Algebraic Structures in Multi-Scale Scattering Amplitudes

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Based on joined work with:

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MathemAmplitudes, Mainz
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1 Introduction

2 Top-pair production with a W-boson

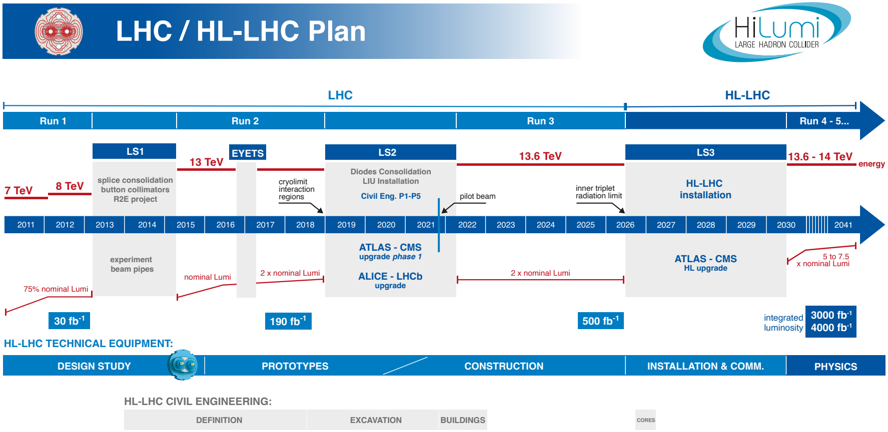
3 Example of algebraic complexity

4 Conclusions

Introduction

Precision era of High-Luminocity LHC

- Future 14 TeV HL-LHC runs $\Longrightarrow$ observables measured at $\mathcal{O}(1\%)$ by early 2040s



- Theory must keep pace! $\Longrightarrow$ NNLO (and more) QCD plus EW corrections

Next-to-next-to-leading order corrections

- Cross sections are basic collider observables

$$\sigma_{\mathsf{in}\to\mathsf{out}} = \int [\mathrm{dPS}]\ |\mathcal{M}_{\mathsf{in}\to\mathsf{out}}|^2$$

- Probability of process is measired by scattering amplitude

$$|\mathcal{M}_{\mathsf{in}\to\mathsf{out}}|^2 = \left|\mathcal{M}^{LO}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}\right|^2 + \dots$$

- Strong coupling constant $\alpha_s(m_W^2) = 0.12$ [Review of Particle Physics '24] [RunDec.m]

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gives order
of magnitude

- Strong coupling constant $\alpha_s(m_W^2) = 0.12$

[Review of Particle Physics '24] [RunDec.m]

Next-to-next-to-leading order corrections

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$\mathcal{O}(10\%)$
uncertainty

- Strong coupling constant $\alpha_s(m_W^2) = 0.12$

[Review of Particle Physics '24] [RunDec.m]

Next-to-next-to-leading order corrections

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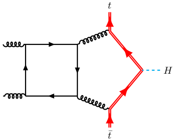
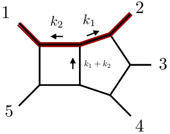
$\mathcal{O}(1\%)$
uncertainty

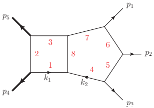
- Strong coupling constant $\alpha_s(m_W^2) = 0.12$ [Review of Particle Physics '24] [`RunDec.m`]
- Next-to-next-to leading order (NNLO) QCD corrections = 2-loop integrals

Current frontier of two-loop integral families

- Five particles with external and internal masses $\Longrightarrow$ 6-7 scales

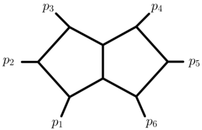


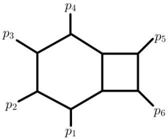


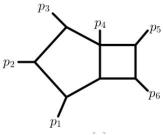


[Febres Cordero, Figueiredo Kraus, Page, Reina '23] [Badger, Becchetti Giraudo, Zoia '24] [Abreu, Chicherin Sotnikov, Zoia '24]

- Six external massless particles $\Longrightarrow$ 8 scales (in 4 dimensions)



