SPQR: A new Package for Polynomial Division and Elimination Theory via Finite Fields

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MathemAmplitudes, Mainz, 23/09/25

Based on upcoming work with

Vsevolod Chestnov

Introduction and Motivation (1/2)

Talk Outline and Goal

Explore technology behind SPQR as well as various applications

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General overview of polynomial division

New methods/implementation: finite fields

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Elimination Theory

Landau Analysis

Super Quick Review of Polynomial Division

Univariate Polynomial Division

Polynomial division allows us to decompose functions as

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$$\updownarrow$$

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$$\begin{aligned} f(x) &= x\,(p(x) + 2x + 1) + a\,(p(x) + 2x + 1) - (4+2a)x + 1 \\ &= p(x)(x+a) + 2x^2 + a - 3x + 1 \\ &= p(x)(x+a) + 2(p(x) + 2x + 1) + a - 3x + 1 \\ &= p(x)(x+a+2) + x + a + 3 \end{aligned}$$

Towards Finite Fields

Polynomial division as row reduction

If all we care about is the remainder, we can reformulate polynomial division as a row reduction problem

$$p(x) = 0 \mod p(x) \longrightarrow x^n p(x) = 0 \mod p(x)$$

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Useful because there exist very quick ways to do row reduction: Sample over finite fields and reconstruct output

Finite Field Reconstruction

Operations on Matrices

$\mathbb{M}(a_1, \cdots, a_n)$

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Algebraic post processing simplification — can become very intensive!

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Substitute numerical values for parameters $\longrightarrow$ Perform all operations numerically mod primes $\longrightarrow$ Reconstruct functional output from numerical sampling

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Complicated cancellations will happen numerically — final reconstructed output already "simplified"

All operations over finite fields handled by the C++/Mathematica package *FiniteFlow* [FiniteFlow, Peraro, 2019]

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$$R = \mathbb{Q}(a)[x]$$

variables

Only need to reconstruct parameters!

Multivariate Polynomial Division (1/3)

Monomial Orderings

For one variable, sorting the monomials from “worst” to “best” is unambiguous

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Is the division unique?

Unfortunately, this is not enough to uniquely determine a multivariate polynomial division

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$xy = x \mod I$ $\qquad$ $xy = y + 1 \mod I$

Problem normally fixed by introducing Groebner Bases

[Buchberger, 1965]

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Groebner Bases

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Avoiding Groebner Bases

Claim: We can explicitly avoid computing a Groebner basis, and still obtain the correct result from polynomial division, using row reduction [Faugére, 1999] [Buchberger, 1985]

Allows us to compute polynomial divisions without needing to reconstruct the “intermediate” Groebner Basis

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Row Reduction Again

We consider again $I = \langle xy - x, xy - y - 1 \rangle$ and let $f(x, y) = xy^2$

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Seed a linear system by multiplying I by $x^n y^m$

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Irreducible monomials

Read off from top row: $f(x,y) = y + 1 \mod I$ No explicit Groebner Basis required!

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f(x) \\ x^2y^2 \\ x^2y \\ x^2 \\ xy^2 \\ xy \\ x \\ y^2 \\ y \\ 1 \end{bmatrix} \xrightarrow{\text{RowRed}} \left[\begin{array}{cccccccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} f(x) \\ x^2y^2 \\ x^2y \\ x^2 \\ xy^2 \\ xy \\ x \\ y^2 \\ \hline y \\ 1 \end{array}\right]$$

Irreducible monomials

Read off from top row: $f(x,y) = y + 1 \mod I$ No explicit Groebner Basis required!

Important to note: this approach does not escape the complexity of computing Groebner Bases

Multivariate Polynomial Division (3/3)

Row Reduction Again

We consider again $I = \langle xy - x, xy - y - 1 \rangle$ and let $f(x,y) = xy^2$

Seed a linear system by multiplying I by $x^n y^m$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\
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0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
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\end{bmatrix}
\begin{bmatrix} f(x) \\ x^2y^2 \\ x^2y \\ x^2 \\ xy^2 \\ xy \\ x \\ y^2 \\ y \\ 1 \end{bmatrix}
\xrightarrow{\text{RowRed}}
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1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
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0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\
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0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\
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Important to note: this approach does not escape the complexity of computing Groebner Bases

For example: the size of the system one needs to generate is unclear a priori and can be very large

[Buchberger, 1985]

Companion Matrices (1/3)

Avoiding Large Systems

If we want to reduce polynomials with large monomial powers, the system to row reduce can become very large very quickly. Solution to this problem: Companion Matrices

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[Sturmfels, 2002]
[Cox, Little O'Shea, 2015]
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Key properties of companion matrices:

$$M_{f_1+f_2} = M_{f_1} + M_{f_2} \qquad M_{f_1\, f_2} = M_{f_1}\, M_{f_2} = M_{f_2}\, M_{f_1} \qquad M_{f_{\text{inv}}} = M_f^{-1}$$

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$$f(x,y) = (0, \cdots, 0, 1) \cdot M_{f(x,y)} \cdot m^T \mod I$$

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Strategy: Build M_x and M_y and you can reduce anything with just matrix multiplication!

Companion Matrices (2/3)

Examples

$$I = \langle xy - x, xy - y \rangle \longrightarrow m = \{y, 1\}$$

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Can now consider any function and build it with companion matrices

$$f(x, y) = xy^2 \longrightarrow M_{f(x,y)} = M_x \cdot M_y^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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Can immediately write down the answer for much more complicated expressions!

$$g(x, y) = a + \frac{x}{y^{100} - 3x + 2} \longrightarrow M_{g(x,y)} = \mathbb{1}\, a + M_x \left(M_y^{100} - 3M_x + 2\,\mathbb{1}\right)^{-1} = \begin{bmatrix} \frac{1}{3}(3a - 1) & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3}(3a - 1) \end{bmatrix}$$

$$g(x, y) = \frac{1}{3}(3a - 1) - \frac{y}{3} \mod I$$

Companion Matrices (3/3)

Variable Elimination

Often one is interested in projecting/eliminating an equation system down onto one variable "slices"

$$I = \langle xy - x, xy - y - 1 \rangle \longrightarrow \{x^* = 0\,,\ y^* = -1\} \cup \{x^* = 2\,,\ y^* = 1\}$$

