Differential Equations for Moving Hyperplane Arrangements

joint work with Anna-Laura Sattelberger

Mainz, 22nd September 2025

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Combinatorial Correlators

Generalized Euler Integral

$$\phi(c_1,\ldots,c_m) = \int_\Gamma (\ell_1(x)-c_1)^{s_1}\ldots(\ell_m(x)-c_m)^{s_m} x_1^{\nu_1}\ldots x_n^{\nu_n}\,\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_n}{x_n}$$

where Γ is a twisted n-cycle of X, and $s \in (\mathbb{C}\setminus\{0\})^m$, $\nu \in (\mathbb{C}\setminus\{1\})^n$ and $\ell_1(x) = a_1^{(1)}x_1 + \cdots + a_n^{(1)}x_n, \ldots, \ell_m(x) = a_1^{(m)}x_1 + \cdots + a_n^{(m)}x_n$ are linear forms.

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Two-Site Cosmological Correlators

$$\phi(c_1,c_2,c_3) = \int_\Gamma \frac{1}{(x_1+c_1)(x_2+c_2)(x_1+x_2+c_3)} x_1^{\varepsilon}x_2^{\varepsilon}\frac{dx_1}{x_1}\wedge\frac{dx_2}{x_2}$$

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Goal of this Paper

For ϕ a combinatorial correlator, we would like to construct combinatorially a D-ideal I annihilating ϕ.

Definitions

Weil algebra

- Weil Algebra $D_m = \mathbb{C}[c_1, \ldots, c_m]\langle \partial_{c_1}, \ldots, \partial_{c_m} \rangle$. Non-commutative but the generators satisfy Leibniz's rule: $\partial_{c_i} c_i - c_i \partial_{c_i} = 1$
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Holonomic Rank

Let I be a left-ideal of D_m. The holonomic rank of I is the dimension of $R_m / R_m I$ as a $\mathbb{C}(c_1, \ldots, c_m)$-vector space.

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Annihilator of a Function

$\mathrm{Ann}_D(f) := \{P \in D_m \mid P \bullet f = 0\}$ is a D_m-ideal.
We call f holonomic if there exists an ideal $I \subseteq Ann_D(f)$ such that I has finite holonomic rank.

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Shift Algebra

$S_n = \mathbb{C}[\nu_1, \ldots, \nu_n]\langle \sigma_{\nu_1}^{\pm 1}, \ldots, \sigma_{\nu_n}^{\pm 1} \rangle$. Non-commutative, but the generators satisfy

$$\sigma_{\nu_i}^{\pm 1} \nu_i = (\nu_i \pm 1) \sigma_{\nu_i}^{\pm 1}.$$

Construction of the ideal

First Replacement Rule

$$\frac{\partial}{\partial c_i}\phi(c_1,\ldots,c_m) = -s_i \cdot \int_\Gamma \frac{\prod_{j=1}^m(\ell_j - c_j)^{s_j}}{(\ell_i - c_i)} x_1^{\nu_1}\cdots x_n^{\nu_n}\frac{dx_1}{x_1}\wedge\cdots\frac{dx_n}{x_n}$$

We can rewrite this as

$$\partial_{c_i}\bullet\phi = -s_i\sigma_{s_i}^{-1}\bullet\phi,$$

and more generally

$$\partial_{c_{i_1}}\cdots\partial_{c_{i_k}}\bullet\phi = (-1)^k s_{i_1}\cdots s_{i_k}\sigma_{s_{i_1}}^{-1}\cdots\sigma_{s_{i_k}}^{-1}\bullet\phi.$$

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Second Replacement Rule

The integration by parts gives

$$-\sum_{i=1}^m a_j^{(i)} \partial_{c_i} \bullet \phi = \int_\Gamma \left(\partial_{x_j} \bullet \prod_{i=1}^m (\ell_i - c_i)^{s_i} \right) x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

$$\overset{\text{IBP}}{=} -(\nu_j - 1) \int_\Gamma \prod_{i=1}^m (\ell_i - c_i)^{s_i} x_1^{\nu_1} \cdots x_j^{\nu_j - 1} \cdots x_n^{\nu_n} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

