Hypergeometric discriminants

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arXiv:2505.13163 and work in progress

Problem Statement: Landau analysis in QFT

The Lee-Pomeransky representation of a regularized Feynman integral:

$$I_\Gamma(z;\nu) = \int_\Gamma f(x;z)^{-\nu_0} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n},$$

where f is a polynomial in two kinds of variables $(x, z) \in (\mathbb{C}^\times)^n \times \mathbb{C}^N$ and $\nu = (\nu_0, \ldots, \nu_n) \in \mathbb{C}^{n+1}$ is a generic complex parameter.

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Conjecture (Fevola-Mizera-Telen)

$$\nabla_\chi(Z) := (\text{Euler discriminant locus}) = (\text{Landau variety}).$$

A hypergeometric view

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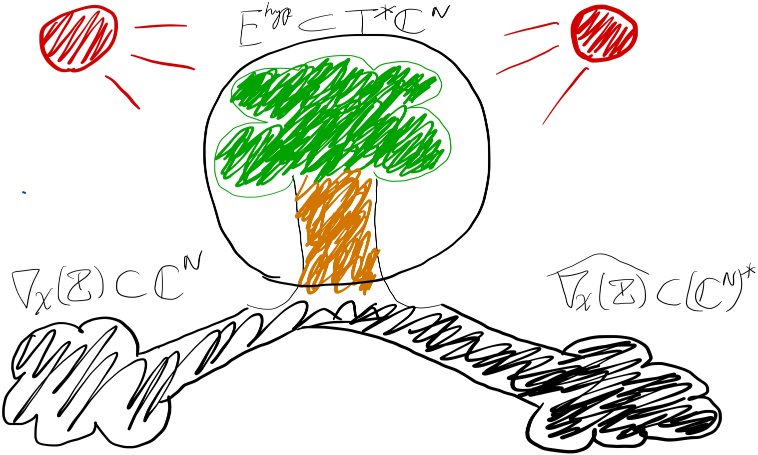

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$$E^{\mathrm{hyp}} = \sum_{Y \subset \mathbb{C}^N} m_Y \, T^*_Y \mathbb{C}^N : \text{Hypergeometric discriminant.}$$

$$\begin{tikzcd}
 & T^*_Y\mathbb{C}^N, \quad E^{\mathrm{hyp}} \arrow[dl, "\text{dual proj., dual int.}"'] \arrow[dr, "\text{projection, integration}"] & \\
Y^\vee, \quad \widehat{\nabla_\chi(Z)} = \mathrm{Sing}(\widehat{M^{\mathrm{hyp}}}) \arrow[rr, leftrightarrow, "\text{Projective duality, Fourier transform}"'] & & Y, \quad \nabla_\chi(Z) = \mathrm{Sing}(M^{\mathrm{hyp}})
\end{tikzcd}$$

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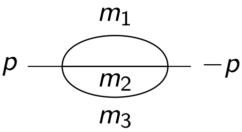
This is a version of Hartogs' theorem.

The running example: sunrise/sunset diagram



$$\Rightarrow \; m_0 := p^2, \quad \nabla_\chi(Z) = \{m_0 m_1 m_2 m_3 \Delta(m_0, m_1, m_2, m_3) = 0\}$$

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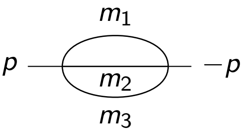

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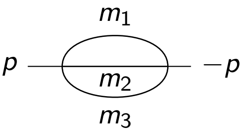

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Two different descriptions of Δ:

1. Direct formula

$$\begin{aligned}\Delta =& m_1^4 - 4m_1^3 m_2 - 4m_1^3 m_3 - 4m_1^3 m_0 + 6m_1^2 m_2^2 + 4m_1^2 m_2 m_3 + 4m_1^2 m_2 m_0 + 6m_1^2 m_3^2 + 4m_1^2 m_3 m_0 + 6m_1^2 m_0^2 - 4m_1 m_2^3 + 4m_1 m_2^2 m_3 \\ &+ 4m_1 m_2^2 m_0 + 4m_1 m_2 m_3^2 - 40m_1 m_2 m_3 m_0 + 4m_1 m_2 m_0^2 - 4m_1 m_3^3 + 4m_1 m_3^2 m_0 + 4m_1 m_3 m_0^2 - 4m_1 m_0^3 + m_2^4 - 4m_2^3 m_3 - 4m_2^3 m_0 \\ &+ 6m_2^2 m_3^2 + 4m_2^2 m_3 m_0 + 6m_2^2 m_0^2 - 4m_2 m_3^3 + 4m_2 m_3^2 m_0 + 4m_2 m_3 m_0^2 - 4m_2 m_0^3 + m_3^4 - 4m_3^3 m_0 + 6m_3^2 m_0^2 - 4m_3 m_0^3 + m_0^4,\end{aligned}$$

