Gravitational Waveforms as Twisted Period Integrals: Analytic 2PM Results via Fourier-Loop IBPs

Giacomo Brunello

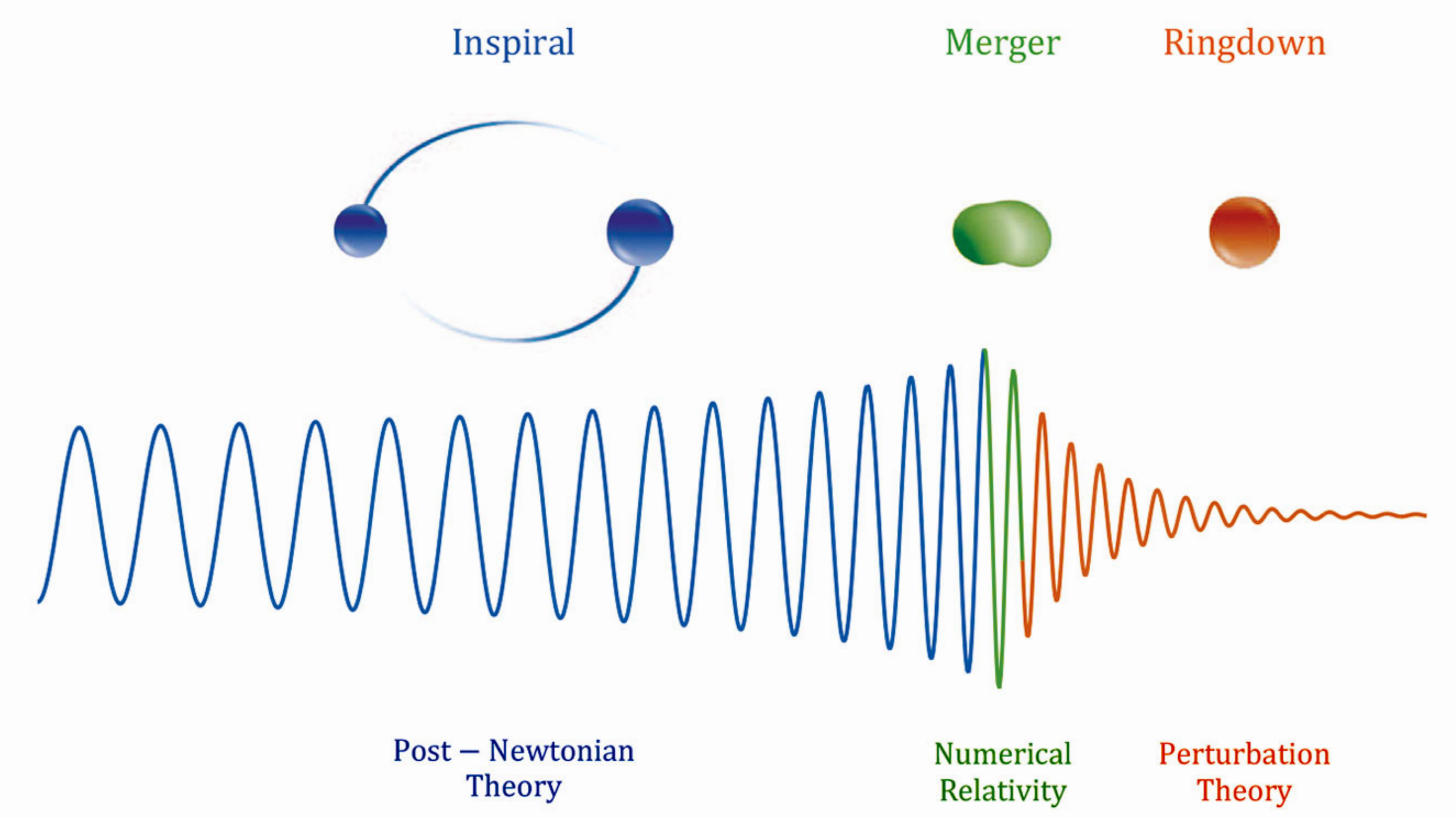
- G.B., S. De Angelis, D. A. Kosower [to appear]
- G.B., S. De Angelis [2403.08009]
- G.B., G. Crisanti, M. Giroux, P. Mastrolia, S. Smith [2311.14432]

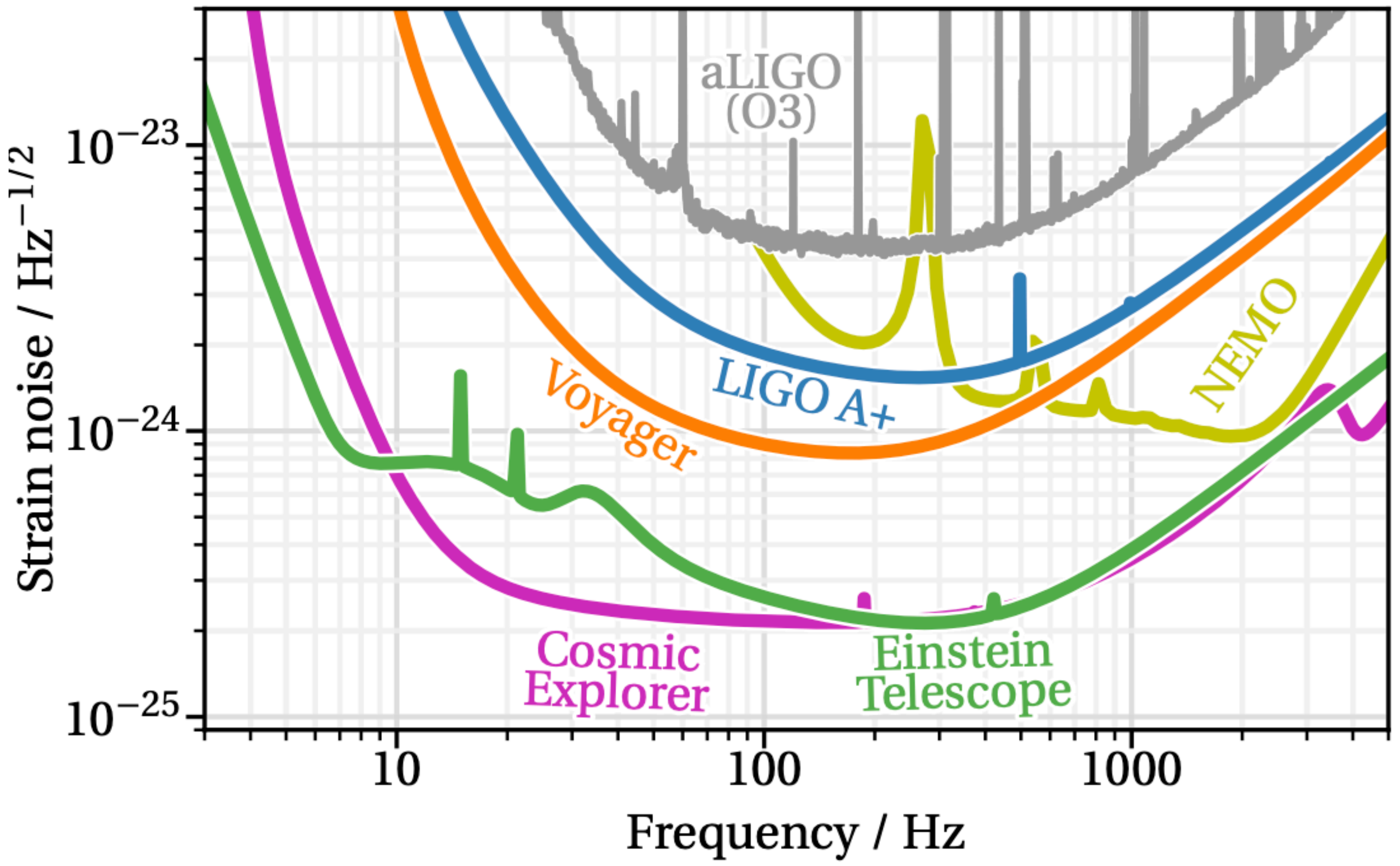
MathemAmplitudes 2025, MITP,

Mainz, 22/09/2025

Precision Era of Gravitational Waves Physics

Ligo-Virgo-Kagra efficiently detect GWs emitted by Coalescing Binary Systems.





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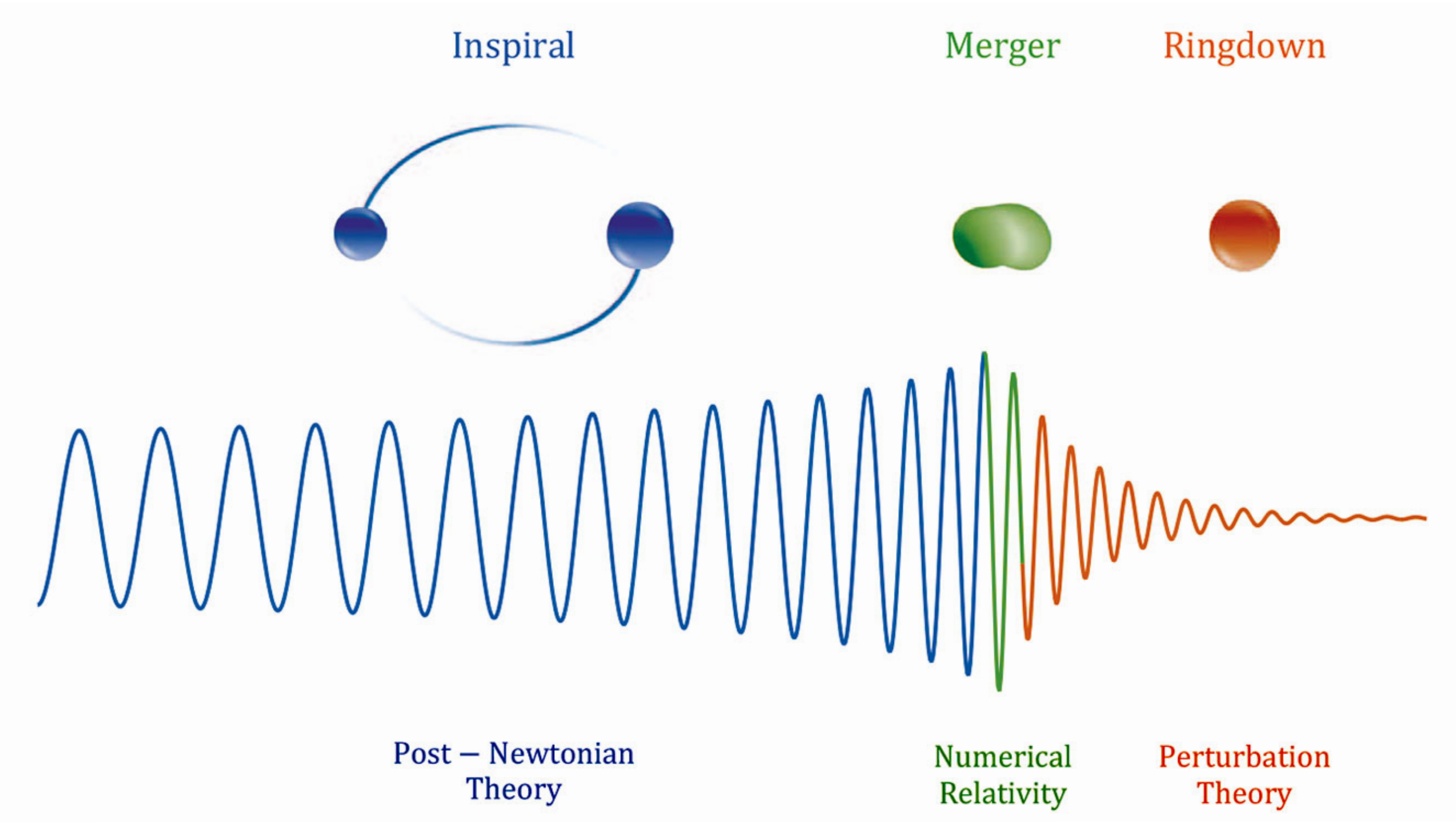