It implies the replacement rules: $(\nu_j - 1)\sigma_{\nu_j}^{-1} \bullet \phi = \sum_{i=1}^m a_j^{(i)} \partial_{c_i} \bullet \phi$

Construction of the ideal

Homogeneity Operator

$\phi(c_1, \dots, c_m)$ is homogeneous, i.e.,

$$\phi(\lambda c_1, \dots, \lambda c_m) = \lambda^{s_1+\dots+s_m+\nu_1+\dots+\nu_n}\phi(c_1, \dots, c_m).$$

This implies that

$$H := c_1\partial_{c_1} + \dots + c_m\partial_{c_m} - (\sum_{i=1}^{n}\nu_i + \sum_{i=1}^{m}s_j) \in \mathrm{Ann}_{D(s,\nu)}(\phi).$$

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Operators from the Hyperplanes

With $\eta = (\prod_{i=1}^{m} (\ell_i - c_i)^{s_i}) x^\nu \frac{dx}{x}$, the relation

$$\int_\Gamma a_1^{(i)} x_1 \eta + \cdots + \int_\Gamma a_n^{(i)} x_n \eta - \int_\Gamma c_i \eta \; = \; \int_\Gamma (\ell_i - c_i) \eta$$

can be rewritten in terms of shift operators:

$$\left[a_1^{(i)} \sigma_{\nu_1} + \cdots + a_n^{(i)} \sigma_{\nu_n} - c_i \right] \bullet \phi \; = \; \sigma_{s_i} \bullet \phi \, .$$

Construction of the ideal

Operators from Circuits and Syzygies

If we can find $\{i_1, \ldots, i_k\} \subseteq [m]$ and k polynomials $p_1, \ldots, p_k \in \mathbb{C}[c_1, \ldots, c_m]$ and $q \in \mathbb{C}[c_1, \ldots, c_m]$, such that

$$p_1(\ell_{i_1} - c_{i_1}) + \cdots + p_k(\ell_{i_k} - c_{i_k}) \;=\; q,$$

we have

$$\begin{aligned}
\sum_{j=1}^{k} p_j s_{i_j} \partial_{\hat{i}_j} \bullet \phi &= \int_\Gamma \sum_{j=1}^{k} p_j s_{i_j} \frac{\prod_{l=1}^{m} (\ell_l - c_l)^{s_j}}{\prod_{j \neq l=1}^{m} (\ell_{i_l} - c_{i_l})} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1}{x_1} \wedge \cdots \frac{dx_n}{x_n} \\
&= \int_\Gamma \sum_{j=1}^{k} p_i s_{i_1} (\ell_{i_j} - c_{i_j}) \frac{\prod_{l=1}^{m} (\ell_l - c_l)^{s_j}}{\prod_{l=1}^{m} (\ell_{i_l} - c_{i_l})} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1}{x_1} \wedge \cdots \frac{dx_n}{x_n} \\
&= q \partial_{c_{i_1}} \cdots \partial_{c_{i_k}} \bullet \phi.
\end{aligned}$$

This implies that

$$p_1 s_{i_1} \partial_{\hat{i}_1} + \cdots + p_k s_{i_k} \partial_{\hat{i}_k} \;-\; q\, \partial_{c_{i_1}} \cdots \partial_{c_{i_k}} \in \mathrm{Ann}_{D(s,\nu)}(\phi).$$