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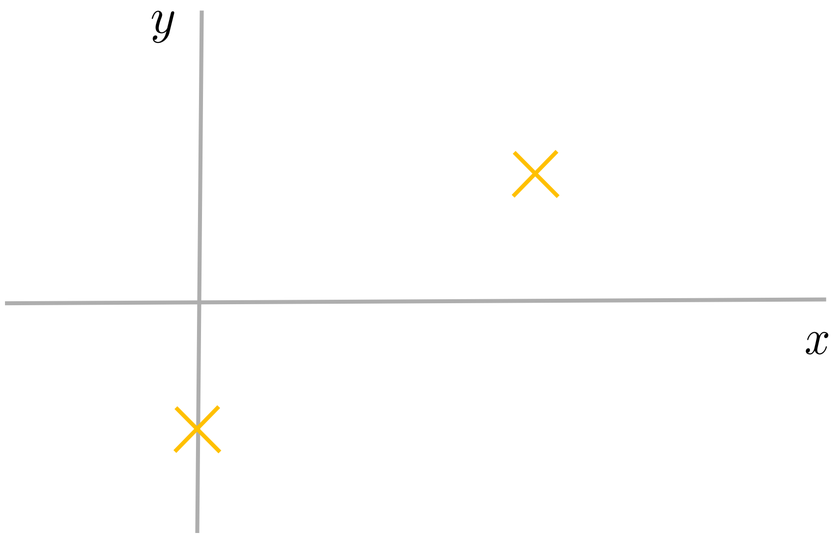

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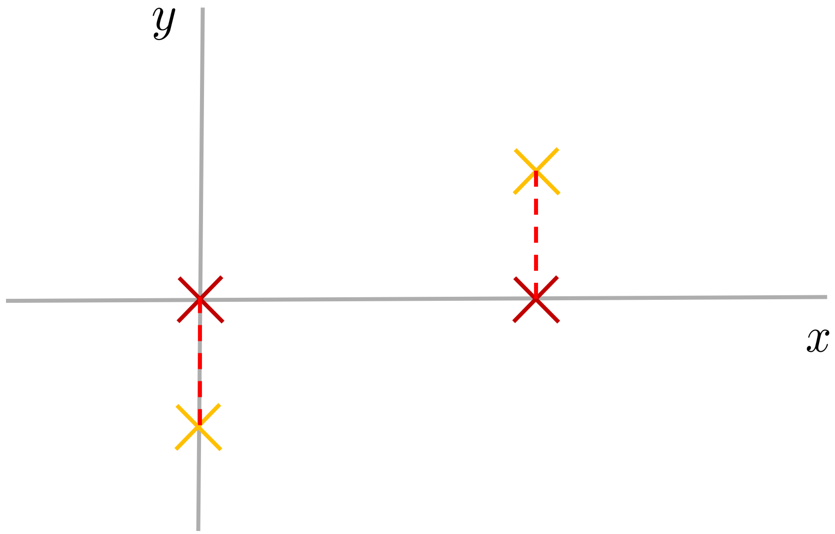

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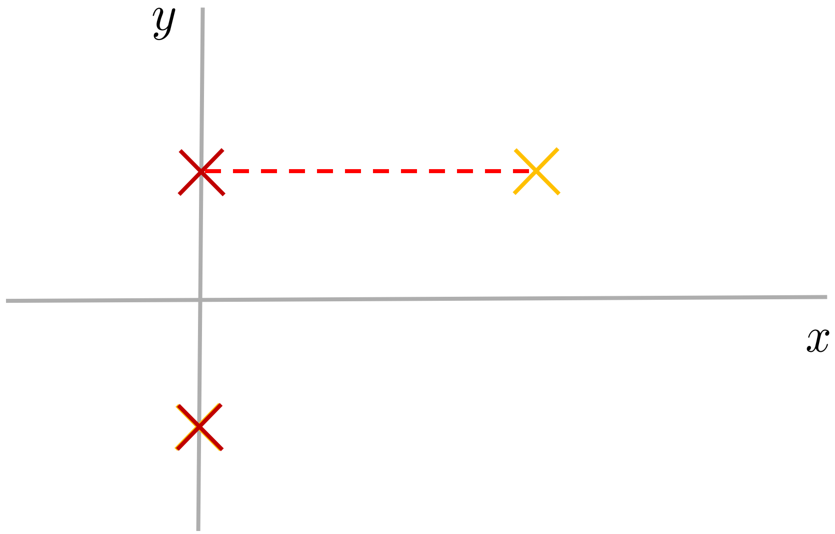

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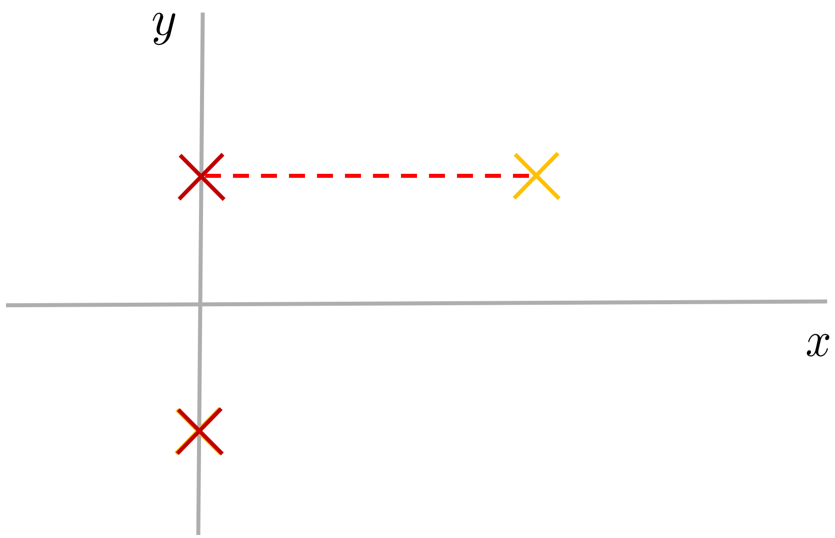

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The eigenvalues of companion matrices encode the roots of the polynomial system. Can be conveniently accessed by computing characteristic polynomials

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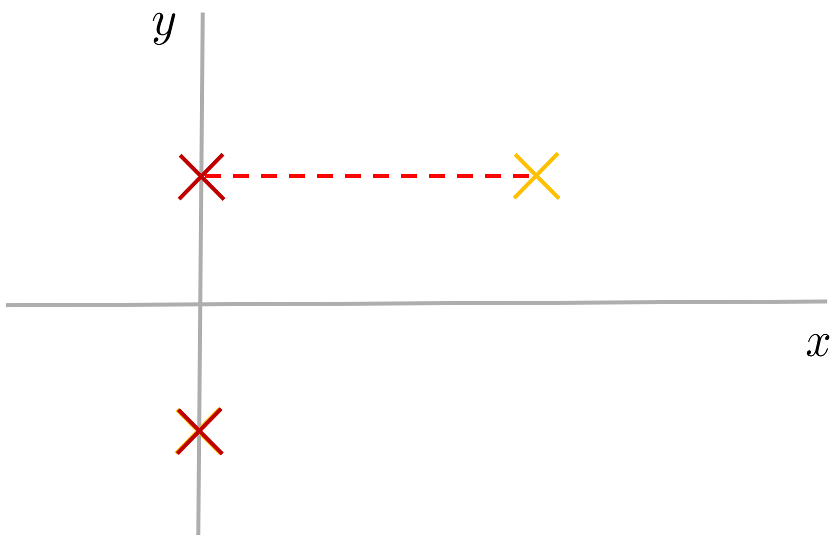

