The calculation of the 4-loop QED contribution to the electron g-2 and the perspectives for the 5-loop calculation

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*M*athemAmplitudes 2025

Mainz

22-26 Sep 2025

In Quantum Electrodynamics the *anomalous magnetic moment*, expressed in Bohr magnetons can be written as a power series in the small quantity$\left(\frac{\alpha}{\pi}\right)$ ($\alpha \approx 1/137$).

$$F_2(0) = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$

Another similar quantity, the so-called *slope* of the Dirac form factor, important for bound states calculation, can be expanded in power series

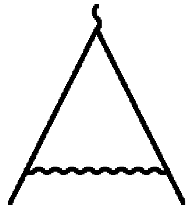
$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$

the coefficients C_i and A_i are pure numbers and can be extracted from the Feynman diagrams of the theory as linear combination of (a large number of) Feynman integrals.

These combinations are to be reduced to a linear combination of (irreducible) master integrals by solving of large systems of IBP identities, or, hopefully in near future, avoiding this step through intersection theory.

For a given n, both C_n and A_n are espressible in term of the same master integrals, and as a consequence have analytical expressions with similar structure.

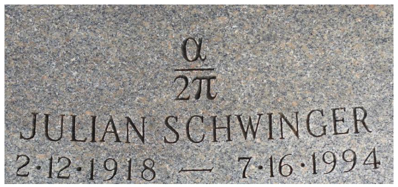
$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \dots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \dots$$



1 diagram $\to$ 1 master integral

$$C_1 = \frac{1}{2}$$

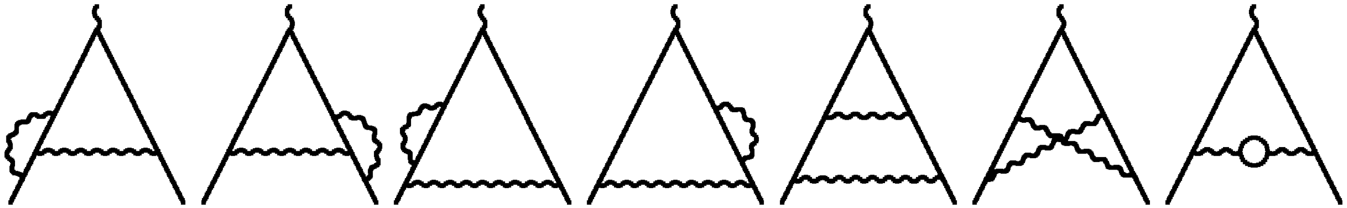
Obtained by Julian Schwinger in 1948



$$A_1 = -\tfrac{1}{8} - \tfrac{1}{3}\ln\tfrac{\Delta E}{m} + \tfrac{5}{18} \qquad \text{(Bethe 1947)}$$

Contributions at two loops

$$
\begin{aligned}
(g-2)/2 =& C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \dots \\
F_1'(0) =& A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \dots
\end{aligned}
$$



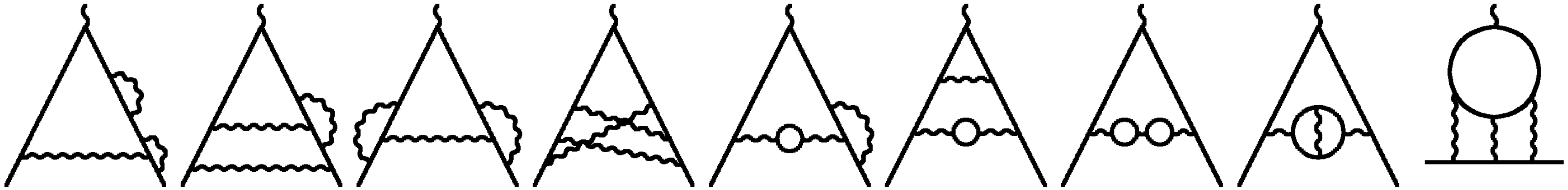
7 diagrams $\rightarrow$ 3 master integrals

$$
\begin{aligned}
C_2 =& \quad \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) = -0.328\,478\,965\dots \\
A_2 =& -\frac{4819}{5184} - \frac{49}{432}\pi^2 + \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta(3) = \ 0.469\,941\,487\dots
\end{aligned}
$$

- C_2 was computed by Petermann and Sommerfeld in 1957.
- A_2 was computed by R. Barbieri, J. A. Mignaco and E. Remiddi in 1972.

Contributions at three loops

$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \ldots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \ldots$$



+63

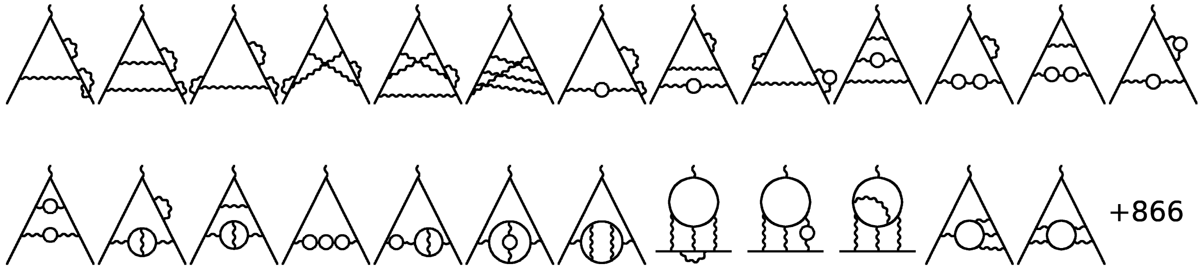
72 diagrams → 17 master integrals

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2\ln^2 2}{24}\right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} = 1.181\ 24$$

$$A_3 = -\frac{17}{24}\pi^2\zeta(3) + \frac{25}{8}\zeta(5) - \frac{217}{9}\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{103}{1080}\pi^2\ln^2 2 + \frac{3899}{25920}\pi^4 - \frac{2929}{288}\zeta(3) + \frac{41671}{2160}\pi^2\ln 2 - \frac{454979}{38880}\pi^2 - \frac{77513}{186624} = 0.171\ 720\ 018\ldots$$

- C_3 was obtained by **S.L.** and Ettore Remiddi in 1996.
- A_3 was obtained by Melnikov and Ritbergen in 1999.

