Parametric Cycles for Feynman Integrals

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- In terms of Feynman parameters, cuts of Feynman integrals can be obtained by a systematic change of integration contour.

- Cuts are related to discontinuities and monodromies around Landau singularities.

- Parameter space reveals implications for sequential discontinuities.

- A Feynman integral takes the form $I \sim \int_\Gamma \frac{\mathrm{d}\alpha}{\mathrm{GL}(1)} \, \mathscr{U}^\kappa(\alpha) \, \mathscr{F}^\lambda(\alpha)$.

- $\alpha = \{\alpha_1, \ldots, \alpha_E\}$ are the Feynman parameters.

- $\mathscr{U}$ and $\mathscr{F}$ are homogeneous graph polynomials in α; $/\mathrm{GL}(1)$ makes it a projective integral.

- Prefactors and exponents κ, λ depend on dimensions, multiplicity of propagators, loop order. $\kappa = E - (L+1)D/2, \quad \lambda = -E + LD/2$

- $$I \sim \int_\Gamma \frac{\mathrm{d}\alpha}{\mathrm{GL}(1)}\, \mathscr{U}^\kappa(\alpha)\, \mathscr{F}^\lambda(\alpha)\,.$$

- The original integration domain Γ is a projective slice of $\{\alpha_i \geq 0\}$. It is bounded by coordinate hyperplanes.

- Claim: discontinuities are obtained by changing the boundaries of Γ to include $\mathscr{F} = 0$ and systematically removing hyperplanes.

New contours for discontinuities

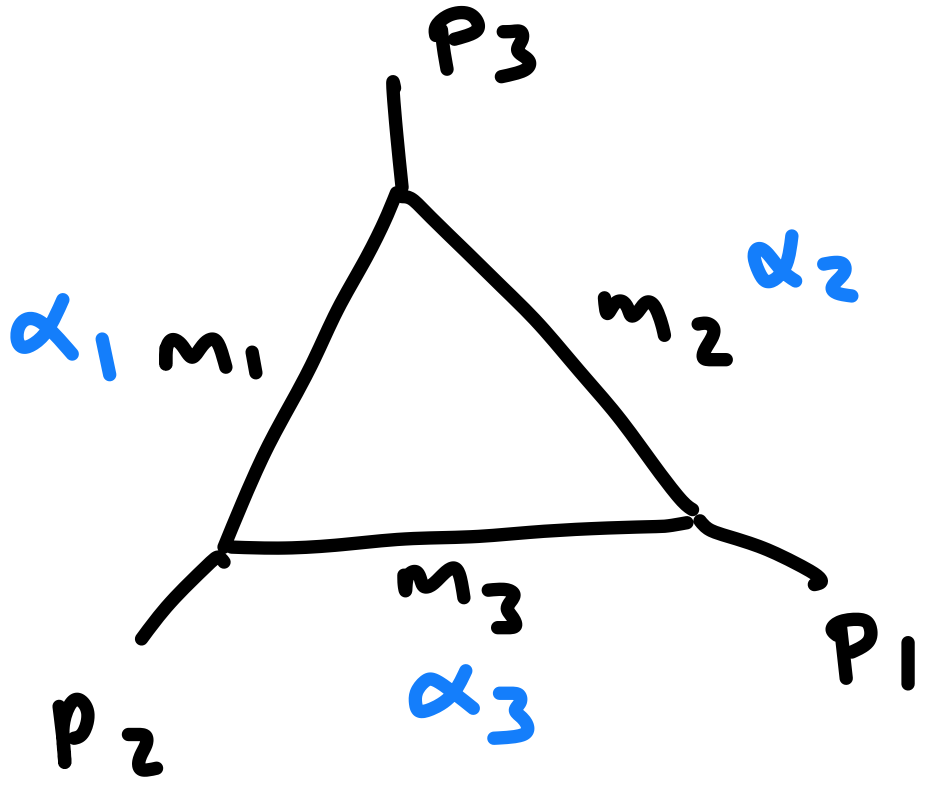
$$I = \int_\Gamma \mathrm{d}\alpha\, \mathcal{U}^\kappa \mathcal{F}^\lambda$$

$$\mathrm{Disc}[I] = \int_\Gamma \mathrm{d}\alpha\, \mathcal{U}^\kappa\, \mathrm{Disc}[\mathcal{F}^\lambda]$$

$$\begin{aligned} \mathrm{Disc}[\mathcal{F}^\lambda] &= (\mathcal{F} - i0)^\lambda - (\mathcal{F} + i0)^\lambda \\ &= \theta[-\mathcal{F}]\, \left[((-\mathcal{F})(e^{-i\pi}))^\lambda - ((-\mathcal{F})(e^{i\pi}))^\lambda\right] \\ &= -\,\theta[-\mathcal{F}]\,[-\mathcal{F}]^\lambda\, 2i\sin(\pi\lambda) \end{aligned}$$

if $\lambda \notin \mathbb{Z}$

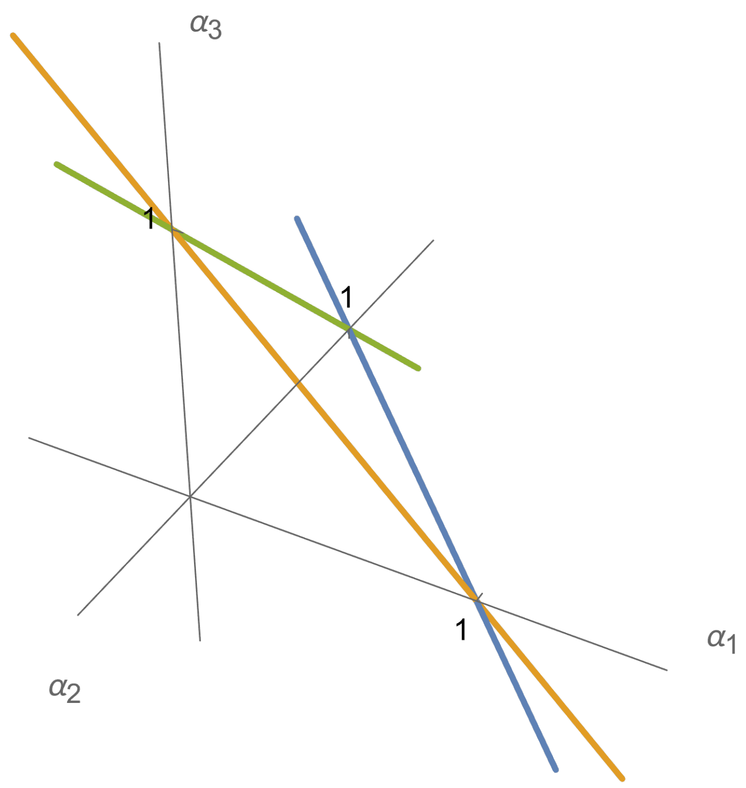
Example: triangle



Feynman parameters $\alpha_1, \alpha_2, \alpha_3$.

$$\mathcal{F} = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2) - \alpha_1\alpha_2 p_3^2 - \alpha_1\alpha_3 p_2^2 - \alpha_2\alpha_3 p_1^2$$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3$$

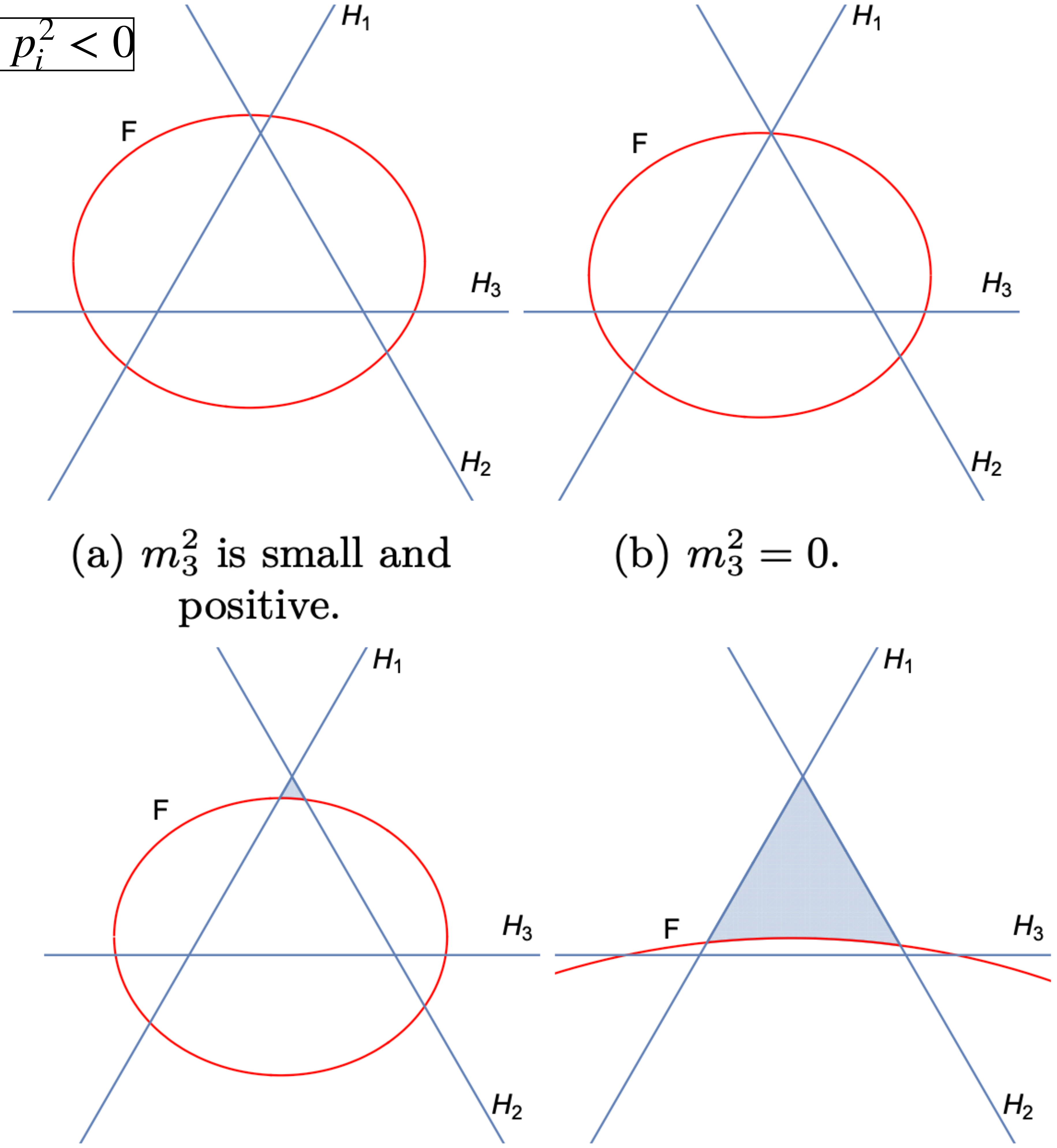


$$I = \int_\Gamma \frac{\mathrm{d}\alpha}{\mathrm{GL}(1)} \mathcal{U}^\kappa \mathcal{F}^\lambda$$

$$\Gamma = \{\alpha_i \geq 0, \sum \alpha_i = 1\}$$

m_3^2 discontinuity of the triangle

$\boxed{m_1^2, m_2^2 > 0,\ p_i^2 < 0}$



(a) m_3^2 is small and positive. (b) $m_3^2 = 0$. (c) m_3^2 is small and negative. (d) m_3^2 is large and negative.