Construction of the ideal

Operators from Circuits

Let

$$A = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(m)} \\ \vdots & \vdots & & \vdots \\ a_n^{(1)} & a_n^{(2)} & \cdots & a_n^{(m)} \end{bmatrix}$$

be the matrix of coefficients of the hyperplane arrangement, $C = \{i_1, \ldots, i_k\} \subset [m]$ be a circuit of A and $A_C = \left[a^{(i_1)} | a^{(i_2)} | \cdots | a^{(i_k)}\right]$ the corresponding submatrix. Let $\begin{bmatrix} p_1 \\ \vdots \\ p_k \end{bmatrix}$ be a column of $\ker(A_C)$ and let $q = c_{i_1} p_1 + \cdots + c_{i_k} p_k$. Then,

$$p_1(\ell_{i_1} - c_{i_1}) + \cdots + p_k(\ell_{i_k} - c_{i_k}) = q\,.$$

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Operators from Syzygies

Let A be the matrix of coefficients of the hyperplane arrangement as before. After performing a syzygy computation, we obtain k polynomials $p_1, \ldots, p_k$ such that

$$p_1(\ell_{i_1} - c_{i_1}) + \cdots + p_k(\ell_{i_k} - c_{i_k}) = 0\,.$$

Example of the Two-Site

Homogeneity Operator

$$H = c_1\partial_{c_1} + \ldots + c_m\partial_{c_m} - (2\varepsilon - 3).$$

Operators from the Hyperplanes

$$\begin{aligned}
L_1 \;&=\; \frac{1}{(\varepsilon-1)^2} \cdot \Big[c_1\partial_{c_1}^3 + c_1\partial_{c_1}^2\partial_{c_2} + c_1\partial_{c_1}^2\partial_{c_3} + c_1\partial_{c_1}\partial_{c_2}\partial_{c_3} \\
&\quad - (2\varepsilon-3)\partial_{c_1}^2 - (\varepsilon-2)\partial_{c_1}\partial_{c_2} - (\varepsilon-2)\partial_{c_1}\partial_{c_3} + \partial_{c_2}\partial_{c_3}\Big], \\
L_2 \;&=\; \frac{1}{\varepsilon-1}\left(c_2\partial_{c_1}\partial_{c_2} + c_2\partial_{c_2}^2 + \partial_{c_1} - (\varepsilon-2)\partial_{c_2}\right), \\
L_3 \;&=\; \frac{1}{\varepsilon-1}\left(c_3\partial_{c_1}\partial_{c_3} + c_3\partial_{c_3}^2 + \partial_{c_1} - (\varepsilon-2)\partial_{c_3}\right).
\end{aligned}$$

Operators from Circuits

$$P = (c_1 - c_2 - c_3)\partial_{c_1}\partial_{c_2}\partial_{c_3} + (\partial_{c_2}\partial_{c_3} - \partial_{c_1}\partial_{c_3} - \partial_{c_1}\partial_{c_2})$$

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L_2 &= \frac{1}{\varepsilon-1}\left(c_2\partial_{c_1}\partial_{c_2} + c_2\partial_{c_2}^2 + \partial_{c_1} - (\varepsilon-2)\partial_{c_2}\right), \\
L_3 &= \frac{1}{\varepsilon-1}\left(c_3\partial_{c_1}\partial_{c_3} + c_3\partial_{c_3}^2 + \partial_{c_1} - (\varepsilon-2)\partial_{c_3}\right).
\end{aligned}$$

Operators from Circuits

$$P = (c_1 - c_2 - c_3)\partial_{c_1}\partial_{c_2}\partial_{c_3} + (\partial_{c_2}\partial_{c_3} - \partial_{c_1}\partial_{c_3} - \partial_{c_1}\partial_{c_2})$$

Proposition

The holonomic rank of this ideal I generated by H, L_1, L_2, L_3 and P is 4. Moreover, I is the restricted GKZ-ideal.

Results and Conjectures

Proposition

If $n = 1$, the combinatorial correlator is holonomic and its holonomic rank at most m.

Proposition

If $n = 1$ or $n = 2$, the singular locus of the D_m-ideal I is contained in the discriminantal arrangement of A.

Conjecture

The D_m-ideal I has finite holonomic rank and its singular locus is contained in the discriminantal arrangement of A.

Conjecture

The D_m-ideal I is the restricted GKZ-ideal.