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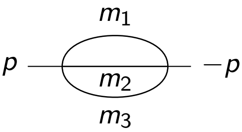

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$$\Delta = \prod_{(\eta_1,\eta_2,\eta_3)\in\{\pm\}^3/\{\pm\}} \left(m_0 - (\eta_1\sqrt{m_1} + \eta_2\sqrt{m_2} + \eta_3\sqrt{m_3})^2\right),$$

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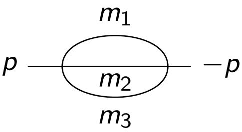

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E^{hyp} is equivalent to the Euler stratification by Telen-Wiesmann.

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for generic $\nu_0, \ldots, \nu_n \in \mathbb{C}$,

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Theorem

The hypergeometric discriminant E^{hyp} is a flat degeneration of the elimination of x-variables of the family of likelihood equations.

Sunrise/Sunset again

$$E^{\mathrm{hyp}}_{sunrise} = 7[T^*_{\mathbb{C}^4}\mathbb{C}^4] + [T^*_{V(\Delta)}\mathbb{C}^4] + 3\sum_{i=0}^{3}[T^*_{V(m_i)}\mathbb{C}^4] + \sum_{0\le i<j\le 3}[T^*_{V(m_i,m_j)}\mathbb{C}^4].$$

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Hypergeometric discriminant gives rise to the dual picture:

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$$\chi_z = 7 - 3Eu_{V(m_0)}(z) - 3Eu_{V(m_1)}(z) + Eu_{V(m_0,m_1)}(z) = 7 - 3 - 3 + 1 = 2.$$

Hypergeometric discriminant gives rise to the dual picture:
$\varpi : T^*\mathbb{C}^4 = \mathbb{C}^4 \times (\mathbb{C}^4)^* \to (\mathbb{C}^4)^*$

Sunrise/Sunset again

$$E^{\mathrm{hyp}}_{sunrise} = 7[T^*_{\mathbb{C}^4}\mathbb{C}^4] + [T^*_{V(\Delta)}\mathbb{C}^4] + 3\sum_{i=0}^{3}[T^*_{V(m_i)}\mathbb{C}^4] + \sum_{0\leq i<j\leq 3}[T^*_{V(m_i,m_j)}\mathbb{C}^4].$$

$z = (m_0 = 0, m_1, m_2, m_3)$: generic

$$\chi_z = 7Eu_{\mathbb{C}^4}(z) - 3Eu_{V(m_0)}(z) = 7 - 3 = 4$$

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$\varpi : T^*\mathbb{C}^4 = \mathbb{C}^4 \times (\mathbb{C}^4)^* \to (\mathbb{C}^4)^*$

$$\varpi(E^{\mathrm{hyp}}_{sunrise}) = \left\{\frac{1}{\mu_0} + \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 0\right\}.$$

Projective duality
(in progress)

Projective duality

$$
\begin{array}{ccc}
 & (\mathbb{P}^{N-1})^* \times \mathbb{P}^{N-1} & \\
 & \cup & \\
 & \mathbb{P}(T^*_{\mathrm{Cone}(Y)}\mathbb{C}^N) = \overline{\{(H,\ell) \in (\mathbb{P}^{N-1})^* \times Y_{sm} \mid T_\ell Y \subset H\}} & \\
\swarrow & & \searrow \\
(\mathbb{P}^{N-1})^* \supset Y^\vee & & Y \subset \mathbb{P}^{N-1}
\end{array}
$$

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\end{array}
$$

$$(\mathbb{P}^{N-1})^* \supset Y^\vee \longleftarrow \qquad \longrightarrow Y \subset \mathbb{P}^{N-1}$$

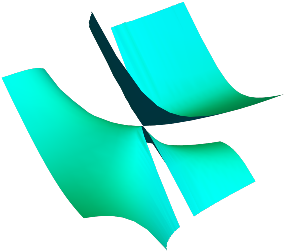

Figure: Dehomonegized $\frac{1}{\mu_0} + \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 0$