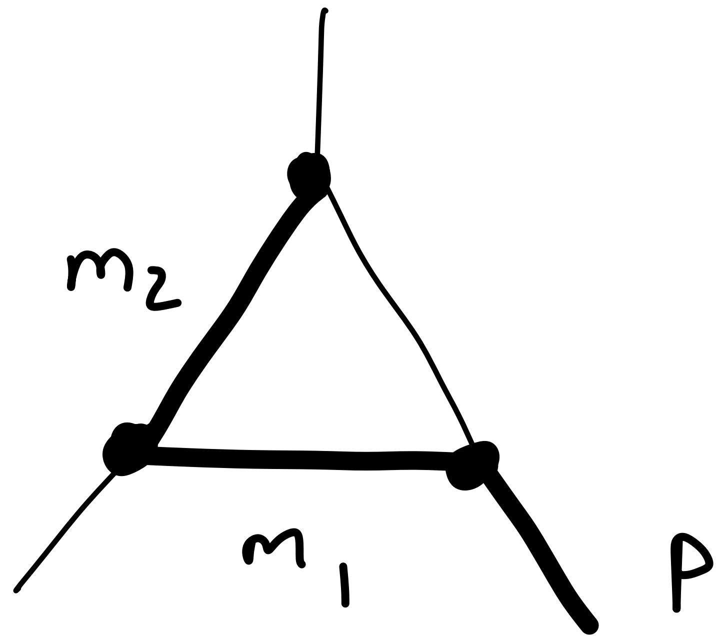
Disc has support where $\mathscr{F} < 0$.

$\mathscr{F} = 0$ becomes a new boundary.

Some original boundaries are lost.

Implement this idea systematically.

The hook triangle



A 3-scale triangle that's finite in 4 dimensions.

Set $m_3^2 = 0$, $p_2^2 = p^2$, $p_1^2 = p_3^2 = 0$.

$$\mathcal{F} = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 m_1^2 + \alpha_2 m_2^2) - \alpha_1 \alpha_3 p^2$$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3$$

$$I_3^{\text{hook}} = \frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)}{\epsilon(1-\epsilon)\left(m_1^2 - m_2^2\right)}\left[\left(m_2^2\right)^{-\epsilon} {}_2F_1\left(1,1;2-\epsilon;\frac{p^2}{m_1^2-m_2^2}\right) - \left(m_1^2\right)^{-\epsilon} F_1\left(1;1,\epsilon;2-\epsilon;\frac{p^2}{m_1^2-m_2^2};\frac{p^2}{m_1^2}\right)\right]$$

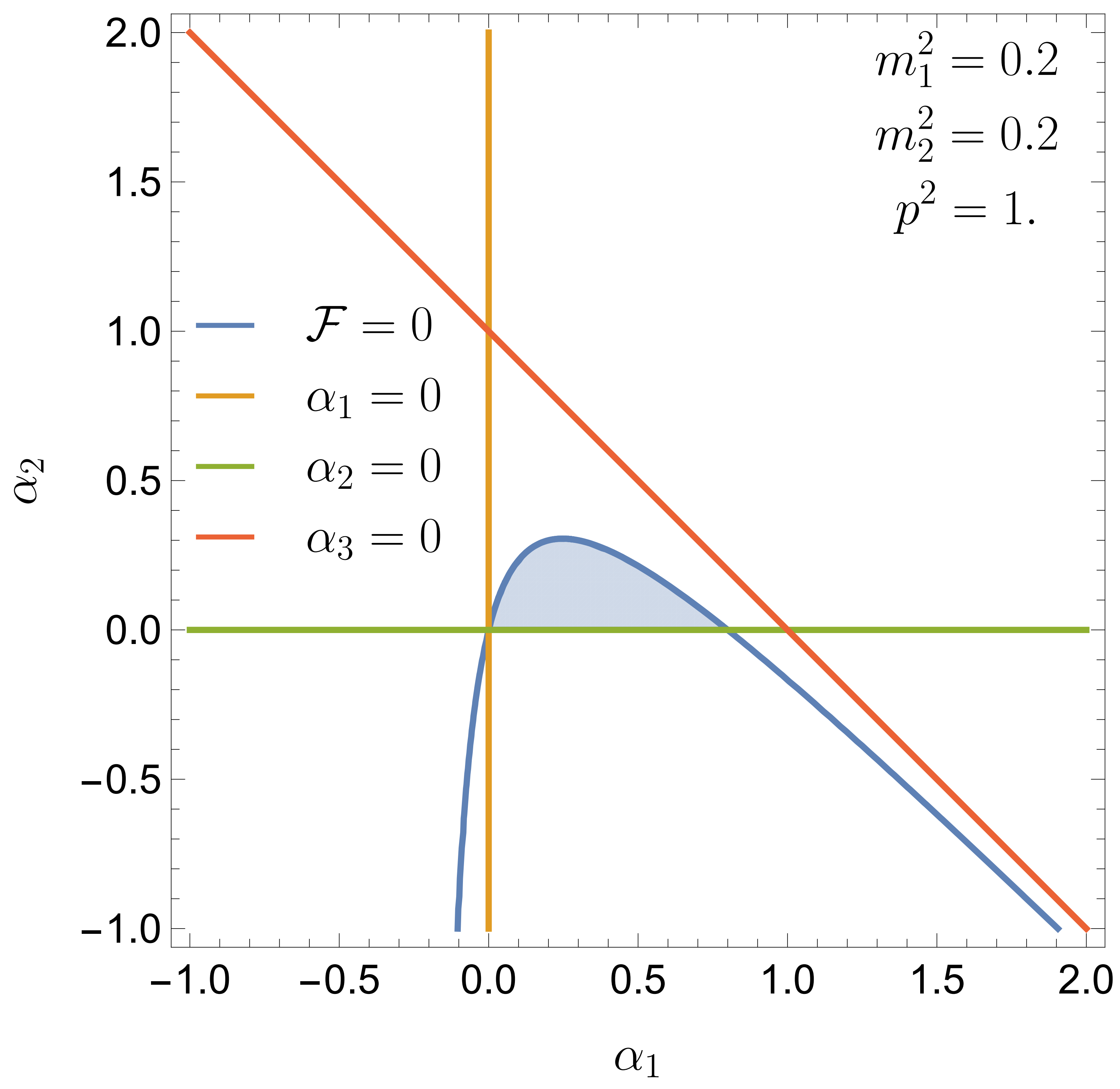
Can work in dim reg, or set $\epsilon = 0$ for $D = 4$.

Symbol of the hook triangle

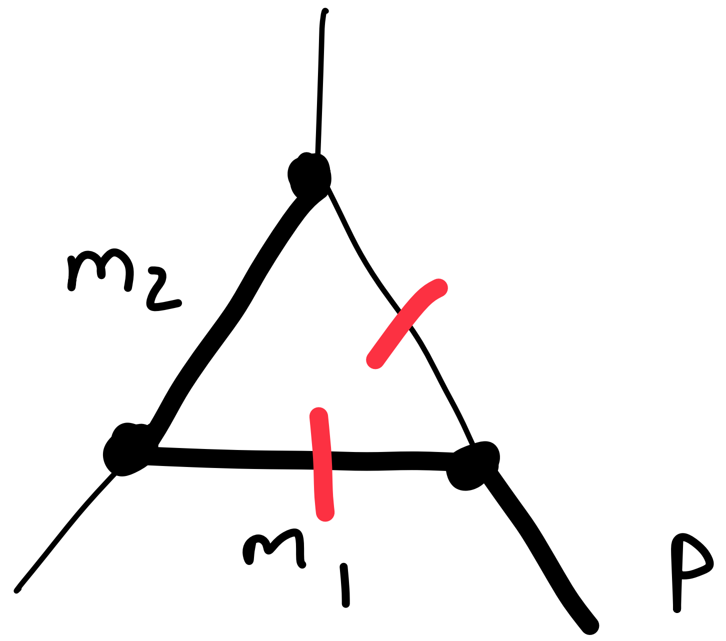
$$\mathcal{S}(I_3^{\text{hook}}) = \mathcal{S}_2(I_3^{\text{hook}}) + \epsilon \mathcal{S}_3(I_3^{\text{hook}}) + \mathcal{O}(\epsilon^2)\,.$$

$$\mathcal{S}_2(I_3^{\text{hook}}) = m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right) - (p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right)$$

