Homology for exponential integrals with multivalued potential

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Based on **arXiv:2410.20973** joint with C. Wheeler

MathemAmplitudes: Co-homology and Combinatorics of GKZ systems, Euler-Mellin-Feynman Integrals and Scattering Amplitudes

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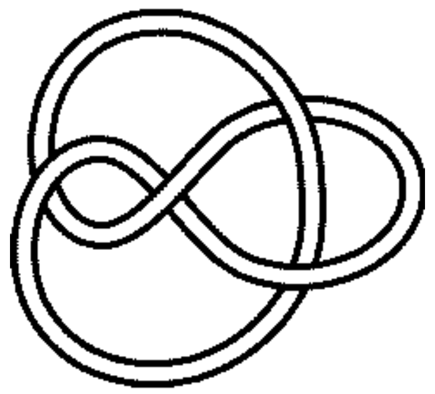
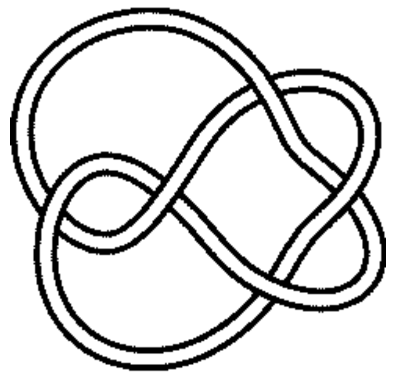
The dilogarithm

$$\mathrm{Li}_2(z) \;=\; \int_0^1 dt_1 \int_0^1 dt_2 \, \frac{z}{1 - z t_1 t_2}\,, \qquad z \in \mathbb{C}\backslash[1,\infty)$$



Period function of a certain relative cohomology

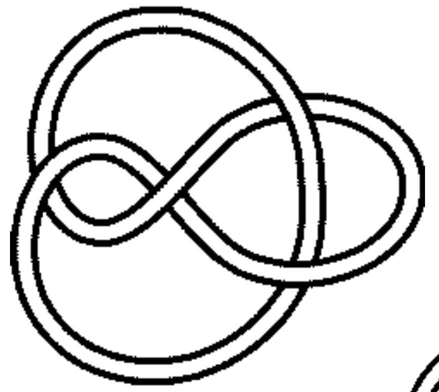
Volume of hyperbolic knots



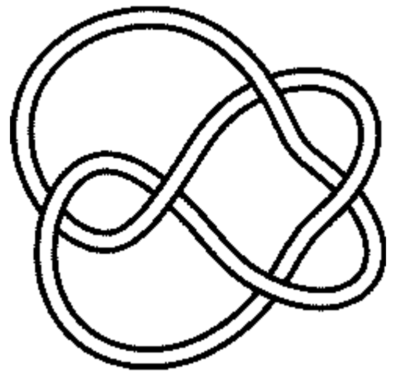
Faddeev's quantum dilogarithm

$$\Phi(z\tau;\tau) \sim \exp\!\left(\frac{\mathrm{Li}_2(\mathbf{e}(z))}{2\pi \mathrm{i}}\,\tau\right) \cdot [\ldots] \text{ as } \tau \to \mathrm{i}\infty$$

Motivation: quantum invariants of hyperbolic knots



$K = 4_1$



$K = 5_2$

Hyperbolic Volume $\xrightarrow{\text{Quantization}}$ **Kashaev invariants** [Kashaev 95]

Asymptotics

Volume Conjecture

Topological invariants as divergent power series / Topological invariants as analytic functions

$\mathrm{SL}_2(\mathbb{C})$ Chern-Simons on $S^3 \backslash K$ [Witten 10]

Localization

Andersen-Kashaev state integral: partition function of a 3D Teichmüller TQFT

Asymptotics

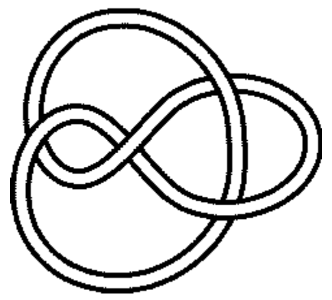
AK volume conjecture

Dimofte-Garoufalidis invariants
asymptotics expansion to all orders of the Kashaev invariant (volume is the leading order)

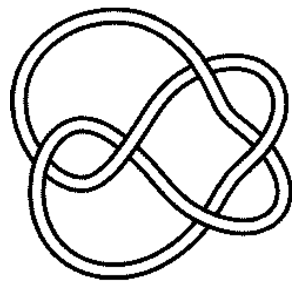
Perturbative Topological invariants of hyperbolic knots

Dimofte-Garoufalidis perturbative invarianats [Dimofte-Garoufalidis 13, Garoufalidis-Strozer-Wheeler 22]

$$\tilde{\Upsilon}_K(\tau) := \int \tilde{\Psi}(z;\tau)^B \, \mathbf{e}\left(\frac{A}{2} z^2 \tau\right) dz$$



$\mathbf{4_1}: (A = 1\,, B = 2)$



$\mathbf{5_2}: (A = 2\,, B = 3)$

$$\tilde{\Psi}(z;\tau) = \mu_8 \, e^{-\frac{\pi \mathrm{i}}{12\tau}} e^{\frac{\pi \mathrm{i}\tau}{12}} \, \mathbf{e}\left(\frac{\mathrm{Li}_2(\mathbf{e}(z))}{(2\pi \mathrm{i})^2}\,\tau\right) \exp\left(-\mathrm{Li}_1(\mathbf{e}(z)) - \sum_{k=2}^{\infty} (2\pi \mathrm{i})^{k-1} \frac{B_k}{k!} \mathrm{Li}_{2-k}(\mathbf{e}(z))\, \tau^{1-k}\right)$$

Exponential term

Factorally divergent series in $1/\tau$

The formal series $\tilde{\Upsilon}_K(\tau)$ is defined by formal Gaussian integration at the critical points of $V(z) = \frac{A}{2} z^2 + B \frac{\mathrm{Li}_2(\mathbf{e}(z))}{(2\pi \mathrm{i})^2}$

A class of formal integrals

$$\int e^{-\tau f(\mathbf{z})} \left[\tilde{\varphi}_0(\mathbf{z}) + \frac{\tilde{\varphi}_1(\mathbf{z})}{\tau} + \frac{\tilde{\varphi}_2(\mathbf{z})}{\tau^2} + \dots\right] d\mathbf{z}$$

$f\colon X \dashrightarrow \mathbb{C}$ is a **multivalued** map from a complex **n-dimensional** variety X

$\tilde{\varphi}_j(\mathbf{z})$ are analytic functions of $\mathbf{z} \in X$

+ technical assumptions on $e^{-\tau f(\mathbf{z})} \left[\tilde{\varphi}_0(\mathbf{z}) + \frac{\tilde{\varphi}_1(\mathbf{z})}{\tau} + \frac{\tilde{\varphi}_2(\mathbf{z})}{\tau^2} + \dots\right]$

Examples: perturbative invariants of hyperbolic knots **[DG 13, GSW 22]**, fermionic traces from toric Calabi-Yau 3-folds **[Kashaev-Mariño 15]**, hemisphere partition functions in GLSM for hypersurfaces in $\mathbb{P}^N$ **[Knapp-Romo-Scheidegger 16]**, exact WKB **[Aoki-Kawai-Takei 01]**,...

Exponential integrals

$$\int_{\mathscr{C}} e^{-\tau f(\mathbf{z})}\,\varphi(\mathbf{z})\,d\mathbf{z}$$

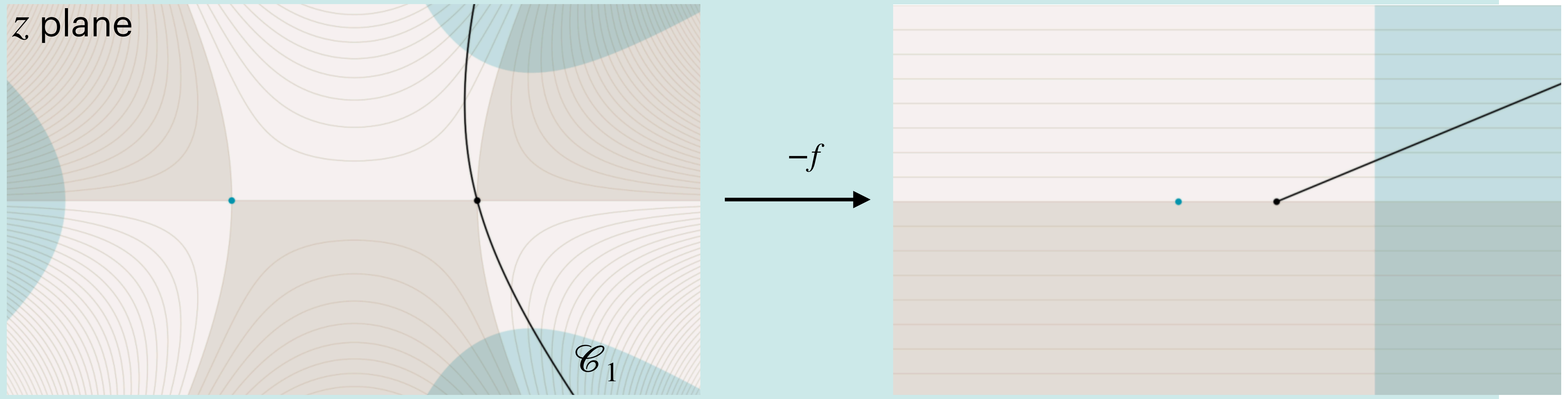
$f\colon X \to \mathbb{C}$ is a **proper algebraic** map from a complex n-dimensional algebraic variety X

$\varphi(\mathbf{z})$ is an analytic function of $\mathbf{z} \in X$

$\mathscr{C}$ defines a class in the relative homology $H_n(X, f)$

Example: the **Airy function**

$$\mathrm{Ai}(\tau) = \int_{\mathscr{C}_1} e^{-\tau\left(\frac{z^3}{3} - z\right)}\,dz$$



The Andersen-Kashaev state integrals

Partition function of a 3D Teichmüller TQFT that conjecturally describes $\mathrm{SL}_2(\mathbb{C})$ Chern-Simons theory on $S^3 \backslash K$

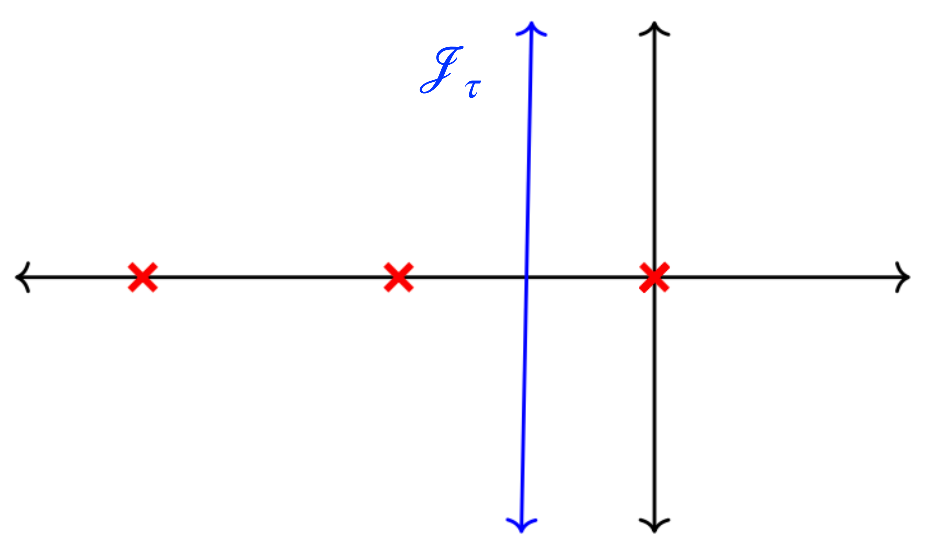
$$I_K(\tau) \;=\; \int_{\mathcal{J}_\tau} \Phi(z_1\,\tau;\tau)^{B_1} \ldots \Phi(z_n\,\tau;\tau)^{B_n}\, \mathbf{e}\big(\mathbf{z}^t A \mathbf{z}\;\tau\big) d\mathbf{z}\,,$$

$A \in \mathrm{Mat}_{n\times n}(\mathbb{Z})\,, B \in \mathbb{Z}^n$ the so-called Neumann-Zagier data given by a triangulation of $S^3 \backslash K$

$\Phi(z;\tau)$ the Faddeev's quantum dilogarithm, which is a meromorphic function of $z \in \mathbb{C}$ and analytic in $\tau \in \mathbb{C} \smallsetminus \mathbb{R}_{\leq 0}$