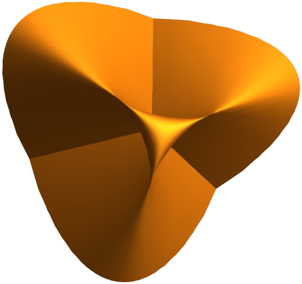

Figure: Dehomogenized $V(\Delta)$

Projective duality

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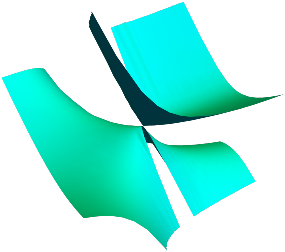

Figure: Dehomonegized $\frac{1}{\mu_0} + \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 0$

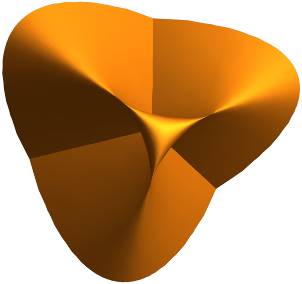

Figure: Dehomogenized $V(\Delta)$

Projective duality does not preserve degree or dimension.

Fourier transform

Assume linearity: $f(x; z) = \sum_{i=1}^{N} z_i f_i(x)$.

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$$I_\Gamma(z;\nu) = \int d\xi \exp\{\sum_{i=1}^{N} z_i \xi_i\} \int_\Gamma \prod_{i=1}^{N} \delta(\xi_i - x_0 f_i(x)) x_0^{\nu_0} \cdots x_n^{\nu_n} \frac{dx_0 \cdots dx_n}{x_0 \cdots x_n}.$$

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$\widehat{I}_\Gamma(\xi;\nu)$ is supported on a SINGLE IRREDUCIBLE VARIETY

$$\widehat{\nabla_\chi(Z)} := \overline{\mathrm{Im}\left((\mathbb{C}^*)^{n+1} \ni (x_0, x) \mapsto (x_0 f_1(x), \ldots, x_0 f_N(x)) \in \mathbb{C}^N\right)}.$$

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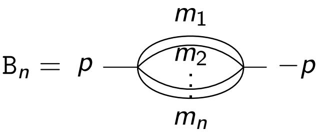
We expect a correspondence

$$(\text{Irreducible decomposition of } \nabla_\chi(Z)) \leftrightarrow (\text{A stratification of } \widehat{\nabla_\chi(Z)}).$$

$(n-1)$-loop banana



Theorem (in progress)

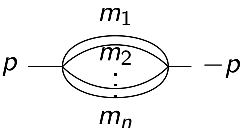
The hypergeometric discriminant of B_n is given by a formal sum

$$E^{\mathrm{hyp}}_{\mathrm{B}_n} = n_0[T^*_Z Z] + [T^*_{V(\Delta_n)} Z] + \sum_{p=1}^{n-1} n_p \sum_{0 \leq i_1 < \cdots < i_p \leq n} [T^*_{V(m_{i_1},\ldots,m_{i_p})} Z],$$

where $Z = \mathbb{C}^{n+1}$ and $n_p = 2^{n-p} - 1$. The variety $V_{\mathbb{P}^n}(\Delta_n)$ is the projective dual of the reciprocal hyperplane $\frac{1}{\mu_0} + \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n} = 0$.

$(n-1)$-loop banana

$\mathrm{B}_n =$



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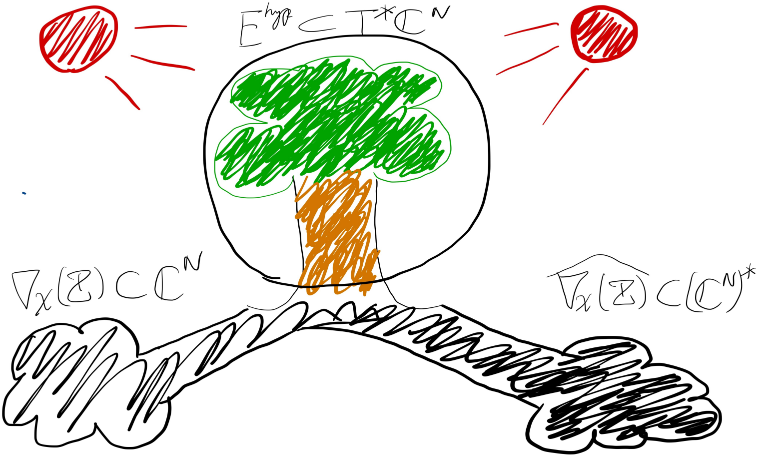
The hypergeometric discriminant of B_n is given by a formal sum

$$E_{\mathrm{B}_n}^{\mathrm{hyp}} = n_0[T_Z^* Z] + [T_{V(\Delta_n)}^* Z] + \sum_{p=1}^{n-1} n_p \sum_{0 \leq i_1 < \cdots < i_p \leq n} [T_{V(m_{i_1},\ldots,m_{i_p})}^* Z],$$