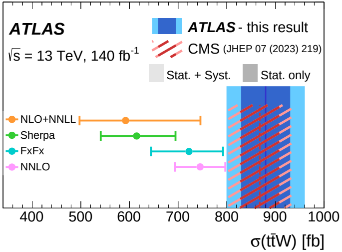


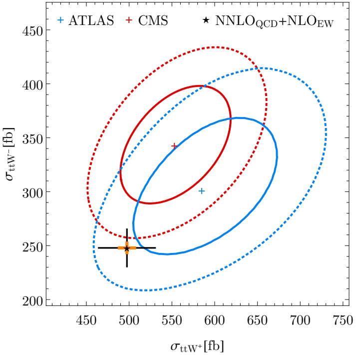


[Abreu, Monni Page, Usovitsch '24] [Henn, Matijašić, Miczajka Peraro, Xu, Zhang '25]

Tension in $t\bar{t}W$ production at NNLO

- Relevant background for Standard Model processes with top quark $t\bar{t}H$ and $t\bar{t}t\bar{t}$
- Multi-lepton signature important for Beyond the SM searches
- Recent measurements at LHC [CMS '23] [ATLAS '24]
- Current theoretical prediction: NLO QCD and EW corrections [Badger, Campbell Ellis '11] [Campbell, Ellis '12] [Maltoni, Pagani Tsinikos '15] [Frixione, Hirschi Pagani, Shao, Zaro '15] [Frederix, Pagani Zaro '17]
- NNLO predictions available in soft approximation in tension with experimental results [Buonocore, Devoto, Grazzini, Kallweit Mazzitelli, Rottoli, Savoini '23]



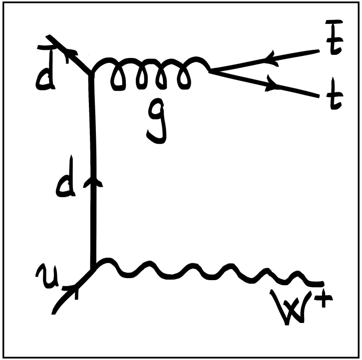


Top-pair production with a W-boson

5-point kinematics and its complexity

- $\bar{t}(p_1) + t(p_2) + \bar{d}(p_3) + W(p_4) + u(p_5) \to 0$
- 7 invariants $\vec{x} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_t^2, m_W^2\}$ with $s_{ij} = (p_i + p_j)^2$
- Dimensional regularization: $d = 4 - 2\varepsilon$
- Physical channel for $t\bar{t}W$ production $(35 \to 124)$

$$
\begin{aligned}
p_3 \cdot p_1 \,, \; p_3 \cdot p_2 \,, \; p_3 \cdot p_4 \,, \; p_5 \cdot p_1 \,, \; p_5 \cdot p_2 \,, \; p_5 \cdot p_4 &< 0 \\
p_3 \cdot p_5 \,, \; p_1 \cdot p_2 \,, \; p_1 \cdot p_4 \,, \; p_2 \cdot p_4 &> 0 \\
p_1^2 = p_2^2 = m_t^2 &> 0 \\
p_4^2 = m_W^2 &> 0 \\
p_3^2 = p_5^2 &= 0
\end{aligned}
$$



- Can we describe all the star-shaped strata? [Bernd's talk yesterday]

Example of integral family

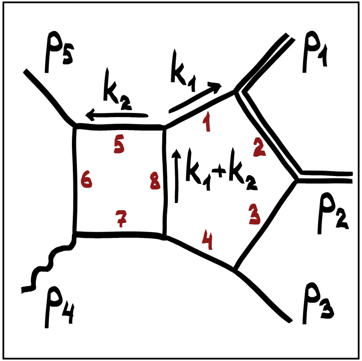
$$I_{a_1 \ldots a_{11}} = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \, \frac{D_9^{a_9} D_{10}^{a_{10}} D_{11}^{a_{11}}}{D_1^{a_1} \ldots D_8^{a_8}}$$

$$D_1 = k_1^2 \, , \; D_2 = (k_1 - p_1)^2 - m_t^2$$
$$D_3 = (k_1 - p_{12})^2 \, , \; D_4 = (k_1 - p_{123})^2$$
$$D_5 = k_2^2 \, , \; D_6 = (k_2 + p_{1234})^2$$
$$D_7 = (k_2 + p_{123})^2 \, , \; D_8 = (k_1 + k_2)^2$$

