Pfaffian Systems and Gröbner Bases

From ideals to Pfaffian systems

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w/ Görlach, Koefler, Sattelberger, Sayrafi, Schroeder, and Zaffalon

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Introduction

Feynman Integrals Are Holonomic Functions

Key object of interest:

$$I(m,p;\nu) = \int_{\{\alpha_i \geq 0\}} \frac{(\alpha_1)^{\nu_1}\dots(\alpha_m)^{\nu_m}}{(\mathcal{G}(\alpha;m,p))^{D/2}} \frac{d\alpha_1}{\alpha_1}\dots\frac{d\alpha_m}{\alpha_m}, \qquad \mathcal{G} \in \mathbb{C}[\alpha, m_i, p_{ij}]$$

Abstractly:

- Restriction of GKZ systems are holonomic.

Concretely:

- Systematic (/ brute force) way via IBP to find family $\mathbb{C}(x)\{I_1, ..., I_r\}$ (*master integrals*), s.th. closed under d.

(Pfaffian System:) $\qquad d(I, I_2, \dots, I_r)^t = A(x)(I, I_2, \dots, I_r)^t$

- Minimal number of master integrals determined by $\mathcal{G}(\alpha;m,p)$.
  - Decomposition of I into basis by *intersection theory*.

D-ideal approach

Can also directly study the differential equations for I:

$$P_i(x, \partial_x) \bullet I(x) = 0, \qquad i = 1, \ldots, N$$

- Encode properties of $I(x)$, i.e. symmetries (scaling)
- Direct study of $I(x)$ (e.g. banana family)
- D-module theory...

Implementation in *Macaulay2*:

Systematically constructing "connection matrices" from D-ideal with Gröbner bases, based on [Saito, Sturmfels, Takayama '00]:

$$\underbrace{\langle P_i(x, \partial_x) \mid i = 1, \ldots, N \rangle}_{\text{Collection of differential operators}} \xrightarrow{\texttt{connectionMatrices}} \underbrace{d\vec{I} = A(x)\vec{I}}_{\text{Pfaffian System}}$$

Weyl Algebras and Gröbner Bases

Weyl algebras and their ideals

Weyl Algebra of linear differential operators

$$\begin{array}{rl} \text{Weyl Algebra} & \mathrm{D}_n := \mathbb{C}[x_1, ..., x_n]\langle \partial_1, ..., \partial_n \rangle \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]) \\ \text{Rational Weyl Algebra} & \mathrm{R}_n := \mathbb{C}(x_1, ..., x_n)\langle \partial_1, ..., \partial_n \rangle \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}(\mathbf{x})) \end{array}$$

Left Ideals in D_n (resp. R_n)

A system of **linear** differential equations with **polynomial** / **rational** coefficients:

$$P_1 \bullet f = \ldots = P_r \bullet f = 0, \qquad P_i \in \mathrm{D}_n \text{ (resp. } \mathrm{R}_n\text{)}.$$

Left ideal has the same solutions

$$\forall\, P \in \mathrm{D}_n\langle P_1, ..., P_k \rangle, \quad P \bullet f = 0$$

$$\mathrm{D}_n\langle P_1, ..., P_k \rangle := \{Q_1 \cdot P_1 + ... + Q_k \cdot P_k \mid Q_i \in \mathrm{D}_n\}$$

D-modules to study properties of solutions to $I \bullet f = 0$

Study D/I to study properties of solutions to $I \bullet f = 0$

- Regularity, local basis expansions,...

Algebraization of solutions:

$$Sol(I) = \mathrm{Hom}_D(D/I, \mathcal{O})$$

$$I \bullet f = 0 \quad \longleftrightarrow \quad \varphi : [1] \mapsto f \quad D\text{-linear}$$

Theorem (Cauchy–Kovalevskaya–Kashiwara)

For a holonomic D-ideal I, and on a nhd away from the singular locus

$$\dim_{\mathbb{C}}(Sol(I)) = \dim_{\mathbb{C}(x)} \frac{R_n}{R_n I}.$$

Pfaffian systems for finite holonomic rank ideals

Assume $\dim_{\mathbb{C}(x_1,..,x_n)} R_n/R_n I < \infty$, and $s_1, \ldots, s_r \in R_n$ form a basis:

Definition (Pfaffian System)

For a D_n ideal I of *finite holonomic rank* r, the **Pfaffian system** of I associated to the basis $s_1, ..., s_r$ is

$$(\textbf{Diff. Eqs.}): \qquad \partial_i \bullet \begin{pmatrix} s_1 \bullet y \\ \vdots \\ s_r \bullet y \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11}^{(i)} & \ldots & a_{1r}^{(i)} \\ \vdots & \ddots & \vdots \\ a_{r1}^{(i)} & \ldots & a_{rr}^{(i)} \end{pmatrix}}_{A_i \in \mathbf{Mat}(\mathbb{C}(x_1,\ldots,x_n))} \begin{pmatrix} s_1 \bullet y \\ \vdots \\ s_r \bullet y \end{pmatrix} \qquad i = 1, \ldots, n \tag{1}$$

such that the **connection matrices** A_i satisfy

$$(\mathrm{R}_n \textbf{ module}): \qquad \partial_i \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} = \begin{pmatrix} a_{11}^{(i)} & \ldots & a_{1r}^{(i)} \\ \vdots & \ddots & \vdots \\ a_{r1}^{(i)} & \ldots & a_{rr}^{(i)} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} \quad \textbf{mod} \quad \mathrm{R}_n\mathbf{I} \tag{2}$$

Pfaffian systems for finite holonomic rank ideals (in short)

Assume $\dim_{\mathbb{C}(x_1,..,x_n)} R_n/R_nI < \infty$, and $s_1, \ldots, s_r \in R_n$ form a basis:

Definition (Pfaffian System)

For a D_n ideal I of *finite holonomic rank* r, the **Pfaffian system** of I associated to the basis $s_1, ..., s_r$ is

$$(\textbf{Diff. Eqs.}): \qquad d\vec{y} = A\vec{y}, \qquad A = \sum A_i \cdot dx_i \tag{3}$$

such that the **connection matrices** A_i satisfy

$$(\mathbf{R}_n \textbf{ module}): \qquad \partial_i \cdot \vec{s} = A_i\vec{s} \quad \textbf{mod} \quad \mathbf{R}_n\mathbf{I} \tag{4}$$

Question: Are there examples for master integrals where we cannot write $\vec{I} = (P_1 \bullet I, ..., P_r \bullet I)$?

Pfaffian Systems via Gröbner Bases

Excourse Gröbner Bases

Ubiquitous in (computational) algebraic geometry, study $\mathbb{C}[x_1, ..., x_n]/\langle f_1, ..., f_k\rangle$

- Images of algebraic morphisms $\varphi(V(I))$ (elimination)
- Complements of varieties $V(I)\backslash V(J)$ (ideal quotient)
- ...

Definition

For $\prec$ a monomial (term) order on $\{x_1^{\alpha_1} \dots x_n^{\alpha_n}\}$

- **Initial Ideal**: $\mathrm{in}_\prec(I) := \langle \mathrm{lt}_\prec(f) \mid f \in I\rangle$, example...
- **Gröbner Basis**: A collection $g_1, \dots, g_N$, such that

$$I = \langle g_1, \dots, g_N\rangle \qquad \text{and} \qquad \mathrm{in}_\prec(I) = \langle \mathrm{lt}_\prec(g_1), \dots, \mathrm{lt}_\prec(g_N)\rangle.$$

Immediate: Standard monomials $s_i \in \{x_1^{\alpha_1} \dots x_n^{\alpha_n}\}\backslash \mathrm{in}_\prec(I)$ provide a $\mathbb{C}$-basis of

$$\mathbb{C}[\mathbf{x}]/\mathbb{C}[\mathbf{x}]I := \frac{\mathbb{C}[x_1, \dots, x_n]}{\langle f_1, ..., f_n\rangle}.$$

Gröbner Bases for D_n and R_n

Gröbner bases exist in some non-commutative settings, too:

D_n		R_n
$\left\{x_1^{\alpha_1}\dots x_n^{\alpha_n}\partial_1^{\beta_1}\dots\partial_n^{\beta_n}\right\}$		$\{\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}\}$
$\mathbb{C}$-coefficients		$\mathbb{C}(x_1,\dots,x_n)$-coefficients
$\{g_1,\dots,g_k\}$ is GB w.r.t. an *elimination order* $\prec$	$\Rightarrow$	$\{g_1,\dots,g_k\}$ is GB w.r.t. $\prec\mid_{\partial^\alpha}$