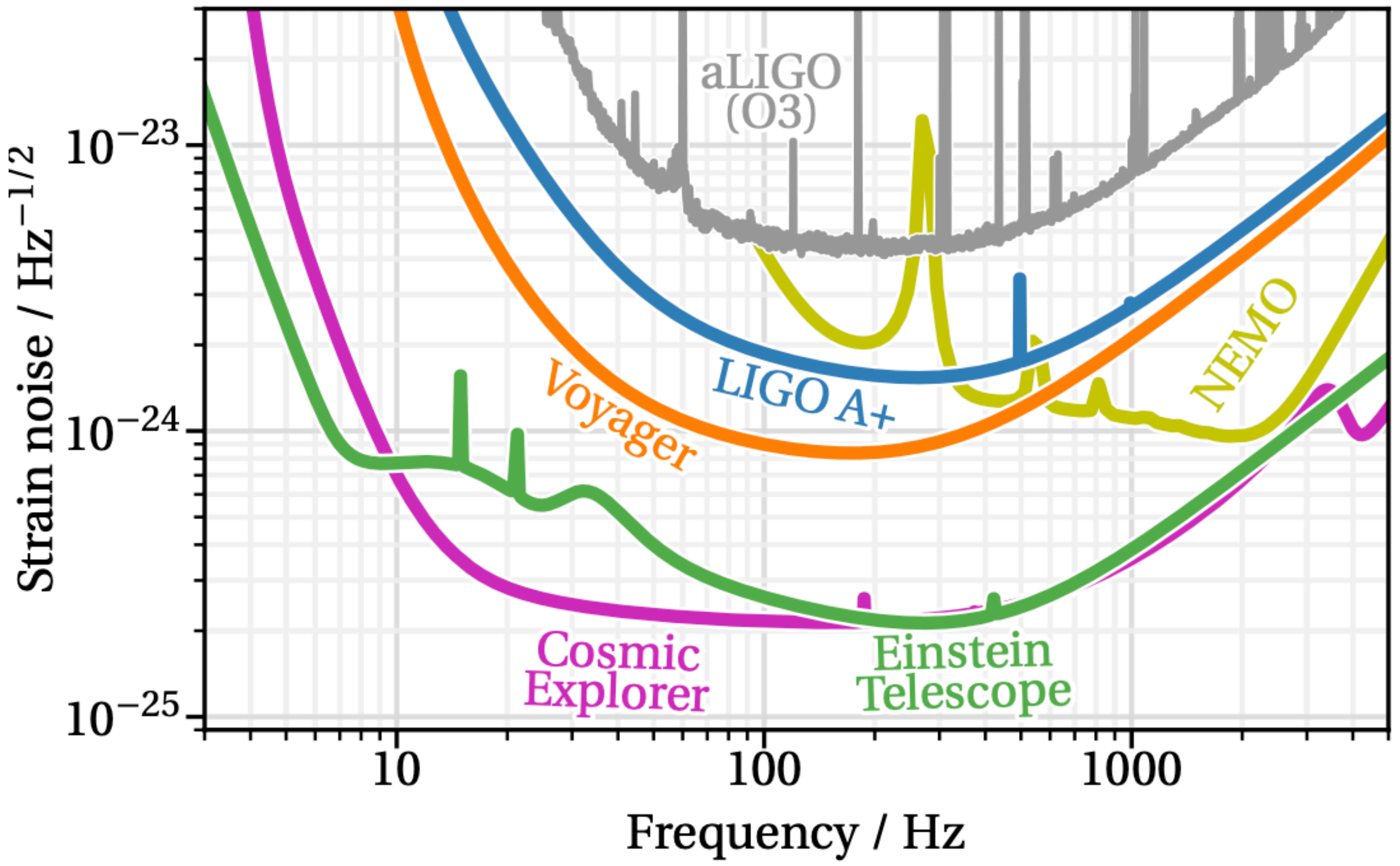

Extreme need for precise theoretical predictions for signal templates for Matched filtering analyses

The gravitational waveform

Scattering Waveform emitted by a system of two scattering black holes

$\kappa = \sqrt{32\pi G}$

$$h_{\mu\nu}\Big|_{|x|\to\infty} = \eta_{\mu\nu} + \frac{\kappa^2 M}{8\pi |x|}\left[\frac{\kappa^2 M}{\sqrt{-b^2}}\hat{h}^{(1)}_{\mu\nu} + \left(\frac{\kappa^2 M}{\sqrt{-b^2}}\right)^2 \hat{h}^{(2)}_{\mu\nu} + \ldots\right]$$

$\mathcal{O}[G]$ result:

Kovacs, Thorne
Jakobsen, Mogull, Plefka,Steinhoff
Mougiakakos, Riva, Vernizzi
Di Vecchia, Heisenberg, Russo, Veneziano

Spin:

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De Angelis, Gonzo, Novichkov
Brandhuber, Brown, Chen, Gowdy, Travaglini
Aoude, Haddad, Heissenberg, Helset
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$\mathcal{O}[G^2]$

momentum space:

Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini
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G.B., De Angelis
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Template for matched filtering analyses

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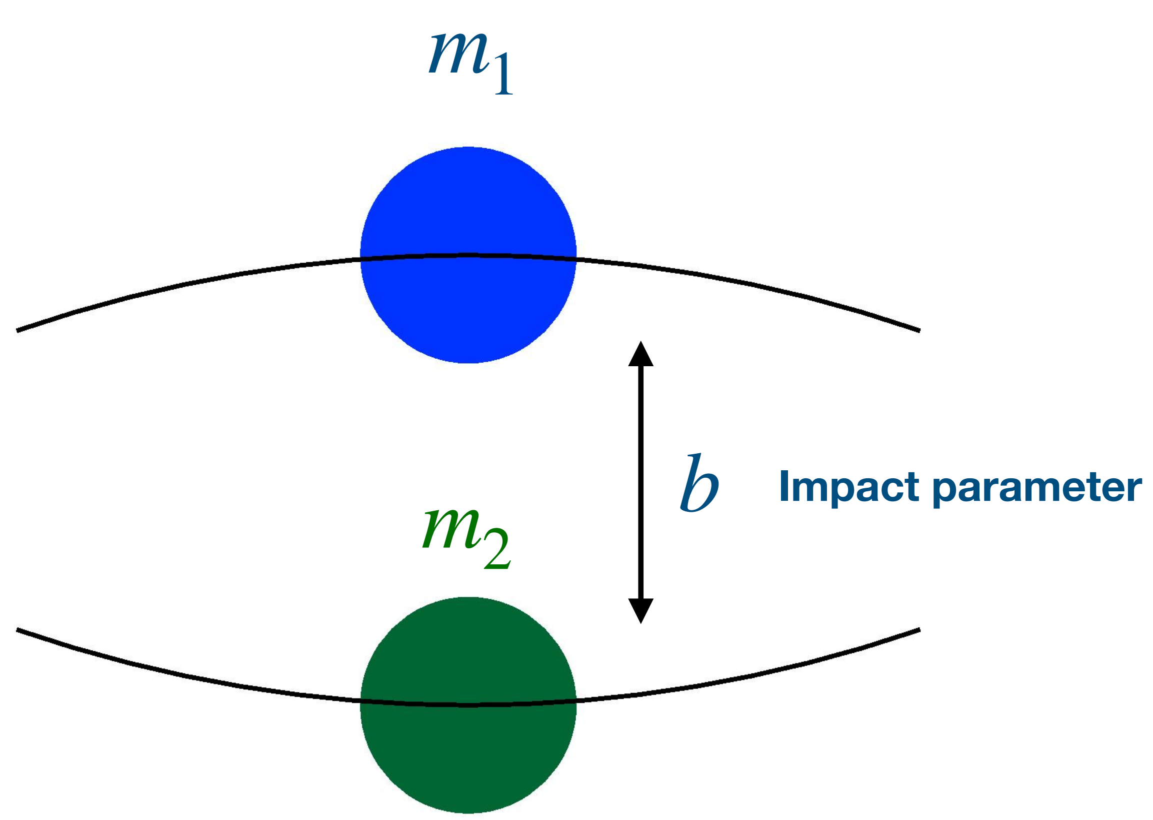
Template for matched filtering analyses