$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \dots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \dots$$



891 diagrams → 334 master integrals

$C_4 =$ `-1.912245764926445574152647167439830054060873390658725345171329848006038443980651706142 7...`

$A_4 =$ `+0.886545673946443145836821730610315359390424032660064745368055909320840316465628927454 8...`

- (**S.L** 2017, **S.L** 2019)
- Numerical values calculated with 1100 digits of precision
- Semi-analytical expressions fitted with the PSLQ algorithm with very high reliability. Components of analytical expressions known with at least 4800 digits.
- Jump in complexity: analytical fits contain ∼120 terms

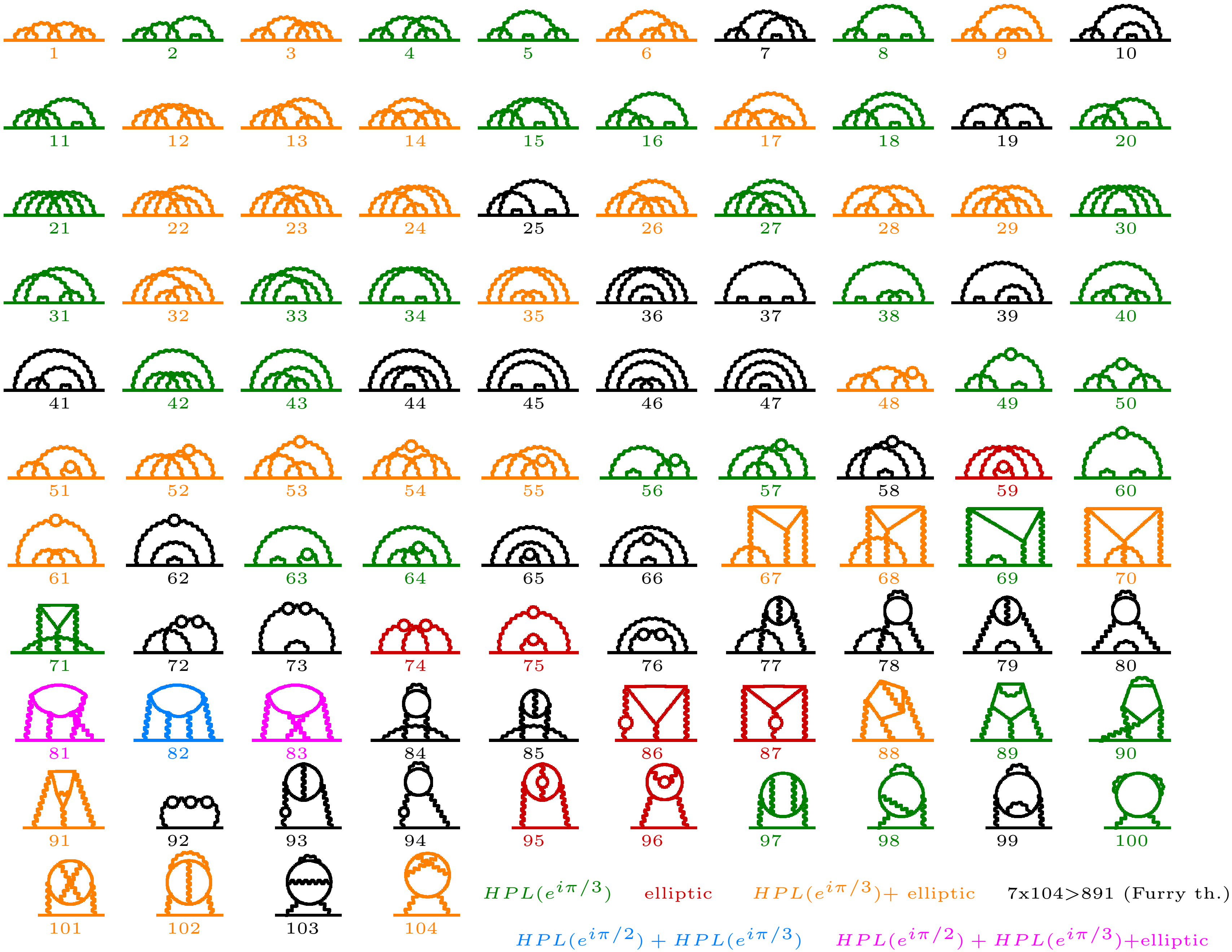
$$
\begin{aligned}
C_4 =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
& - \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 \\
& + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) \\
& + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 + \frac{407771}{432}\zeta^2(3)\ln 2 \\
& - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2 \\
& + \sqrt{3}\Big[-\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + 19\,\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi - \frac{223}{243}\zeta(4)\pi\ln 2 \\
& + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 - \frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2\Big] + \frac{13487}{60}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) \\
& + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\,\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi + \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
& + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
& + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\left(4\,\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\,\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) \\
& + \sqrt{3}\Big[\pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) + \pi\left(\frac{4715}{1944}\ln 2\, f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
& + \pi\Big(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\, f_2(0,0,2) + \frac{5525}{162}\ln 2\, f_2(0,1,1) - \frac{5525}{243}\ln 2\, f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) \\
& - \frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) + \frac{69245}{124416}f_2(2,1,0)\Big)\Big] - \frac{4715}{1458}\zeta(2)f_1(0,0,1) + \zeta(2)\Big(\frac{2541575}{82944}f_1(0,0,2) \\
& - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\Big) - \frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}.
\end{aligned}
$$