Immediate: Standard monomials $s_i \in \{\partial_1^{\alpha_1}\dots x_n^{\alpha_n}\}\setminus \mathrm{in}_\prec(R_n I)$ provide a $\mathbb{C}(x_1,...,x_n)$-basis of

$$\mathrm{R}_n/\mathrm{R}_n I := \frac{\mathbb{C}(x_1,\dots,x_n)\langle\partial_1,...,\partial_n\rangle}{\langle P_1,...,P_k\rangle}.$$

Normal Form via Gröbner Basis

Goal: Choose a 'good' representative for $[P] \in \mathrm{R}_n/\mathrm{R}_n I$.
$\Rightarrow$ e.g. of $[\partial s_j]$ in terms of $[s_1], \ldots, [s_r]$.

Lemma

Fix $R_n I$ and $\prec$. For all $[P] \in \boldsymbol{R}_n/I$, there exists a unique normal form $R \in \boldsymbol{R}_n$, such that

$$P = R \mod \boldsymbol{R}_n I, \quad \textit{and} \quad \mathrm{monomials}(R) \cap \mathrm{in}_\prec(\boldsymbol{R}_n I) = \emptyset.$$

- Normal form algorithm for D_n described in [Saito, Sturmfels, Takayama; '00]. Extends to R_n.

Consequence: $\mathrm{normalForm}_{I,\prec}(P)$ decomposes as a $\mathbb{C}(\mathrm{x})$-linear combination of the **standard monomials**.

Connection matrices for standard monomials

Compute connection matrices via normal form:

$$
\begin{aligned}
\mathrm{normalForm}_I(\partial_i s_1) &= \mathsf{a}_{\mathsf{11}}(x)s_1 + \ldots + \mathsf{a}_{\mathsf{1r}}(x)s_r \\
\mathrm{normalForm}_I(\partial_i s_2) &= \mathsf{a}_{\mathsf{21}}(x)s_1 + \ldots + \mathsf{a}_{\mathsf{2r}}(x)s_r \\
&\vdots \\
\mathrm{normalForm}_I(\partial_i s_r) &= \mathsf{a}_{\mathsf{r1}}(x)s_1 + \ldots + \mathsf{a}_{\mathsf{rr}}(x)s_r
\end{aligned}
$$

$$
\Rightarrow A_i = (\mathsf{a}_{\mathsf{kl}}(x))_{kl=1,\ldots,r}
$$

Computation for other bases via gauge transform.

ConnectionMatrices in Macaulay2

Grayson, Daniel R. and Stillman, Michael E.

Macaulay2, a software system for research in algebraic geometry.

Available at `http://www2.macaulay2.com`

Görlach, Koefler, Sattelberger, Sayrafi, Schroeder, W, and Zaffalon.

Connection Matrices in Macaulay2.

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Some of the **implemented methods** for $R_n I$:

```
normalForm    standardMonomials    connectionMatrices
gaugeMatrix   gaugeTransform       ...
```

Going backwards

Going back?

$$\partial_i \bullet \begin{pmatrix} s_1 \bullet y \\ \vdots \\ s_r \bullet y \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11}^{(i)} & \dots & a_{1r}^{(i)} \\ \vdots & \ddots & \vdots \\ a_{r1}^{(i)} & \dots & a_{rr}^{(i)} \end{pmatrix}}_{A_i \in \mathbf{Mat}(\mathbb{C}(x_1,\dots,x_n))} \begin{pmatrix} s_1 \bullet y \\ \vdots \\ s_r \bullet y \end{pmatrix}, \qquad \forall i$$

Each row, is a differential eq./operator annihilating y:

$$\left(\partial_i s_j - (a_{j1}^{(i)} s_1 + \cdots + a_{jr}^{(i)} s_j)\right) \bullet y = 0$$

How much of $D_n I$ (rather $R_n I$) can we recover? Ongoing work by Sattelberger and Rodriguez

Thank you for listening!

Questions?