Example: the 4_1 **knot** $(A = 1\,, B = 2)$

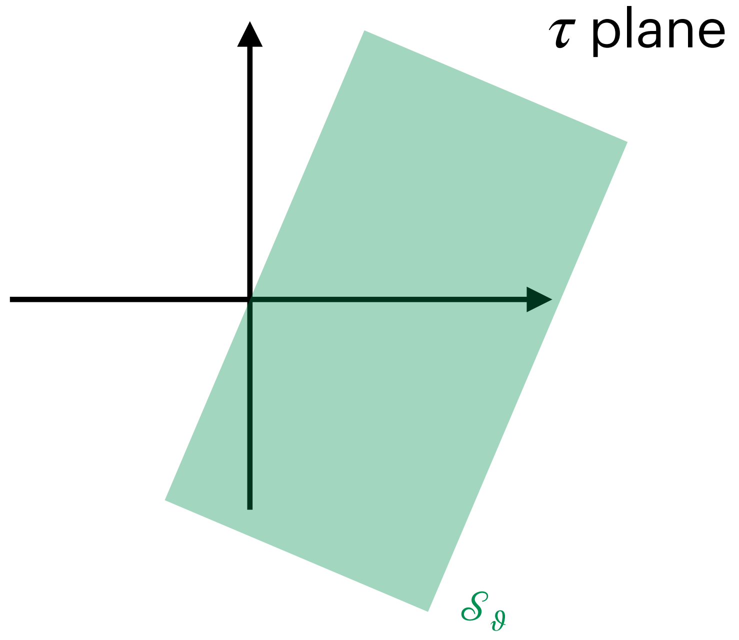
$$I_{4_1}(\tau) \;=\; \mu_8^2\, e^{-\frac{\pi i}{6\tau}}\, e^{\frac{\pi i \tau}{6}} \int_{\mathcal{J}_\tau} \Phi(z\,\tau;\tau)^2\, \mathbf{e}\left(\frac{1}{2} z(z\tau + \tau + 1)\right) dz$$



Goal

- **Borel resum** divergent asymptotic series and investigate the **effectiveness of Borel summation**

$$
\begin{tikzcd}
 & \Upsilon(\tau)\ \text{analytic} \arrow[dl, "\text{asymptotics}"'] \arrow[dr, equal, "?"] & \\
\tilde{\Upsilon}(\tau) \in \mathbb{C}[\![\tau^{-1}]\!] \arrow[rr, shift right, "\textbf{Borel resum}"'] & & \hat{\Upsilon}_\vartheta(\tau)\ \text{analytic for } \tau \text{ in a given sector } \mathcal{S}_\vartheta \arrow[ll, shift right, "\text{asymptotics}"']
\end{tikzcd}
$$



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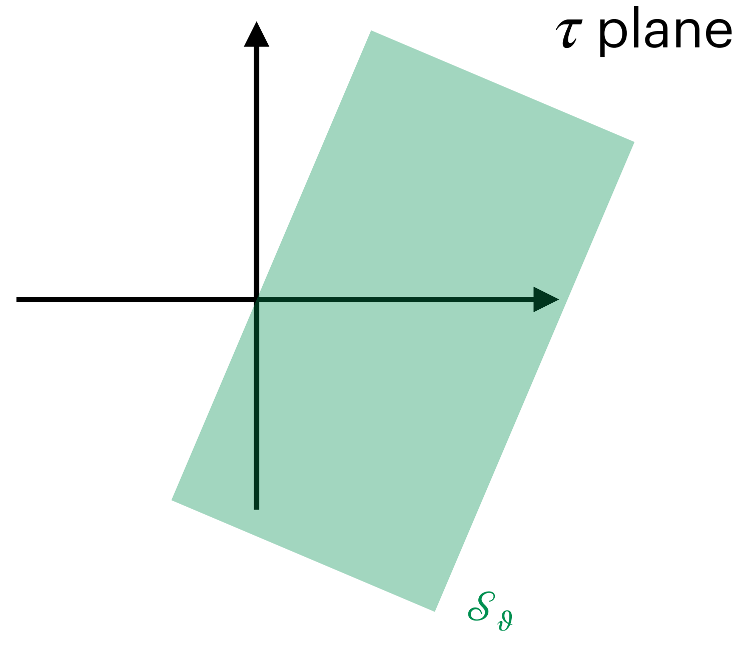
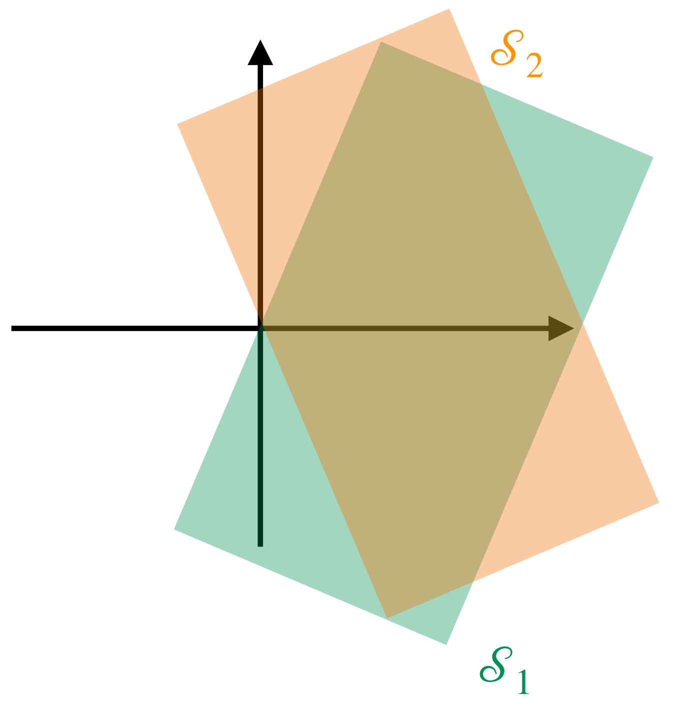


- Describe the **Stokes phenomenon**

$$\hat{\Upsilon}_1(\tau) - \hat{\Upsilon}_2(\tau) \;=\; \mathbf{S}_{12}\hat{\Upsilon}_2(\tau) \;\text{ when } \tau \in \mathcal{S}_1 \cap \mathcal{S}_2 \text{ for some } \mathbf{S}_{12} \in \mathbb{C}$$

and the constant $\mathbf{S}_{12}$ describing the jump is called the **Stokes constant**



Airy function

- Exponential integrals along thimbles are the Borel sum of their asymptotics (in 1-dimension **[VF-Fenyes 24]**)

$$
\begin{tikzcd}
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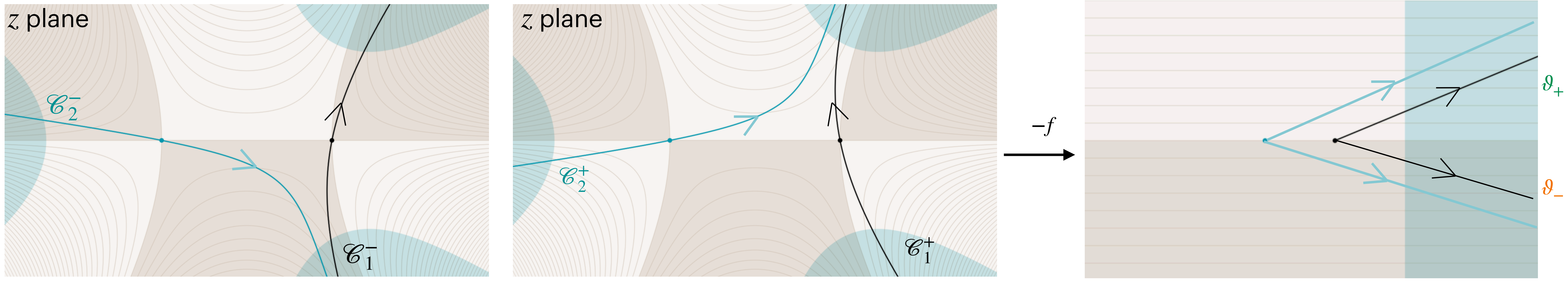
\tilde{\Upsilon}(\tau) \in \mathbb{C}[[\tau^{-1}]] \arrow[rr, shift right, "\text{Borel resum}"'] & & \hat{\Upsilon}_{\vartheta}(\tau) \text{ analytic} \arrow[ll, shift right, "\text{asymptotics}"']
\end{tikzcd}
$$

- The Stokes phenomenon can be described geometrically, and the Stokes constants can be computed via **Picard-Lefschetz formulas**



After choosing a basis of $H_1(X, f)$ and the orientation of the thimbles, we find

$$\mathscr{C}_1^- = \mathscr{C}_1^+$$

$$\mathscr{C}_2^- = \mathscr{C}_2^+ - \mathbf{1} \cdot \mathscr{C}_1^+$$

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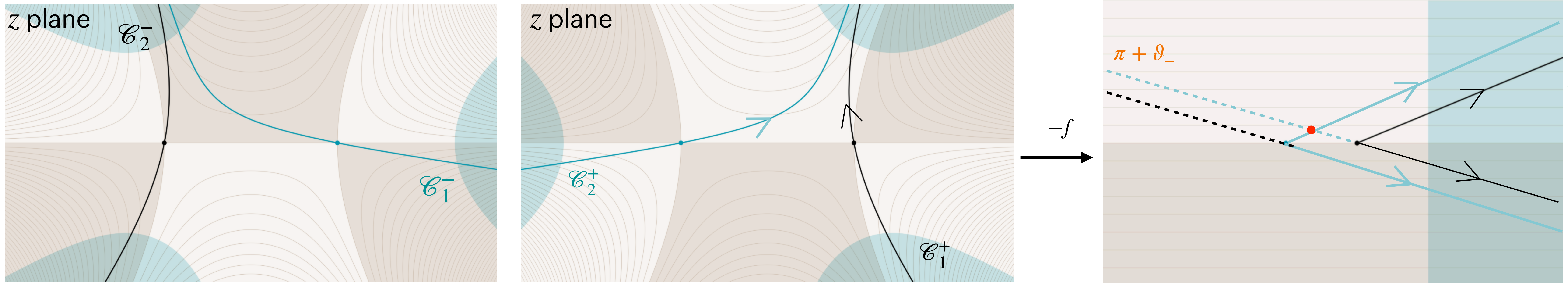

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$$\mathscr{C}_1^- = \mathscr{C}_1^+$$

$$\mathscr{C}_2^- = \mathscr{C}_2^+ - 1 \cdot \mathscr{C}_1^+$$

Intersection number of thimbles

Airy function

- Exponential integrals along thimbles are the Borel sum of their asymptotics (in 1-dimension **[VF-Fenyes 24]**)