(polylogarithms) (harmonic polylogarithms) (elliptic) (unknown elliptic)

$$
\begin{aligned}
A_4 = & -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) + \frac{4572662443}{12247200}\ln^2 2\zeta(2) - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5) \\
& + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 - \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 + \frac{18215}{27}a_5\ln 2 \\
& - \frac{18215}{27}\zeta(5)\ln 2 + \frac{402152509}{189000}a_4\zeta(2) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 - \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) \\
& - \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) - \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
& - \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 + \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 \\
& + \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2 + \sqrt{3}\Bigg[-\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi \\
& + \frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 - \frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 + \frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 \\
& + \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2\Bigg] + \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) - \frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& - 507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& - \frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) \\
& - 118\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) + \sqrt{3}\Bigg[\pi\left(+\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) \\
& - \frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\left(\frac{751}{972}\ln 2\, f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0) + \frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\right) - \frac{751}{729}\zeta(2)f_1(0,0,1) \\
& + \pi\Bigg(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1) + \frac{1105}{108}\ln 2\, f_2(0,0,2) - \frac{2210}{81}\ln 2\, f_2(0,1,1) + \frac{4420}{243}\ln 2\, f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2) - \frac{4011005}{497664}f_2(0,2,1) \\
& + \frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0) - \frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\Bigg)\Bigg] + \zeta(2)\Bigg(-\frac{4629335}{165888}f_1(0,0,2) \\
& + \frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\Bigg) + \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}
\end{aligned}
$$

(polylogarithms) (harmonic polylogarithms) (elliptic) (unknown elliptic)

A coloured view of the 104 self-mass diagrams



The slope at 4 loops: analytical fit part 1

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+ \frac{4572662443}{12247200}\zeta(2)\ln^2 2 - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&- \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 \\
&+ \frac{402152509}{189000}a_4\zeta(2) + \frac{18215}{27}a_5\ln 2 - \frac{18215}{27}\zeta(5)\ln 2 + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 \\
&- \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) \\
&- \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
&- \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 \\
&+ \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 + \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2
\end{aligned}$$

$$\zeta(n) = \sum_{i=1}^{\infty} i^{-n} \quad a_n = \mathrm{Li}_n(1/2) \ b_6 = H_{0,0,0,0,1,1}(1/2) \ b_7 = H_{0,0,0,0,0,1,1}(1/2) \ d_7 = H_{0,0,0,0,1,-1,-1}(1)$$

$H_{i_1,i_2,\dots}(x)$ harmonic polylogarithms $\quad H_{a_1,a_2,\dots a_n}(x) = \int^{x} \frac{dy_1}{y_1 - a_1} \dots \int^{y_{n-2}} \frac{dy_{n-1}}{y_{n-1} - a_{n-1}} \int^{y_{n-1}} \frac{dy_n}{y_n - a_n} \qquad a_i \in \{-1, 0, 1\}$

The slope at 4 loops: analytical fit part 2

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_a =& -\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi + \frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+ \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+ \frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 \\
&- \frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
&+ \frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

$\mathrm{Cl}_n(\theta) = \mathrm{Im}\ \mathrm{Li}_n(e^{i\theta})$ $\quad H_{i_1,i_2,\ldots}(x)$ harmonic polylogarithms

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_b =& \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) - \frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&-507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+\frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&-\frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&-\frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
&-\frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
W =& -\frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) - 118\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\
&-2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg),
\end{aligned}$$

The slope at 4 loops: analytical fit part 5

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$E_a = \pi\left(\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) - \frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\left(\frac{751}{972}\ln 2\ f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0)\right.$$
$$\left. + \frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\right) - \frac{751}{729}\zeta(2)f_1(0,0,1) + \pi\left(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1)\right.$$
$$+ \frac{1105}{108}\ln 2\ f_2(0,0,2) - \frac{2210}{81}\ln 2\ f_2(0,1,1) + \frac{4420}{243}\ln 2\ f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2)$$
$$- \frac{4011005}{497664}f_2(0,2,1) + \frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0)$$
$$\left. - \frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\right)$$

$$E_b = \zeta(2)\left(-\frac{4629335}{165888}f_1(0,0,2) + \frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\right)$$

$$f_1(i,j,k) = \int_1^9 ds\ D_1^2\left[s - \frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s) \qquad f_2(i,j,k) = \int_1^9 ds\ D_1(s)\sqrt{3}\mathrm{Re}D_m(s)\left[s - \frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s)$$

$$D_1(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \qquad D_2(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right)$$

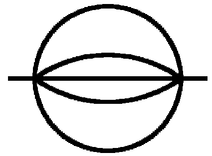
$K(x)$ complete elliptic integral of the first kind

The slope at 4 loops: analytical fit part 6

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$U = \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}$$

C_{xy} known only numerically; they contain two elliptic kernels at weight six.