Properties of higher-points scattering amplitudes

Classical Physics from Scattering Amplitudes



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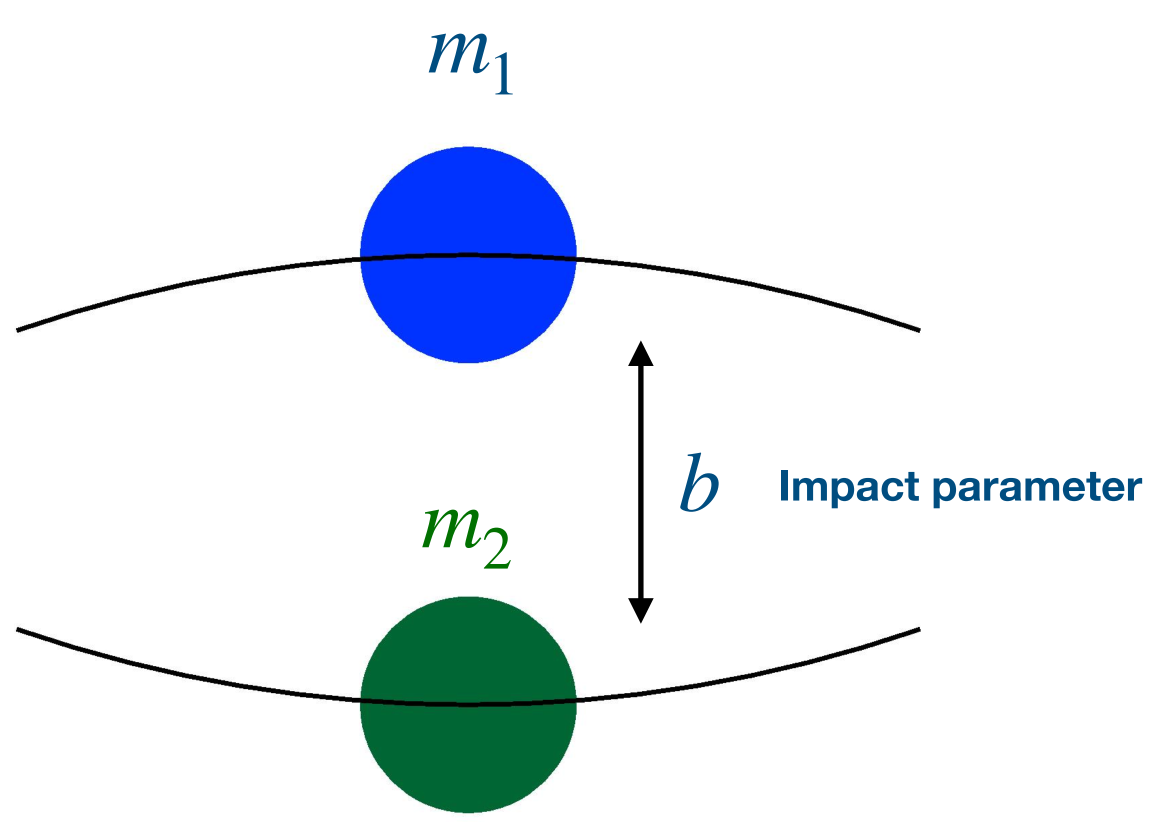

Effective Field Theory approach: Black-holes as point particles Goldberger, Rothstein

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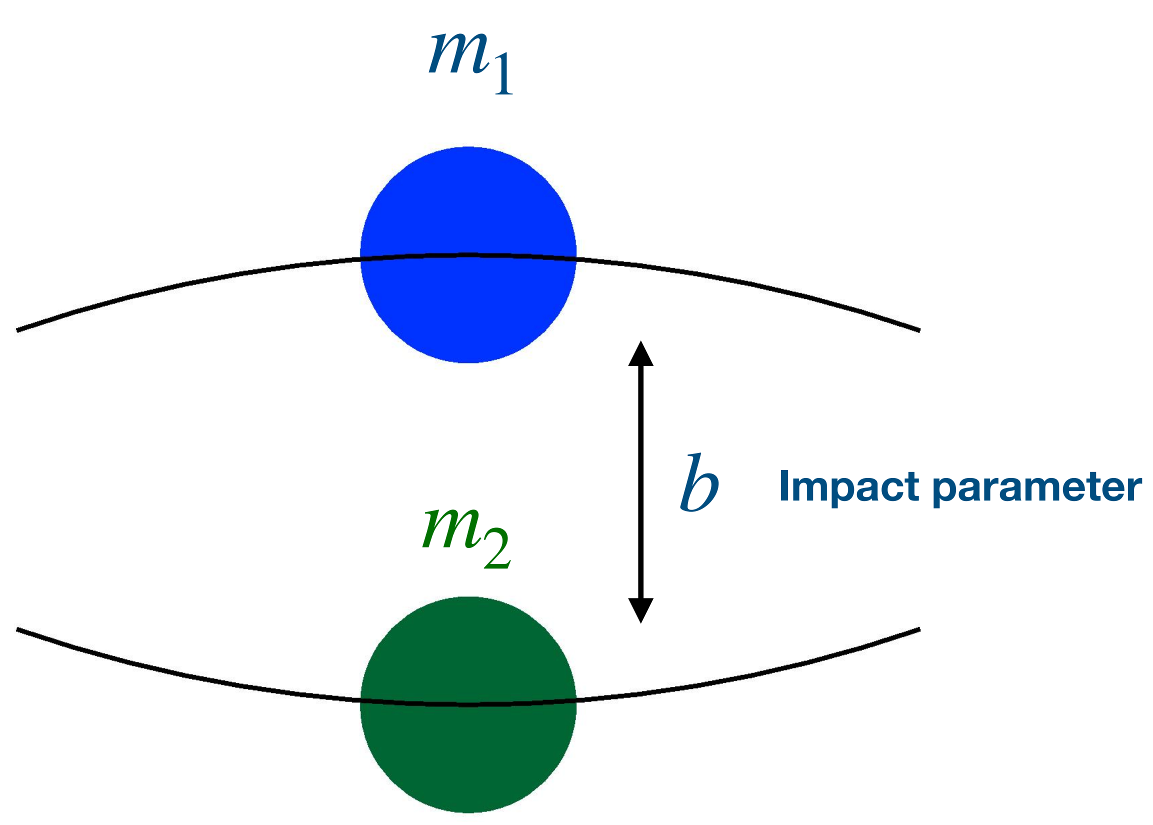

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Classical Physics is governed by long-range contributions: Local interactions are quantum

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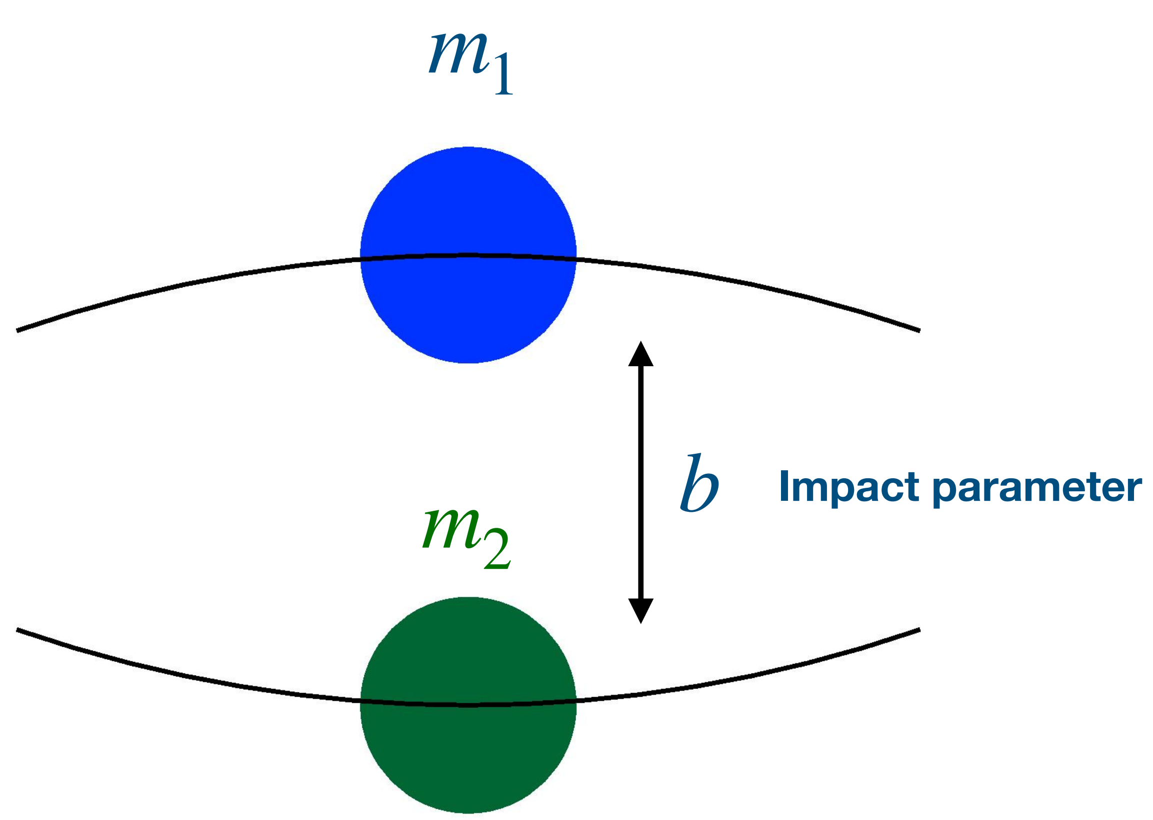

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The scales of the problem

$$\frac{1}{m} \ll G\,m \ll |b| \ll r$$

Compton wavelength   Schwarzschild radius   Impact parameter   Distance

$$\frac{Gm}{b}$$

Post-Minkowskian expansion parameter

Physical Observables from Scattering Amplitudes

Kosower, Maybee, O'Connell

Asymptotic states for two compact objects: wavefunctions peaked around their classical values

$$|\psi\rangle_{in} = \int \underbrace{d\phi(p_1)d\phi(p_2)}_{\text{On-shell phase space integral}} \; \underbrace{\phi_1(p_1)\phi_2(p_2)}_{\text{wavefunction}} \; e^{i(b_1\cdot p_1+b_2\cdot p_2)} \; \underbrace{|p_1,p_2\rangle_{in}}_{\text{Two particle momentum eigenstates}}$$

$$|\psi\rangle_{out} = S|\psi\rangle_{in} \qquad S = 1 + i\ T$$

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Expectation value of a physical observable

$$\Delta\langle\mathcal{O}\rangle = {}_{out}\langle\psi|\mathcal{O}|\psi\rangle_{out} - {}_{in}\langle\psi|\mathcal{O}|\psi\rangle_{in} = {}_{in}\langle\psi|S^\dagger[\mathcal{O}, S]|\psi\rangle_{in}$$

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The gravitational waveform

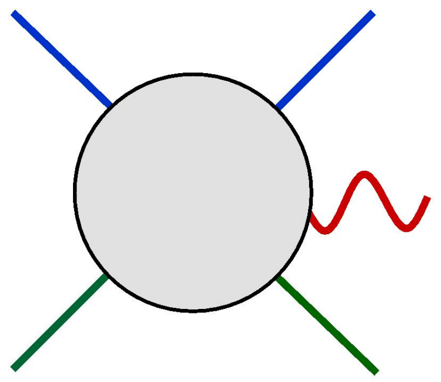
$$\mathcal{O} = \mathcal{O}_{\mathscr{W}} = \kappa\;\epsilon_\lambda^{\mu\nu} h_{\mu\nu} \qquad\qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa\; h_{\mu\nu}$$

Scattering Waveforms from Amplitudes

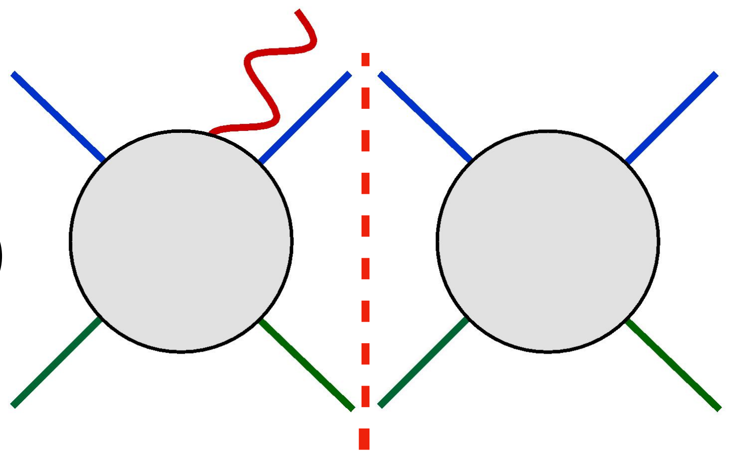
Cristofoli, Gonzo, Kosower, O'Connell

The Fourier transform of $2 \to 3$ scattering amplitudes.

