No
- What is the minimal set of monomial needed in this approach?

Massless Box one more time

- There are four integration variables $\alpha_i, i = 1, \dots, 4$.
- For an order three operator we have: $\deg(N_2) = 4 \Rightarrow 35$ monomials for $p(\alpha)$
- N_3 contains only the following four monomials:
 $\{\alpha_1^3\alpha_3^3, \alpha_1^2\alpha_2\alpha_3^2\alpha_4, \alpha_1\alpha_2^2\alpha_3\alpha_4^2, \alpha_2^3\alpha_4^3\}$
- Divide these four monomials by all monomials of degree two $\Rightarrow 19$ monomials $\Rightarrow$ sufficient to find non-trivial solutions.
- Can we do better?
- Only five monomials are needed:

$$\begin{aligned} &\{\alpha_1^2\alpha_3^2, \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_2^2\alpha_4^2, \alpha_1\alpha_3^2\alpha_4, \alpha_2\alpha_3\alpha_4^2\} \\ &\{\alpha_1^2\alpha_3^2, \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_2^2\alpha_4^2, \alpha_1\alpha_2\alpha_3^2, \alpha_2^2\alpha_3\alpha_4\} \\ &\{\alpha_1^2\alpha_3^2, \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_2^2\alpha_4^2, \alpha_1^2\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_4^2\} \\ &\{\alpha_1^2\alpha_3^2, \alpha_1\alpha_2\alpha_3\alpha_4, \alpha_2^2\alpha_4^2, \alpha_1^2\alpha_2\alpha_3, \alpha_1\alpha_2^2\alpha_4\} \end{aligned}$$

Banana integrals



Symanzik polynomials:

$$U = \left(\prod_{k=1}^n \alpha_k \right) \left(\sum_{j=1}^n \frac{1}{\alpha_j} \right)$$
$$F = \prod_{i=1}^n \alpha_i s_0 - U \sum_{i=1}^n \alpha_i s_i,$$

where $\kappa = n - \frac{\ell+1}{2}d$, $\eta = n - \frac{\ell d}{2}$ and $n = \ell + 1$.

Banana integrals

Theorem

The following differential operators annihilate the ℓ -loop banana integral

$$1) \quad \mathcal{A}_E = \sum_{i=0}^n s_i \partial_i + \left(n - \frac{\ell d}{2} \right),$$

$$2) \quad \mathcal{A}_i = -s_0 \partial_0^2 + s_i \partial_i^2 - \frac{d}{2} \partial_0 + \left(2 - \frac{d}{2} \right) \partial_i, \quad i = 1, \dots, n,$$

$$3) \quad \mathcal{A}_s = \left(\prod_{i=0}^n \partial_i \right) \left(\sum_{i=0}^n \frac{1}{\partial_i} \right) \equiv \sum_{i=0}^n \partial_0 \dots \hat{\partial}_i \dots \partial_n.$$

Where do these operators come from?

- Is there an associated symmetry?
- Commutation relations

Lemma

Operators $\mathcal{A}_E$, $\mathcal{A}_i$ and $\mathcal{A}_S$ satisfy the following commutation relations

$$\begin{aligned} [\mathcal{A}_E, \mathcal{A}_i] &= -\mathcal{A}_i, & [\mathcal{A}_E, \mathcal{A}_S] &= -n\mathcal{A}_S, \\ [\mathcal{A}_i, \mathcal{A}_j] &= 0, & [\mathcal{A}_i, \mathcal{A}_S] &= (\partial_0 - \partial_i)\mathcal{A}_S. \end{aligned}$$

- How should we interpret the last commutation relation?

Banana integrals

Theorem

The singular locus of the ideal $\mathcal{I}_B$ generated by the operators $\mathcal{A}_E$, $\mathcal{A}_s$ and $\mathcal{A}_i$, $i = 1, \dots, n$, is contained in

$$\text{sing}(\mathcal{I}_B) \subseteq \prod_{i=0}^n s_i \prod_{\{\delta_{\pm}\}} \left(s_0 - \left(\sum_{j=1}^n \delta_{\pm} \sqrt{s_j} \right)^2 \right), \text{ where } \delta_{\pm} = \pm 1,$$

and the second product runs over all projectively nonequivalent ways to assign $\delta_{\pm}$ in the sum.