Denominators

$$D_9 = (k_1 - p_{1234})^2 \, , \; D_{10} = (k_2 + p_1)^2 - m_t^2$$
$$D_{11} = (k_2 + p_{12})^2$$

ISPs



Integration-by-parts and reduction to master integrals

- IBPs for Feynman integrals is an old story

[Tkachov '81] [Chetyrkin Tkachov '81]

$$0 = \int \mathrm{d}^d k_j \, \frac{\partial}{\partial k_j^\mu} \left(v^\mu \, \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} \right)$$

$$I_j = \sum_k \underbrace{c_{jk}}_{\text{rational coefficients}} \underbrace{G_k}_{\text{master integrals}}$$

- The Laporta algorithm: non-commutative analog of the Macaulay system

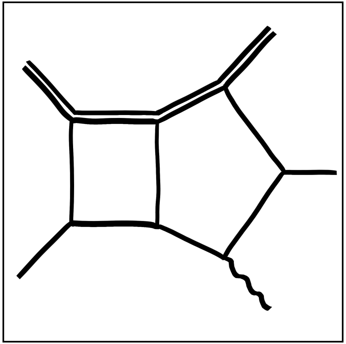
[Stefano Laporta 2000] [Francis Sowerby Macaulay 1902]

Some more modern takes on this motive:

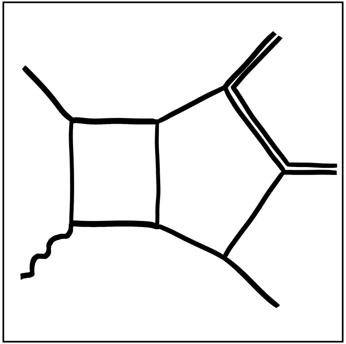
- Finite field arithmetics and rational function reconstruction [von Manteuffel Schabinger '14] [Peraro '16]
- FiniteFlow computational framework [Peraro '19]
- NeatIBP: syzygies, Gröbner bases [Wu, Boehm, Ma Xu, Zhang '23] [Singular] [Rourou's talk on Thursday]
- CALICO: parametric annihilators, syzygies from linear algebra

[Bertolini, Fontana Peraro '25] [Gaia's talk tomorrow]

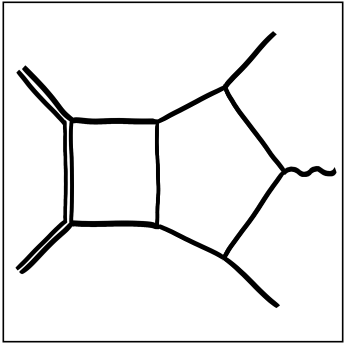
Integral families of $t\bar{t}W$



F1: 141 MIs



F2: 122 MIs



F3: 131 MIs

Method of differential equations

- Feynman integrals are holonomic functions. They satisfy integrable linear first-order PDEs (Pfaffian systems) [Regge '67] [Kashiwara Kawai '77] [Nicolas' talk today]
- In perturbative QFT, this is one of the main computational tools [Kotikov '91] [Bern, Dixon Kosower '94] [Remiddi '97] [Gehrmann Remiddi '00] ... [Henn '13] ...
- Idea: use IBPs to find connection 1-form Ω in the differential equations

$$\partial_x G_i(\vec{x},\varepsilon) = \sum_{\vec{a}} c_{i,\vec{a}}(\vec{x},\varepsilon)\, I_{\vec{a}}(\vec{x},\varepsilon)$$

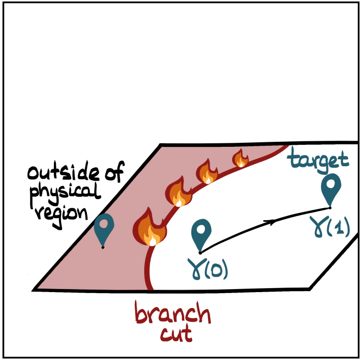
IBP reduction

$$\mathrm{d}\vec{G}(\vec{x},\varepsilon) = \Omega(\vec{x},\varepsilon)\cdot\vec{G}(\vec{x},\varepsilon)$$

- Our contribution: set of differential equations [Tiziano's talk on Thursday]
- **Mistery**: degree 14 polynomial appears among the denominators of Ω. Is it spurious?