The basic elliptic constants come from the sunrise diagram $S_4 =$  with one line differentiated and/or cutted. We can define 4 constants with a relatively simple integral representation: A_3, B_3, C_3, D_3.

$$A_3 = \int_0^1 dx \frac{K_c(x) K_c(1\text{-}x)}{\sqrt{1-x}} \qquad B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} \qquad K_c(x) = {}_2F_1\left({\tfrac{1}{3}\ \tfrac{2}{3} \atop 1} ; x \right)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} \qquad D_3 = \int_0^1 dx \frac{E_c(x) E_c(1-x)}{\sqrt{1-x}} \qquad E_c(x) = {}_2F_1\left({\tfrac{1}{3}\ -\tfrac{1}{3} \atop 1} ; x \right)$$

$$S_4(D{=}4\text{-}2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + \frac{\pi\sqrt{3}}{240}\left(297 B_3 - 1477 C_3\right) - \frac{2320981}{20736} + O(\epsilon)$$

$$S_4(D{=}2) = \sqrt{3}\pi B_3$$

A_3 disappears in the final results. D_3 appears only cutting a line.

This constants have a hypergeometric expression (**S.L.** 2017, Y.Zhou 2018)

$$A_3 = \frac{\pi}{54}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2}\\ \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}\end{matrix};1\right) + {}_4\tilde{F}_3\left(\begin{matrix}\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2}\\ \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}\end{matrix};1\right)\right] \quad \text{sum}$$

$$B_3 = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2}\\ \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}\end{matrix};1\right) - {}_4\tilde{F}_3\left(\begin{matrix}\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2}\\ \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}\end{matrix};1\right)\right] \quad \text{difference}$$

$$C_3 = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2}\\ -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}\end{matrix};1\right) - {}_4\tilde{F}_3\left(\begin{matrix}-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2}\\ -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}\end{matrix};1\right)\right] \quad \text{difference}$$

$$D_3 = \frac{-\pi}{180}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2}\\ -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}\end{matrix};1\right) + {}_4\tilde{F}_3\left(\begin{matrix}-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2}\\ -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}\end{matrix};1\right)\right] + \frac{3}{10}A_3 + \frac{2}{9}\pi\sqrt{3} \quad \text{sum}$$

The hypergeometric regularized ${}_4\tilde{F}_3$ is

$${}_4\tilde{F}_3\left(\begin{matrix}a_1\ a_2\ a_3\ a_4\\ b_1\ b_2\ b_3\end{matrix};x\right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\,{}_4F_3\left(\begin{matrix}a_1\ a_2\ a_3\ a_4\\ b_1\ b_2\ b_3\end{matrix};x\right)$$

A_3 and D_3 do not appear in 4-loop QED g-2 and slope integrals.

(Bailey, Borwein, Broadhurst, Glasser 2008)

$$M(a,b,c) = \int_0^\infty I_0^a(x) K_0^b(x) x^c \, dx$$

$I_0(x)$, $K_0(x)$: modified Bessel functions.

4-loop sunrise in $D = 2$ dimensions

$$S_4(1) = 2^4 M(1,5,1) = 2^4 \int_0^\infty I_0(x) K_0^5(x) \, dx$$

(Broadhurst, Mellit 2016) (Zhou 2018)

$$\mathrm{Det} \begin{bmatrix} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{bmatrix} = \frac{\pi^4}{24^2}$$

$$7\ {}_4F_3\left({\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}};1\right){}_4F_3\left({-\frac{1}{2}\ \frac{1}{3}\ \frac{4}{3}\ \frac{1}{6} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right)+10\ {}_4F_3\left({\frac{1}{2}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{6} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}};1\right){}_4F_3\left({-\frac{1}{2}\ -\frac{1}{3}\ \frac{5}{6}\ -\frac{4}{3} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right)=40$$

introducing the regularized ${}_4\tilde{F}_3$

$${}_4\tilde{F}_3\left({\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ \frac{1}{3}\ \frac{4}{3}\ \frac{1}{6} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right)-{}_4\tilde{F}_3\left({\frac{1}{2}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{6} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ -\frac{1}{3}\ \frac{5}{6}\ -\frac{4}{3} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right)=\frac{240\sqrt{3}\pi}{7}$$

Conjecture: fractions appearing in the indices are actually $\frac{1}{2}\pm k\delta$ or $1\pm k\delta$ with small integer k and $\delta=\frac{1}{6}$, that is, a deformation of the "simplest" hypergeometric

$${}_4F_3\left({\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} \atop 1\ 1\ 1};1\right)=\frac{4}{\pi^4}M(0,4,0) \tag{1}$$

i.e.

$$\begin{aligned}\frac{1}{3}&\to\frac{1}{2}-\delta\\ \frac{2}{3}&\to\frac{1}{2}+\delta\\ \frac{5}{6}&\to\frac{1}{2}+2\delta\\ -\frac{4}{3}&\to-\frac{3}{2}+\delta\ \text{or}\ -1-2\delta\ldots\end{aligned}$$

Experimenting a bit we found

$$
{}_4\tilde{F}_3\left({\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ \frac{1}{3}\ \frac{4}{3}\ \frac{1}{6} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right)
- {}_4\tilde{F}_3\left({\frac{1}{2}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{6} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ -\frac{1}{3}\ \frac{5}{6}\ -\frac{4}{3} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right) = \frac{240\sqrt{3}\pi}{7}
$$

corresponds to the $\delta = 1/6$ case of the general conjecture

$$
{}_4\tilde{F}_3\left({\frac{1}{2}\ \ \frac{1}{2}+\delta\ \ \frac{1}{2}+\delta\ \ \frac{1}{2}+2\delta \atop 1+\delta\ \ 1+\delta\ \ 1+2\delta};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ \ \frac{1}{2}-\delta\ \ \frac{3}{2}-\delta\ \ \frac{1}{2}-2\delta \atop -\delta\ \ 1-\delta\ \ 2-2\delta};1\right)
- {}_4\tilde{F}_3\left({\frac{1}{2}\ \ \frac{1}{2}-\delta\ \ \frac{1}{2}-\delta\ \ \frac{1}{2}-2\delta \atop 1-\delta\ \ 1-\delta\ \ 1-2\delta};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ \ -\frac{1}{2}+\delta\ \ \frac{1}{2}+\delta\ \ -\frac{3}{2}+2\delta \atop -\delta\ \ 1-\delta\ \ 2-2\delta};1\right) =
$$