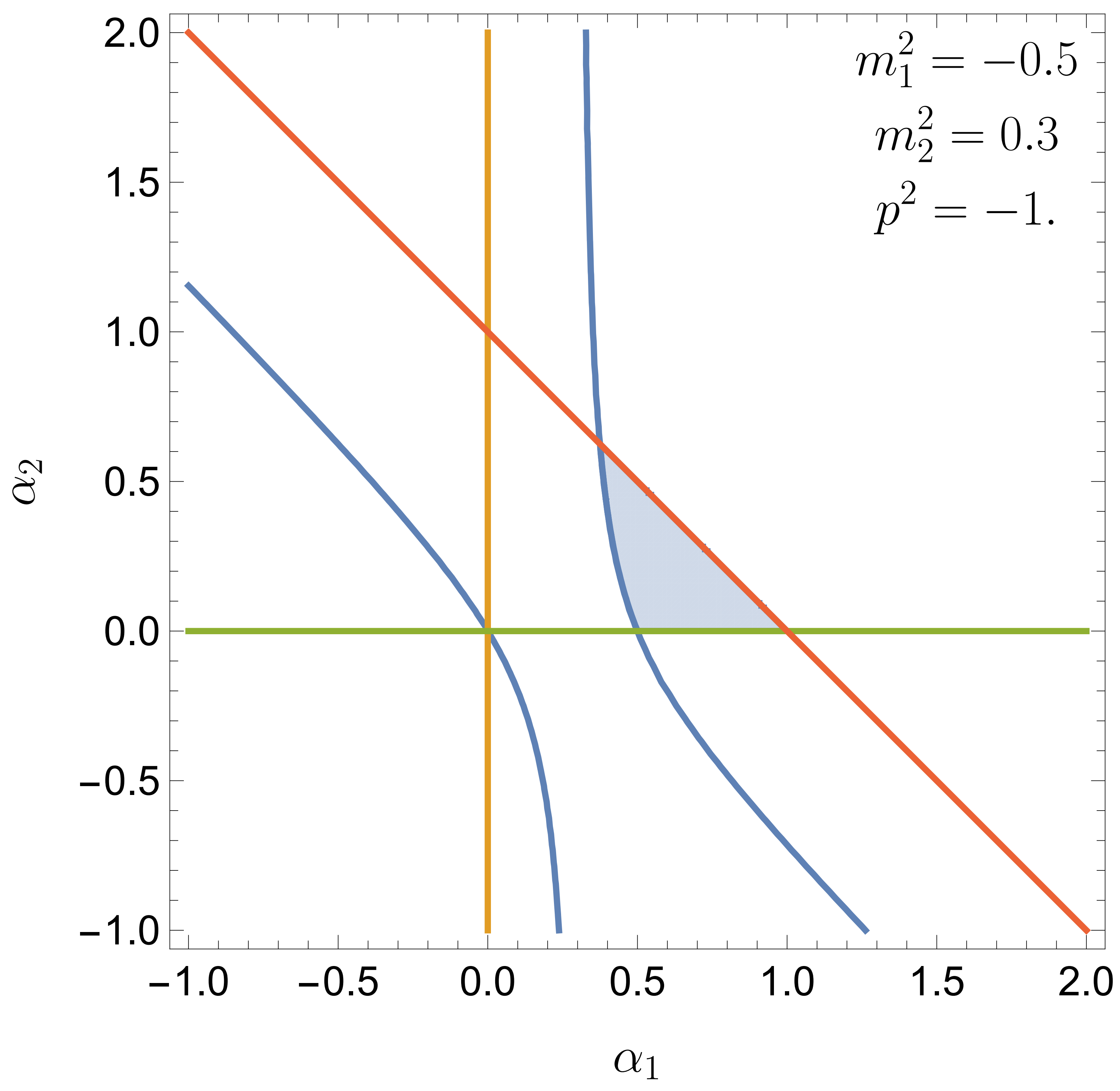
$$\begin{aligned}
\mathcal{S}_3(I_3^{\text{hook}}) = {} & m_1^2 \otimes \frac{p^2}{p^2 - m_1^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} + m_1^2 \otimes m_1^2 \otimes \frac{m_2^2}{m_1^2 - m_2^2} + m_1^2 \otimes \frac{m_1^2 - m_2^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\
& + m_2^2 \otimes m_2^2 \otimes \frac{m_1^2 - m_2^2}{m_1^2 - m_2^2 - p^2} + m_2^2 \otimes \frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\
& - \left(p^2 - m_1^2\right) \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} + \left(p^2 - m_1^2\right) \otimes \frac{\left(p^2 - m_1^2\right)^2}{p^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2}\,.
\end{aligned}$$



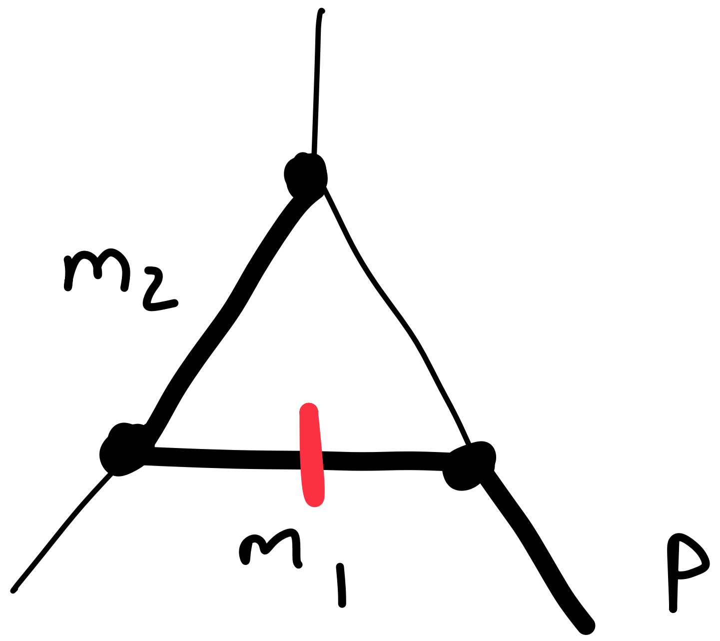
$\Gamma_{p^2 - m_1^2}$



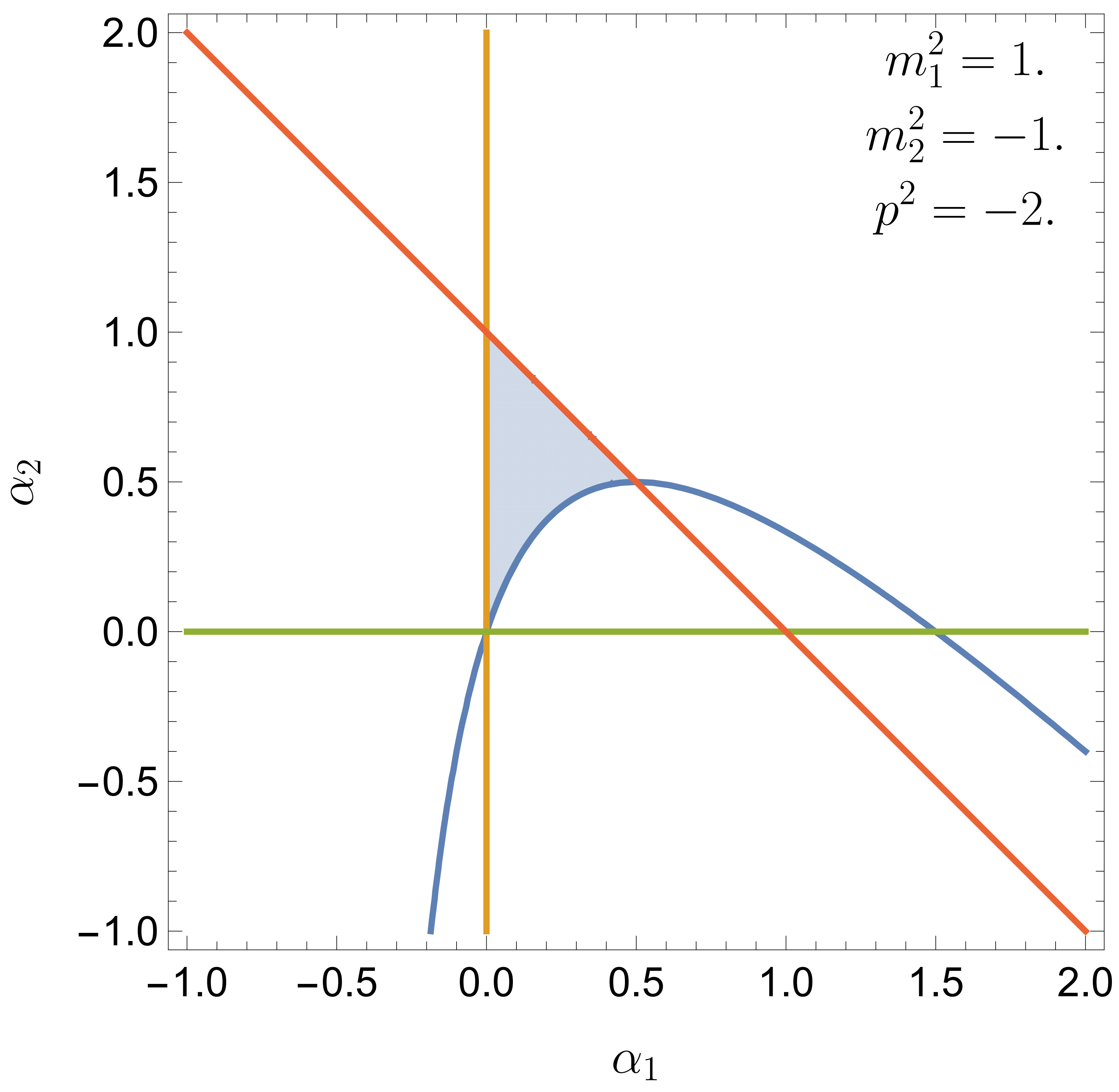
$\alpha_1 = 0$ and $\alpha_3 = 0$ ' are not boundaries.



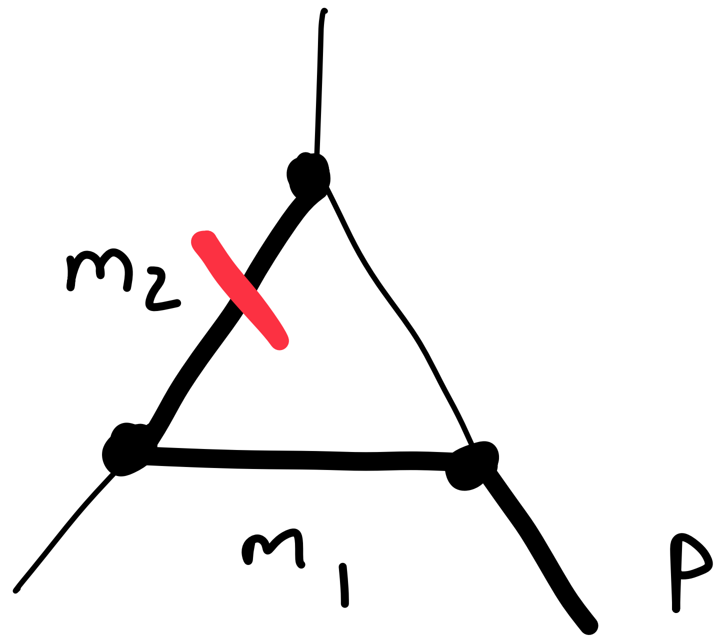
$\Gamma_{m_1^2}$



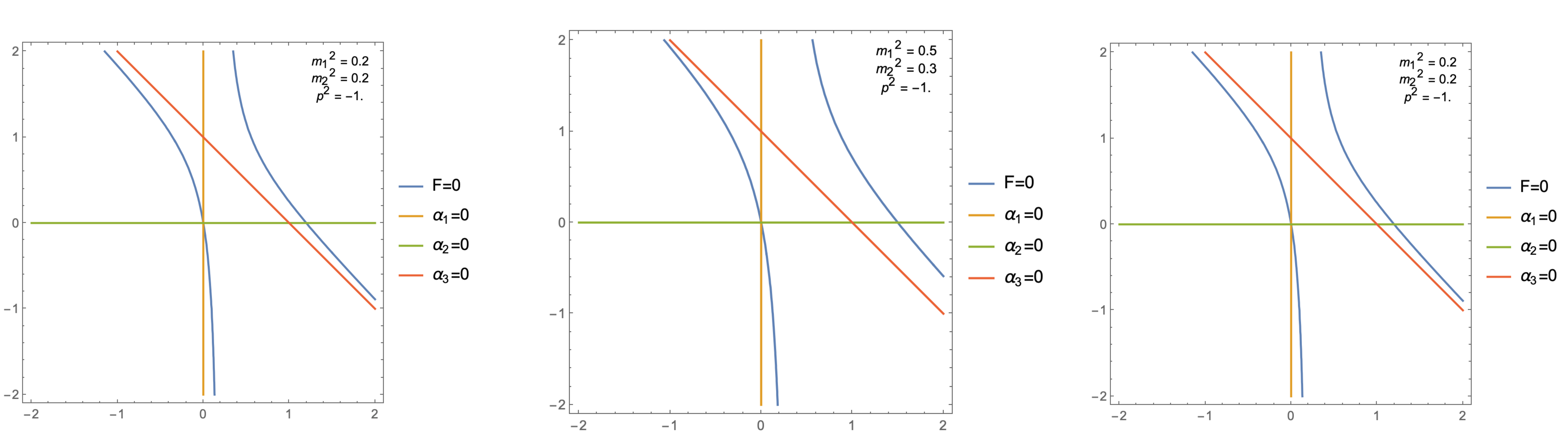
$\alpha_1 = 0$ is not a boundary.