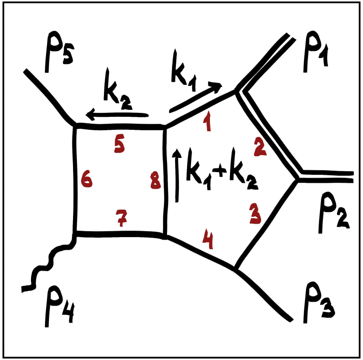
Integrating differential equation along lines

- Numerical method for solving systems of DEs
- Linear path $\gamma(t)$: boundary $\gamma(0)$ to target $\gamma(1)$
- Expand, solve, and glue together patches
- Boundary conditions from AMFLOW [Liu, Ma '22]
- Solvers: DIFFEXP, SEASYDE, LINE [Hidding '20] [Armadillo, Bonciani, Devoto Rana, Vicini '22] [Prisco, Ronca Tramontano '25]



Example of algebraic complexity

The 5-point $t\bar{t}W$ elliptic curve



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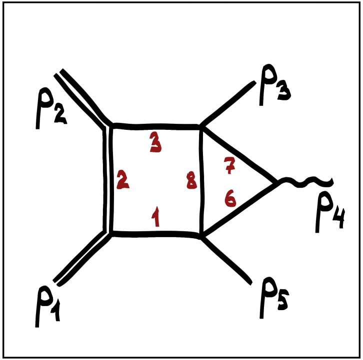
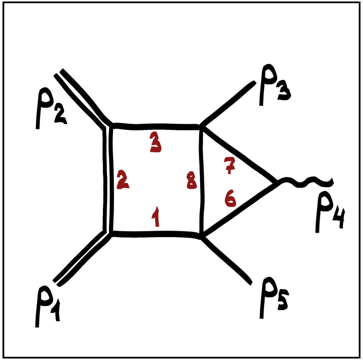


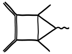
The 5-point $t\bar{t}W$ elliptic curve



- Elliptic curve for a 5-point kinematics with dependence on all 7 invariants
- 7 MIs, only the first two couple in $\varepsilon \to 0$

$$
\mathrm{d}\vec{G} = \begin{bmatrix}
* + \varepsilon * & * + *\varepsilon + *\varepsilon^2 & *\varepsilon + *\varepsilon^2 & *\varepsilon + *\varepsilon^2 & *\varepsilon + *\varepsilon^2 & *\varepsilon + *\varepsilon^2 & *\varepsilon + *\varepsilon^2 \\
* & * + *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon \\
* & * + *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon \\
* & * + *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon \\
* & * + *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon \\
* & * + *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon \\
0 & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon & *\varepsilon
\end{bmatrix} \cdot \vec{G}
$$

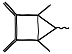
Leading singularity analysis: parametric representation

- We've been promised that  is elliptic

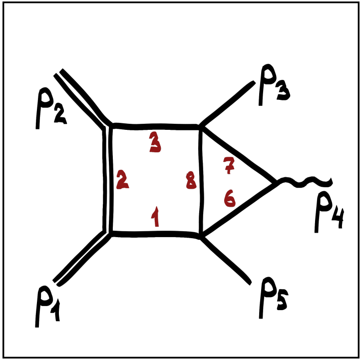
[Hjalte's talk yesterday]

- Loop-by-loop Baikov parametrization

[Baikov '96] [Frellesvig '17 Papadopoulos] [BaikovPackage] [Frellesvig '24]



$$= \int \frac{\mathrm{d}z_{123678}}{z_{123678}} \wedge \mathrm{d}z_4 \wedge \mathrm{d}z_9 \, \frac{G_2^{-1/2+\varepsilon}}{G_3^{\varepsilon}} \frac{G_4^{1/2+\varepsilon}}{G_5^{1+\varepsilon}}$$