Holonomic Rank:

ℓ	1	2	3	4	5	6	7	8
$r(\mathcal{I}_B)$	3	7	15	31	63	127	255	511

Number of master integrals [14]: $2^{\ell+1} - 1$

Banana integrals

Conjecture

The holonomic $\mathcal{D}$ -ideal of rank $2^{\ell+1} - 1$ for the ℓ -loop banana integral is generated by the following $\ell + 3$ operators

$$\mathcal{A}_E = \sum_{i=0}^n x_i \partial_i + \left(n - \frac{\ell d}{2} \right),$$

$$\mathcal{A}_i = -x_0 \partial_0^2 + x_i \partial_i^2 - \frac{d}{2} \partial_0 + \left(2 - \frac{d}{2} \right) \partial_i, \quad i = 1, \dots, n,$$

$$\mathcal{A}_S = \left(\prod_{i=0}^n \partial_i \right) \left(\sum_{i=0}^n \frac{1}{\partial_i} \right) \equiv \sum_{i=0}^n \partial_0 \dots \hat{\partial}_i \dots \partial_n.$$

Summary

- We provided an algorithm based on the Griffiths-Dwork reduction together with an additional boundary condition to construct differential operators annihilating Feynman integrals.
- Constructed operators are of the form: $\mathbf{A}_p \equiv \mathbf{D}_p + \mathbf{D}_{p-1}$.
- We applied the algorithm to Feynman integrals, hypergeometric functions and Witten diagrams.
- For the examined examples the operators form a generating $\mathcal{D}$ -ideal whose holonomic rank agrees with the number of master integrals.
- We found a set of simple differential operators that annihilate banana integrals.
- Singular locus of the banana operators is contained in the set of Landau singularities of the first and second type.
- We checked the holonomic rank up to $\ell = 8$ and it agrees with the formula for the number of master integrals for banana graphs.

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Extra slides

Massless box - Full expressions

$$\mathbf{A}_3 = p_{3,0} \partial_s^3 + p_{2,1} \partial_s^2 \partial_t + p_{1,2} \partial_s \partial_t^2 + p_{0,2} \partial_t^2 ,$$

where

$$p_{3,0} = -4s^2 ,$$

$$p_{2,1} = s((d-8)^2 s + 2(26-3d)t) ,$$

$$p_{1,2} = -t(2(28-3d)s + (d-8)^2 t) ,$$

$$p_{0,2} = 2(d-12)t .$$

Massless box - Full expressions

$$\mathbf{A}_{\text{PF}} = (\partial_s + r_1) \cdot (\partial_s + r_2) \cdot (\partial_s + r_3) ,$$

where

$$r_1 = -\frac{d-10}{2s} - \frac{1}{u} ,$$

$$r_2 = \frac{1}{s} - \frac{1}{u} ,$$

$$r_3 = -\frac{1}{2} \left(\frac{d-4}{u} + \frac{d-6}{s} \right) ,$$

and $u = -s - t$.

Massless box - Full expressions

$$B = \begin{pmatrix} 1 & 0 & 0 \\ \frac{dt-2s-6t}{2s(s+t)} & \frac{2(d-3)}{s^2(s+t)} & -\frac{2(d-3)}{st(s+t)} \\ \frac{(d^2-14d+48)t^2-8(d-6)st+8s^2}{4s^2(s+t)^2} & \frac{(d-3)((d-12)s+2(d-7)t)}{s^3(s+t)^2} & \frac{(d-3)(6s-(d-8)t)}{s^2t(s+t)^2} \end{pmatrix}$$

Idea of the proof

- 1) $\mathcal{A}_E$ annihilates the integrand
- 2) We use the twisted Griffiths-Dwork reduction with the following coefficients:

$$\lambda_j = \frac{1}{n} \left(n - \frac{n-1}{2} d \right) \left(n - \frac{n-1}{2} d + 1 \right) \alpha_n \alpha_j \left(U - \frac{1}{\alpha_n} \prod_{k=1}^n \alpha_k \right) \quad j = 1, \dots, n-1,$$
$$\lambda_n = -\frac{1}{n} \left(n - \frac{n-1}{2} d \right) \left(n - \frac{n-1}{2} d + 1 \right) \alpha_n^2 \left((n-1)U + \frac{1}{\alpha_n} \prod_{k=1}^n \alpha_k \right).$$

- 3) $\mathcal{A}_S$ annihilates the integrand