$$\widetilde{\mathscr{W}}_h(u,\mathbf{n}) = \frac{1}{4\pi r}\int_0^\infty \hat{d}\omega \int d\mu \Big\{ \hat{\delta}^D(q_1+q_2-k)\, e^{-i\omega u} \Big[$$



$$- i \int d(LIPS)$$



$$\Big] + c.c. \Big\}$$

Fourier Transform

$$d\mu = \prod_{i=1}^{2} \hat{d}^D q_i \delta(2p_i \cdot q_i + q_i^2) e^{ib_i \cdot q_i}$$

Five-points amplitude

$$\mathscr{M}(p_1 p_2 \to p_1' p_2' k^h)$$

Iteration terms

$$\mathscr{M}^*(\tilde{p}_1' \tilde{p}_2' \to \tilde{X}) \otimes \mathscr{M}(p_1 p_2 \to X k^{-h})$$

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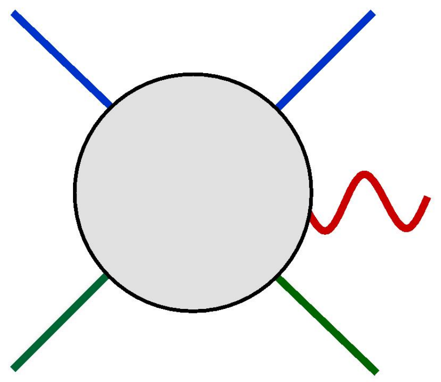

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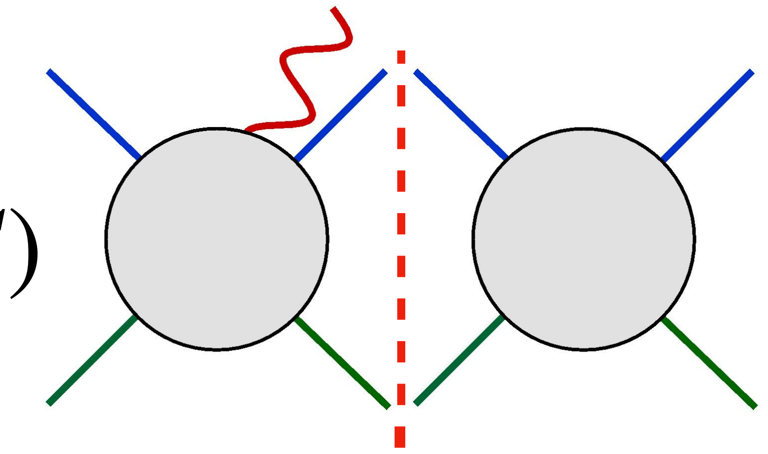

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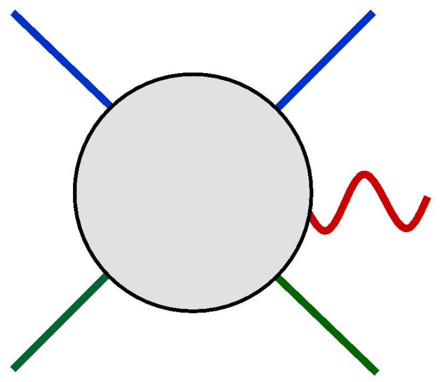

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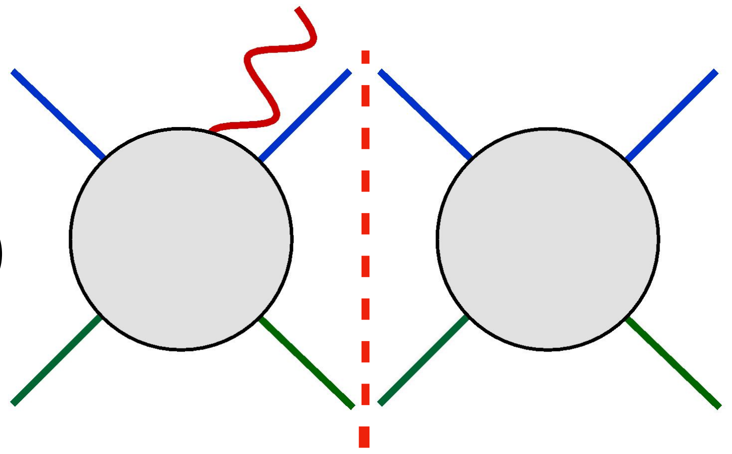

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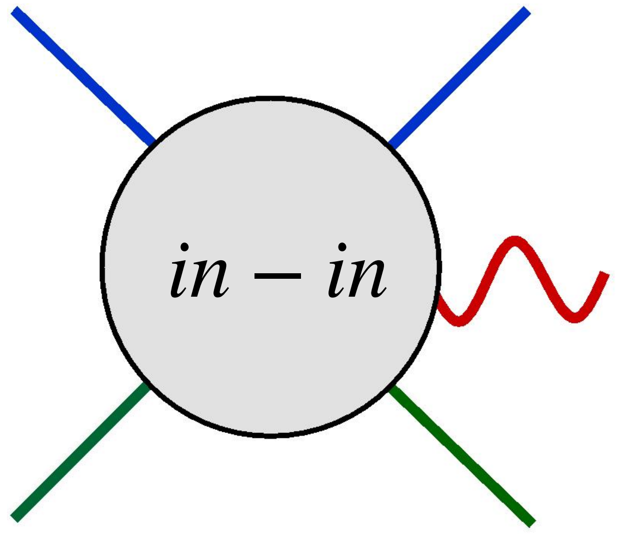
Scattering amplitudes are in-out: Iteration terms restore in-in prescription

Caron-Huot, Giroux, Hannesdottir, Mizera

How to compute waveforms?

Loop and Fourier integration are intertwined together: loop-by-loop approach is very cumbersome

$$\mathscr{W}_h(\omega, \mathbf{n}) = \int d\hat{\mu}\ e^{ib\cdot q} \Bigg\{$$



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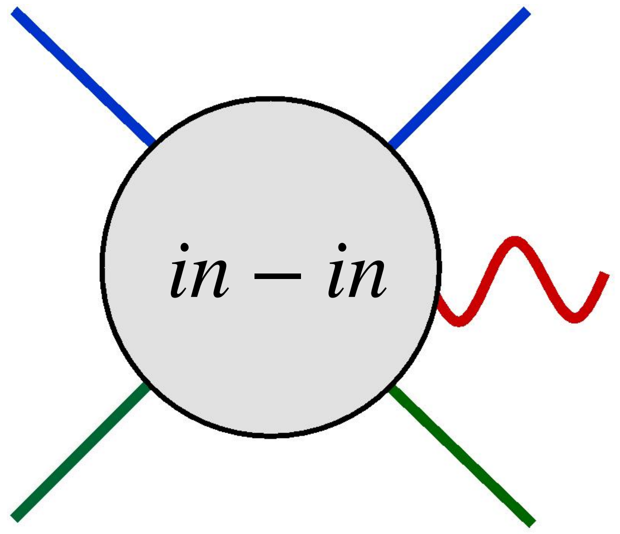

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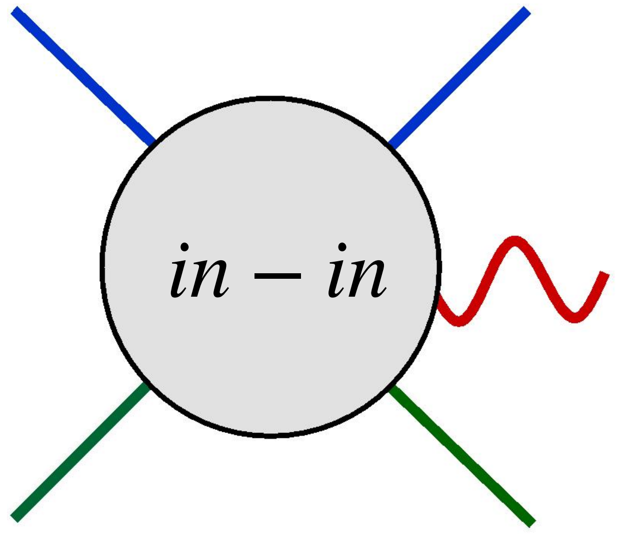

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GB, De Angelis

GB, De Angelis, Kosower

A combined Fourier-Loop approach: Scattering Amplitudes techniques in impact parameter space

Integrand Generation

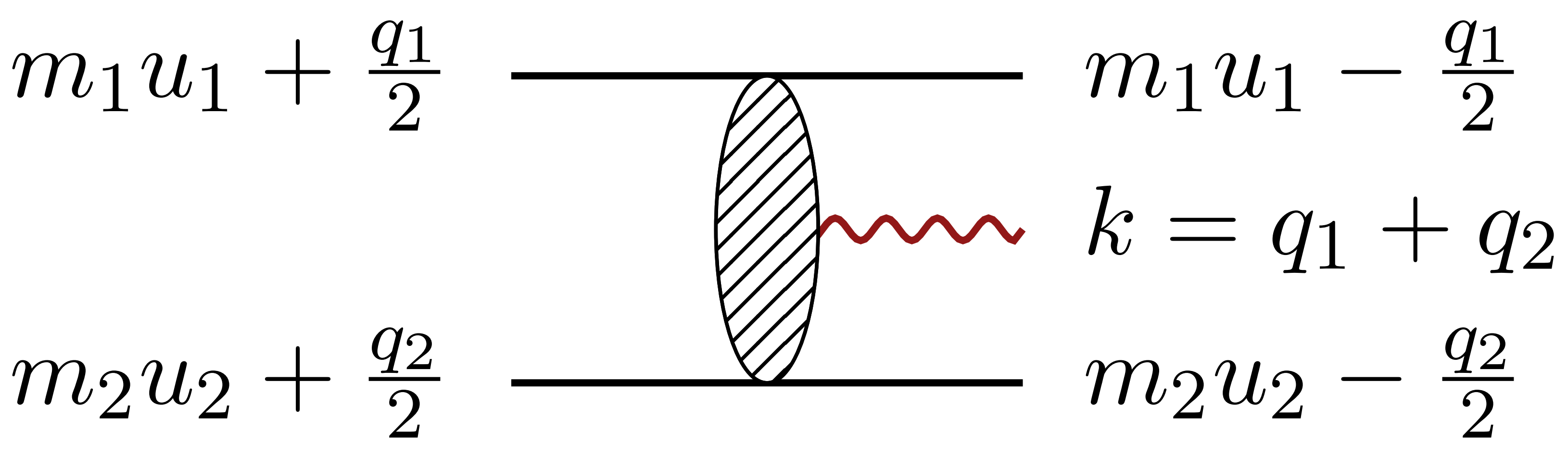
Integral Reduction

Integral Evaluation

Where are classical contributions?