- Gram determinants after taking maximal cut $z_{123678} \to 0$ are either quadratic

$$G_2 = G(p_4, k_1 + p_5) = -m_W^4 - (z_4 - z_9)^2 + 2m_W^2(z_4 + z_9)$$

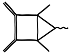
$$G_5 = G(k_1, p_1, p_2, p_3, p_4) = \ldots + \ldots z_4 + \ldots z_9 + \ldots z_4 z_9 + \ldots z_4^2 + \ldots z_9^2$$

or constant

$$G_4 = G(p_1, p_2, p_3, p_4) = -4m_t^4 m_W^2 s_{12} + \ldots \text{40 other terms} \ldots + s_{34}^2 s_{45}^2$$

Leading singularity analysis: towards the elliptic curve

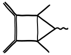
- To integrate the leading singularity in z_9

$$\mathrm{LS}\Big($$

$$\Big) = \sqrt{G_4} \int \mathrm{d}z_4 \wedge \mathrm{d}z_9 \, \frac{1}{\sqrt{G_2(z_4, z_9)}\, G_5(z_4, z_9)}$$

we find the roots of $G_5(z_4, z_9)$ using the discriminant

$$\mathrm{Disc}_{z_9}(G_5) = G_4 \;\cdot\; s_{12} \;\cdot\; (\ldots z_4^2 + \ldots z_4 + \ldots)$$

- Rationalizing the square root $\sqrt{\mathrm{Disc}_{z_9}(G_5)}$ we arrive

$$\mathrm{LS}\Big($$

$$\Big) = \int \frac{\mathrm{d}z}{\sqrt{\mathcal{Q}_4(z)}} \wedge \mathrm{d}\log(\alpha(y,z)) + \frac{\mathrm{d}z}{\sqrt{\mathcal{Q}_4(z)^\dagger}} \wedge \mathrm{d}\log(\alpha(y,z)^\dagger)$$

with $f^\dagger = f\big|_{\sqrt{G_4} \to -\sqrt{G_4}}$

- $\mathcal{Q}_4(z)$ is quartic in z and degree 14 in $\vec{x}$, has $\sqrt{G_4}$ and 2787 terms

The elliptic curve and its discriminant

- An implicit Möbius transformation

$$z \to \frac{a_1 + a_2 z}{a_3 + a_4 z}$$

brings both quartics $\mathcal{Q}_4(z)$ and $\mathcal{Q}_4(z)^\dagger$ to the same Weierstraß form

$$\mathcal{Q}^{\mathrm{W}}(z) = 4z^3 + g_1 z + g_0$$

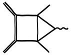
- No hope to find the root(s)!
- Compute the discriminant (or j-invariant) to find a degree-14 singularity $\mathcal{P}_{14}$

$$\mathcal{P}_{14} = -16 m_t^{12} m_W^{12} s_{12}^2 + 64 m_t^{14} m_W^8 + 32 m_t^{12} m_W^{10} s_{12}^3 + 8 m_t^{10} m_W^{12} s_{12}^3 - 16 m_t^{12} m_W^8 s_{12}^4 + \text{2542 terms}$$

with integer coefficients ranging from -576 to 744.

Polynomial Reduction and Singularities of Integrals

- Fubini reduction algorithm [Brown '08] [Brown '09] [Panzer '15] [Erik's lectures at Amplitudes'23 summer school]

- For  we find the $\mathcal{P}_{14}$ factor in

$$\mathrm{Disc}_{z_9}\Big(\mathrm{Res}_{z_4}(G_2, G_5)\Big) = \mathrm{const} \cdot m_W^4 \cdot s_{12}^2 \cdot G_4^2 \cdot (16 m_t^{10} m_W^6 s_{12}^3 + 1183 \text{ terms})^2 \cdot \mathcal{P}_{14}$$

- Implemented in HyperInt and SOFIA [Panzer '14] [Correia, Giroux, Mizera '25]