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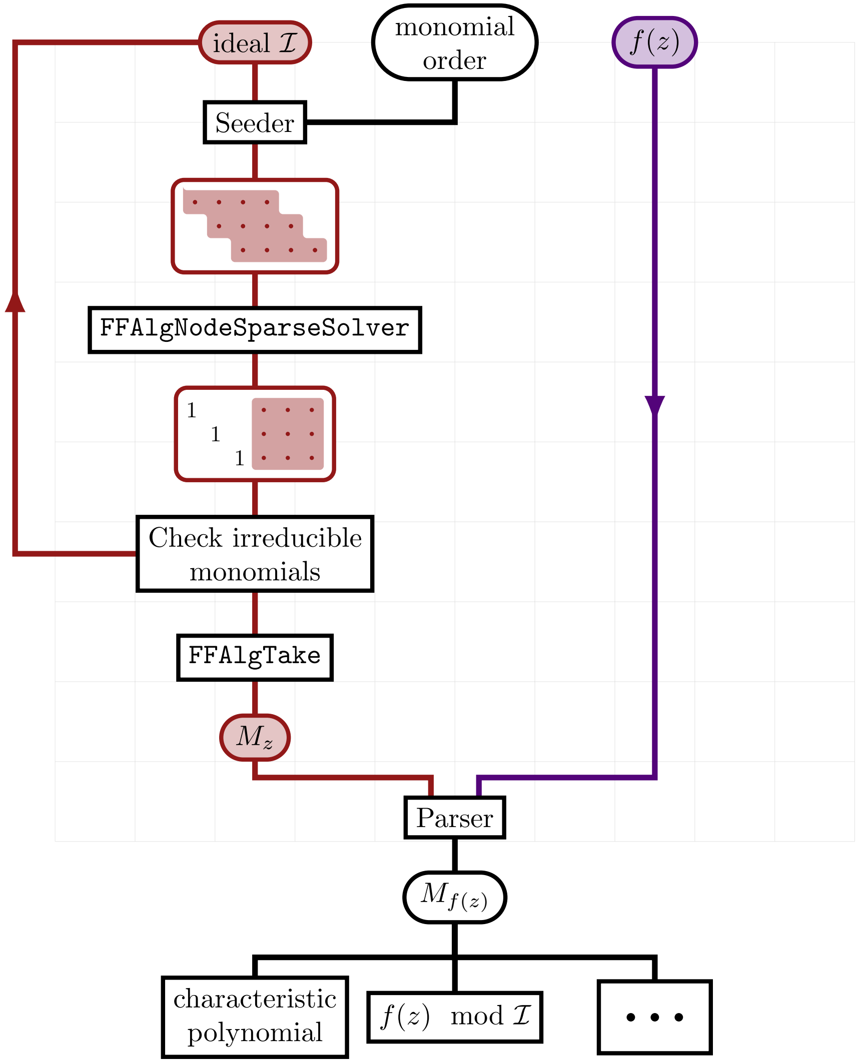
$$\mathrm{Elim}_x(\mathcal{I}) = \det(M_y - y\mathbb{1}) = \det\begin{bmatrix} -y & 1 \\ 1 & -y \end{bmatrix} = y^2 - 1$$

$$\mathrm{Elim}_y(\mathcal{I}) = \det(M_x - x\mathbb{1}) = \det\begin{bmatrix} 1-x & 1 \\ 1 & 1-x \end{bmatrix} = x(x-2)$$

SPQR Summary

A (Mathematica) package that performs polynomial division and reconstruction over finite fields



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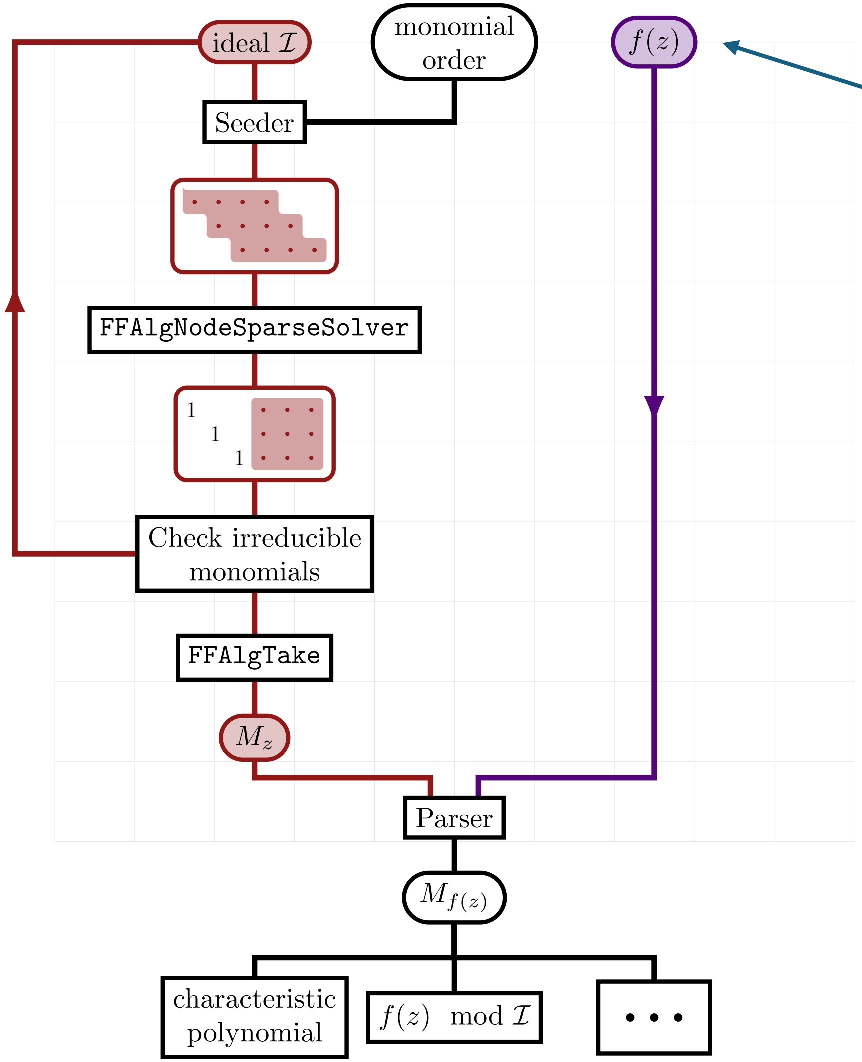