$$\begin{tikzcd}
 & \mathrm{Ai}(\tau) \text{ analytic} \arrow[dl, "\text{asymptotics}"'] \arrow[dr, equal, "\checkmark"] & \\
\tilde{\Upsilon}(\tau) \in \mathbb{C}[\![\tau^{-1}]\!] \arrow[rr, shift right, "\text{Borel resum}"'] & & \hat{\Upsilon}_{\vartheta}(\tau) \text{ analytic} \arrow[ll, shift right, "\text{asymptotics}"']
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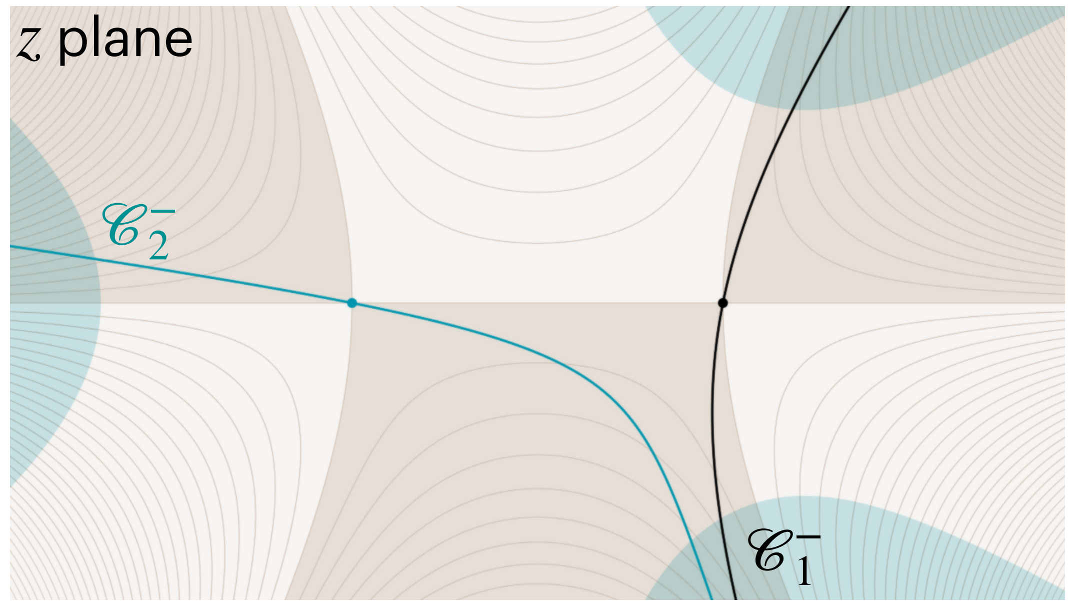
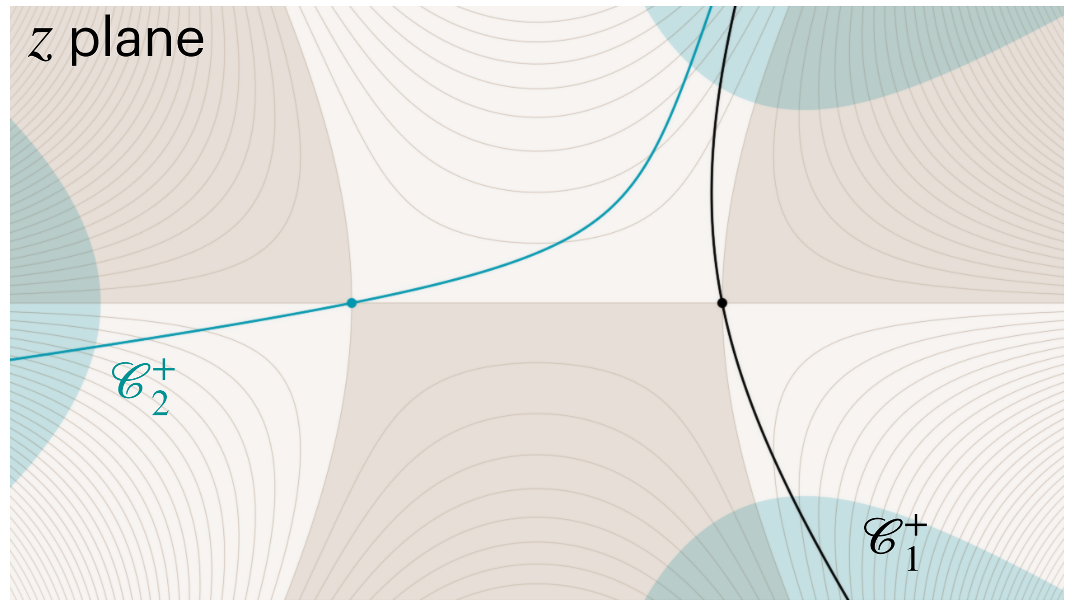
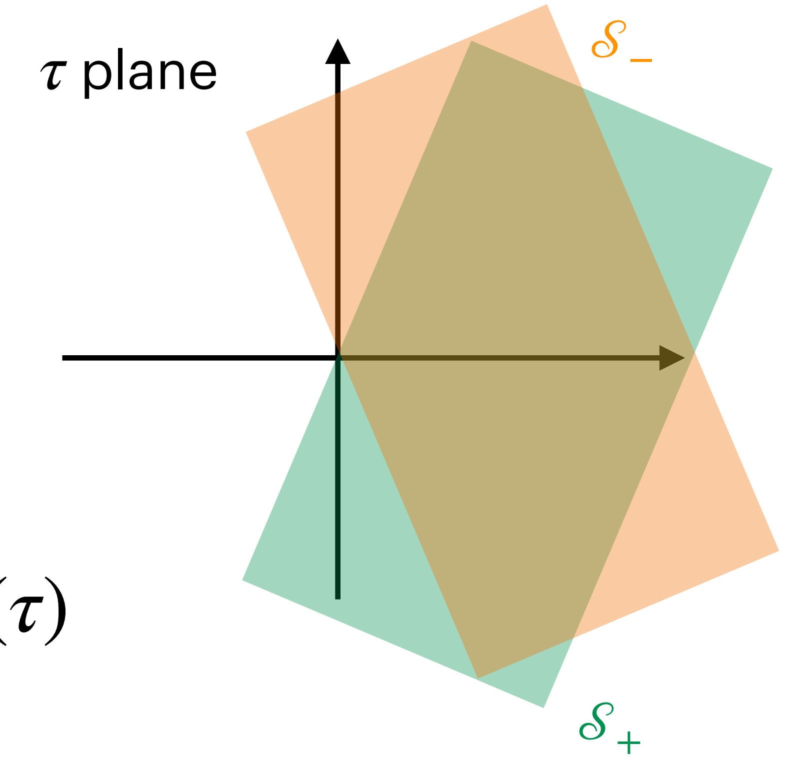




After choosing a basis of $H_1(X, f)$ and the orientation of the thimbles, we find

$$\mathscr{C}_1^- = \mathscr{C}_1^+ \qquad \hat{\Upsilon}_1^-(\tau) = \hat{\Upsilon}_1^+(\tau)$$

$$\mathscr{C}_2^- = \mathscr{C}_2^+ - \mathbf{1} \cdot \mathscr{C}_1^+ \qquad \hat{\Upsilon}_2^-(\tau) = \hat{\Upsilon}_2^+(\tau) - \mathbf{1} \cdot \hat{\Upsilon}_1^+(\tau)$$



The Andersen-Kashaev state integrals

- **Conjecture** [Garoufalidis-Gu-Mariño 21] Borel resummation of the perturbative invariants $\tilde{\Upsilon}_K$ is a linear combination of the Andersen-Kashaev integral I_K and its descendants $I_{K,m,\ell}$

$$
\begin{tikzcd}
 & I_K(\tau)\ \text{analytic} \arrow[dl, "\text{asymptotics}"'] \arrow[dr, equal, "\text{Not always!}"] & \\
\tilde{\Upsilon}_K(\tau) \in \mathbb{C}[[\tau^{-1}]] \arrow[rr, shift right, "\text{Borel resum}"'] & & \hat{\Upsilon}_{K,\vartheta}(\tau)\ \text{analytic} \arrow[ll, shift right, "\text{asymptotics}"']
\end{tikzcd}
$$

where $I_{K,m,\ell}(\tau) := \mu_8^B\, e^{-\frac{\pi i B}{12\tau}} e^{\frac{\pi i B\tau}{12}} \int_{\mathcal{J}_{\ell,\tau}} \Phi((z-\ell)\tau;\tau)^B\, \mathbf{e}\left(\frac{A}{2} z(z\tau+\tau+1)+mz\tau\right) dz\,,$ with $m = 0,\ldots,A-1\,,\ \ell \in \mathbb{Z}$ and

$I_{K,0,0} = I_K$

The Andersen-Kashaev state integrals

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Perturbative invariants

$$\tilde{\Upsilon}_K(\tau) := \int \tilde{\Psi}(z;\tau)^B \, \mathbf{e}\left(\frac{A}{2} z^2 \tau\right) dz$$

Andersen-Kashaev state integrals

$$I_{K,m,\ell}(\tau) := \int_{\mathcal{J}_{\ell,\tau}} \Phi((z-\ell)\tau;\tau)^B \, \mathbf{e}\left(\frac{A}{2} z(z\tau+\tau+1) + mz\tau\right) dz\,,$$

with $m = 0,\ldots,A-1$ et $\ell \in \mathbb{Z}$

$$\tilde{\Psi}(z;\tau) \underset{\text{asymptotics}}{\overset{\text{Borel sum}}{\rightleftarrows}} \Phi(z;\tau)$$

Borel summation does not always commute with integration on the parameter space

Main result

Theorem [VF-Wheeler 24] Borel resummation of the perturbative invariants $\tilde{\Upsilon}_K$ is a linear combination of the Andersen-Kashaev integral and its descendants $I_{K,m,\ell}$ for $K = 4_1$ and 5_2

Main result

Theorem **[VF-Wheeler 24]** Borel resummation of the perturbative invariants $\tilde{\Upsilon}_K$ is a linear combination of the Andersen-Kashaev integral and its descendants $I_{K,m,\ell}$ for $K = 4_1$ and 5_2

- The functions $I_{K,m,\ell}$ **are not thimble integrals**; indeed, the contours $\mathcal{J}_{\ell,\tau}$ are not of steepest descent for the function

$$V(z) = B\,\frac{\mathrm{Li}_2(\mathbf{e}(z))}{(2\pi \mathrm{i})^2} + \frac{B}{24} + \frac{A}{2} z(z+1)$$

- However, they form a basis for a ***relative*** **homology with coefficients**

- Hence, the steepest descent contours can be written in terms of the $\mathcal{J}_{\ell,\tau}$, allowing us to compute the Stokes constants

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V **is multivalued**

- However, they form a basis for a ***relative*** **homology with coefficients**

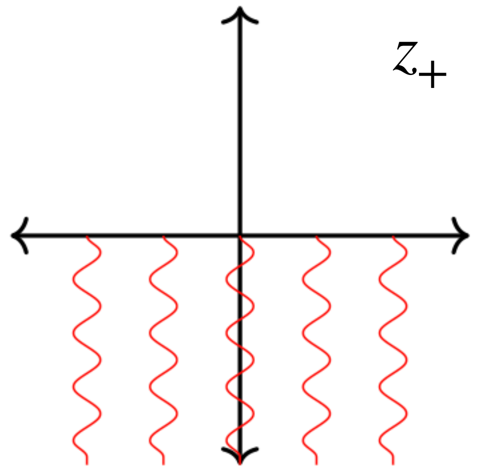
The **system of coefficients** is given by the **Stokes phenomenon** of the **Faddeev's dilogarithm**

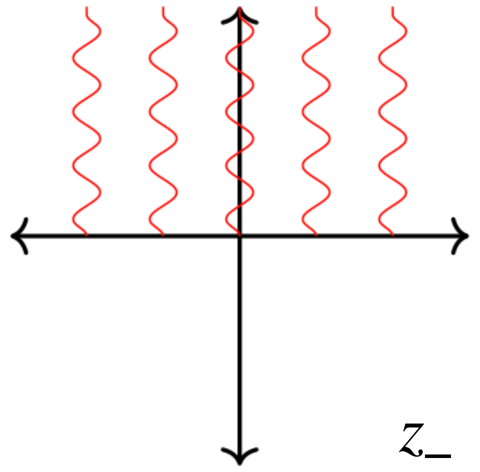
- Hence, the steepest descent contours can be written in terms of the $\mathcal{J}_{\ell,\tau}$, allowing us to compute the Stokes constants

An homology theory for state integral contours

An homology theory for state integral contours

- **The Riemann surface of V**





  - Fix a branch of $\mathrm{Li}_2(\mathbf{e}(z))$ and restrict the function V to the Riemann surface Σ

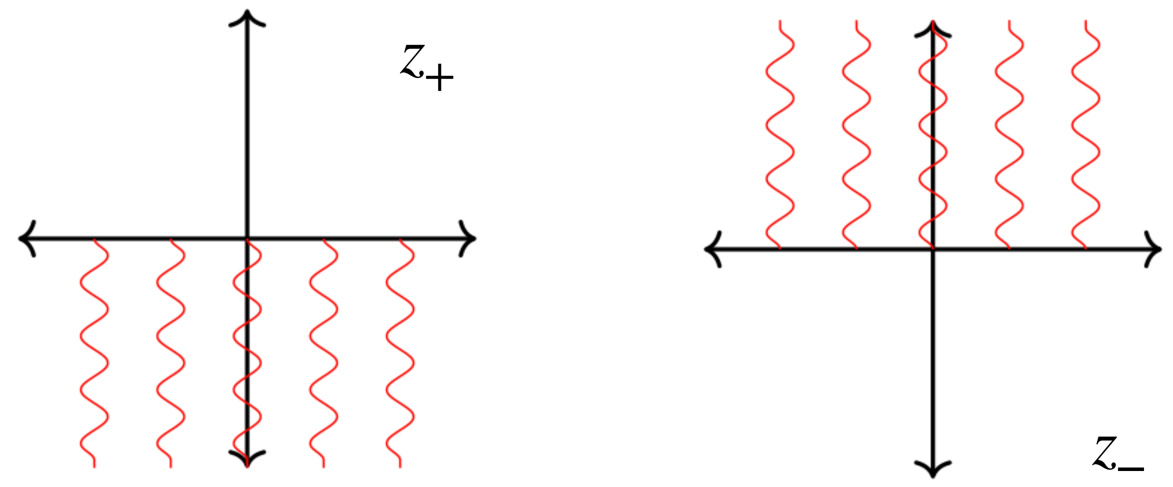
$$V(z_+, m, n) = B\,\frac{\mathrm{Li}_2(\mathbf{e}(z_+))}{(2\pi \mathrm{i})^2} + \frac{B}{24} + \frac{A}{2} z_+(z_+ + 1) + m\, z_+ + n\,,$$

$$\Lambda(z_-, m, n) = B\frac{\mathrm{Li}_2(\mathbf{e}(-z_-))}{-(2\pi i)^2} + \frac{B}{12} - \frac{B}{2}\left(z_- - \frac{1}{2}\right)^2 + \frac{A}{2} z_-(z_- + 1) + m\, z_- + n\,,$$

where $m = 0,\ldots, A-1$ and $n \in \mathbb{Z}$

An homology theory for state integral contours

- **The Riemann surface of V**



  - Fix a branch of $\mathrm{Li}_2(\mathbf{e}(z))$ and restrict the function V to the Riemann surface Σ

$$V(z_+, m) = B\,\frac{\mathrm{Li}_2(\mathbf{e}(z_+))}{(2\pi \mathrm{i})^2} + \frac{B}{24} + \frac{A}{2} z_+(z_+ + 1) + m\,z_+\,,$$