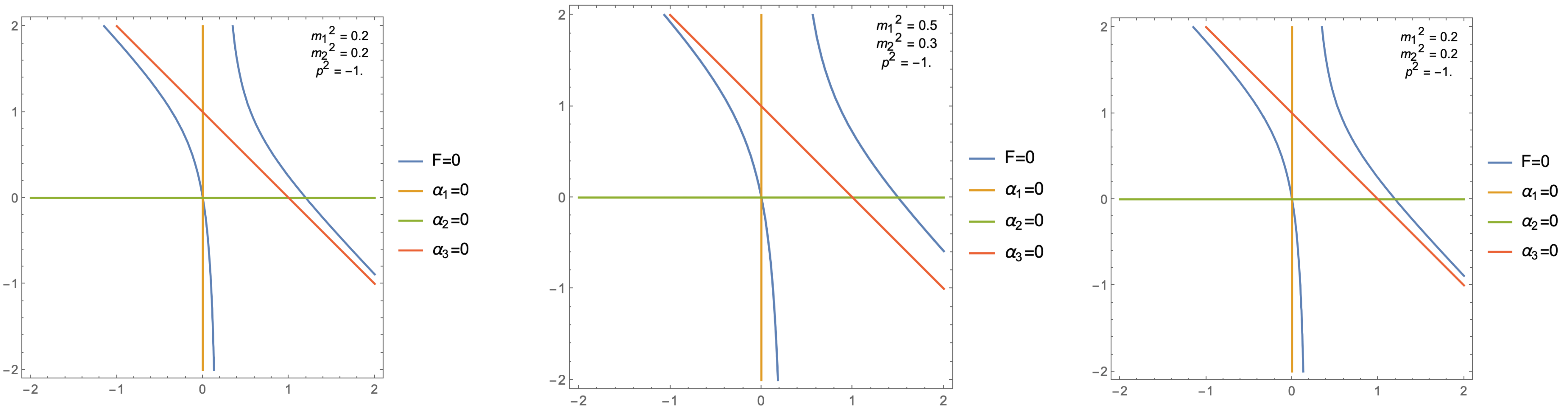
$\Gamma_{m_2^2}$



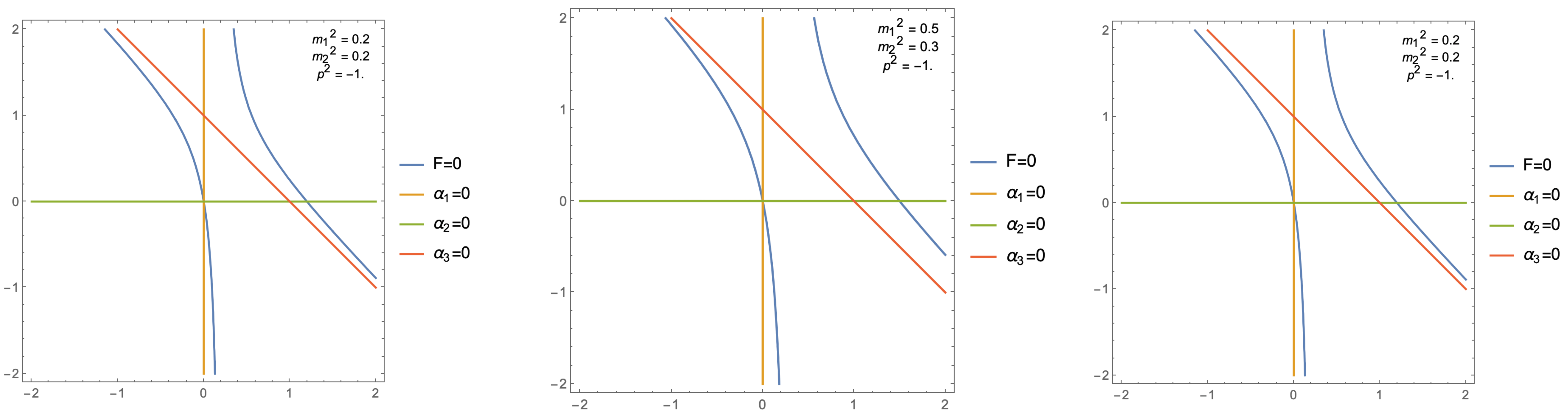
$\alpha_2 = 0$ is not a boundary.



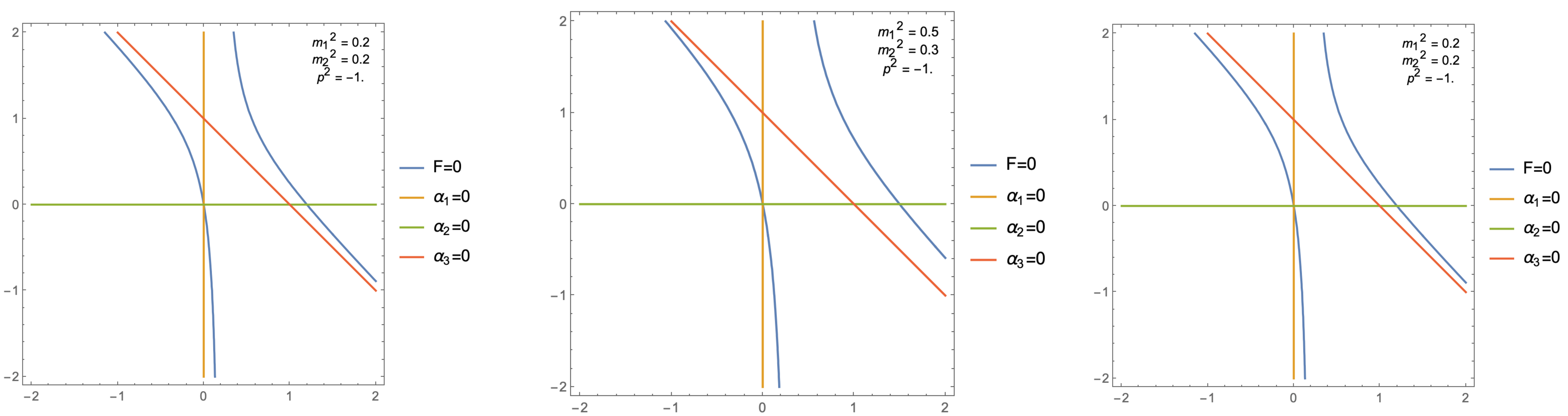
m1^2 = 0.2
m2^2 = 0.2
p^2 = −1.
F=0
α1=0
α2=0
α3=0
m1^2 = 0.5
m2^2 = 0.3
p^2 = −1.
F=0
α1=0
α2=0
α3=0
m1^2 = 0.2
m2^2 = 0.2
p^2 = −1.
F=0
α1=0
α2=0
α3=0



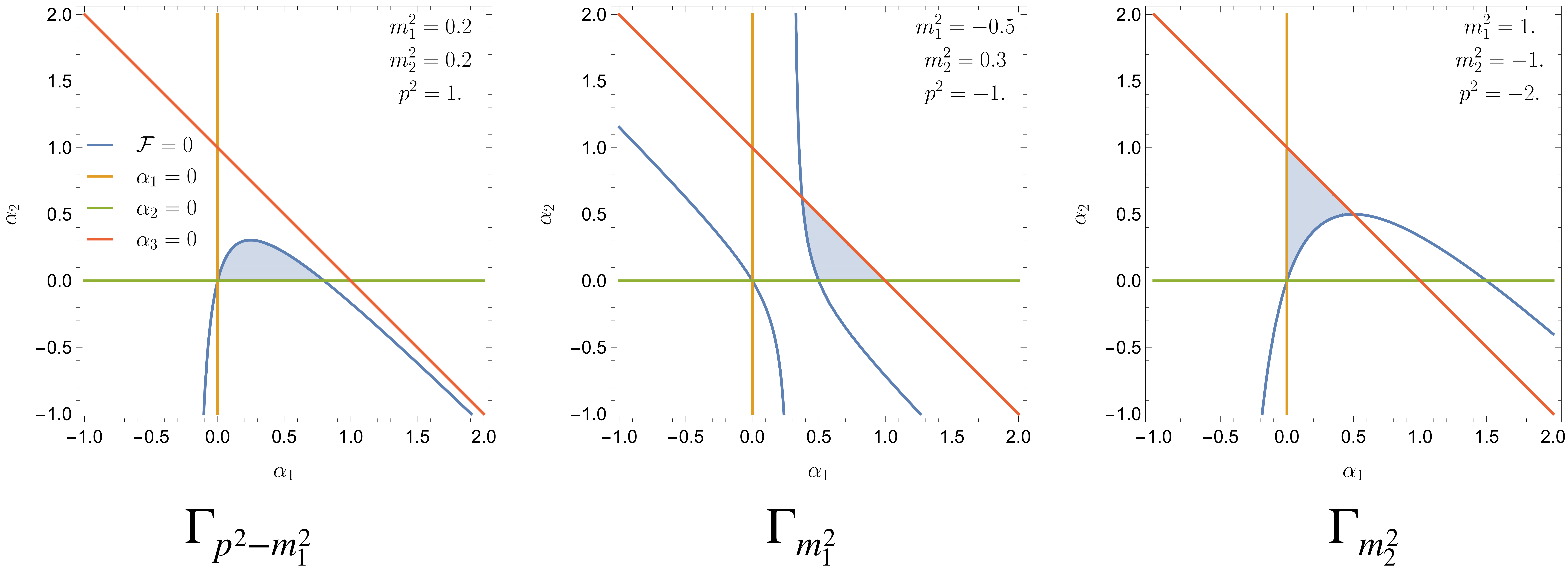
m1^2 = 0.2
m2^2 = 0.2
p^2 = −1.
F=0
α1=0
α2=0
α3=0
m1^2 = 0.5
m2^2 = 0.3
p^2 = −1.
F=0
α1=0
α2=0
α3=0
m1^2 = 0.2
m2^2 = 0.2
p^2 = −1.
F=0
α1=0
α2=0
α3=0



$m_1^2 = 0.2$
$m_2^2 = 0.2$
$p^2 = -1.$
F=0
α_1=0
α_2=0
α_3=0
$m_1^2 = 0.5$
$m_2^2 = 0.3$
$p^2 = -1.$
F=0
α_1=0
α_2=0
α_3=0
$m_1^2 = 0.2$
$m_2^2 = 0.2$
$p^2 = -1.$
F=0
α_1=0
α_2=0
α_3=0



$m_1^2 = 0.2$
$m_2^2 = 0.2$
$p^2 = -1.$
$m_1^2 = 0.5$
$m_2^2 = 0.3$
$p^2 = -1.$
$m_1^2 = 0.2$
$m_2^2 = 0.2$
$p^2 = -1.$
F=0
α_1=0
α_2=0
α_3=0



These are the first-entry letters of the symbol. Which letters can follow them in the second entries?