$$
= \frac{4\Gamma(1/2-2\delta)\Gamma(1/2-\delta)\Gamma(-(1/2)+\delta)\Gamma(-(3/2)+2\delta))}{\Gamma(1-\delta)\Gamma(-1+\delta)}
$$

$$
= 128\pi\frac{\delta-1}{(4\delta-1)(2\delta-3)(2\delta-1)}\frac{\sin(\pi\ \delta)^2}{\sin(4\pi\ \delta)^2}
$$

numerically checked

$$
\begin{aligned}
&{}_4\tilde{F}_3\left(\begin{smallmatrix} \frac{1}{2} & \frac{1}{2}+d & \frac{1}{2}+d & \frac{1}{2}+2d \\ & 1+d & 1+d & 1+2d \end{smallmatrix};1\right){}_4\tilde{F}_3\left(\begin{smallmatrix} -\frac{1}{2} & \frac{1}{2}-d & \frac{3}{2}-d & \frac{1}{2}-2d \\ & -d & 1-d & 2-2d \end{smallmatrix};1\right) \\
&\qquad - {}_4\tilde{F}_3\left(\begin{smallmatrix} \frac{1}{2} & \frac{1}{2}-d & \frac{1}{2}-d & \frac{1}{2}-2d \\ & 1-d & 1-d & 1-2d \end{smallmatrix};1\right){}_4\tilde{F}_3\left(\begin{smallmatrix} -\frac{1}{2} & -\frac{1}{2}+d & \frac{1}{2}+d & -\frac{3}{2}+2d \\ & -d & 1-d & 2-2d \end{smallmatrix};1\right) = \\
&\qquad = \frac{4\Gamma(1/2-2d)\Gamma(1/2-d)\Gamma(-(1/2)+d)\Gamma(-(3/2)+2d))}{\Gamma(1-d)\Gamma(-1+d)} \\
&\qquad = 128\pi \frac{d-1}{(4d-1)(2d-3)(2d-1)} \frac{\sin(\pi\, d)^2}{\sin(4\pi\, d)^2}
\end{aligned}
$$

This conjecture can be further generalized introducing at least another free parameter. It can be rigorously proved likely by using intersections theory along the same lines of similar identities for the simpler hypergeometric functions ${}_2F_1$ and ${}_3F_2$.

$$+{}_2F_1\left({\tfrac{1}{2}+\lambda \;\; -\tfrac{1}{2}-\nu \atop 1+\lambda+\mu};r\right){}_2F_1\left({\tfrac{1}{2}-\lambda \;\; \tfrac{1}{2}+\nu \atop 1+\nu+\mu};1-r\right)+{}_2F_1\left({\tfrac{1}{2}+\lambda \;\; \tfrac{1}{2}-\nu \atop 1+\lambda+\mu};r\right){}_2F_1\left({-\tfrac{1}{2}-\lambda \;\; \tfrac{1}{2}+\nu \atop 1+\nu+\mu};1-r\right)$$

$$-{}_2F_1\left({\tfrac{1}{2}+\lambda \;\; \tfrac{1}{2}-\nu \atop 1+\lambda+\mu};r\right){}_2F_1\left({\tfrac{1}{2}-\lambda \;\; \tfrac{1}{2}+\nu \atop 1+\nu+\mu};1-r\right)=\frac{\Gamma\left(1+\lambda+\mu\right)\Gamma\left(1+\nu+\mu\right)}{\Gamma\left(\frac{3}{2}+\lambda+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\mu\right)}$$

for $\lambda=\mu=\nu=0 \to$ Legendre's identity (easily obtainable with intersections)

Key points:

- i.b.p. identities for reduction to M.I.
- i.b.p. identities for determining the systems of difference/differential equations satisfied by the M.I.
- high-precision solution of the systems of difference/differential equations
- high-precision analytical fit with PSLQ

- used the program `SYS`
- systems of i.b.p. identities generated for reduction to master integrals
- exact arithmetic with polynomials in 1 variable
- systems of $\sim 10^6$ i.b.p. at 4 loop

- system of i.b.p. identities solved for finding system of difference/differential equations satisfied by the M.I.
- exact arithmetic with polynomials in 2 variables
- slower arithmetic, it requires more cpu
- "anti"-canonical choice of master integrals for difference/differential equations: it maximizes the cancellation of spurious poles in ϵ for the sake of internal checks. In the worst case spurious poles up ϵ^{-37} were generated. Drawback: (much) slower numerical solution. At 5 loop it could be not necessary.
- for topologies with several M.I., a single higher-order equation for the "scalar" M.I. is generated and solved.
- it needs fast cpu

A simple example of 4-loop PSLQ analytical fit

$$
\begin{aligned}
G_7 = -\,&2342.2075141060230754235225405907927098853287320565594708 07\\
&3594814835713846916806455916973185992614831948904197343569 86\\
&6405364828391809277375993763069797378291106083117076717679 35\\
&9831391259607669183299238838719305848684965160728687292431 83\\
&3178005196947599399147517611412834358100307911368387937080 71\\
&1573460997870203023575268524120954362873328464489262424305 03\\
&2364495474744073075812911236379210785864186765175498779728 67\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&= \frac{1671597}{512} - \frac{4381}{96}\pi^2 - \frac{22193}{24}\zeta(3) - 144\,\pi^2\ln 2 - \frac{3617}{240}\pi^4 - \frac{71}{2}\zeta(5)\\
&- \frac{393}{2}\pi^2\zeta(3) - \frac{869}{162}\pi^6 - 24\pi^4\ln^2 2 + 576\pi^2 a_4 + 24\pi^2\ln^4 2 - \frac{803}{2}\zeta(3)^2\\
&+ 504\pi^2\zeta(3)\ln 2 - \frac{1735}{4}\zeta(7) + \frac{799}{6}\pi^2\zeta(5) - \frac{661}{180}\pi^4\zeta(3)
\end{aligned}
$$

black: ansatz (the input)