Can handle polynomials or multivariate rational functions as input to arbitrary nested depth

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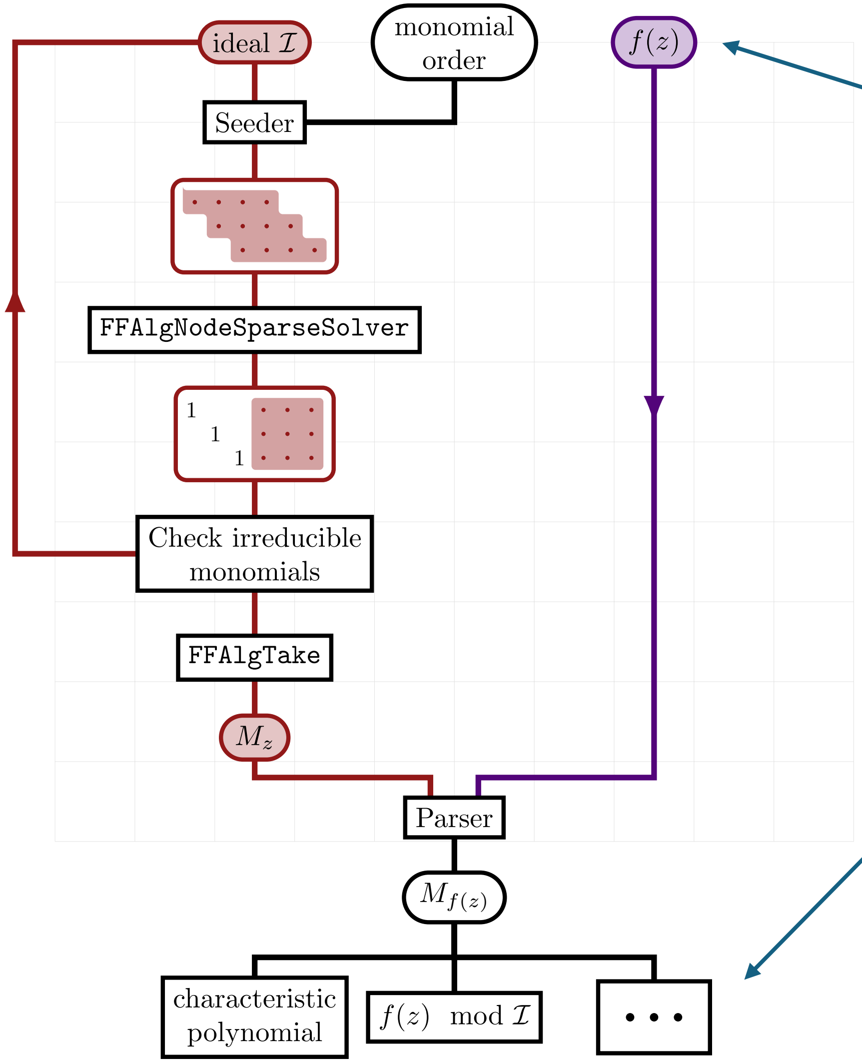

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No intermediate reconstructions required ensuring the numerical cancellations of complex intermediate stages

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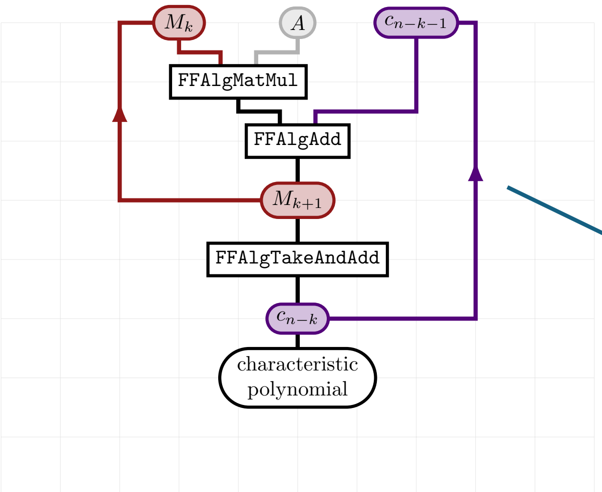
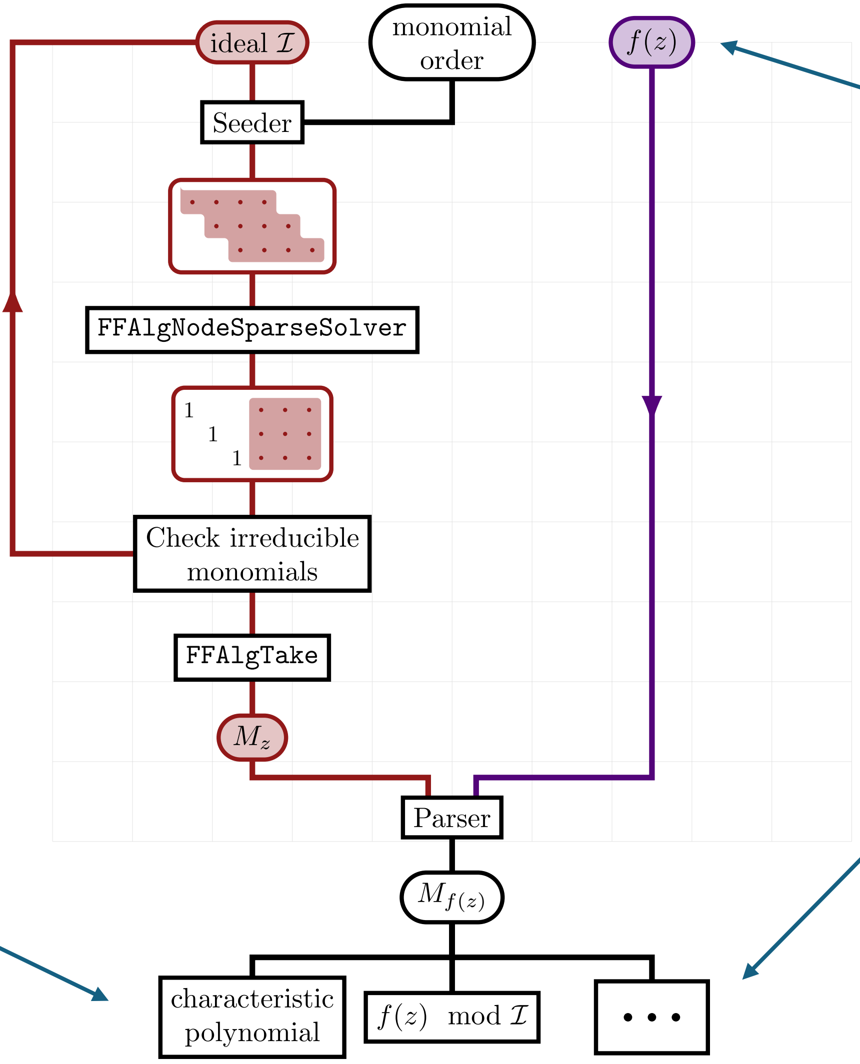
Characteristic polynomial also built inside FiniteFlow with the Faddeev–LeVerrier algorithm