$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right)$$

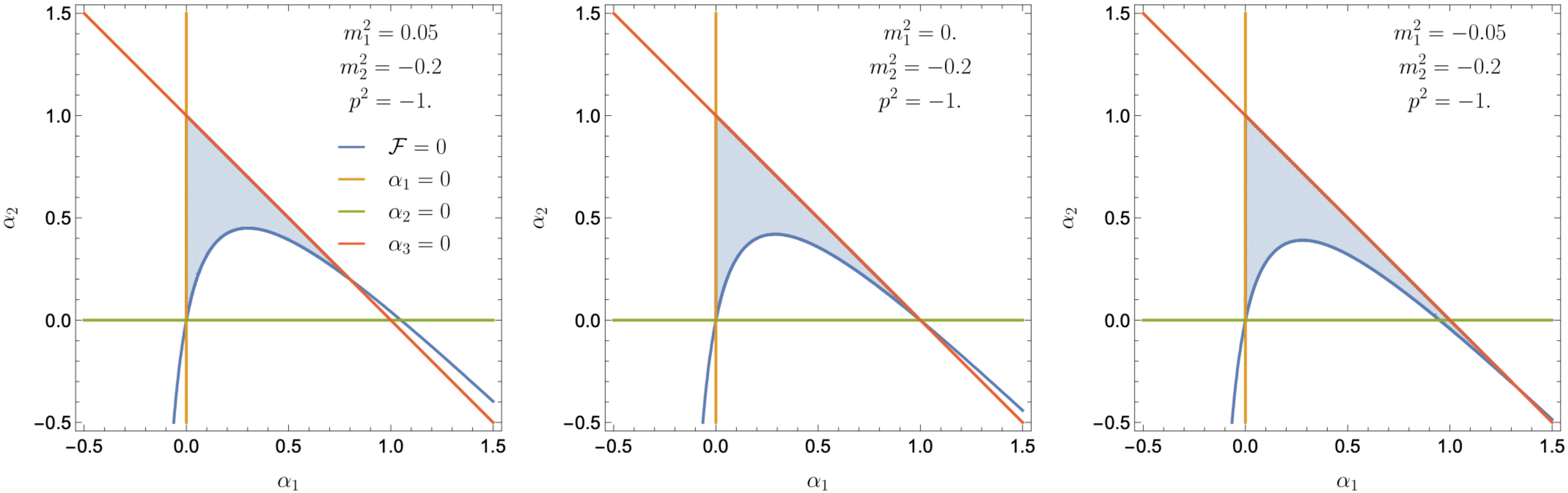
Any region can disappear the same way it appeared. Repeated letters are generically present - but absent for integer dimensions.

$m_1^2 = -0.1$
$m_2^2 = 0.4$
$p^2 = -1.$
$\mathcal{F} = 0$
$\alpha_1 = 0$
$\alpha_2 = 0$
$\alpha_3 = 0$
$m_1^2 = 0.$
$m_2^2 = 0.4$
$p^2 = -1.$
$m_1^2 = 0.1$
$m_2^2 = 0.4$
$p^2 = -1.$
α_1
α_2

$m_1{}^2 = -0.1$
$m_2{}^2 = 0.4$
$p^2 = -1.$
F=0
$\alpha_1=0$
$\alpha_2=0$
$\alpha_3=0$
$m_1^2 = 0.1$
$m_2^2 = 0.4$
$p^2 = -1.$
α_1
α_2

$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2} \right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2} \right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2} \right)$$

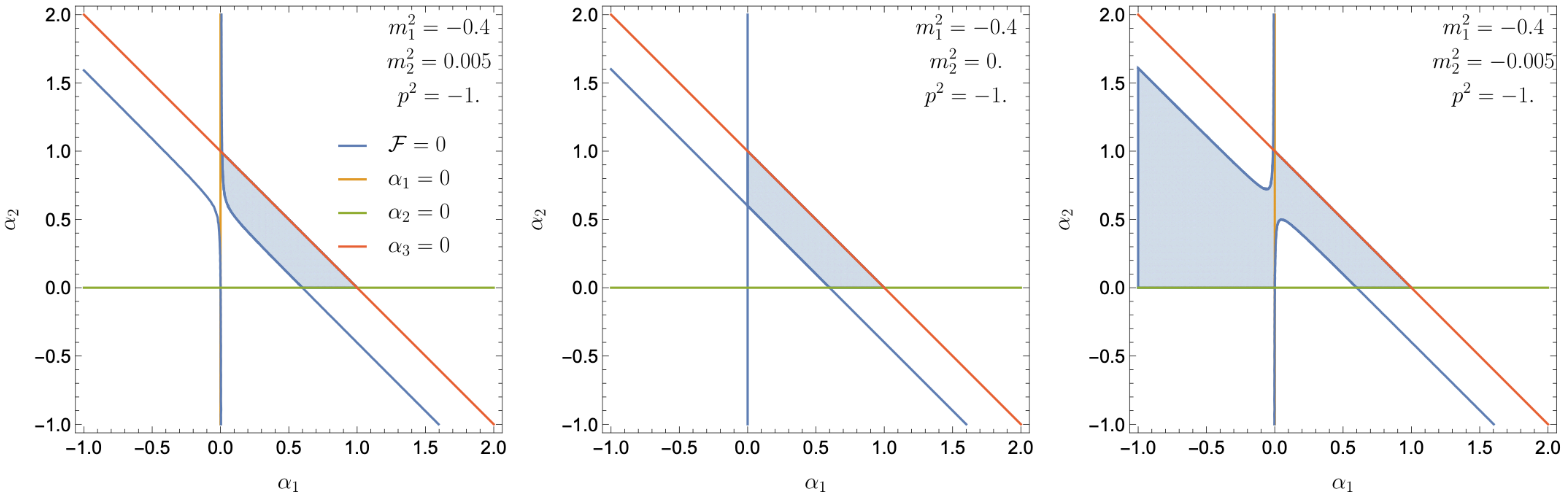
$$
\begin{aligned}
\mathcal{S}_3(I_3^{\text{hook}}) = {} & m_1^2 \otimes \frac{p^2}{p^2 - m_1^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} + m_1^2 \otimes m_1^2 \otimes \frac{m_2^2}{m_1^2 - m_2^2} + m_1^2 \otimes \frac{m_1^2 - m_2^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\
& + m_2^2 \otimes m_2^2 \otimes \frac{m_1^2 - m_2^2}{m_1^2 - m_2^2 - p^2} + m_2^2 \otimes \frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\
& - \left(p^2 - m_1^2\right) \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} + \left(p^2 - m_1^2\right) \otimes \frac{\left(p^2 - m_1^2\right)^2}{p^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} .
\end{aligned}
$$



$\Gamma_{m_2^2}$ varies smoothly through the singularity at $m_1^2 = 0.$

Hence $m_2^2 \otimes m_1^2$ is absent.

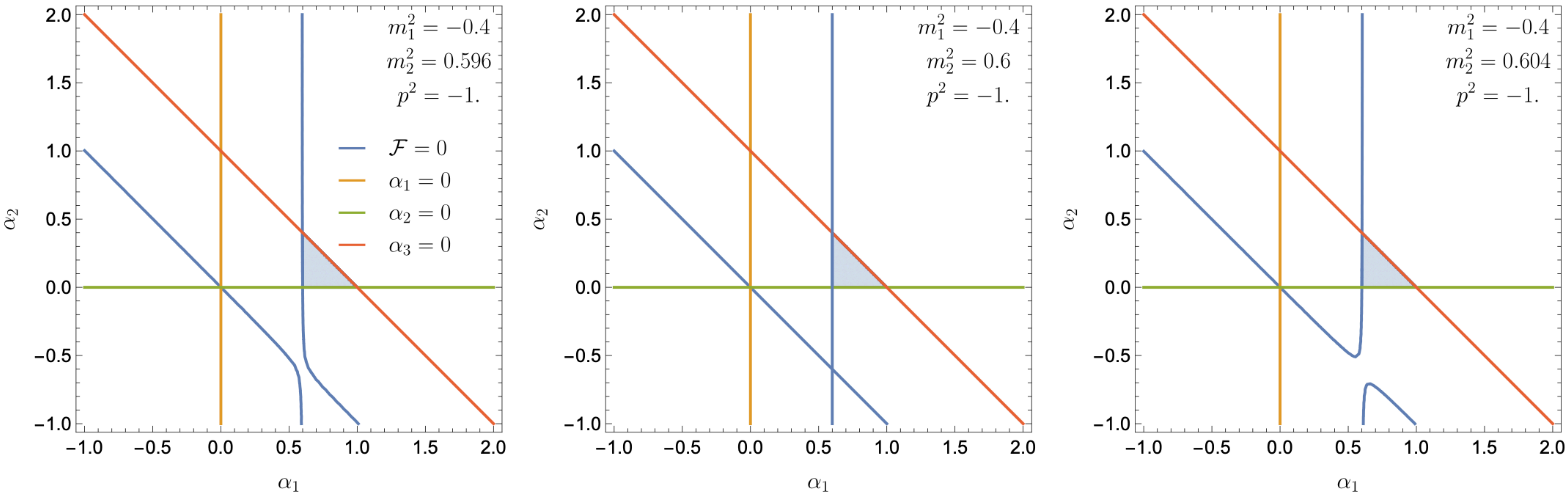
$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right)$$



However, $m_1^2 \otimes m_2^2$ is present. There is a singularity of $\Gamma_{m_1^2}$ at $m_2^2 = 0$.