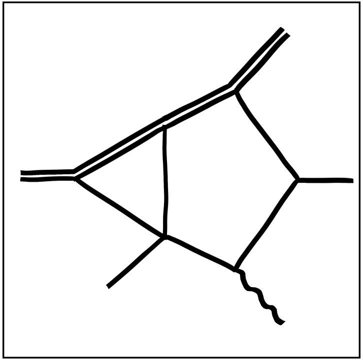
- In practice: taking resultants and discriminants. Can be done pleasantly with SPQR [SPQR] [Giulio's talk today]

Nested square root

- Leading singularities with nested square roots observed for $t\bar{t}H$ and $t\bar{t}j$
  [Cordero, Figueiredo Kraus, Page, Reina '23] [Badger, Becchetti Giraudo, Zoia '24] [Becchetti Dlapa, Zoia '25]
- Also the case for $t\bar{t}W$

$$\mathrm{NR}_\pm = \sqrt{q_1(\vec{x}) \pm q_2(\vec{x})\,\sqrt{G_4}}$$

- Currently not supported by differential equation solvers, so we do not use it



One approach to algebraic complexity

> The most effective way for solving polynomial equation systems is just to interpret such a system as a tool for solving itself . . .
> Therefore, the best way for solving is to return the equations. . . shouting sufficiently loudly that that is the solution. . .
> That is the Kronecker–Duval Philosophy.
> — R.F. Ree, The foundational crisis, a crisis of computability?

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Solving Polynomial Equation Systems I

The Kronecker–Duval Philosophy

TEO MORA

University of Genoa

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One approach to algebraic complexity

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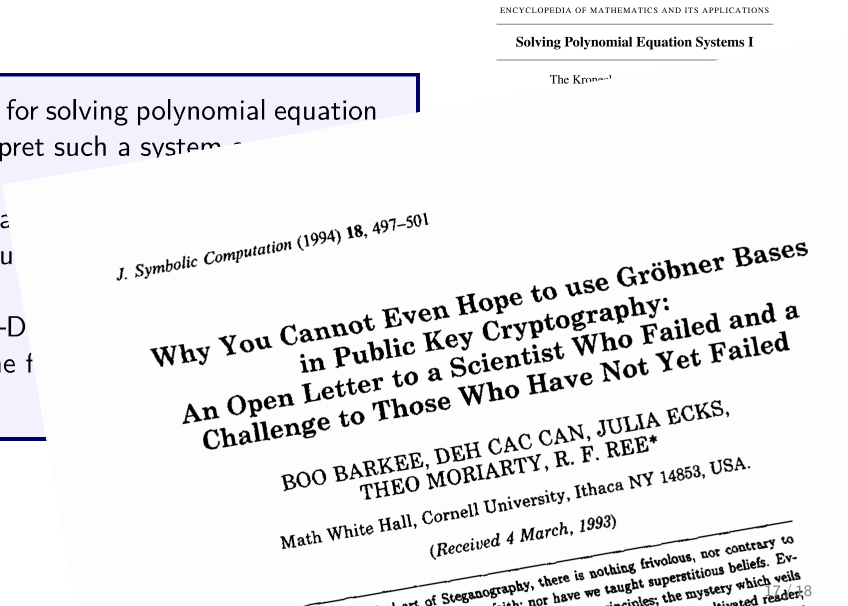
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Solving Polynomial Equation Systems I

The Kronec...

J. Symbolic Computation (1994) **18**, 497–501

Why You Cannot Even Hope to use Gröbner Bases in Public Key Cryptography: An Open Letter to a Scientist Who Failed and a Challenge to Those Who Have Not Yet Failed

BOO BARKEE, DEH CAC CAN, JULIA ECKS, THEO MORIARTY, R. F. REE*

Math White Hall, Cornell University, Ithaca NY 14853, USA.

(Received 4 March, 1993)

Conclusions

Conclusions

- Achieving percent-level precision in theoretical predictions is essential to match the accuracy of modern collider experiments
- Basis of master integrals and differential equations for integral families relevant for $t\bar{t}W$ production at 2-loop and leading color
- New level of algebraic complexity from 5-point kinematics and elliptic geometry interplay requires new computational ideas