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Advantages and Disadvantages (1/3)

Complexity of polynomial ideals

$$\underbrace{R = \mathbb{Q}[a_1 \cdots a_n]}_{\text{parameters}}\underbrace{[x_1 \cdots x_n]}_{\text{variables}}$$

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$$R = \mathbb{Q}[a_1 \cdots a_n][x_1 \cdots x_n]$$

parameters

variables

“Variable complexity” : How many equations are needed for finding a Groebner Basis

Groebner Basis algorithms have fine tuned procedures for keeping this down to a minimum

Our approach (currently) relies on a much coarser and overcomplete system of equations

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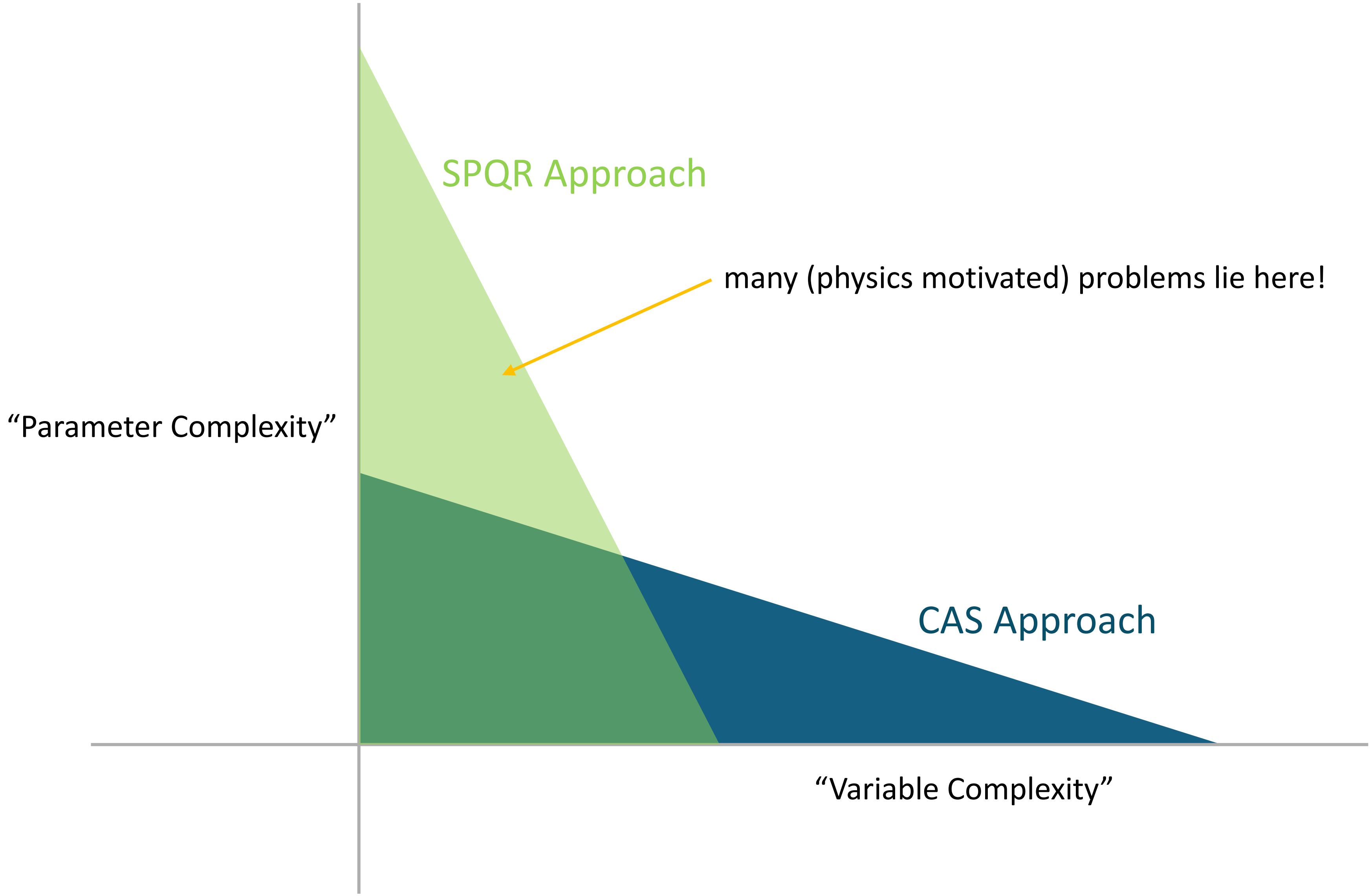
"Parameter complexity" : How complicated are the *coefficients* of the Groebner Basis polynomials

Groebner Basis algorithms suffer from complicated symbolic processing at intermediate stages

Reconstruction over finite fields completely avoids this problem as cancellations happen numerically!

Can speed up computations by orders of magnitude

Advantages and Disadvantages (2/3)



Advantages and Disadvantages (3/3)

Resultant Benchmarking (Preliminary)

$$R = \mathbb{Q}(a, b, c, d)[x, y, z]$$

$$I = \langle a + x^2y^2 + y^3 + z - 1, ax + cxy^2 + cy + z^2 - 2, a + bxy^2 + b + x^2y^2, -c + dxz + xyz + 1 \rangle$$

Task: Eliminate $\{x, y, z\} \longrightarrow \mathcal{R}(a, b, c, d)$

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Resultant	Singular	Macaulay 2	Mathematica (+ tricks)	SPQR
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$\mathcal{R}(3,5,c,d)$				
$\mathcal{R}(3,b,c,d)$				
$\mathcal{R}(a,b,c,d)$				

Advantages and Disadvantages (3/3)

Resultant Benchmarking (Preliminary)

$$R = \mathbb{Q}(a,b,c,d)[x,y,z]$$

$$I = \langle a + x^2y^2 + y^3 + z - 1, ax + cxy^2 + cy + z^2 - 2, a + bxy^2 + b + x^2y^2, -c + dxz + xyz + 1 \rangle$$