Where are classical contributions?

Five-points amplitudes in momentum space



$$y = u_1 \cdot u_2 > 1$$

$$w_i = u_i \cdot k > 0$$

$$q_i^2 < 0$$

Where are classical contributions?

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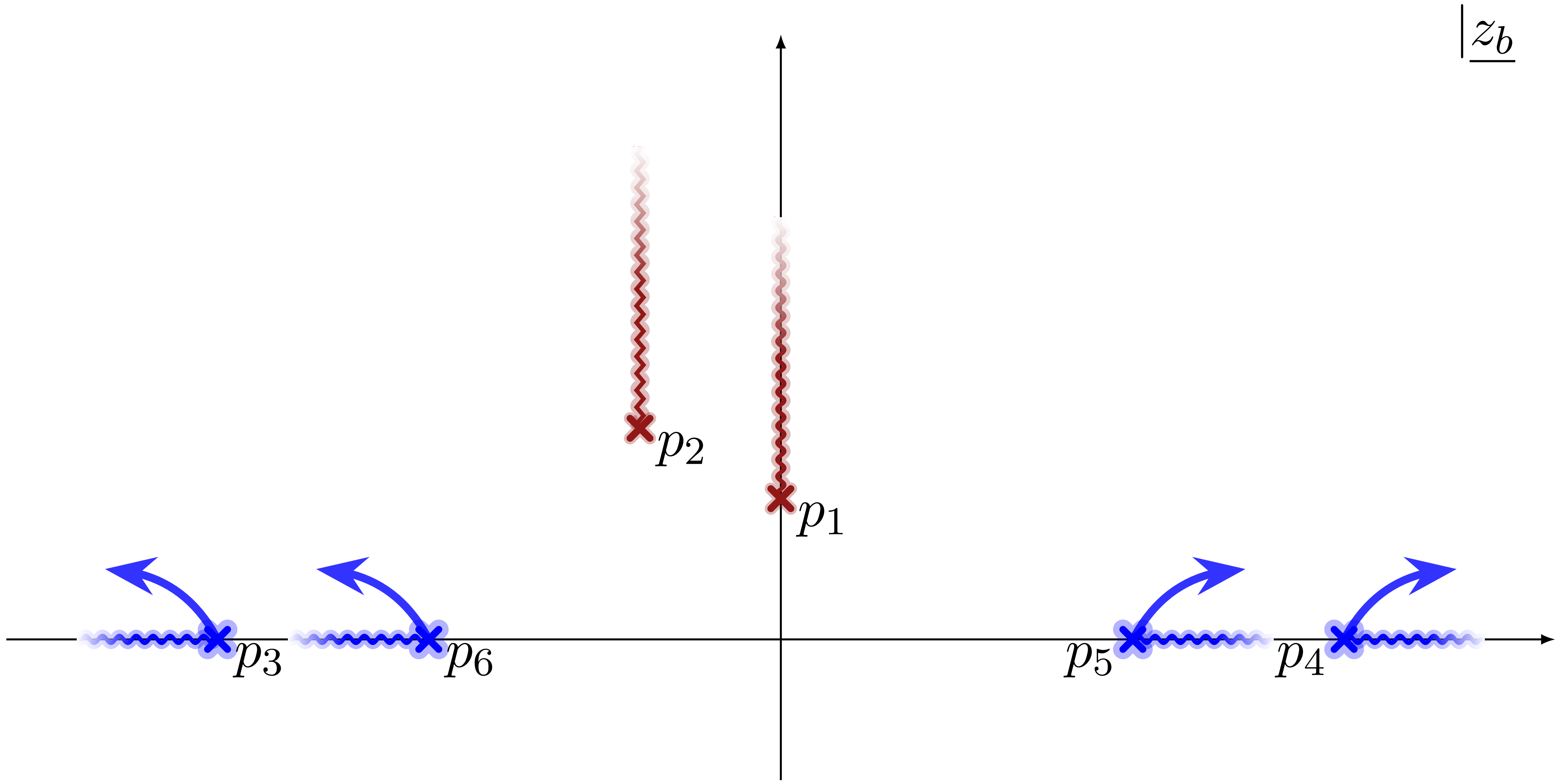
The singularities of five-points scattering amplitudes



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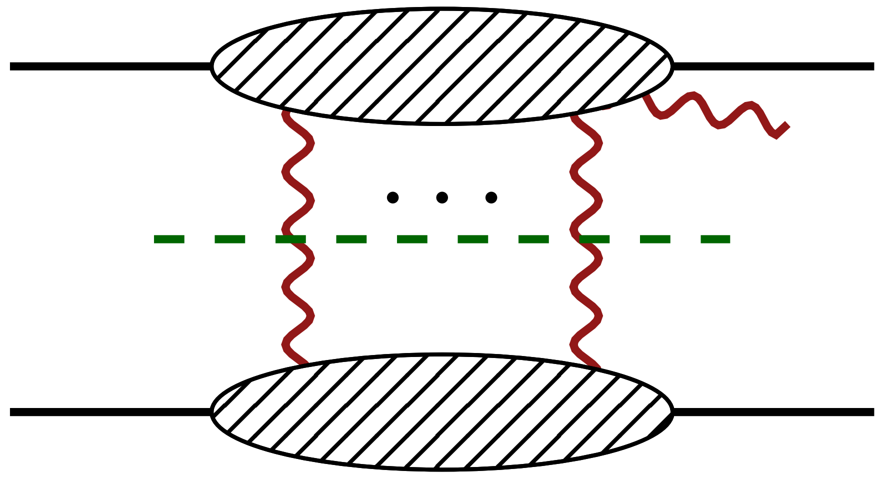
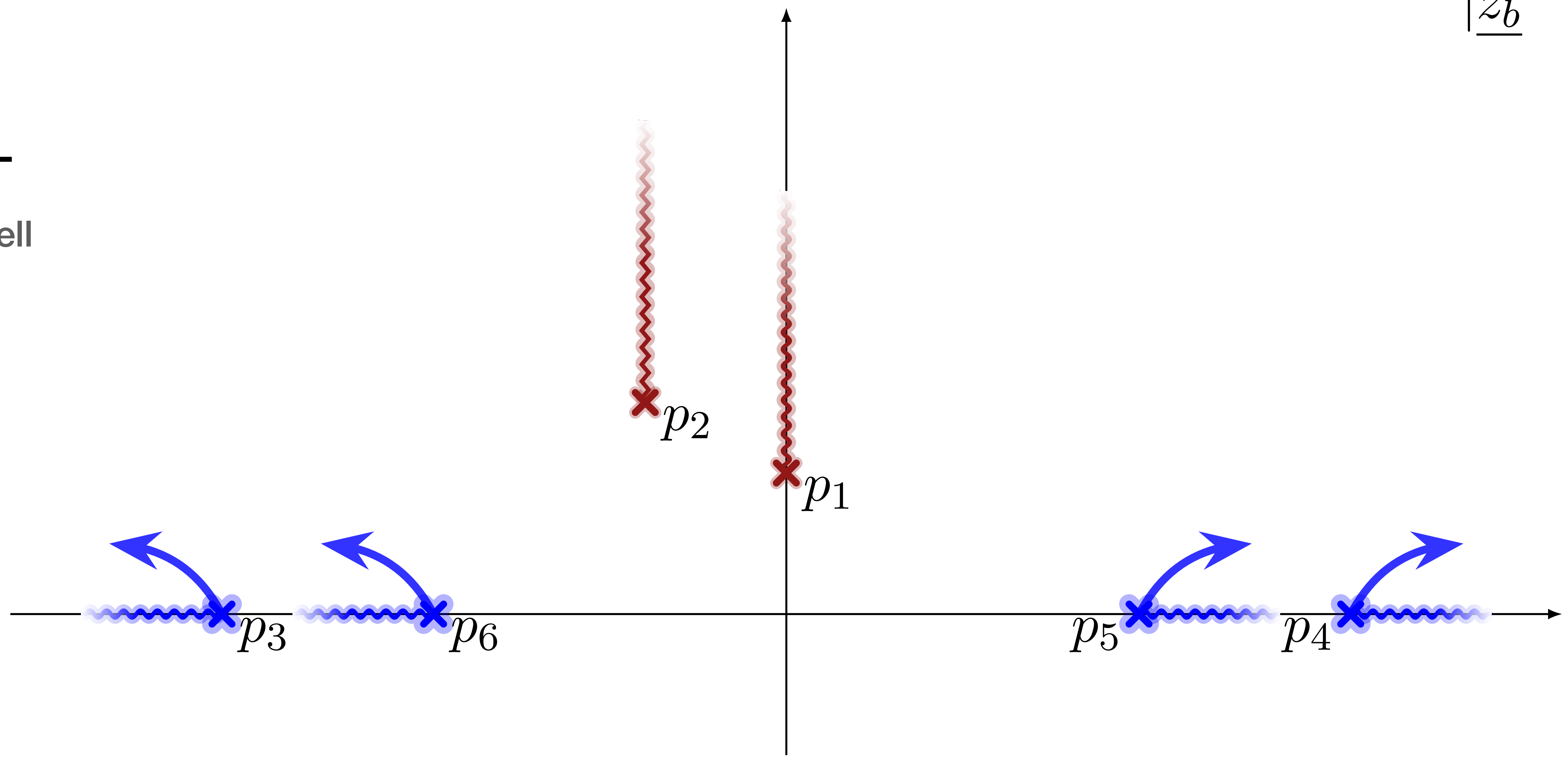


Intermediate gravitons going on-shell

$q_i^2 = 0$



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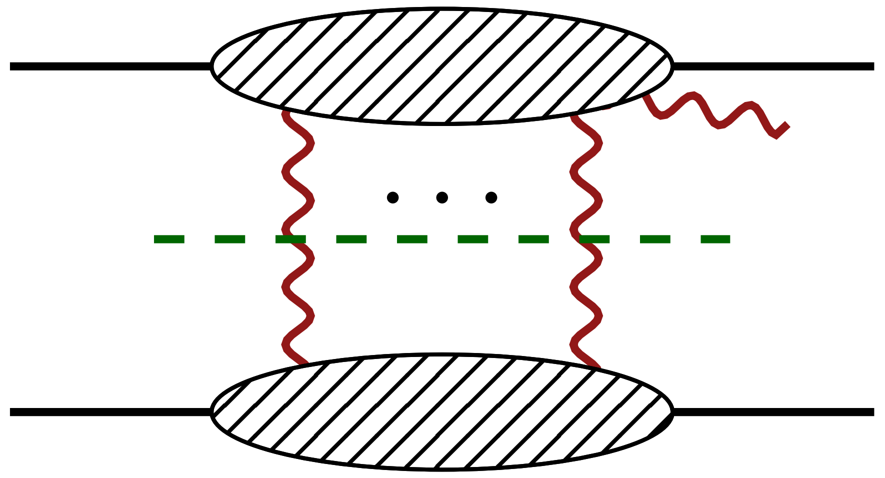