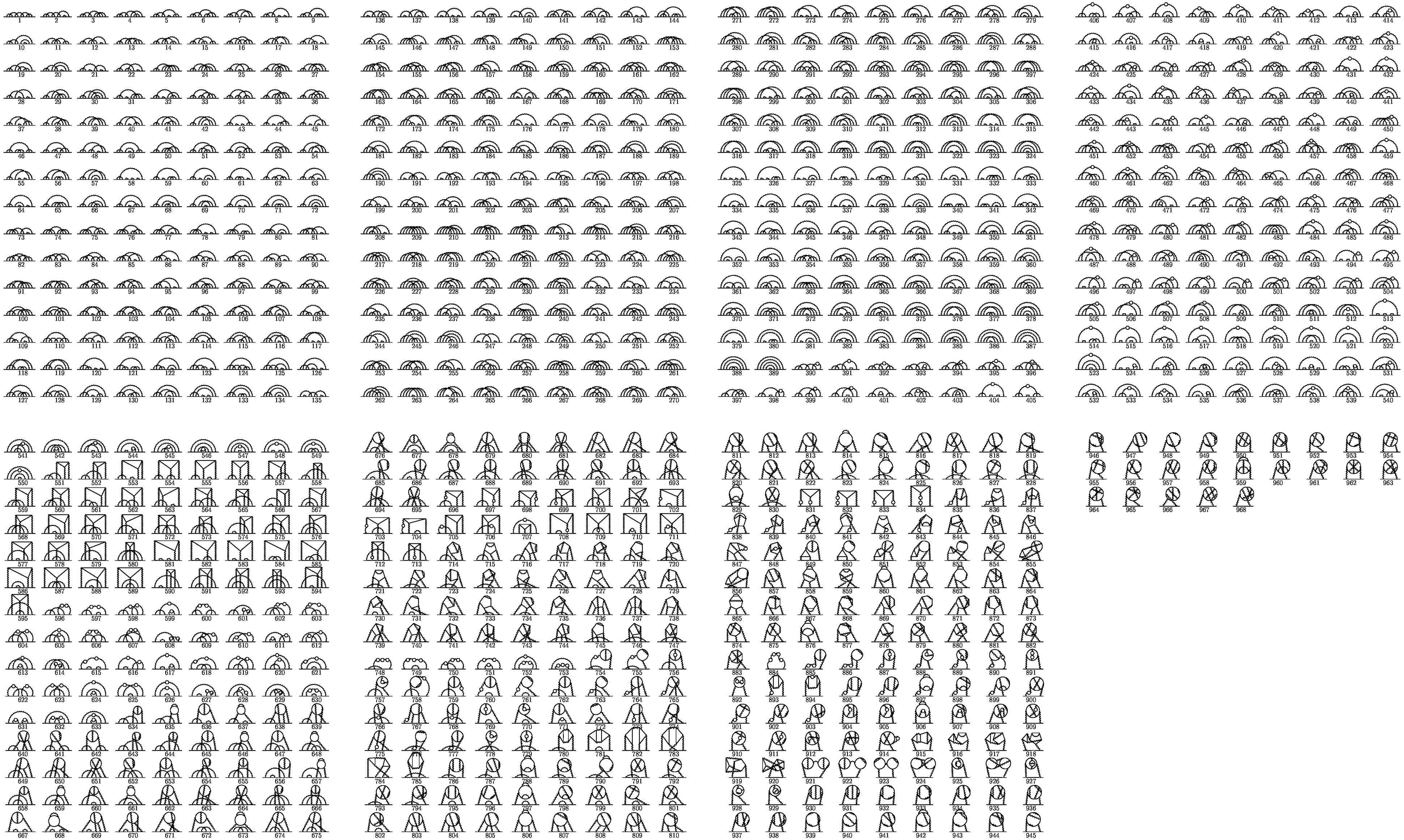
brown: coefficients found by PSLQ (the output)

This particular fit can be found using input data with a minimum precision of 415 digits.

- with ~ 120 analytical terms and ~ 10 digits in the numerical coefficients, minimum number of numerical precision is $\sim 10 * 120$; some analytical constants appear only in particular combinations. Using the combinations where possible, 1200 digits did suffice.
- Used a modified parallel version of PSLQ (Bailey,Broadhurst); tested up to ~ 10000 digits.