Task: Eliminate $\{x, y, z\} \longrightarrow \mathcal{R}(a,b,c,d)$

Resultant	Singular	Macaulay 2	Mathematica (+ tricks)	SPQR
$\mathcal{R}(3,5,7,d)$	$\approx 0.01s$	$\approx 0.06s$	$\approx 0.03s$	$\approx 0.18s$
$\mathcal{R}(3,5,c,d)$	$\approx 53.29s$	$\approx 12.40s$	$\approx 5.03s$	$\approx 0.20s$
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$\mathcal{R}(a,b,c,d)$	$?$	$?$	$> 7d$	$\approx 1.89s$

9 MB degree 34 polynomial

The Finite Fields approach is solving a less refined set of equations, but isn't slowed down by intermediate cancellations

Physics Motivated Examples (1/3)

Landau Singularities of Feynman (Euler) Integrals

Feynman integrals are Euler/Twisted Period integrals

Twist: polynomial raised to a non integer (generic) power

$$I(s_{ij}, m) = \int_0^\infty \mathcal{G}(x, s_{ij}, m)^{-d/2} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

[Lee, 2013]

Parameters (Mandelstam variables)

Integration variables

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χ can be computed by solving systems of polynomial equations

Physics Motivated Examples (2/3)

Computing Euler Characteristics for Landau Analysis

Computing χ: $\omega = \mathrm{d}\log\left(G(z)^{-d/2}\right)$ $\omega = -\frac{d}{2}\left(\frac{\partial_1 G}{G}dx_1 + \cdots + \frac{\partial_n G}{G}dx_n\right)$ $\chi = \#\text{ solutions to }\omega = 0$

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$$\omega = 0 \longrightarrow \boxed{\begin{array}{c} \frac{\partial_1 G}{G} = 0 \\ \vdots \\ \frac{\partial_n G}{G} = 0 \end{array}}$$

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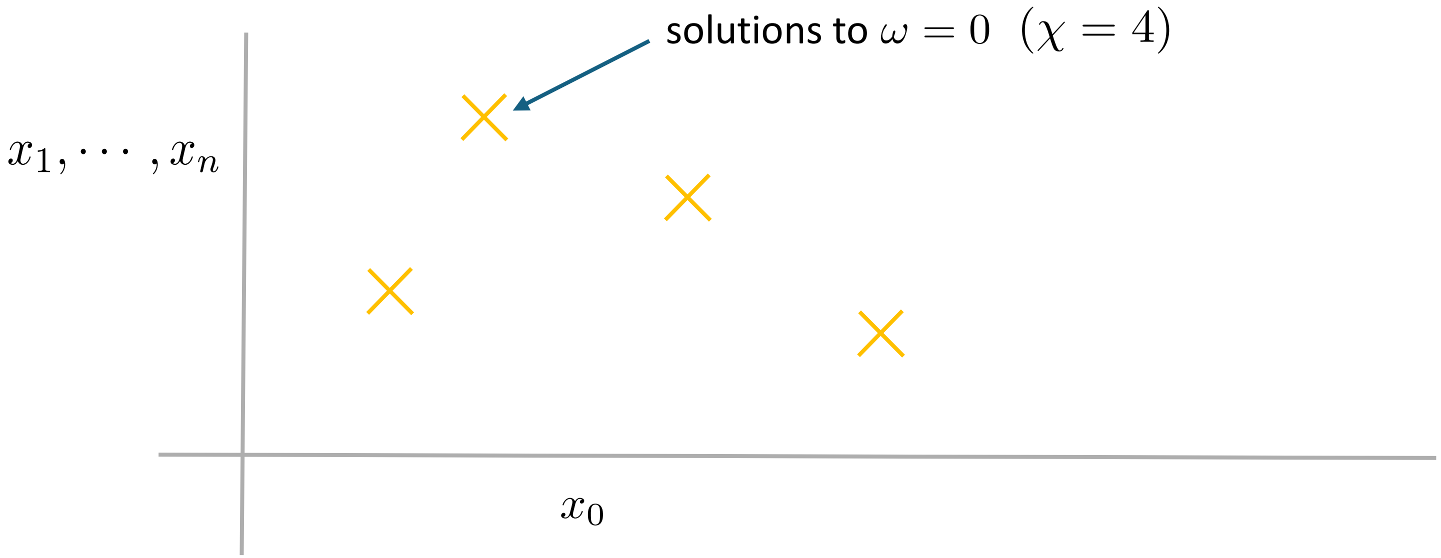

Physics Motivated Examples (2/3)

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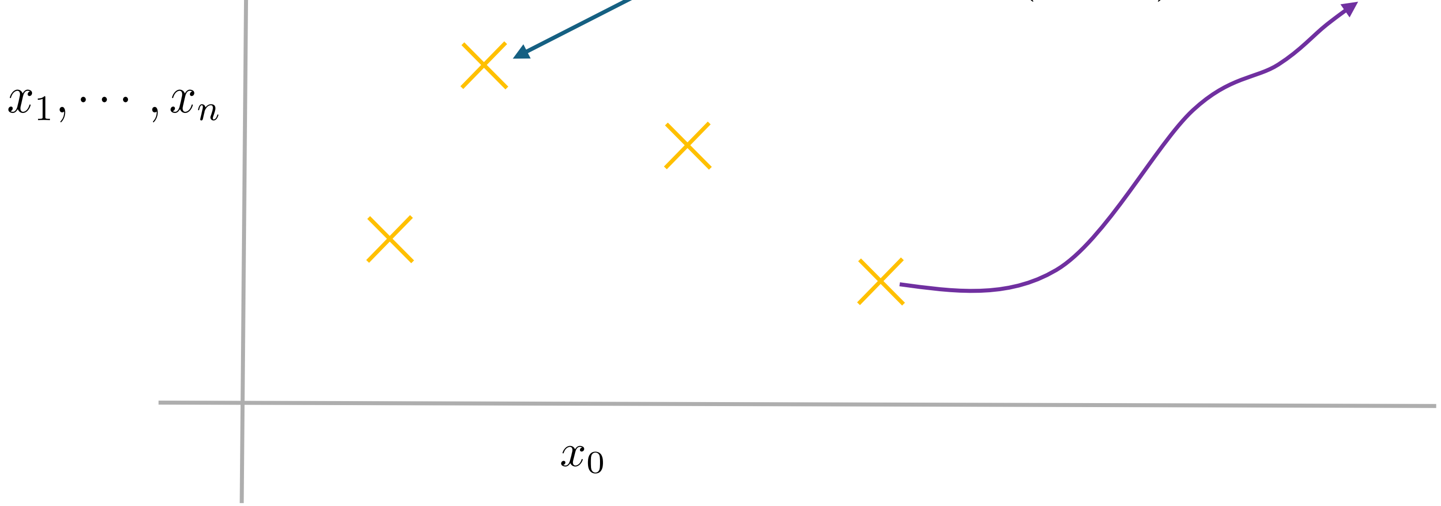
solutions to $\omega = 0 \quad (\chi = 4)$

as the degenerate configuration is approached some solutions fly off to infinity



Physics Motivated Examples (2/3)