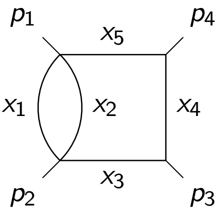
where $Z = \mathbb{C}^{n+1}$ and $n_p = 2^{n-p} - 1$. The variety $V_{\mathbb{P}^n}(\Delta_n)$ is the projective dual of the reciprocal hyperplane $\frac{1}{\mu_0} + \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n} = 0$.

This result repackages Euler characteristic computation of B_n ([Bitoun-Bogner-Klausen-Panzer '18]).

Thank you for your attention!



Example: $\texttt{debox}$ and a rational cubic threefold



$$f_{\texttt{debox}}(x;z) =$$

$$M_1x_1x_2x_5 + M_2x_1x_2x_3 + M_3x_1x_3x_4 + M_3x_2x_3x_4$$

$$+M_4x_1x_4x_5 + M_4x_2x_4x_5 + sx_1x_3x_5 + sx_2x_3x_5 + tx_1x_2x_4$$

$$+x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_5$$

Set $M_4 = 0$. The Euler discriminant locus is

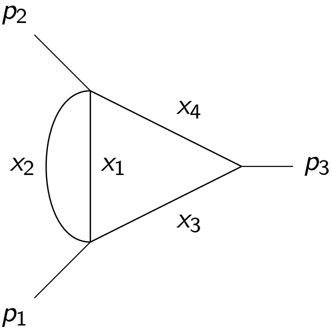
$$M_1M_2M_3s(M_3-s)(M_1-t)(M_1M_3-st)\Delta_2(s,M_1,M_2)\Delta_2(t,M_2,M_3)\lambda(s,t,M_1,M_2,M_3) = 0.$$

Δ_2 is quadratic and $\lambda(s,t,M_1,M_2,M_3)$ is a cubic polynomial given by

$$M_1^2M_3 - M_1M_2M_3 + M_1M_2s + M_1M_3^2 - M_1M_3s - M_1M_3t - M_1st + M_2M_3t - M_2st - M_3st + s^2t + st^2.$$

Smooth cubic threefolds are irrational (Clemens-Griffith's theorem).

Example:parachute



$$
\begin{aligned}
f(x;z) =& (1 - \sum_{i=1}^{4} m_i x_i)((x_1 + x_2)(x_3 + x_4) + x_1 x_2) \\
& + x_1 x_2 (M_1 x_3 + M_2 x_4) + M_3 x_3 x_4 (x_1 + x_2)
\end{aligned}
$$

Simplification: $m_2 = m_3 = M_2 = 0$

List of (Y, m_Y)

$\operatorname{codim} Y$	(Y, m_Y)
0	$(\{0\}, 8)$
1	$(\{m_4 M_1 - m_1 M_3\}, 1), (\{m_1 - m_4 - M_1 + M_3\}, 1), (\{m_1 - m_4\}, 1),$ $(\{m_1 - M_1\}, 1), (\{m_1\}, 4), (\{m_4 + M_1 - M_3\}, 1), (\{m_4 - M_3\}, 2),$ $(\{m_4\}, 2), (\{M_1 - M_3\}, 2), (\{M_1\}, 2), (\{M_3\}, 2)$
2	$(\{m_4 + M_1 - M_3, m_1\}, 1), (\{m_4 - M_3, m_1 - M_1\}, 1), (\{m_4 - M_3, m_1\}, 1),$ $(\{m_1, m_4\}, 2), (\{M_1 - M_3, m_1 - m_4\}, 1), (\{M_1 - M_3, m_1\}, 1),$ $(\{M_1 - M_3, m_4\}, 1), (\{m_1, M_1\}, 2), (\{M_1, m_4 - M_3\}, 1), (\{m_1, M_3\}, 1),$ $(\{m_4, M_3\}, 2), (\{M_1, M_3\}, 2)$
3	$(\{M_1 - M_3, m_4, m_1\}, 1), (\{M_1, m_4 - M_3, m_1\}, 1), (\{m_1, m_4, M_3\}, 1),$ $(\{m_1, M_1, M_3\}, 1), (\{m_4, M_1, M_3\}, 1)$
4	$(\{m_1, m_4, M_1, M_3\}, 1)$