5 loops: the 954 self-masses

12672 Feynman diagrams generated by external photon insertions in 954 self-mass diagrams



We show also the 14 non-contributing diagrams containing 2 electron loops with 3 or 5 vertices (Furry's theorem)

Initial 5σ discrepancy on the set "V" (diagrams without electron loops)

$$C_5(\text{Set V})\ \text{Volkov2024} = 6.828(60)$$

$$C_5(\text{Set V})\ \text{AHKN2019} = 7.604(140) \qquad \leftarrow \text{old}$$

$$C_5(\text{Set V})\ \text{AHKN2024} = 6.800(128) \qquad \leftarrow \text{new, higher statistics}$$

Discrepancy *solved.*

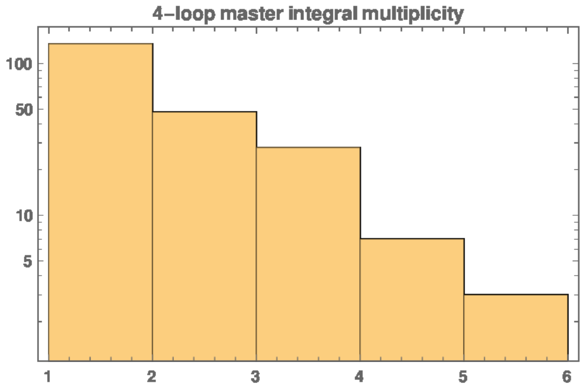
adding the contribution of diagrams with fermion loops, the total 5-loop contribution becomes

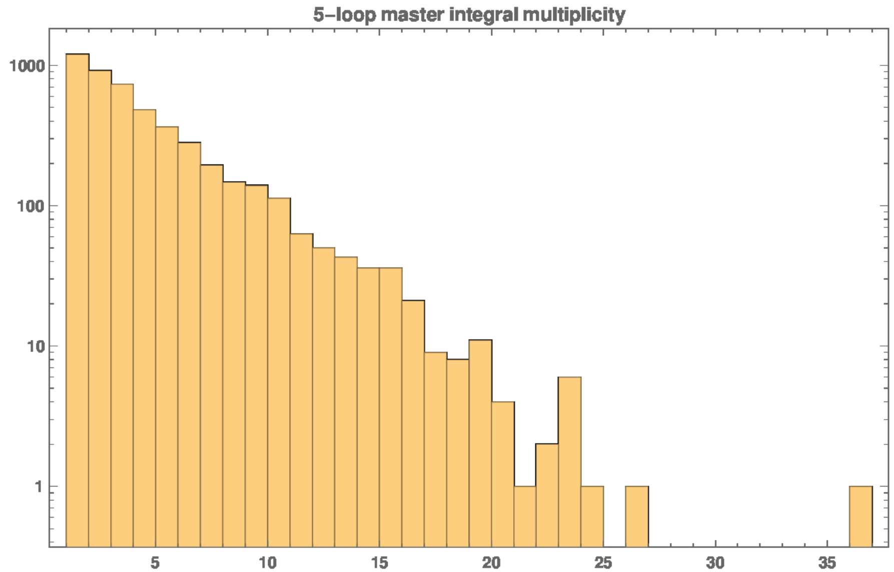
$$C_5(\text{all})\ \text{Volkov2024} = 5.891(61)$$

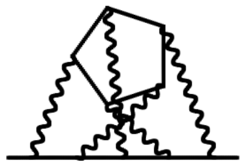
$$C_5(\text{all})\ \text{AHKN2024} = 5.870(128)$$

loop	# i.s.p.	# vertex	# self-mass	estimate # M.I.	actual # M.I.
1	0	1	1	1	1
2	0	7	3	3	3
3	1	72	15	18	17
4	3	891	104	358	325
5	6	12672	954	19670	?

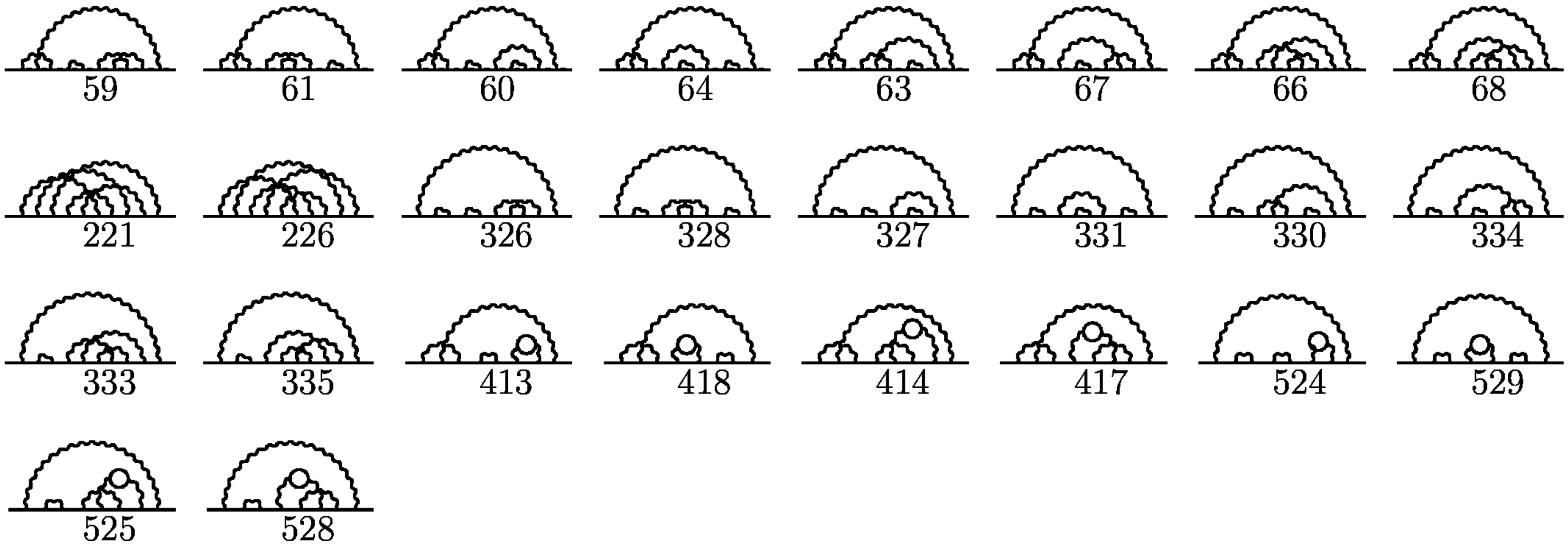
The estimate of the number of M.I. has been done solving (limited) systems of i.b.p. identities on the maximal cut. It may slightly overestimates the real number. Number of master integrals at 5 loops is greater than expected; it is due to the high number of i.s.p. Worst example: the maximal cut of this diagram seems to have 36 M.I.