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where $m = 0, \ldots, A-1$

  - Critical points of the function $V : \Sigma \to \mathbb{C}/\mathbb{Z}$ are solutions of an algebraic equation $(-x)^A = (1-x)^B$ and $x = \mathbf{e}(z)$

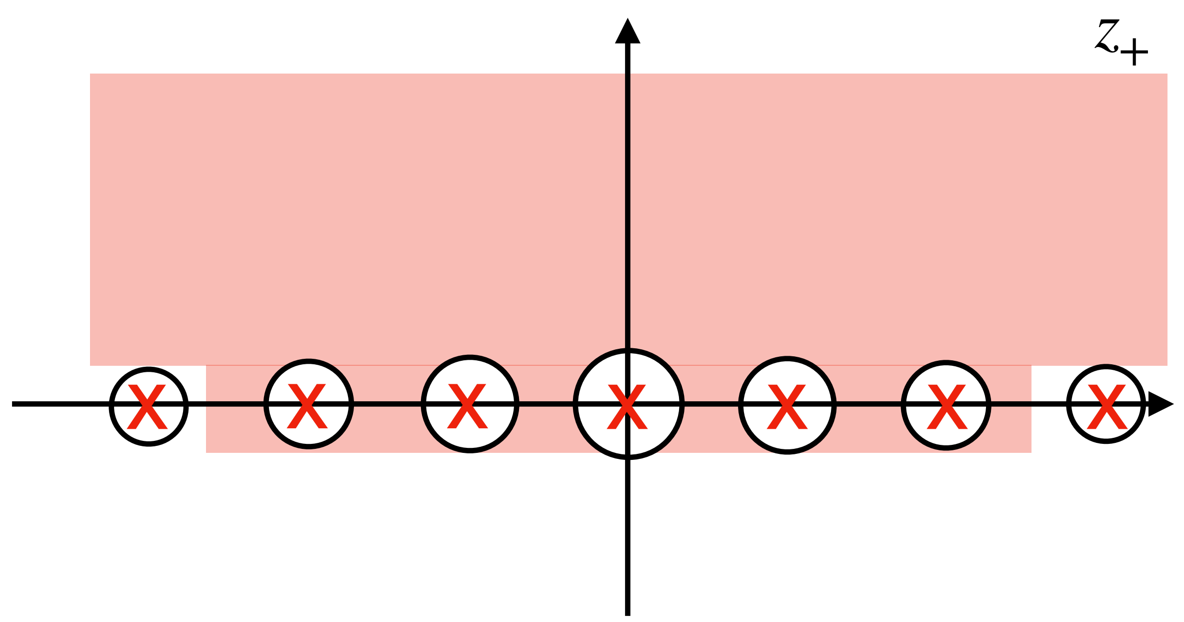
An homology theory for state integral contours

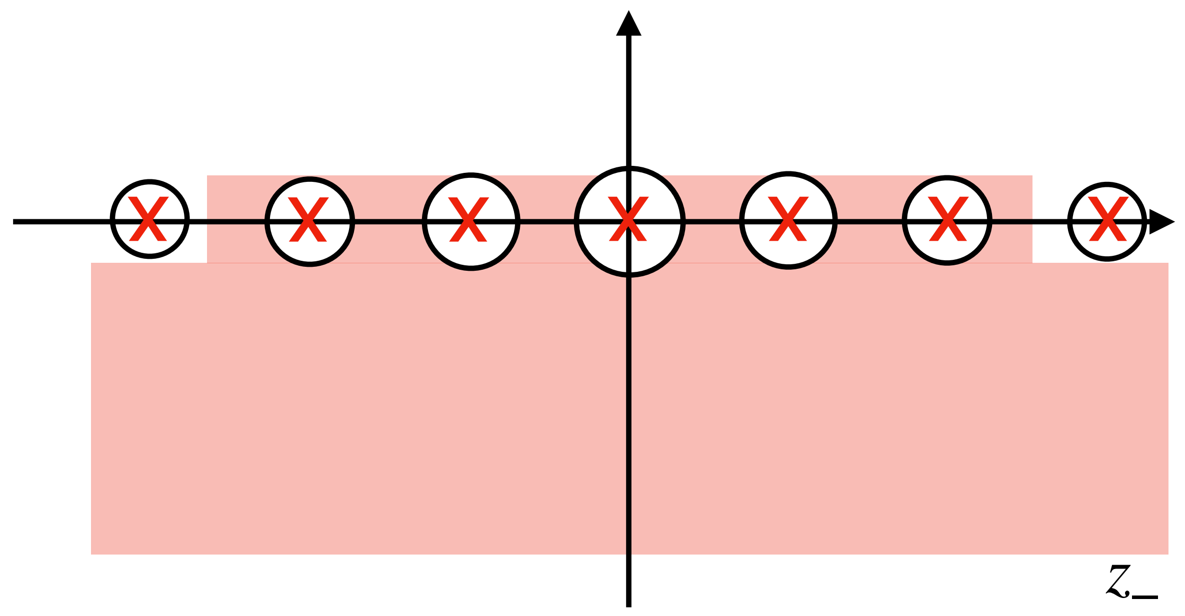
- **A relative homology for** $V \colon \Sigma \to \mathbb{C}/\mathbb{Z}$

  - For z large enough, the function V (resp. Λ) is dominated by the **Gaussian term** $f_+(z) = \frac{A}{2}z^2$ (resp. $f_-(z) = \frac{(A-B)}{2}z^2$)

  - For generic directions $\vartheta \in [0,2\pi)$, the steepest descent contours of V (resp. Λ) are ϵ-bounded away from the branch points

  - Define the surface $X_{M,\epsilon} \subset \Sigma$ by gluing different coordinate charts





An homology theory for state integral contours

- **A *relative* homology for** $V: \Sigma \to \mathbb{C}/\mathbb{Z}$

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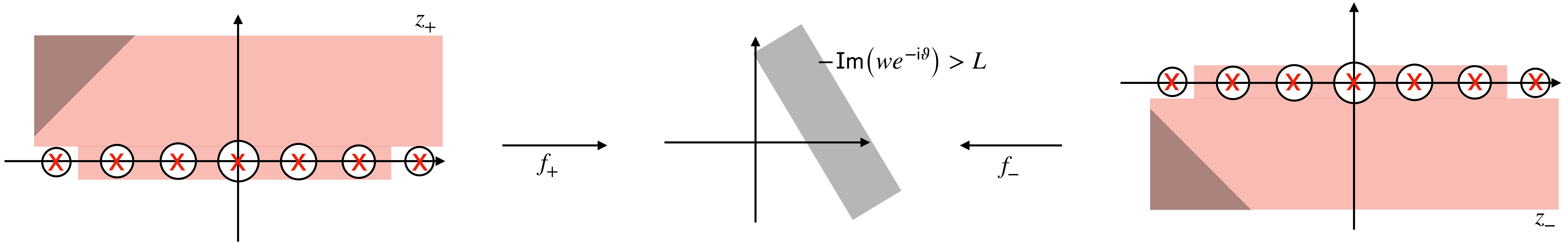


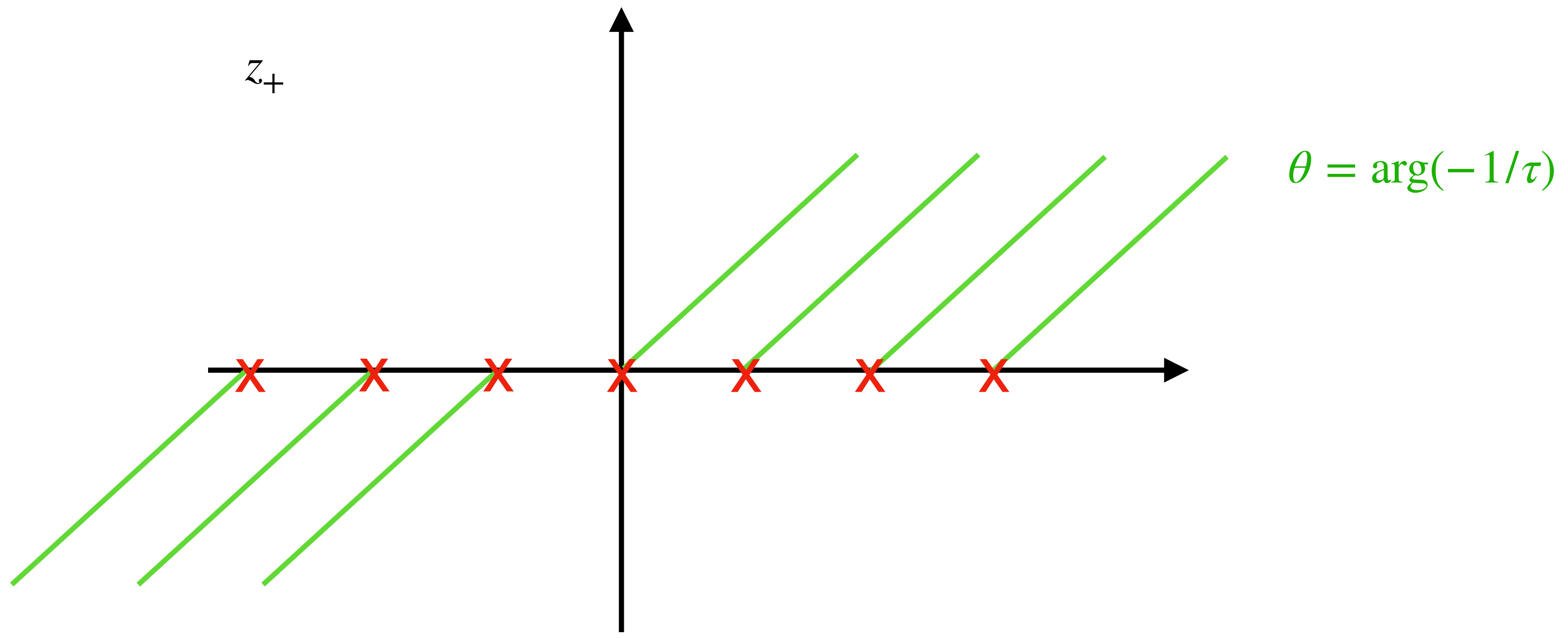
Proposition [VF-Wheeler 24] The relative homology $H_1(X_{M,\epsilon}, f)$ is finite-dimensional, and a basis is given by the state integrals contours $\mathcal{J}_{\ell,\tau}$ with $\arg(\tau) = \vartheta$

An homology theory for state integral contours

- **The system of coefficients**

  - The integral defining the formal series $\tilde{\Upsilon}_K$ depends on τ both in the exponential term $\mathbf{e}(\tau\, V(z))$ and the 1-form
  $\exp\Big(-\mathrm{Li}_1(\mathbf{e}(z)) - \sum_{k=2}^{\infty} (2\pi \mathrm{i})^{k-1} \frac{B_k}{k!} \mathrm{Li}_{2-k}(\mathbf{e}(z))\, \tau^{1-k} \Big)\, dz \in \Omega^1(\Sigma)[\![\tau^{-1}]\!]$

  - The Faddeev's dilogarithm $\Phi(z;\tau)$ jumps crossing these green lines, which represents the Stokes lines for the Borel sum of $\tilde{\Psi}(z;\tau)$