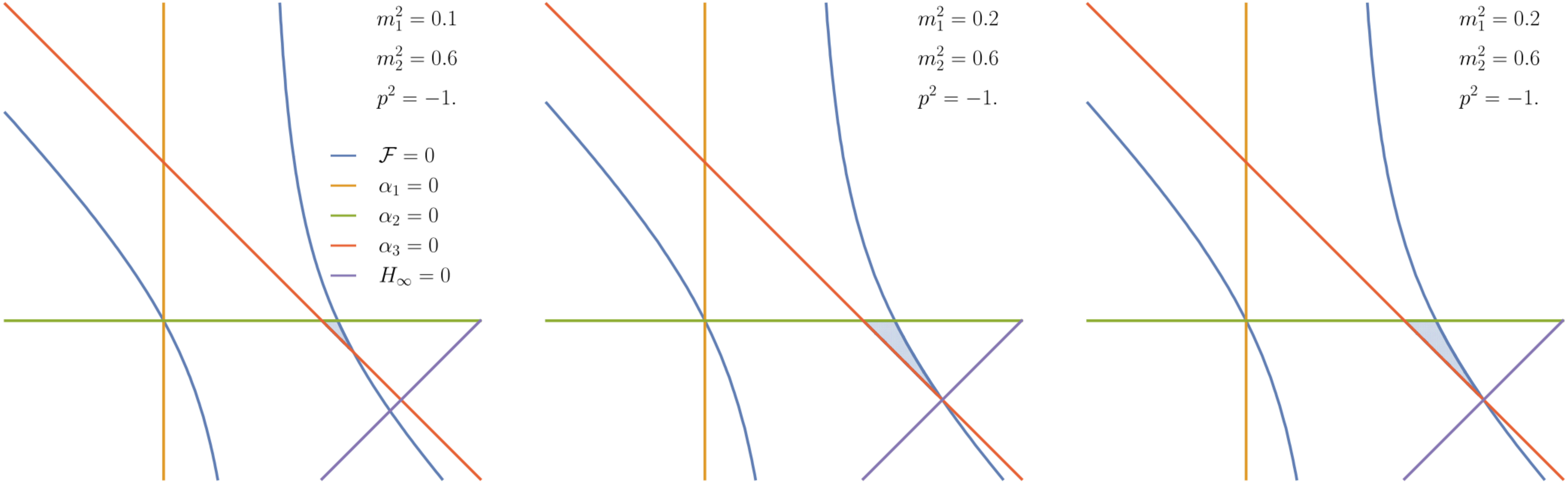
In fact, m_2^2 is a factor of the discriminant of $\mathscr{F}$.

$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right)$$



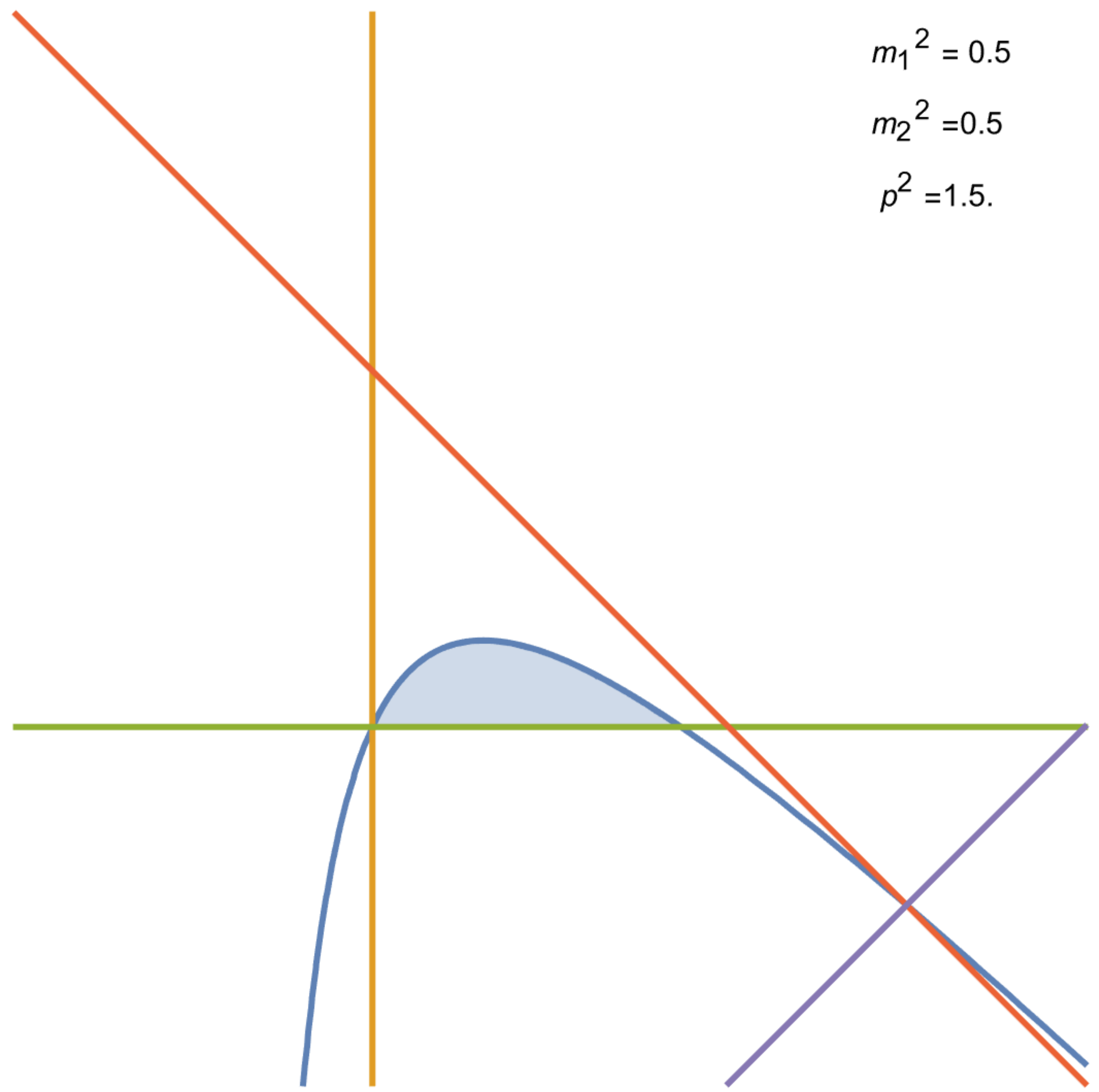
Although $m_1^2 - m_2^2 - p^2$ is another factor of the discriminant of $\mathscr{F}$, $\Gamma_{m_1^2}$ varies smoothly through this singularity. The sequence $m_1^2 \otimes (m_1^2 - m_2^2 - p^2)$ is absent. The other contours are affected.

$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right)$$



$m_1^2 - m_2^2$ is a special letter from a singularity at infinity. The intersection of $\mathscr{F} = 0$ and $\alpha_3 = 0$ goes to infinity when $m_1^2 - m_2^2 = 0$. The purple line is the hyperplane at infinity. It becomes a boundary of $\Gamma_{m_1^2}$. The sequence $m_1^2 \otimes (m_1^2 - m_2^2)$ is present.

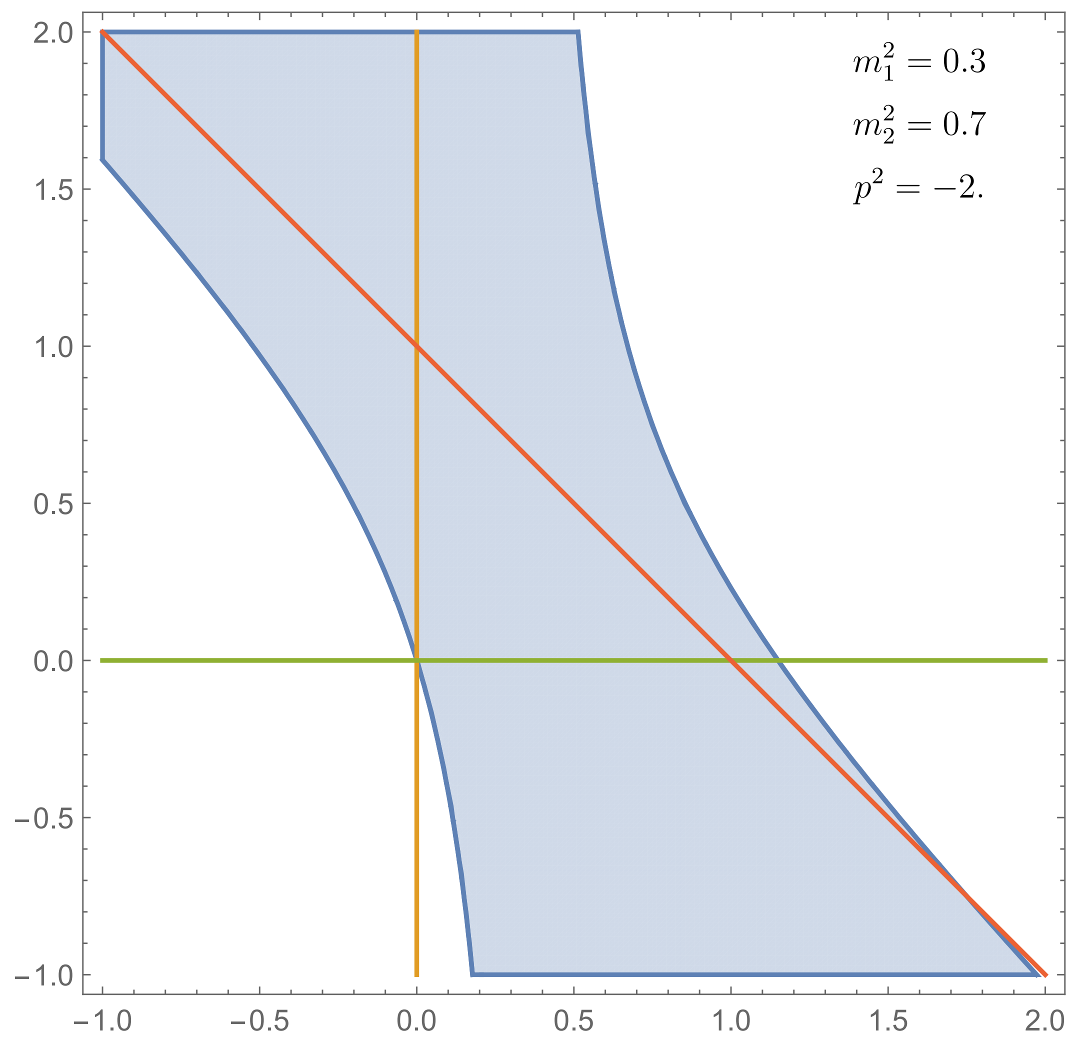
$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right)$$



However, the sequence $(p^2 - m_1^2) \otimes (m_1^2 - m_2^2)$ is absent.