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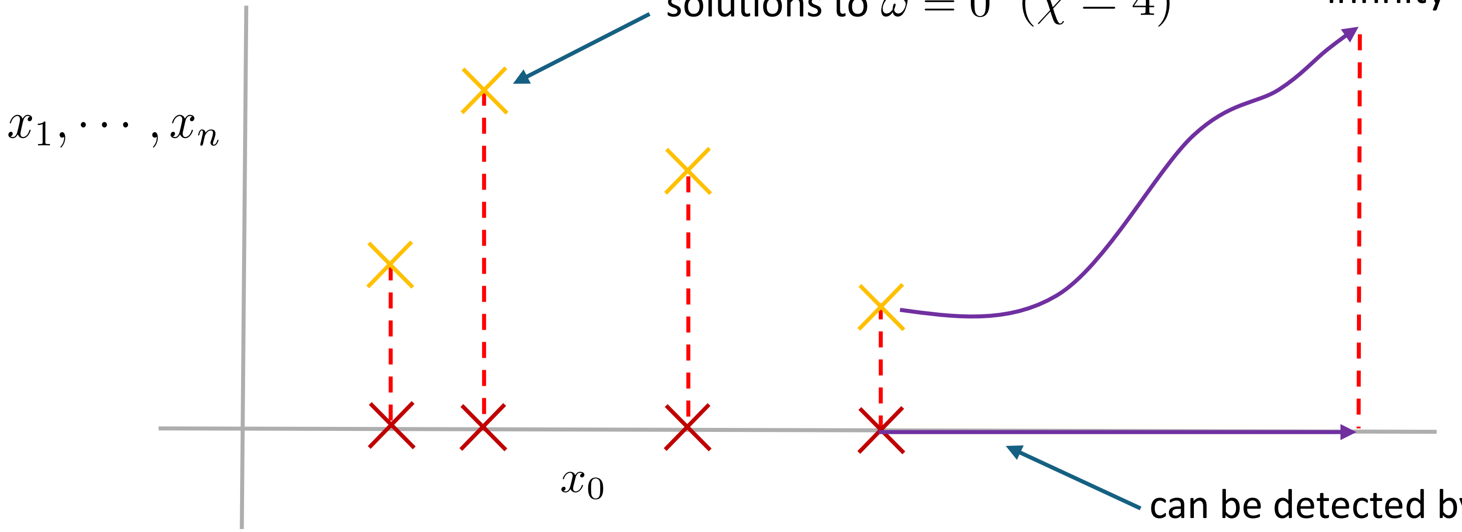
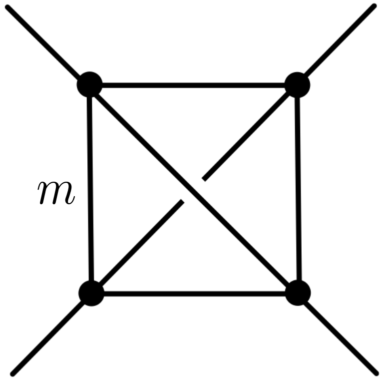


can be detected by checking coefficients of the remaining one variable polynomial

Physics Motivated Examples (3/3)

Computing Euler Characteristics for Landau Analysis: Three loop envelope (preliminary)



[Correia, Sever, Zhibodeev, 2021]

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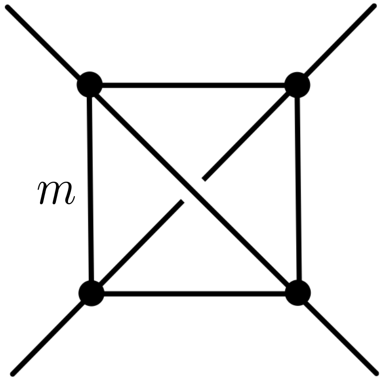

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Horrendous integral: $\chi = 60(!)$ in the top (max cut) sector alone

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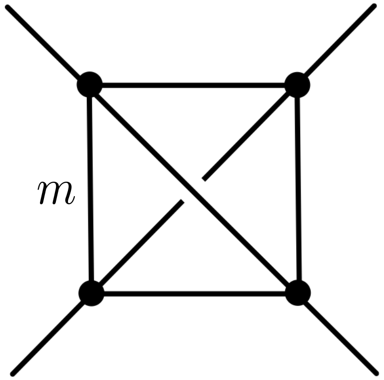

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PLD/SOFIA most complicated letter found: $27(m^2)^3 + 4s^2t + 4st^2$

[Fevola, Mizera, Telen, 2023]

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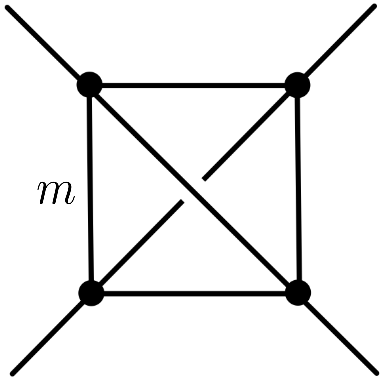

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Euler characteristic strategy

Two new simple letters: $\{s^2 + st + t^2, m^2s^2 + m^2st + s^2t + m^2t^2 + st^2\}$

Physics Motivated Examples (3/3)

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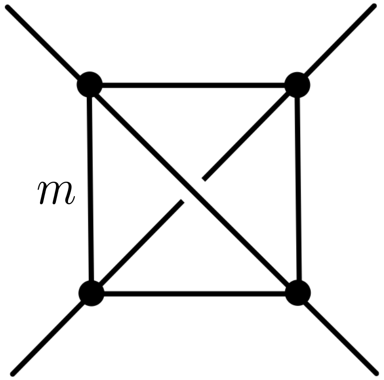