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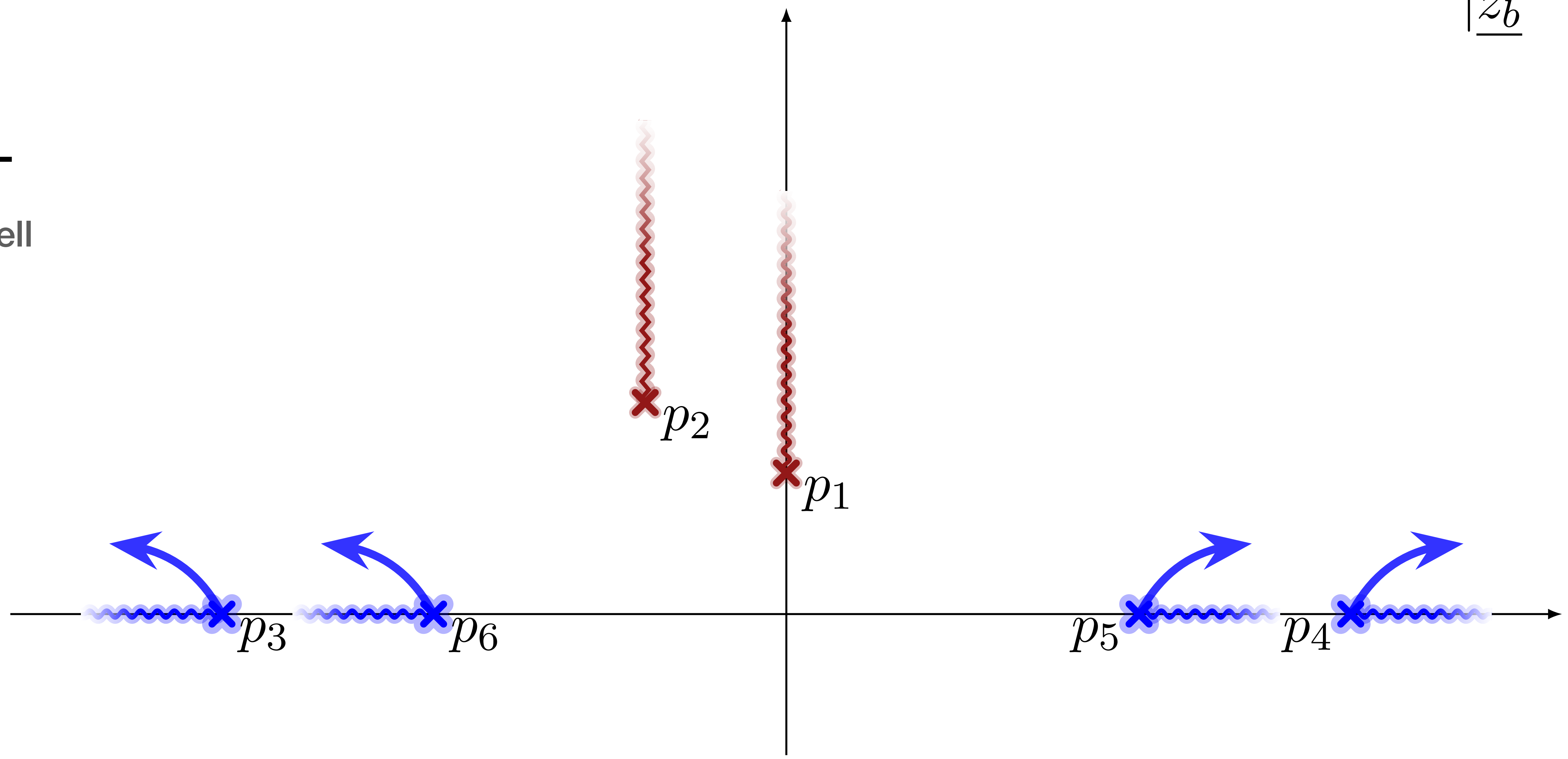

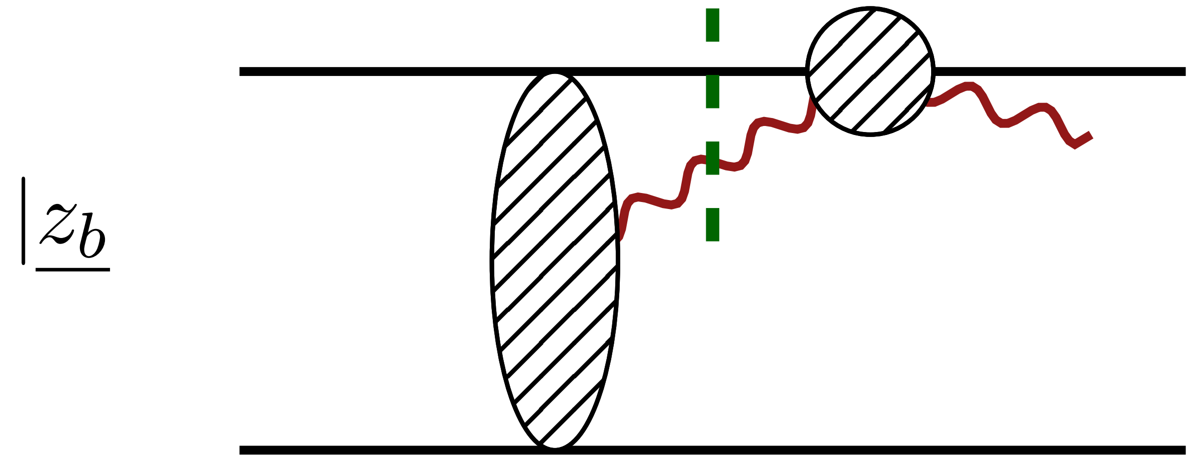

Massive (classical) particles going on-shell

$$\left(m_i u_i \pm \frac{q_i}{2} - k\right)^2 - m_i^2 = 0$$

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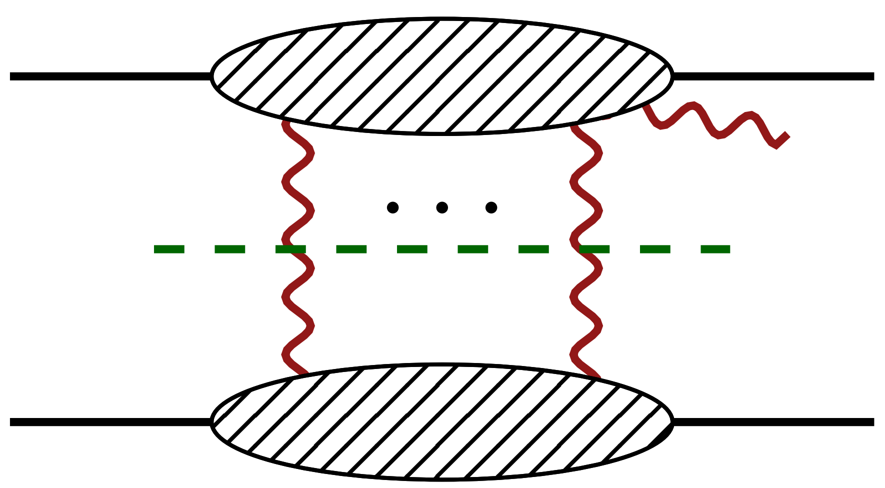
The singularities of five-points scattering amplitudes



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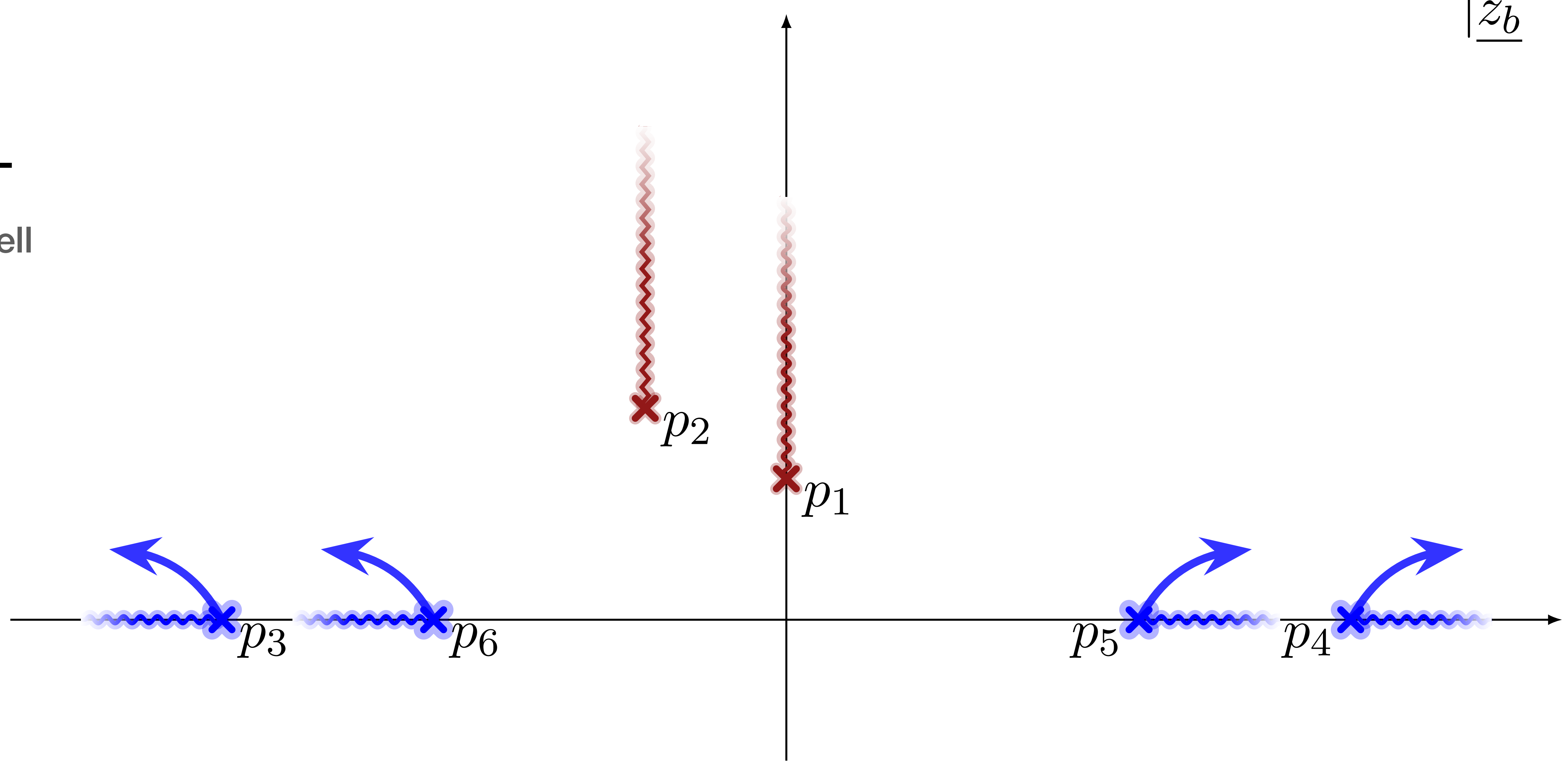