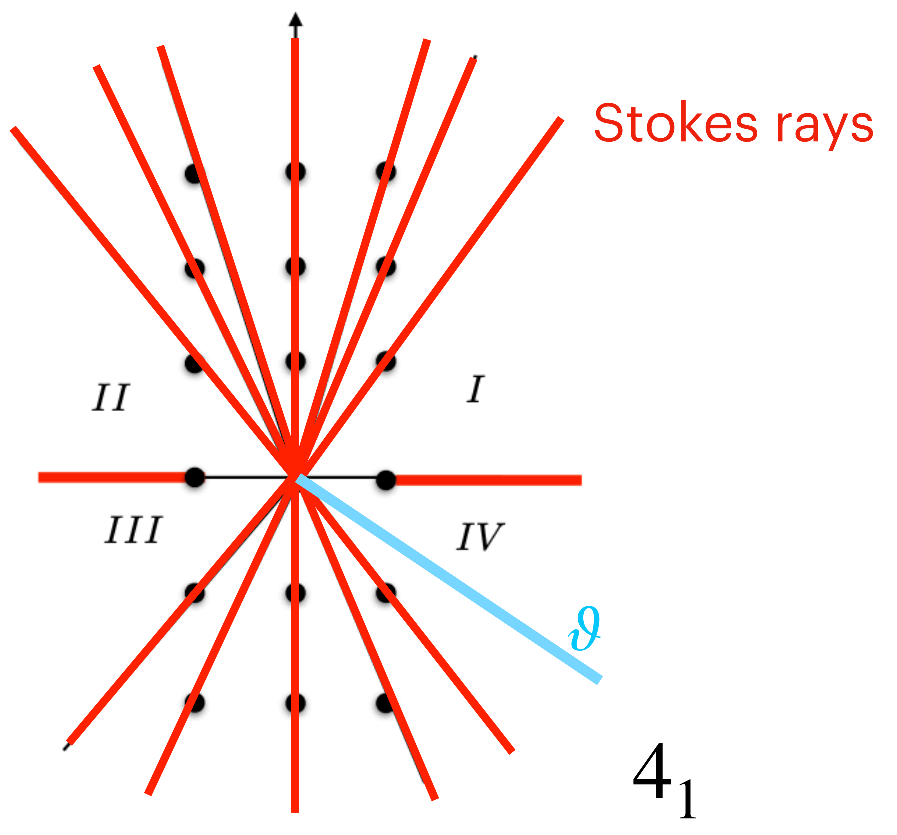


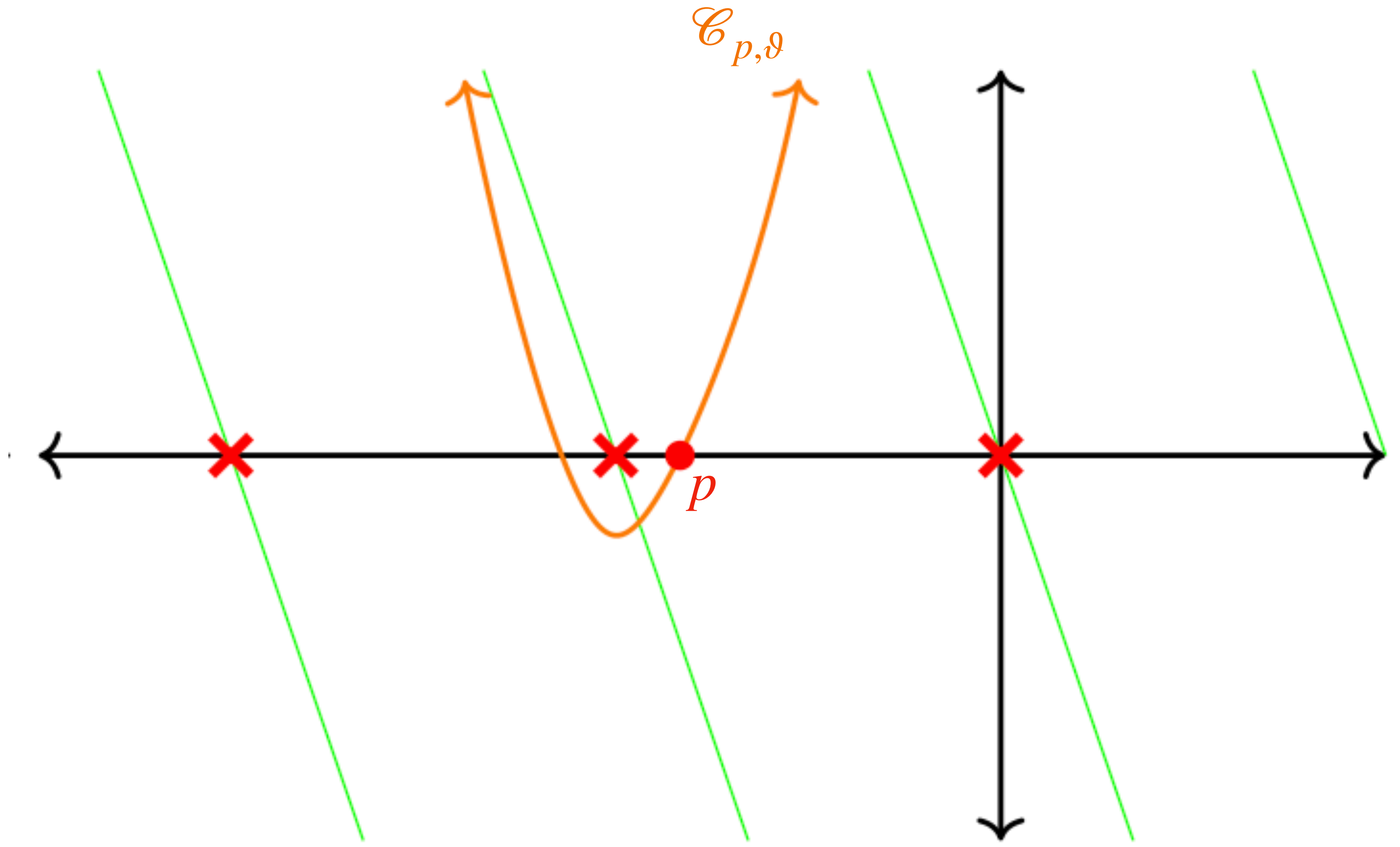
  - Build a sheaf $\mathscr{V} \to \Sigma$ that includes these contributions (*sheaf of resurgent structure*), and define the sheaf homology $H_1(\Sigma, \mathscr{V})$ [*in progress* Andersen-**VF**-Kontsevich-Wheeler]

The algorithm

The algorithm to compute the decomposition of thimble integrals (i.e. the Borel sum of $\tilde{\Upsilon}_K$) in a given direction ϑ in the basis of state integrals $I_{K,m,\ell}$

- Consider the steepest descent contour $\mathscr{C}_{p,\vartheta}$ through the critical point p





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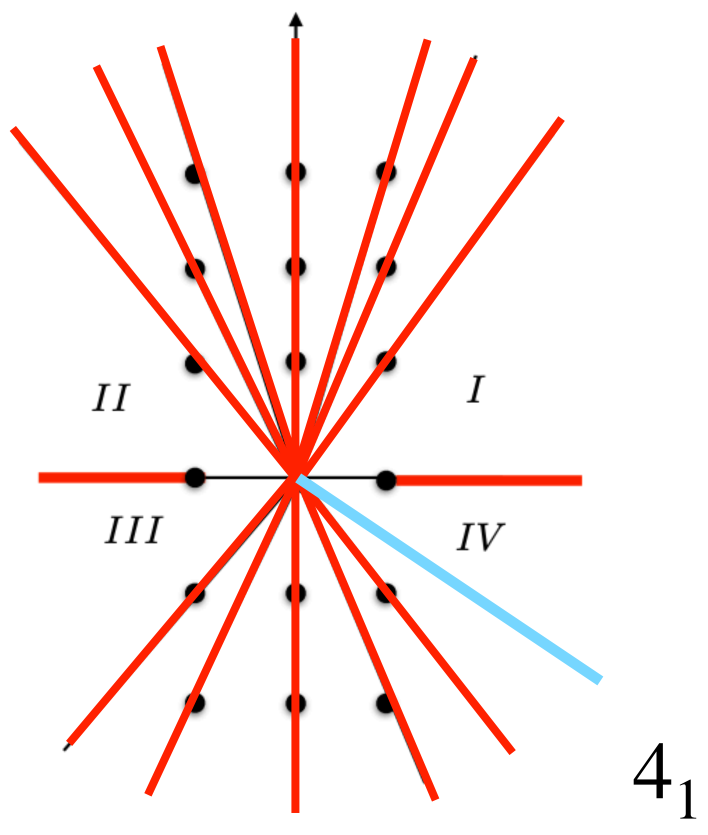
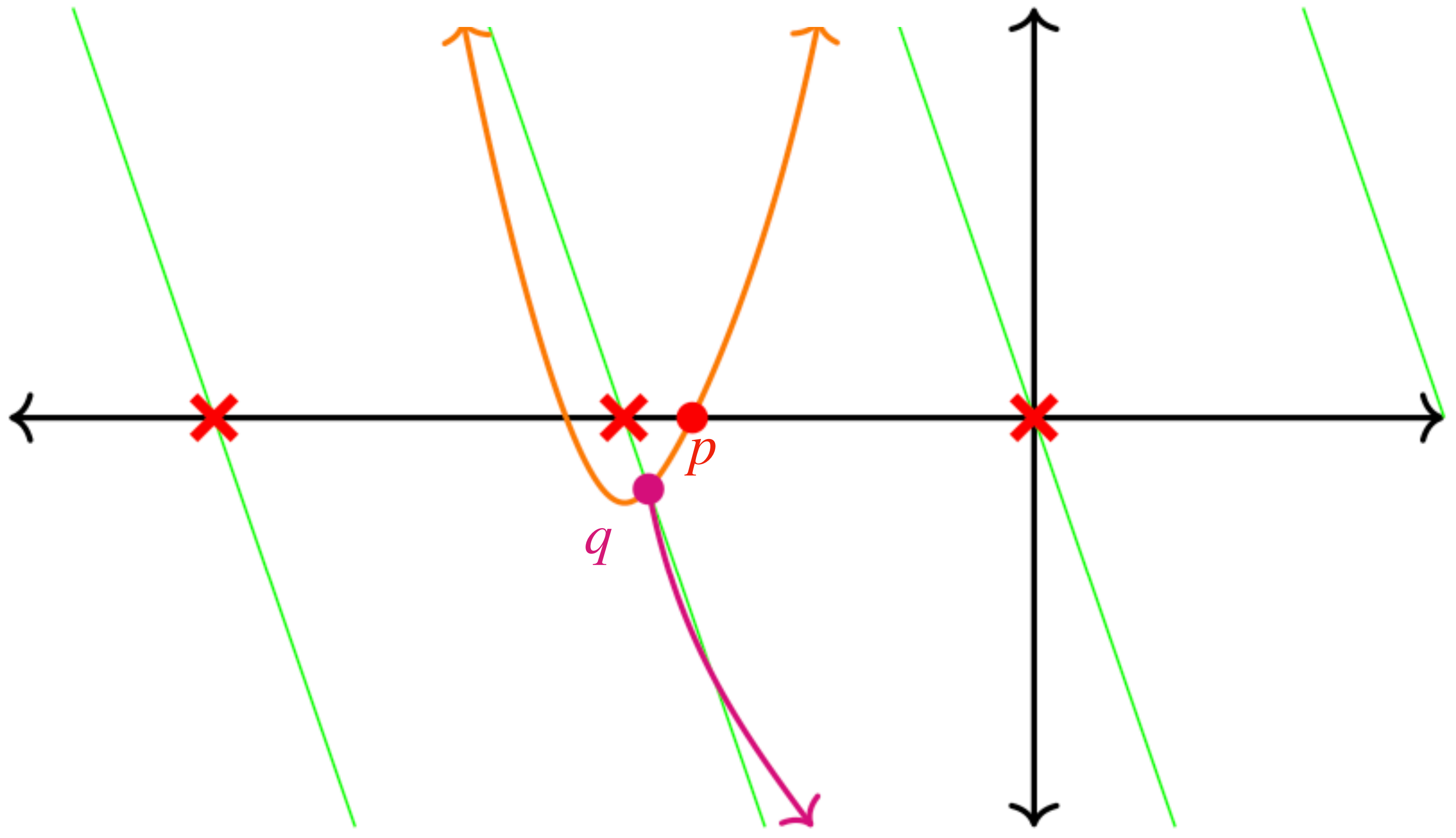
- When $\mathscr{C}_{p,\vartheta}$ intersects the green lines, the function V (resp. Λ) changes, so we need to flow again and consider a new steepest descent contour $\mathscr{C}_{q,\vartheta}$