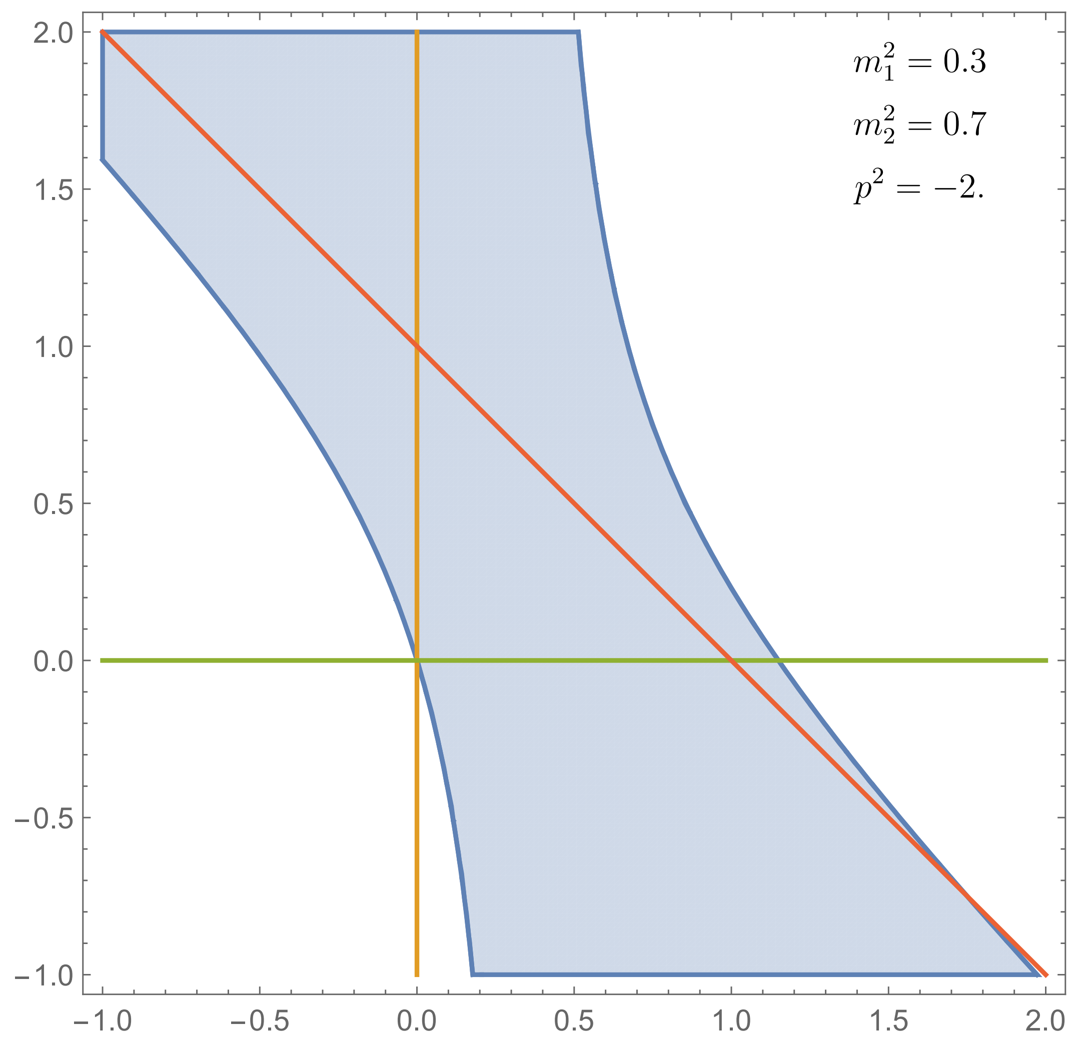
$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right)$$

The maximal cut



Γ_{max}

The maximal cut

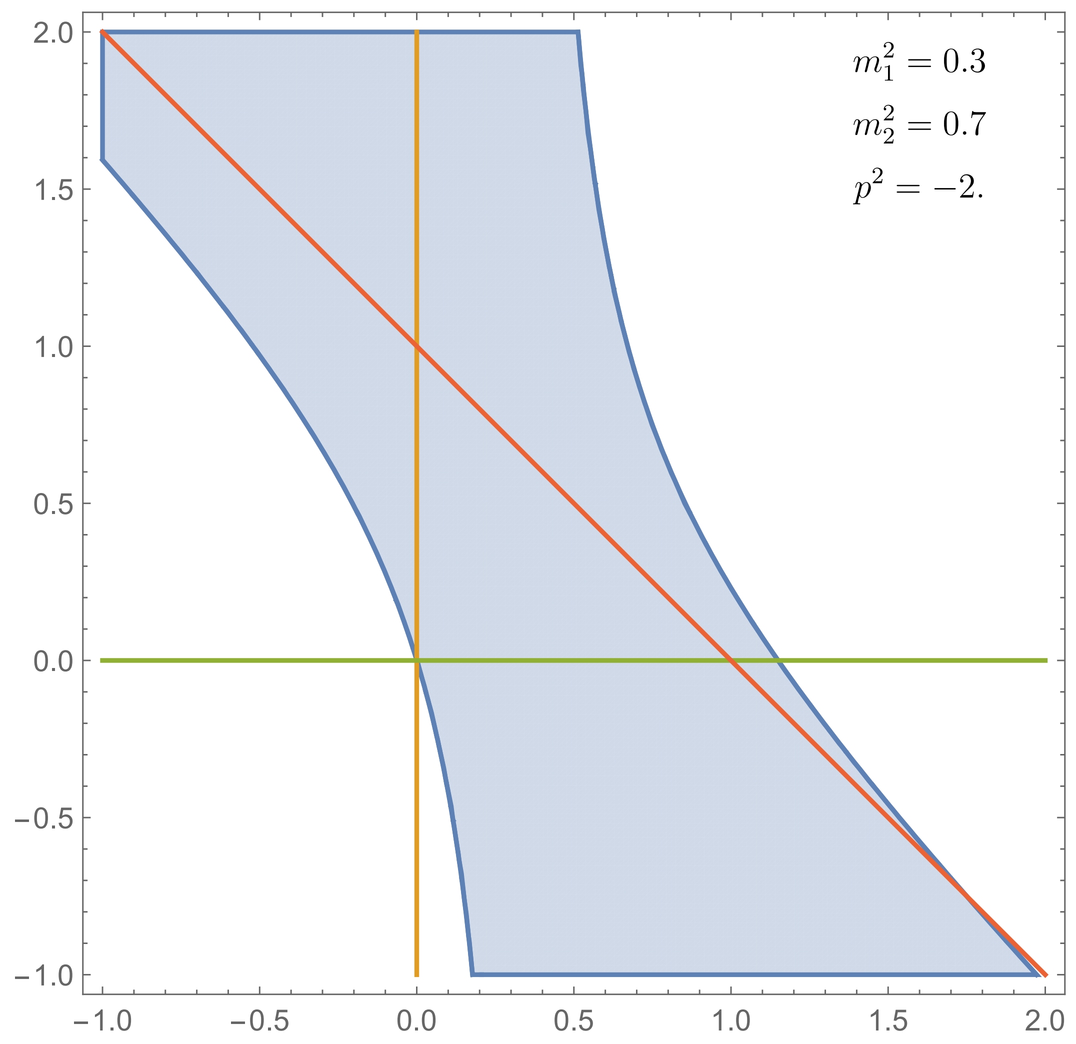


$\Gamma_{\rm max}$

Comes with the “first entry” that is of weight two, equal to the symbol of the uncut triangle at weight 2.

$$\mathcal{S}_3(I_3^{\rm hook}) = m_1^2 \otimes \frac{p^2}{p^2 - m_1^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} + m_1^2 \otimes m_1^2 \otimes \frac{m_2^2}{m_1^2 - m_2^2} + m_1^2 \otimes \frac{m_1^2 - m_2^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)}$$
$$+ m_2^2 \otimes m_2^2 \otimes \frac{m_1^2 - m_2^2}{m_1^2 - m_2^2 - p^2} + m_2^2 \otimes \frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)}$$
$$- \left(p^2 - m_1^2\right) \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} + \left(p^2 - m_1^2\right) \otimes \frac{\left(p^2 - m_1^2\right)^2}{p^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2}\,.$$