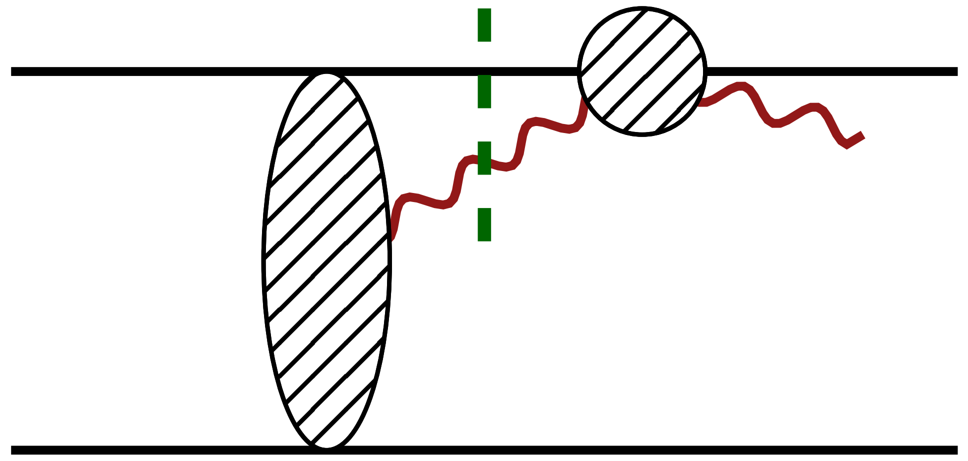

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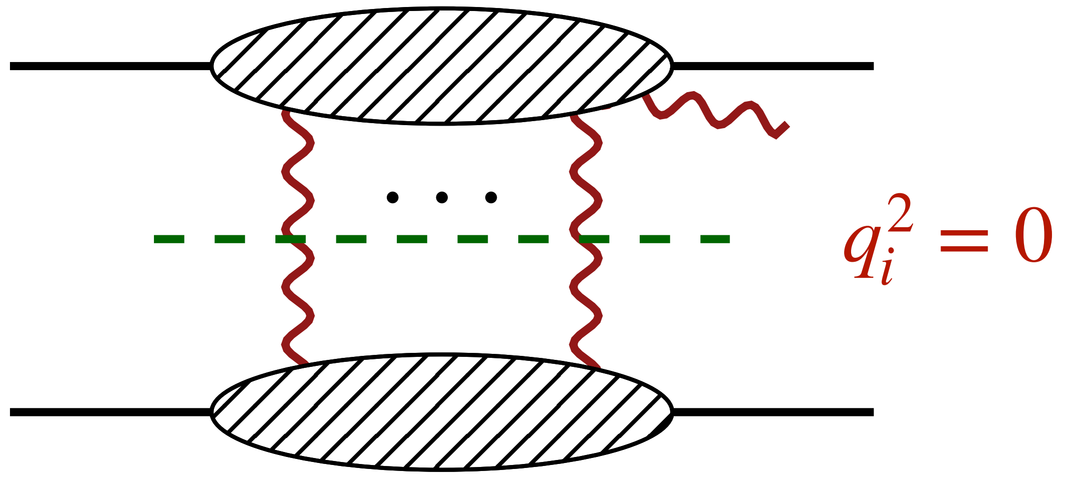
Classical limit captured by on-shell gravitons, corresponding to long-range interactions

Integrand generation: diagrammatics



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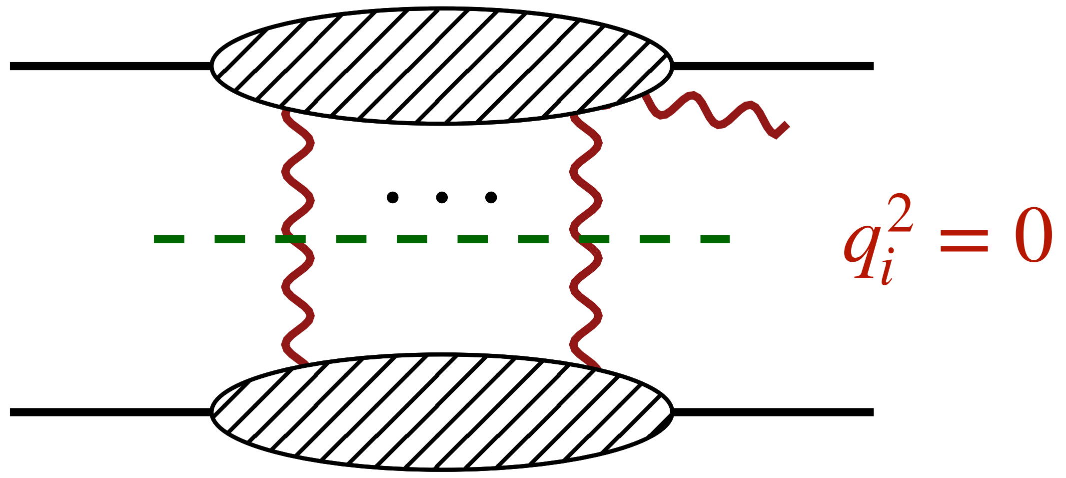
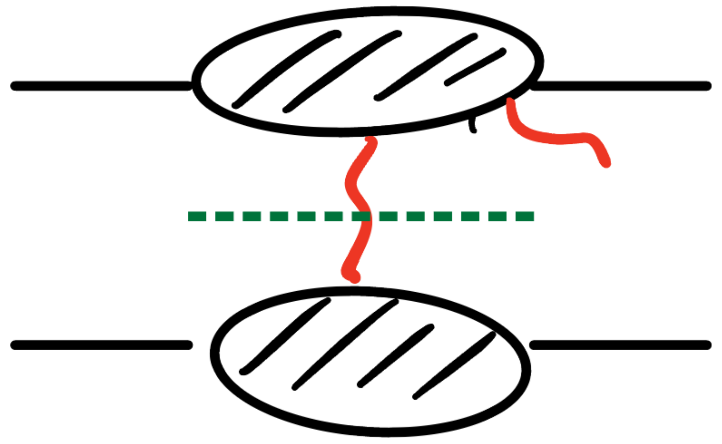


Intermediate gravitons going on-shell

LO



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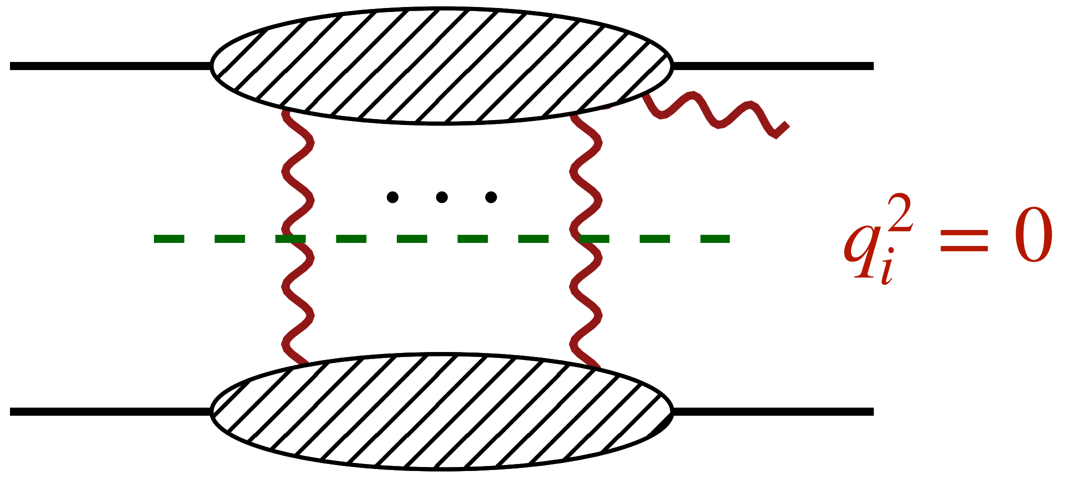

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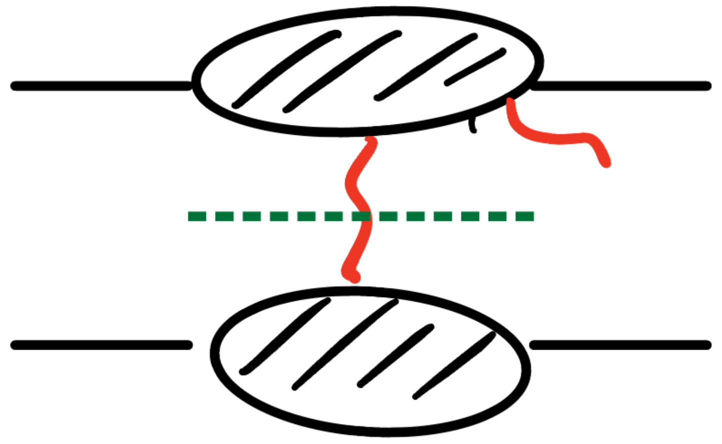
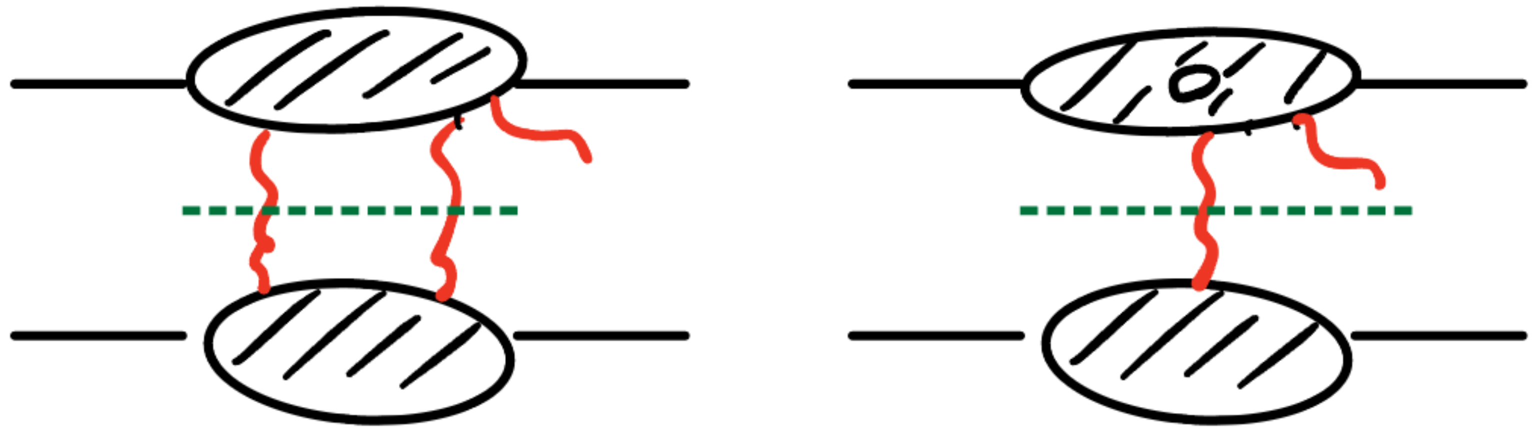