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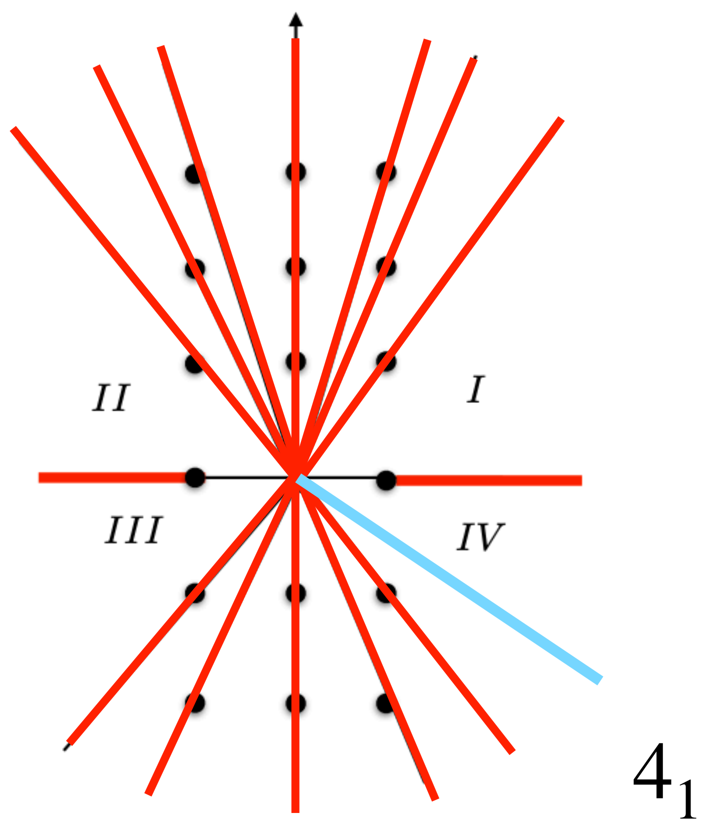
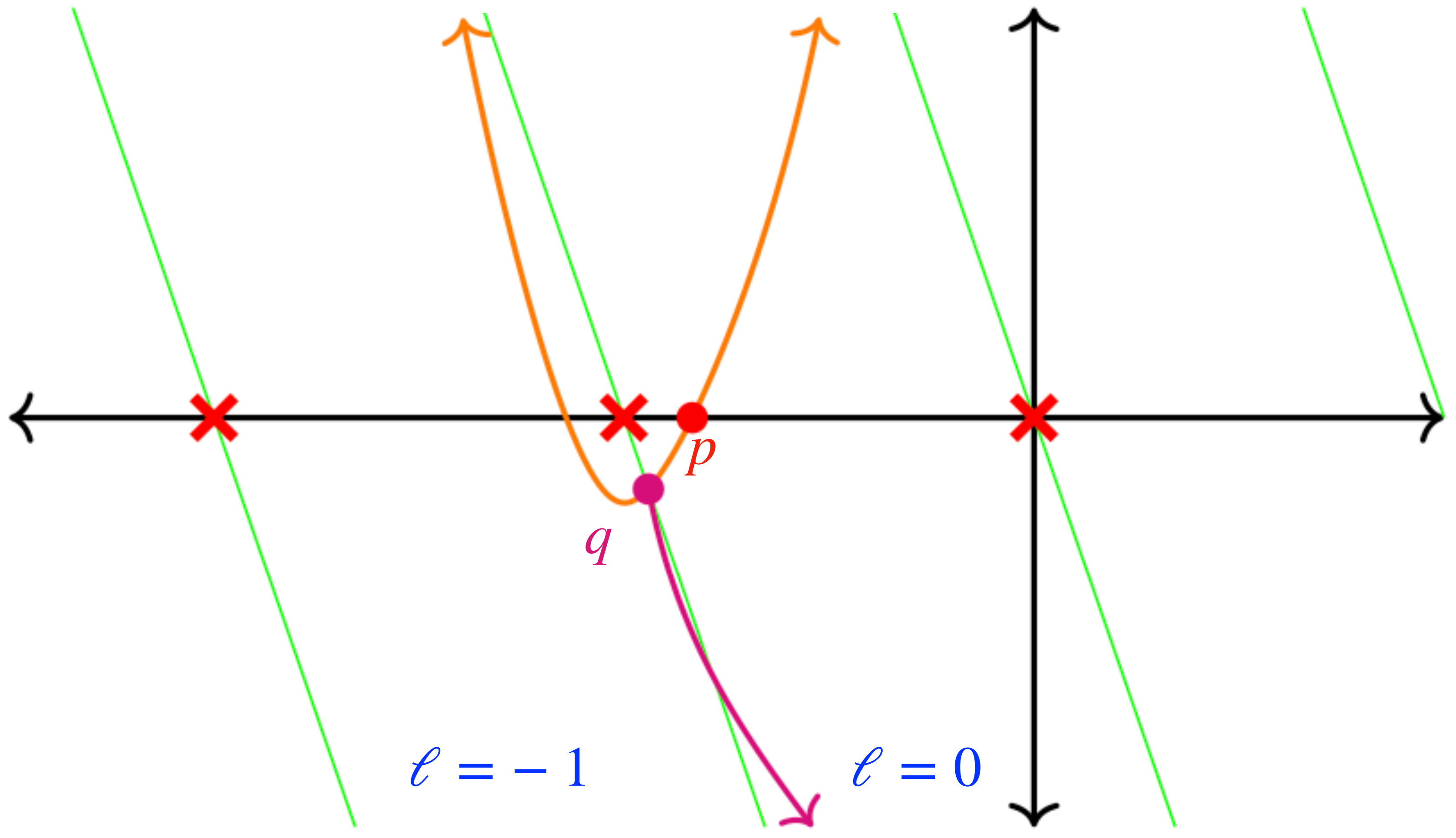
- Only the contours intersecting the reals contribute to the decomposition





$$\hat{\Upsilon}_\vartheta = I_{0,0} + q^2 I_{2,-1} = I_{0,0} + I_{1,0}$$

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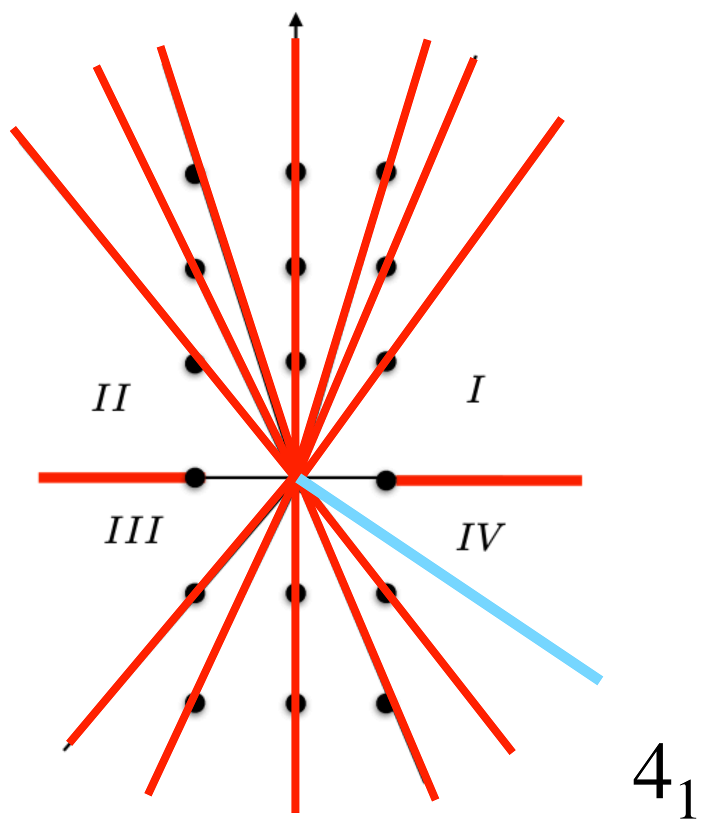

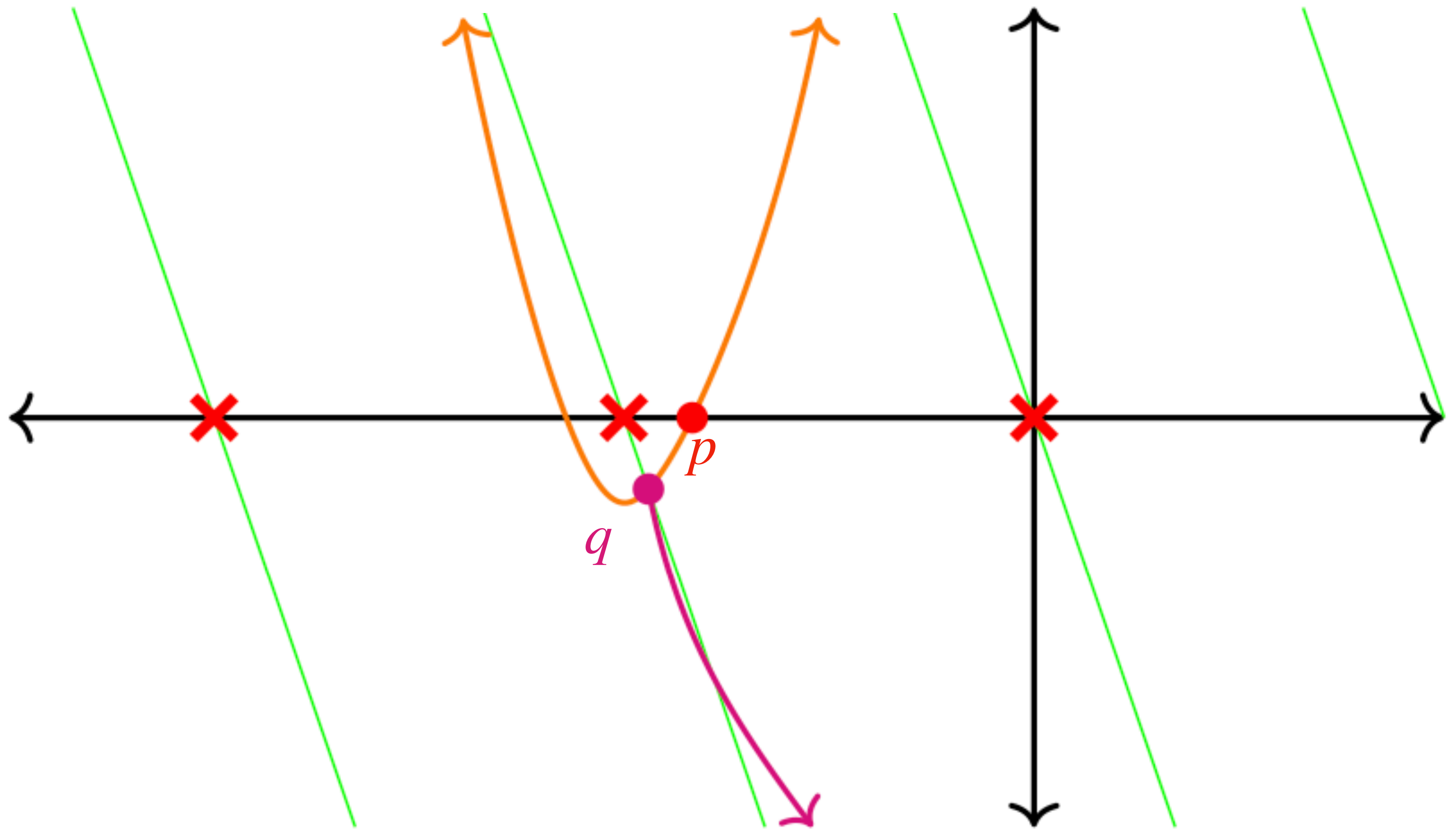

Theorem [VF-Wheeler 24] The algorithm ends after finitely many steps

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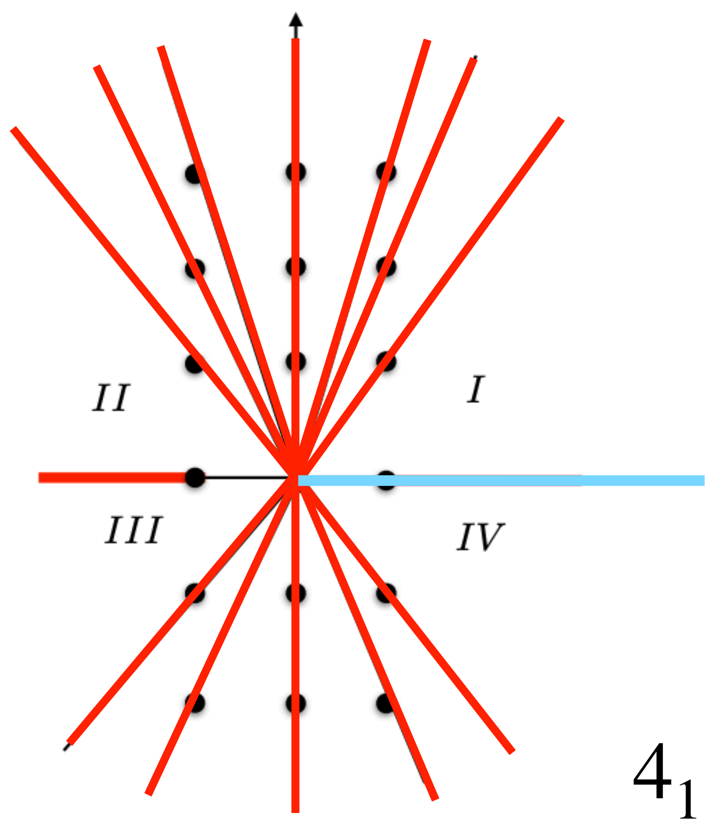
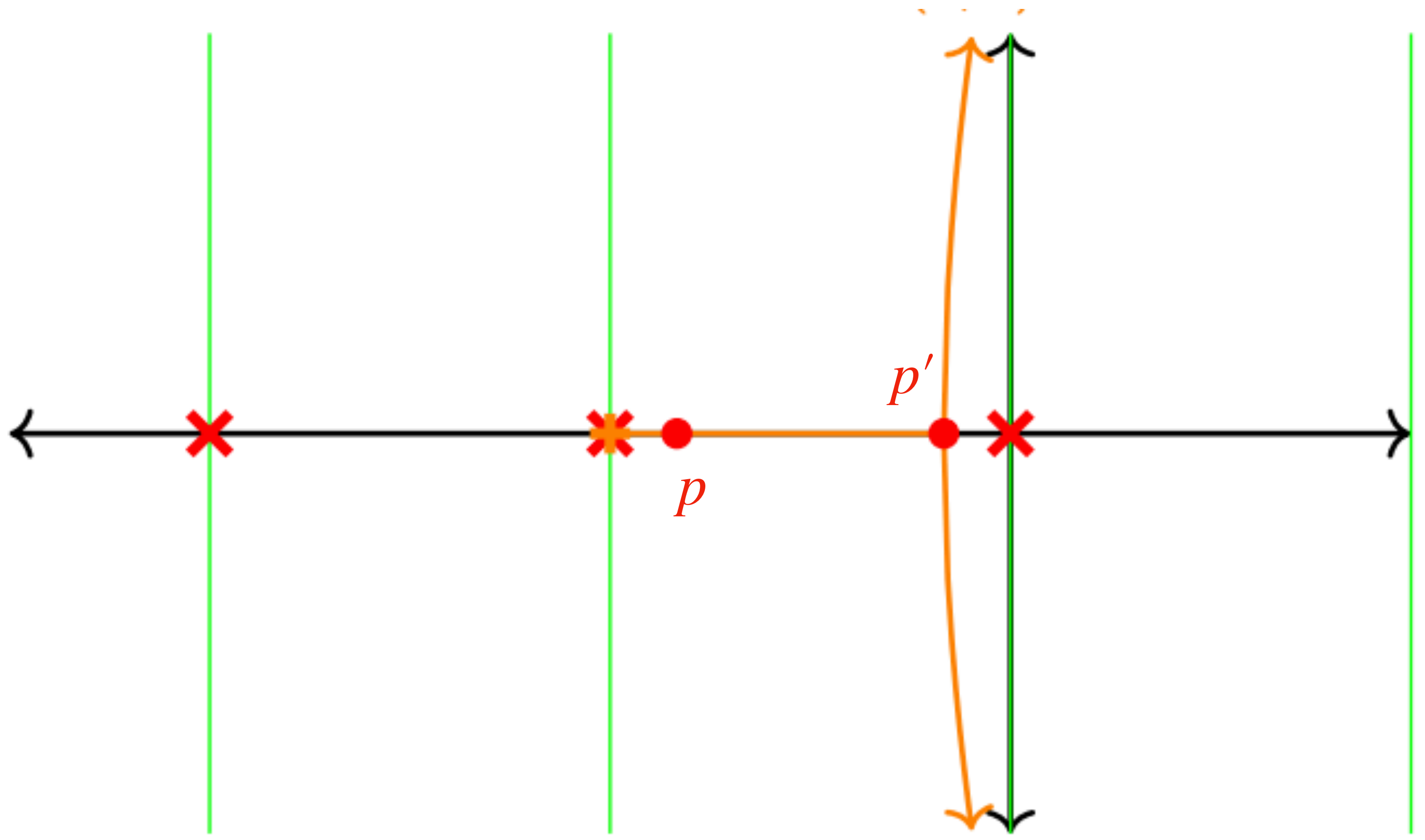
- If ϑ agrees with the direction of a Stokes ray, there is a **saddle connection** between two critical points p , p'