The maximal cut

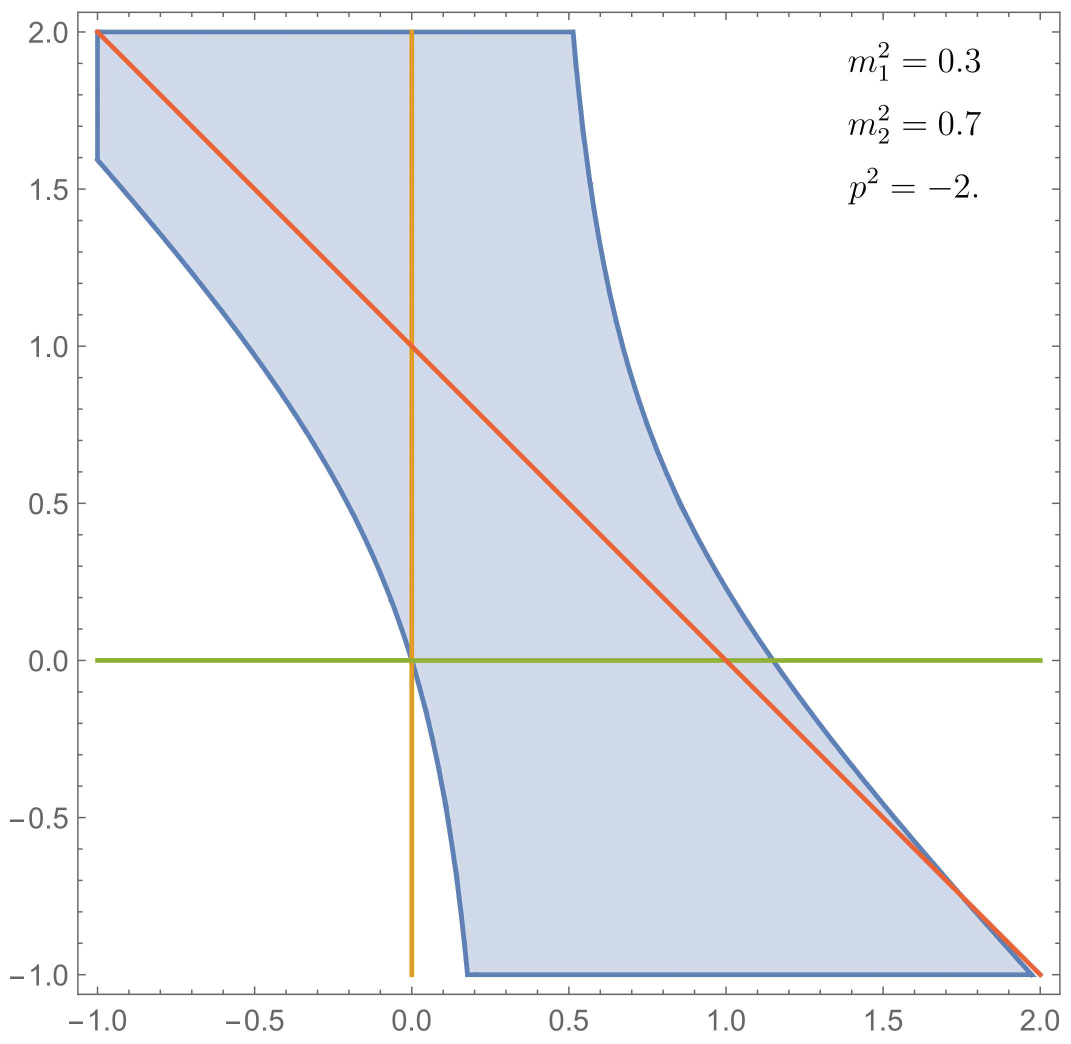


$\Gamma_{\max}$

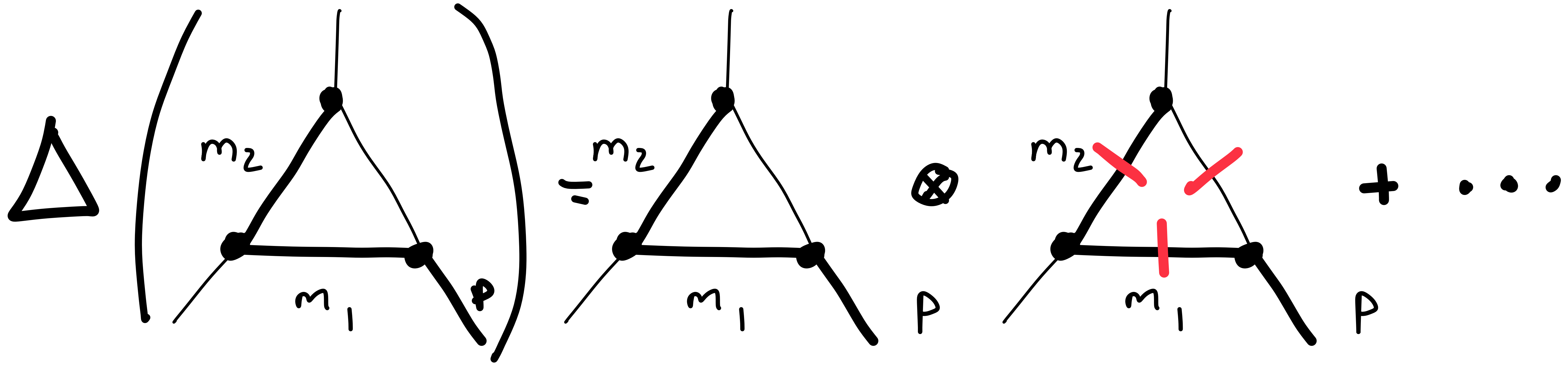
$$\mathcal{S}_2(I_3^{\text{hook}}) = -(p^2 - m_1^2) \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_2^2}\right) + m_1^2 \otimes \left(\frac{m_1^2 - m_2^2}{m_2^2}\right) + m_2^2 \otimes \left(\frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2}\right)$$

$$\begin{aligned}\mathcal{S}_3(I_3^{\text{hook}}) = {}& m_1^2 \otimes \frac{p^2}{p^2 - m_1^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} + m_1^2 \otimes m_1^2 \otimes \frac{m_2^2}{m_1^2 - m_2^2} + m_1^2 \otimes \frac{m_1^2 - m_2^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\ & + m_2^2 \otimes m_2^2 \otimes \frac{m_1^2 - m_2^2}{m_1^2 - m_2^2 - p^2} + m_2^2 \otimes \frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\ & - (p^2 - m_1^2) \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} + (p^2 - m_1^2) \otimes \frac{(p^2 - m_1^2)^2}{p^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} .\end{aligned}$$

The maximal cut



$\Gamma_{\max}$



$$
\begin{aligned}
\mathcal{S}_3(I_3^{\text{hook}}) = {} & m_1^2 \otimes \frac{p^2}{p^2 - m_1^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} + m_1^2 \otimes m_1^2 \otimes \frac{m_2^2}{m_1^2 - m_2^2} + m_1^2 \otimes \frac{m_1^2 - m_2^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\
& + m_2^2 \otimes m_2^2 \otimes \frac{m_1^2 - m_2^2}{m_1^2 - m_2^2 - p^2} + m_2^2 \otimes \frac{m_1^2 - m_2^2 - p^2}{m_1^2 - m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} \\
& - \left(p^2 - m_1^2\right) \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2} \otimes \frac{p^2}{m_2^2(m_1^2 - m_2^2 - p^2)} + \left(p^2 - m_1^2\right) \otimes \frac{\left(p^2 - m_1^2\right)^2}{p^2} \otimes \frac{m_1^2 - m_2^2 - p^2}{m_2^2}\,.
\end{aligned}
$$

Integer dimensions

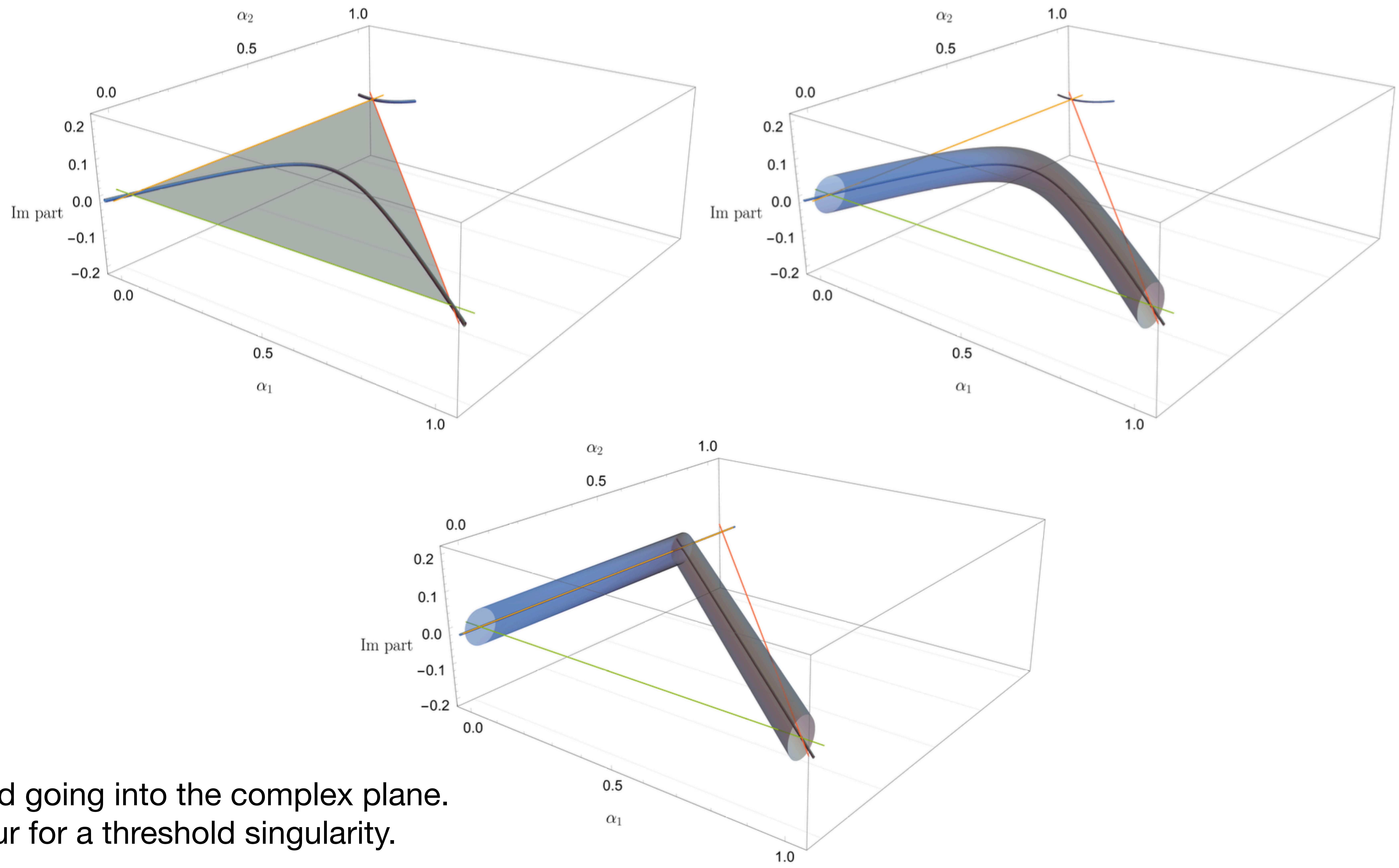
- $$(\mathscr{F} - i\varepsilon)^{-\lambda} - (\mathscr{F} + i\varepsilon)^{-\lambda} \propto \begin{cases} \delta^{\lambda-1}(\mathscr{F}) & \lambda \in \mathbb{Z} \\ \theta(-\mathscr{F})(-\mathscr{F})^{-\lambda} & \lambda \notin \mathbb{Z} \end{cases}$$

- Discontinuities become residues.

- Repeated letters are typically absent.

- Contours are tubular neighborhoods of $\mathscr{F} = 0$, bounded by a subset of coordinate hyperplanes $\alpha_i = 0$.

Example: 3-mass triangle



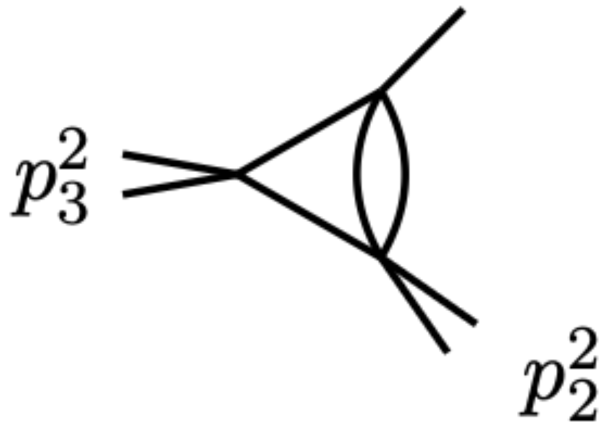
Tubular neighborhood going into the complex plane.
This is the contour for a threshold singularity.

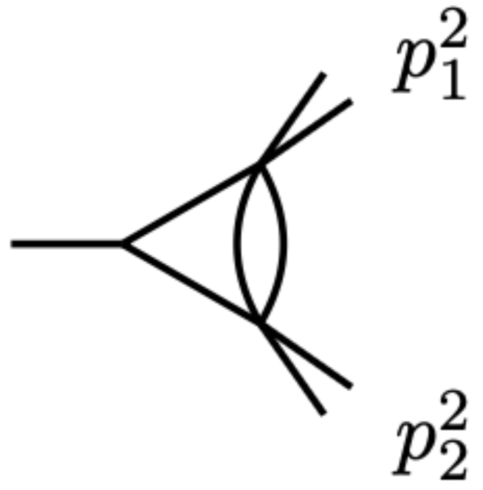
Symbol bootstrap

- Assuming knowledge of the alphabet, we used these constraints on sequential discontinuities, together with other typical considerations (integrability, first-entry, dimensionality) to fix the symbols in several examples.

  - various triangles

  - 2-loop ice cream cones





Summary and Outlook

- Homology classes for Feynman integrals are visible as regions bounded by $\mathscr{F} = 0$ and subsets of the coordinate hyperplanes, full-dimensional in dimensional regularization.

- Sequential discontinuities correspond to singularities of the integration contours.

- Some overlap with the concept of "genealogical constraints."

- To explore: how does this information interplay with diagrammatic coaction? Is it better formulated at the level of coaction?

- What is the full story of the homology of Feynman integrals? What relations are visible?