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Four new complicated letters:

$$
\begin{aligned}
&\{27\,\mathrm{m2}^4 s^2 + 108\,\mathrm{m2}^4 s\,t + 162\,\mathrm{m2}^3 s^2 t + 54\,\mathrm{m2}^2 s^3 t + 4\,\mathrm{m2}\,s^4 t + 108\,\mathrm{m2}^4 t^2 + 162\,\mathrm{m2}^3 s\,t^2 + 45\,\mathrm{m2}^2 s^2 t^2 - \\
&\quad 6\,\mathrm{m2}\,s^3 t^2 - s^4 t^2 - 18\,\mathrm{m2}^2 s\,t^3 - 20\,\mathrm{m2}\,s^2 t^3 - 2\,s^3 t^3 - 9\,\mathrm{m2}^2 t^4 - 10\,\mathrm{m2}\,s\,t^4 - s^2 t^4,\ 108\,\mathrm{m2}^4 s^2 - 9\,\mathrm{m2}^2 s^4 + 108\,\mathrm{m2}^4 s\,t + 162\,\mathrm{m2}^3 s^2 t - \\
&\quad 18\,\mathrm{m2}^2 s^3 t - 10\,\mathrm{m2}\,s^4 t + 27\,\mathrm{m2}^4 t^2 + 162\,\mathrm{m2}^3 s\,t^2 + 45\,\mathrm{m2}^2 s^2 t^2 - 20\,\mathrm{m2}\,s^3 t^2 - s^4 t^2 + 54\,\mathrm{m2}^2 s\,t^3 - 6\,\mathrm{m2}\,s^2 t^3 - 2\,s^3 t^3 + 4\,\mathrm{m2}\,s\,t^4 - s^2 t^4, \\
&27\,\mathrm{m2}^4 s^2 - 54\,\mathrm{m2}^4 s\,t + 162\,\mathrm{m2}^3 s^2 t - 54\,\mathrm{m2}^2 s^3 t + 4\,\mathrm{m2}\,s^4 t + 27\,\mathrm{m2}^4 t^2 + 162\,\mathrm{m2}^3 s\,t^2 - 117\,\mathrm{m2}^2 s^2 t^2 + 22\,\mathrm{m2}\,s^3 t^2 - s^4 t^2 - 54\,\mathrm{m2}^2 s\,t^3 + 22\,\mathrm{m2}\,s^2 t^3 - 2\,s^3 t^3 + 4\,\mathrm{m2}\,s\,t^4 - s^2 t^4, \\
&65\,536\,\mathrm{m2}^{12} + 270\,336\,\mathrm{m2}^{10} s^2 + 33\,024\,\mathrm{m2}^8 s^4 + 1024\,\mathrm{m2}^6 s^6 + 270\,336\,\mathrm{m2}^{10} s\,t - 458\,752\,\mathrm{m2}^9 s^2 t + 66\,048\,\mathrm{m2}^8 s^3 t - 1\,276\,416\,\mathrm{m2}^7 s^4 t + 3072\,\mathrm{m2}^6 s^5 t - 137\,472\,\mathrm{m2}^5 s^6 t - \\
&\quad 4096\,\mathrm{m2}^3 s^8 t + 270\,336\,\mathrm{m2}^{10} t^2 - 458\,752\,\mathrm{m2}^9 s\,t^2 + 99\,072\,\mathrm{m2}^8 s^2 t^2 - 2\,552\,832\,\mathrm{m2}^7 s^3 t^2 - 3\,427\,584\,\mathrm{m2}^6 s^4 t^2 - 412\,416\,\mathrm{m2}^5 s^5 t^2 + 149\,472\,\mathrm{m2}^4 s^6 t^2 - 16\,384\,\mathrm{m2}^3 s^7 t^2 + \\
&\quad 768\,\mathrm{m2}^2 s^8 t^2 + 66\,048\,\mathrm{m2}^8 s\,t^3 - 2\,552\,832\,\mathrm{m2}^7 s^2 t^3 - 6\,860\,288\,\mathrm{m2}^6 s^3 t^3 - 687\,360\,\mathrm{m2}^5 s^4 t^3 + 448\,416\,\mathrm{m2}^4 s^5 t^3 - 49\,888\,\mathrm{m2}^3 s^6 t^3 + 3072\,\mathrm{m2}^2 s^7 t^3 - 48\,\mathrm{m2}\,s^8 t^3 + \\
&\quad 33\,024\,\mathrm{m2}^8 t^4 - 1\,276\,416\,\mathrm{m2}^7 s\,t^4 - 3\,427\,584\,\mathrm{m2}^6 s^2 t^4 - 687\,360\,\mathrm{m2}^5 s^3 t^4 + 597\,888\,\mathrm{m2}^4 s^4 t^4 - 92\,320\,\mathrm{m2}^3 s^5 t^4 + 6144\,\mathrm{m2}^2 s^6 t^4 - 192\,\mathrm{m2}\,s^7 t^4 + s^8 t^4 + \\
&\quad 3072\,\mathrm{m2}^6 s\,t^5 - 412\,416\,\mathrm{m2}^5 s^2 t^5 + 448\,416\,\mathrm{m2}^4 s^3 t^5 - 92\,320\,\mathrm{m2}^3 s^4 t^5 + 7680\,\mathrm{m2}^2 s^5 t^5 - 336\,\mathrm{m2}\,s^6 t^5 + 4\,s^7 t^5 + 1024\,\mathrm{m2}^6 t^6 - 137\,472\,\mathrm{m2}^5 s\,t^6 + 149\,472\,\mathrm{m2}^4 s^2 t^6 - \\
&\quad 49\,888\,\mathrm{m2}^3 s^3 t^6 + 6144\,\mathrm{m2}^2 s^4 t^6 - 336\,\mathrm{m2}\,s^5 t^6 + 6\,s^6 t^6 - 16\,384\,\mathrm{m2}^3 s^2 t^7 + 3072\,\mathrm{m2}^2 s^3 t^7 - 192\,\mathrm{m2}\,s^4 t^7 + 4\,s^5 t^7 - 4096\,\mathrm{m2}^3 s\,t^8 + 768\,\mathrm{m2}^2 s^2 t^8 - 48\,\mathrm{m2}\,s^3 t^8 + s^4 t^8\}
\end{aligned}
$$

Conclusions and Outlook

SPQR is a new open-source Mathematica package for performing polynomial division. It leverages finite field sampling and reconstruction to process Macaulay matrices from complex polynomial systems with high efficiency.

This approach avoids heavy symbolic cancellations, often leading to significant speedups compared to traditional Groebner basis approaches.

The companion matrix formalism helps contain the Macaulay system size, making it possible to handle larger and more complicated problems

Future development directions: A way to generate a smaller set of equations for the row reduction would allow for the tackling of even more complicated problems — perhaps possible to build equation systems based on the steps carried out by Groebner Basis algorithms/interface to current Groebner basis tools?

A reconstruction algorithm that factors expressions could save many sample points for sparse rational function outputs

(Beta) version is available now at github.com/giu989/spqr

```
ResourceFunction["GitHubInstall"]["giu989","SPQR"];
```

Paper coming soon!

Thank you for listening!

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