- Observe also that $\mathscr{C}_{p,\vartheta}$ can end in a branch point





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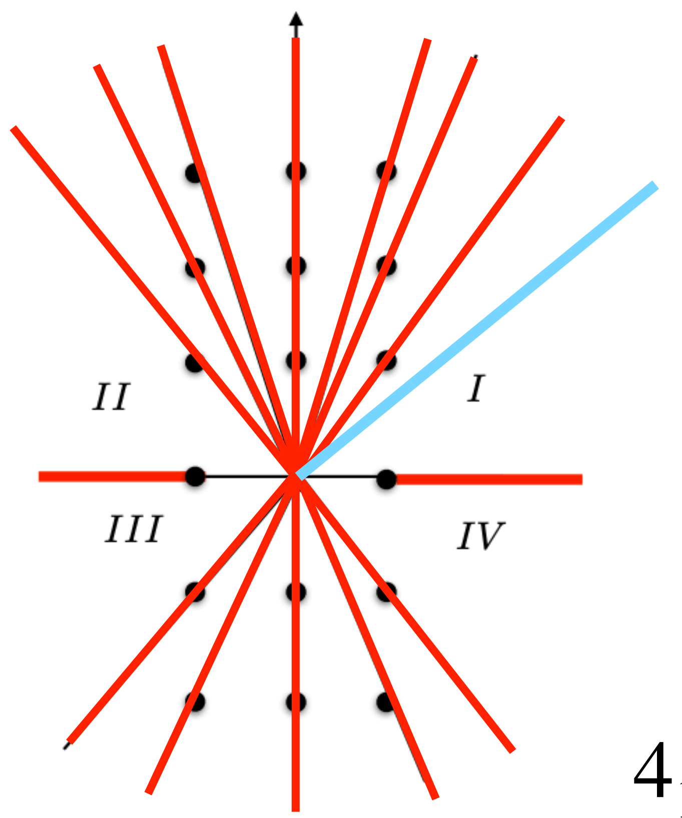
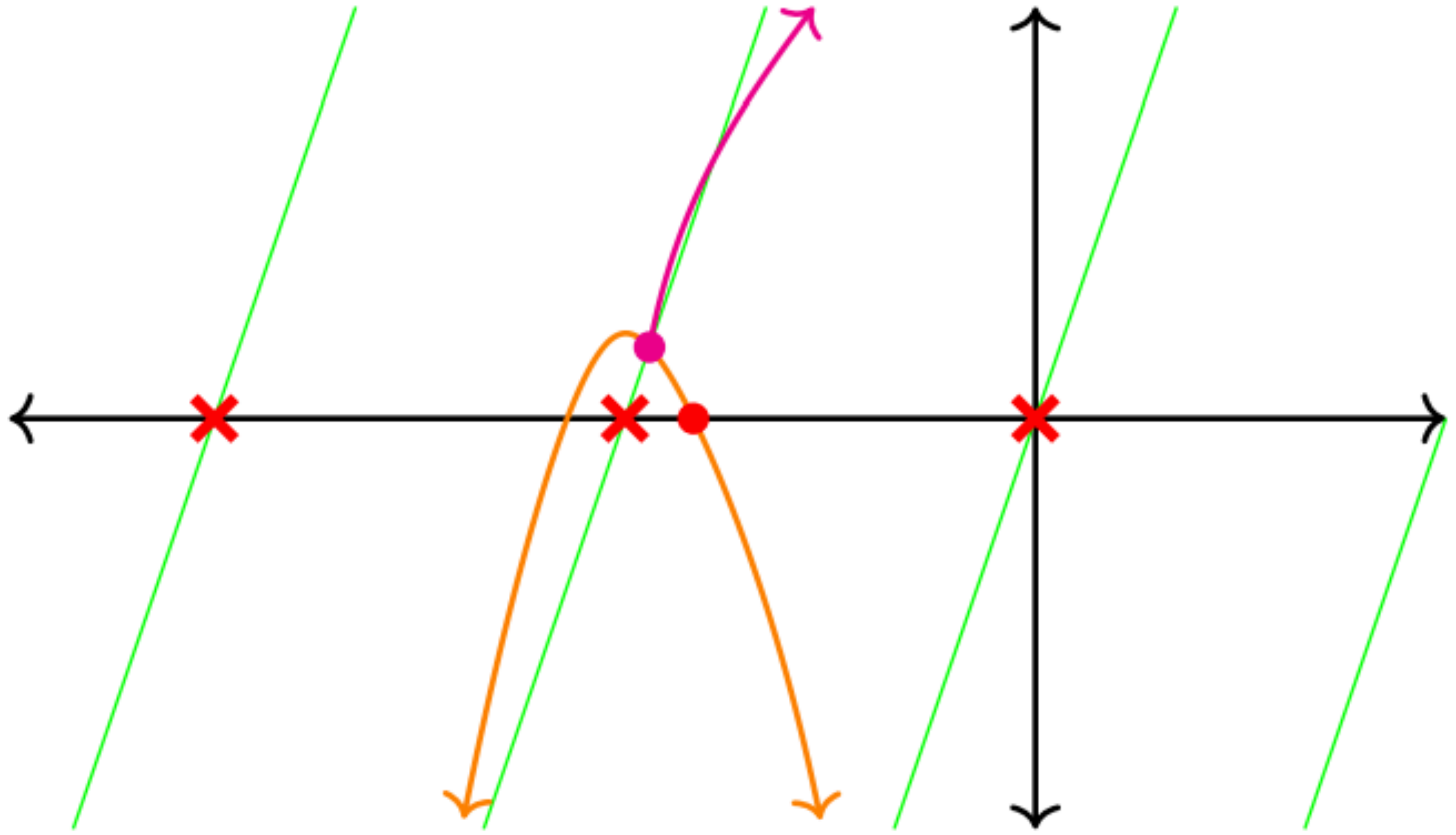


4_1



$$\hat{\Upsilon}_\vartheta = -I_{0,0} + I_{0,-1} = -I_{0,0} - I_{-1,0}$$

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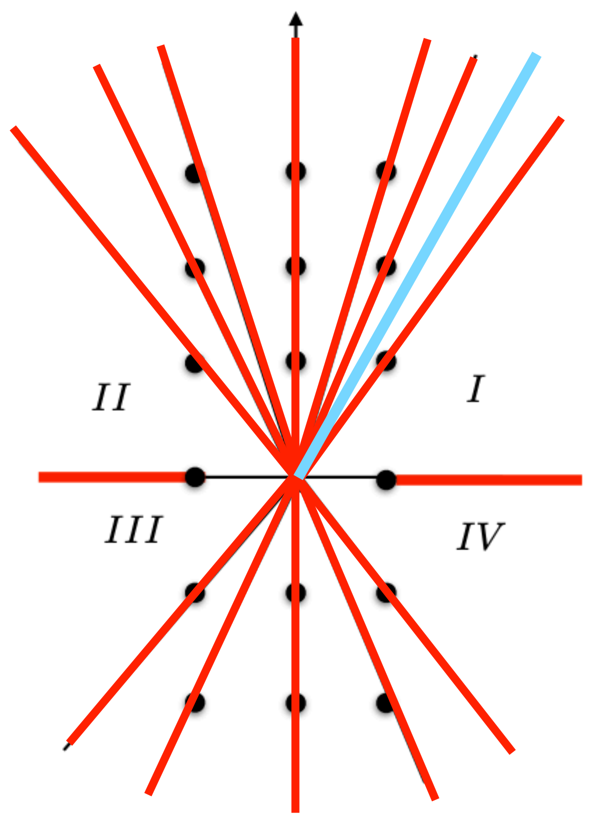

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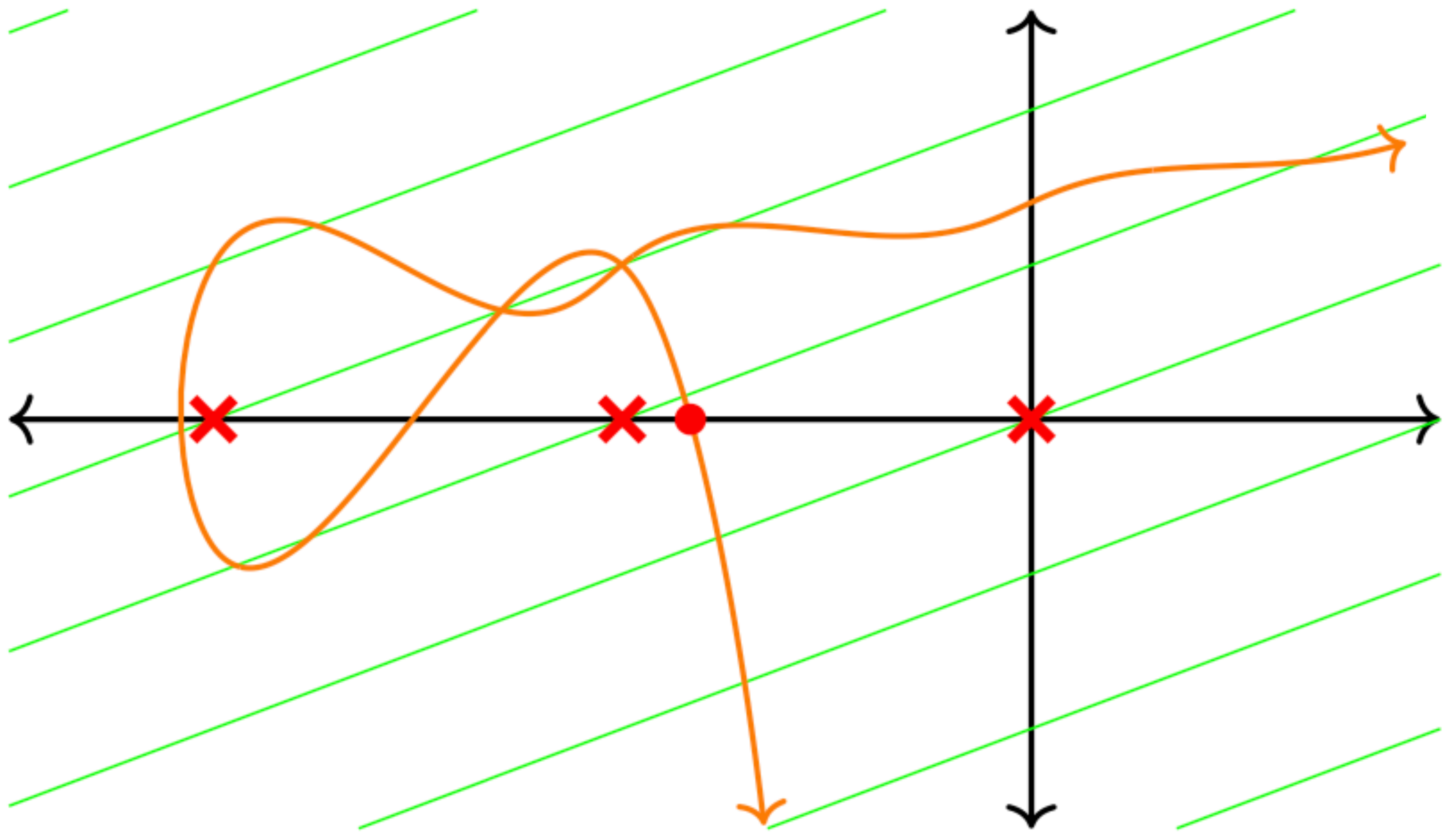

$$\hat{\Upsilon}_\vartheta = -I_{0,0} + I_{0,-1} - q^4 I_{2,-2} + 2q^2 I_{1,-1} + 2q^2 I_{1,-1} - 4q I_{0,0}$$
$$= -I_{0,0} - I_{-1,0} - 9q I_{0,0}$$