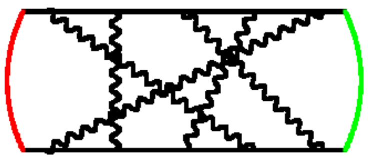



741

New at 5 loops:



The above pairs of self-mass diagrams have *all* the same master integrals, but the photon insertions generate different vertices which are not mirror images.

The pair 221-226 is a bit special  closing on the left-hand side (red), one gets 221; closing on the other side gives the self-mass 226. The main master integral is the same for both diagrams. There are some differences in the master integrals of the subdiagrams.

- systems of i.b.p. identities for reduction to master integrals
- systems of $\sim 10^9$ i.b.p. at 5 loops
- already solved ~ 30 *small* 5-loop systems of 5×10^8 identities
- larger systems will need several TB memory and tens of TB of *fast* disk space

- numerical solution of systems of difference/differential equations
- "anti"-canonical choice of master integrals: (much) slower numerical solution. Not strictly necessary, at 5 loops it could be avoided.
- The high number of M.I. (> 10)for some topologies makes difficult to generate a single high-order equation for the "scalar" M.I, due to blow-up of the expression; numerical solution of the system of coupled first-order difference/differential equation is needed.
- 5-loop renormalization requires (re)calculations of some/all 4-loop M.I. with longer expansions in ϵ: currently work in progress, 70% done, enough terms for 5,6,7 loop, with a slightly higher precision of 1600 digits)

- PSLQ analytical fit
- at 5 loops, my guess is $\sim 5 \times 10^3$ terms and ~ 20 digits coefficient, so a numerical precision of $\sim 10^5$ digits could be necessary.
- Unfortunately, runtime goes as the power $\sim 3-4$ of the number of digits so 10^5 digits seem to be unreachable.
- A moderate precision of a few hundreds of digits seem to be more realistic (but no analytical fit!)

- Extension to 5-loop of the 4-loop approach seems to be possible, with the likely exclusion of the PSLQ analytical fit.
- Huge amounts of runtime/memory/cpu seem to be necessary.

The End

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+ \frac{4572662443}{12247200}\zeta(2)\ln^2 2 - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&- \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 \\
&+ \frac{402152509}{189000}a_4\zeta(2) + \frac{18215}{27}a_5\ln 2 - \frac{18215}{27}\zeta(5)\ln 2 + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 \\
&- \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) \\
&- \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
&- \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 \\
&+ \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 + \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2
\end{aligned}$$

$$\zeta(n) = \sum_{i=1}^{\infty} i^{-n} \quad a_n = \mathrm{Li}_n(1/2) \quad b_6 = H_{0,0,0,0,1,1}(1/2) \quad b_7 = H_{0,0,0,0,0,1,1}(1/2) \quad d_7 = H_{0,0,0,0,1,-1,-1}(1)$$

$H_{i_1,i_2,\ldots}(x)$ harmonic polylogarithms $\quad H_{a_1,a_2,\ldots a_n}(x) = \int^{x} \frac{dy_1}{y_1 - a_1} \ldots \int^{y_{n-2}} \frac{dy_{n-1}}{y_{n-1} - a_{n-1}} \int^{y_{n-1}} \frac{dy_n}{y_n - a_n} \qquad a_i \in \{-1, 0, 1\}$

Backup: The slope at 4 loops: analytical fit part 2

$$A_4 = T + \sqrt{3} V_a + V_b + W + \sqrt{3} E_a + E_b + U$$

$$\begin{aligned}
V_a = & -\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + \frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi + \frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
& + \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& + \frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 \\
& - \frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
& + \frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

$\mathrm{Cl}_n(\theta) = \mathrm{Im}\ \mathrm{Li}_n(e^{i\theta})$ $H_{i_1,i_2,\ldots}(x)$ harmonic polylogarithms

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_b =& \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) - \frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& -507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& +\frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& -\frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
& -\frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
& -\frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
W =& -\frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) - 118\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\
& -2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg),
\end{aligned}$$

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}E_a =&\pi\left(\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) - \frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\left(\frac{751}{972}\ln 2\ f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0)\right.\\ &\left.+\frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\right) - \frac{751}{729}\zeta(2)f_1(0,0,1) + \pi\left(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1)\right.\\ &+\frac{1105}{108}\ln 2\ f_2(0,0,2) - \frac{2210}{81}\ln 2\ f_2(0,1,1) + \frac{4420}{243}\ln 2\ f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2)\\ &-\frac{4011005}{497664}f_2(0,2,1) + \frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0)\\ &\left.-\frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\right)\end{aligned}$$

$$E_b =\zeta(2)\left(-\frac{4629335}{165888}f_1(0,0,2) + \frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\right)$$

$$f_1(i,j,k) =\int_1^9 ds\ D_1^2\left[s - \frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s) \qquad f_2(i,j,k) =\int_1^9 ds\ D_1(s)\sqrt{3}\mathrm{Re}D_m(s)\left[s - \frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s)$$

$$D_1(s) =\frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \qquad D_2(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right)$$

$K(x)$ complete elliptic integral of the first kind

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$U = \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}$$

C_{xy} known only numerically; they contain two elliptic kernels at weight six.