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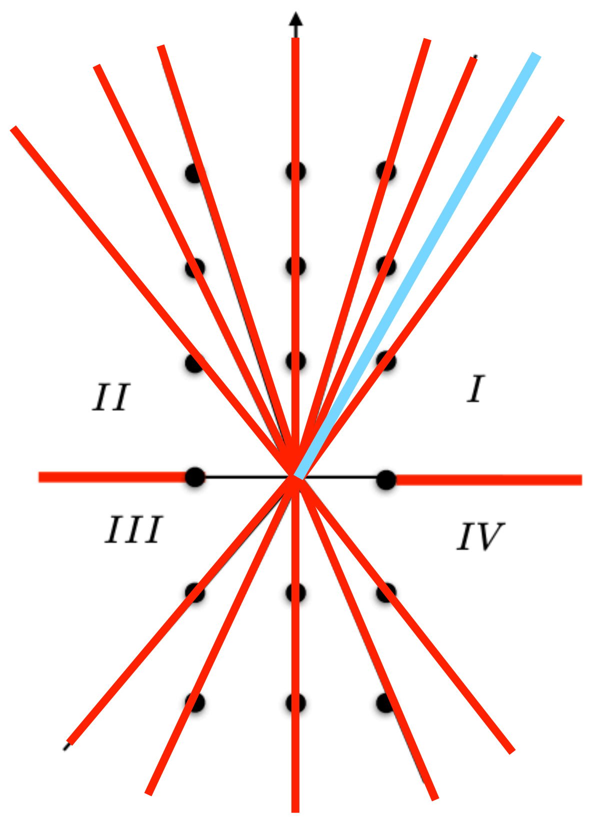

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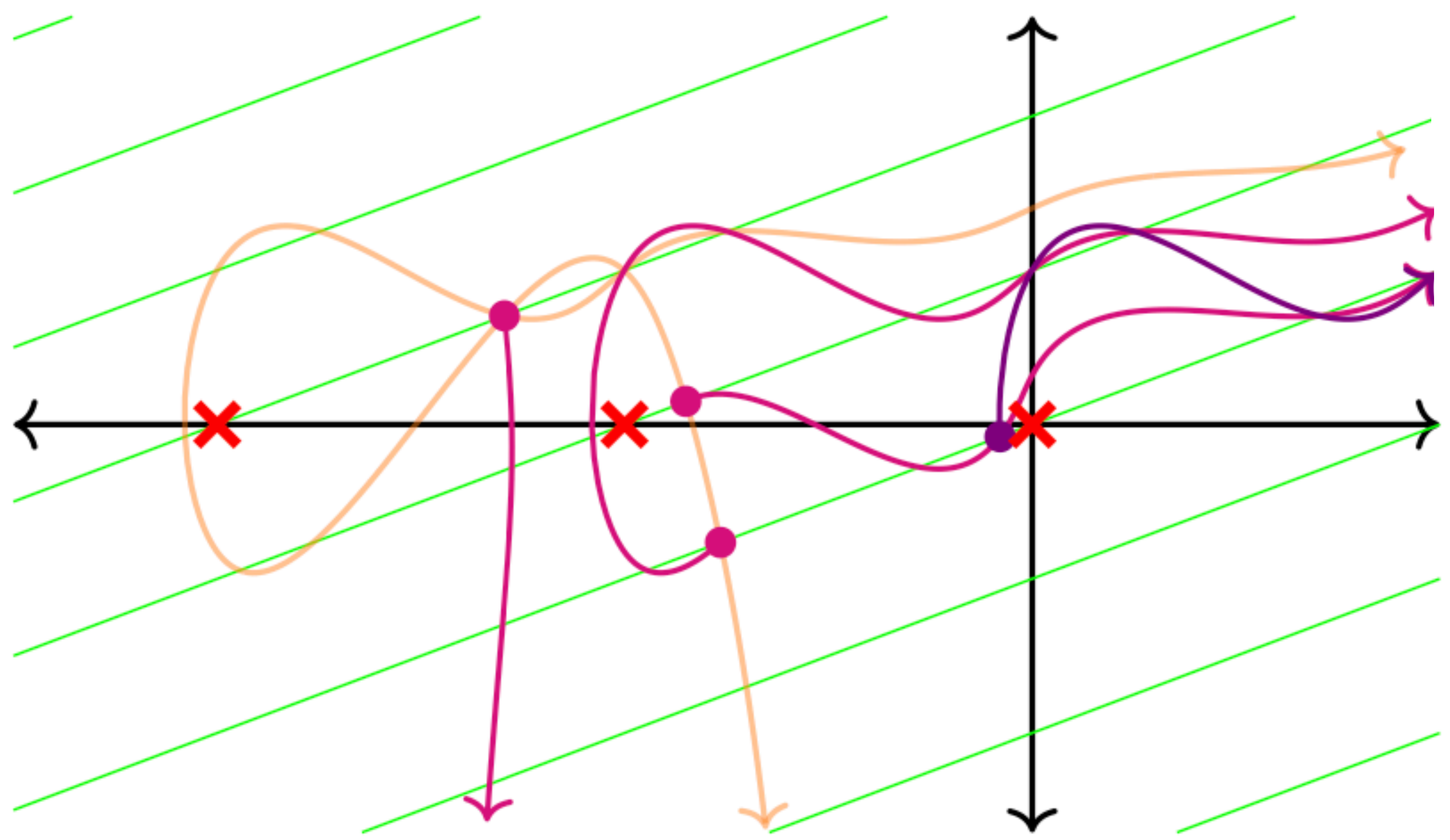

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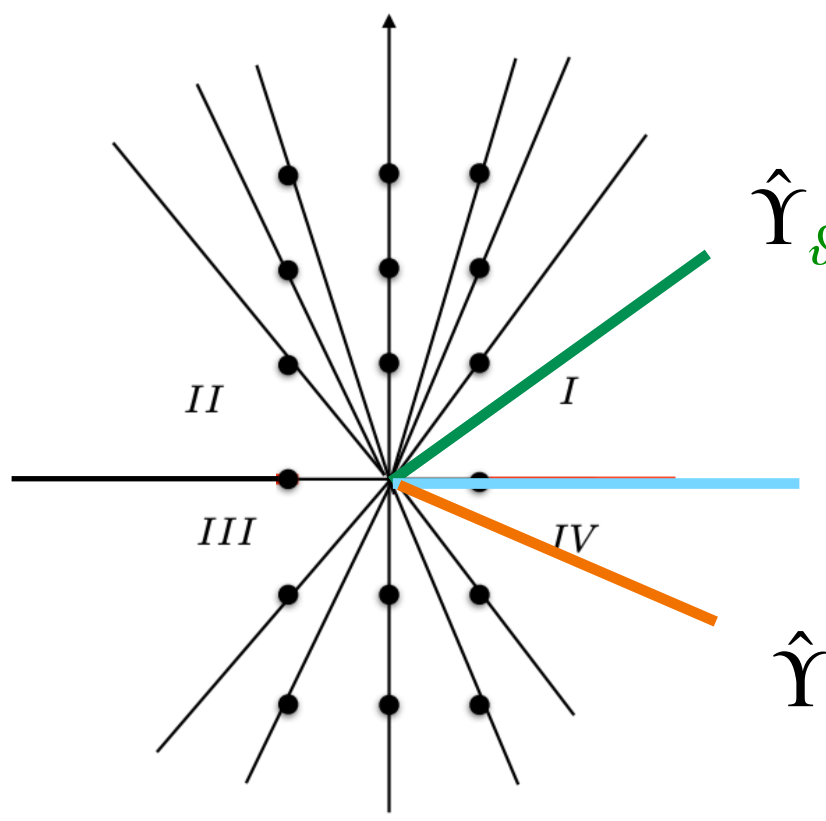
The Stokes phenomenon

Comparing the Borel sum in two different sectors separated by a Stokes line, we compute the Stokes constants

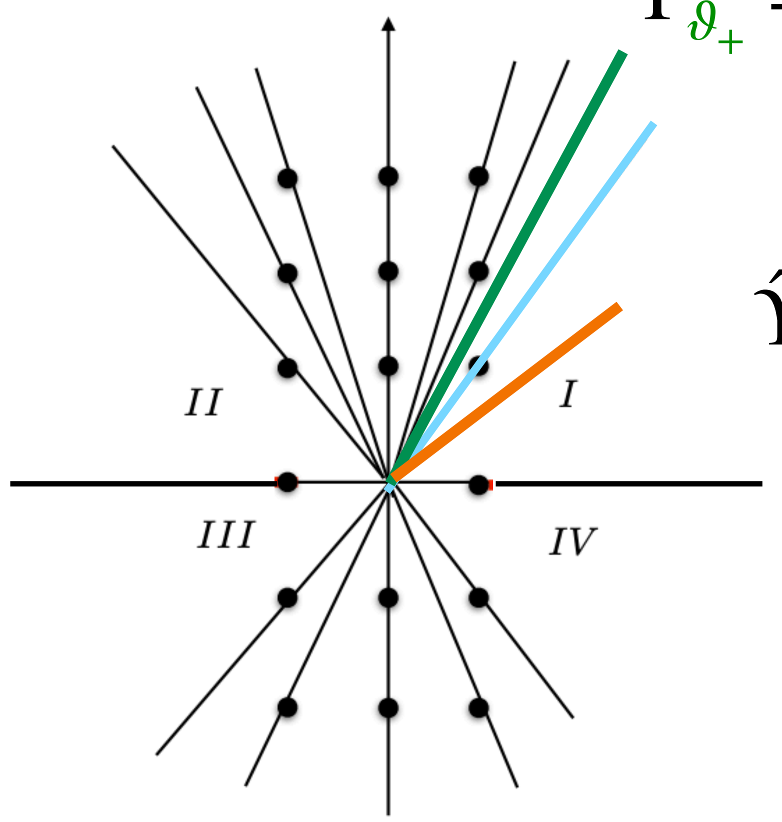
- The thimble integral from the other critical point $\hat{\Upsilon}_{p',\vartheta}$ is equal to $I_{0,0}$



$$\hat{\Upsilon}_{\vartheta_+} = -I_{0,0} - I_{-1,0}$$

$$\hat{\Upsilon}_{\vartheta_-} = I_{0,0} + I_{1,0}$$

$$\hat{\Upsilon}_{p,\vartheta_+} - \hat{\Upsilon}_{p,\vartheta_-} = \mathbf{3}\, I_{0,0}$$



$$\hat{\Upsilon}_{\vartheta_+} = -I_{0,0} - I_{-1,0} - 9qI_{0,0}$$

$$\hat{\Upsilon}_{\vartheta_-} = -I_{0,0} - I_{-1,0}$$

$$\hat{\Upsilon}_{p,\vartheta_+} - \hat{\Upsilon}_{p,\vartheta_-} = -\mathbf{9q}\, I_{0,0}$$

- C. Wheeler also checked the computation using Picard-Lefschetz formulas

Conclusions

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Perturbative topological invariants of the hyperbolic knots 4_1 and 5_2 are described by one-dimensional integrals $\tilde{\Upsilon}_K$

- Their Borel sum $\hat{\Upsilon}_{K,\vartheta}$ is an exponential integral along a **steepest descent contour** for the **multivalued function** V
- The steepest descent contours for V give classes in a **relative homology theory with coefficients**
- The **Stokes data** of the Faddeev's quantum dilogarithm define the **sheaf of coefficients**
- A **basis** for the homology is given by the Andersen-Kashaev state integrals $I_{0,0}$ and their descendants $I_{m,\ell}$

Our construction is somehow **"ad hoc" for Faddeev's dilogarithm**, but the same idea can be generalized to the study of other exponential integrals, which is part of our joint project with Andersen and Kontsevich

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Thank you for your attention