NLO



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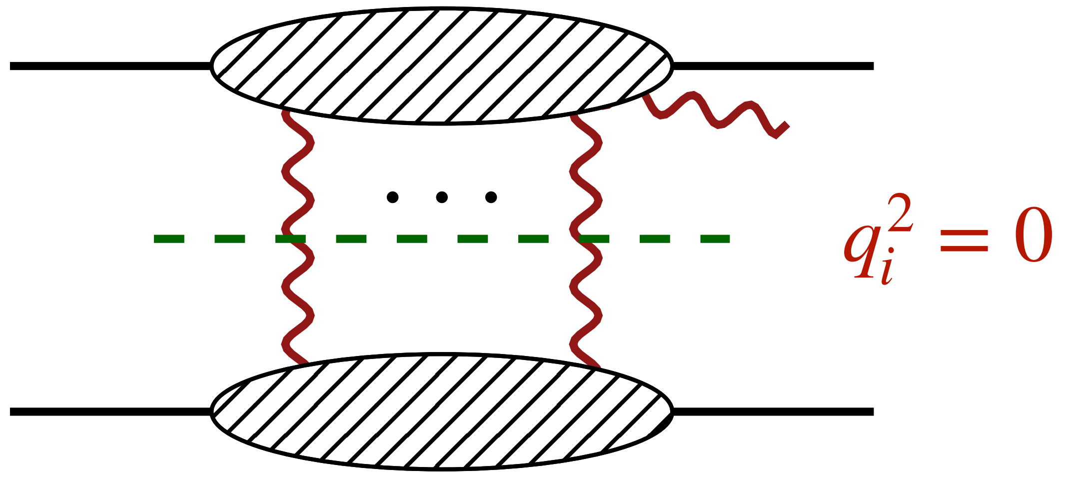

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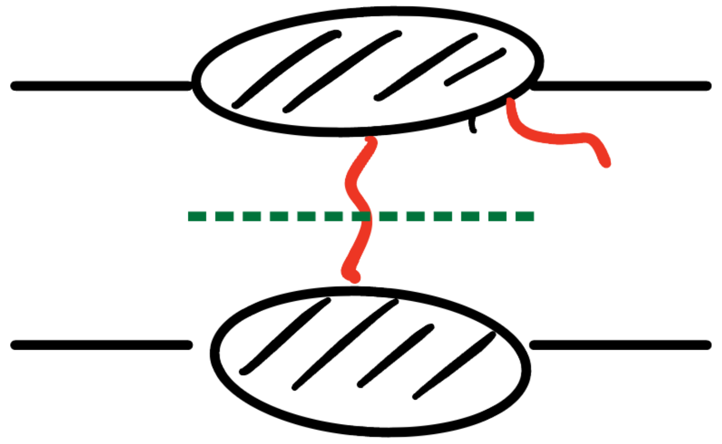

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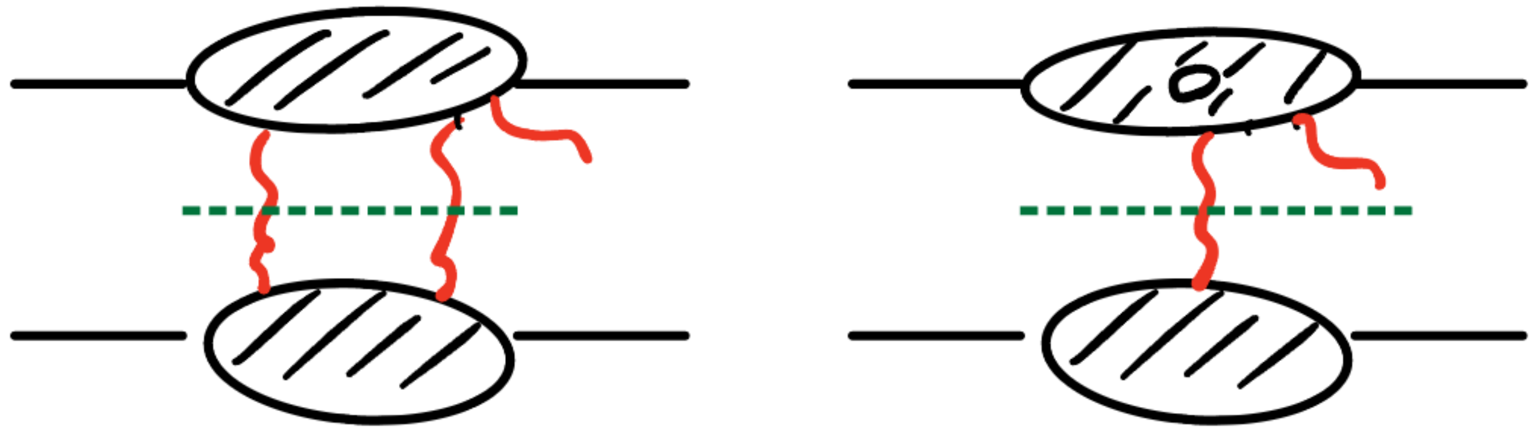
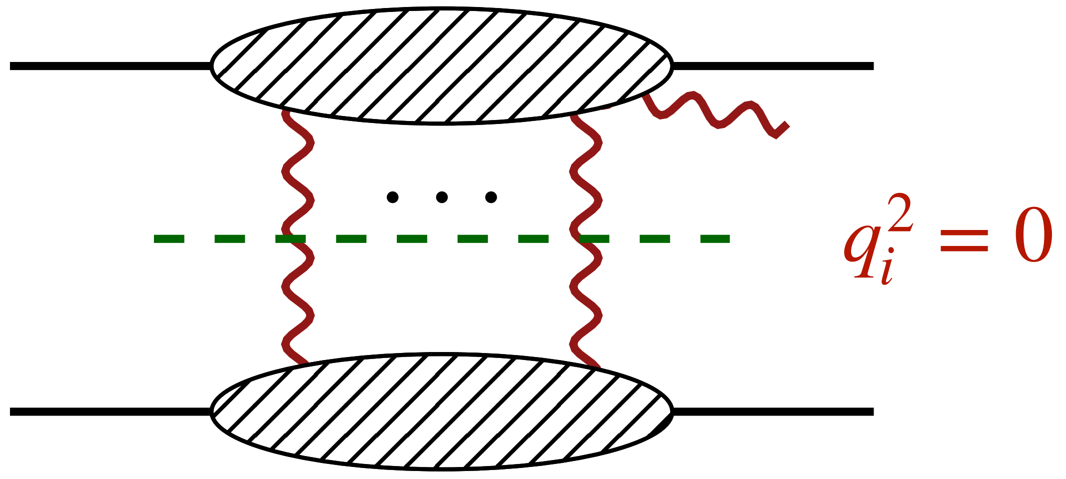


One-loop Compton amplitude

Integrand generation: diagrammatics



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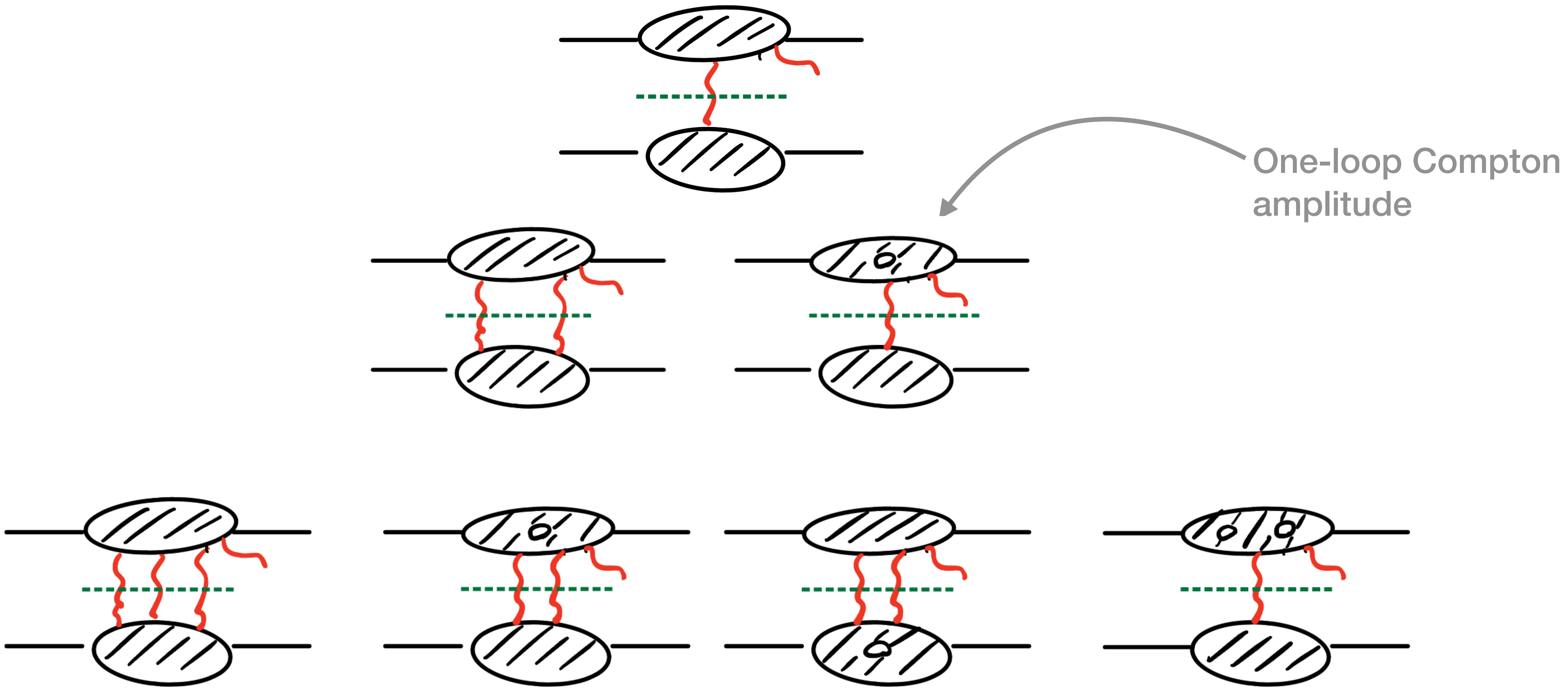
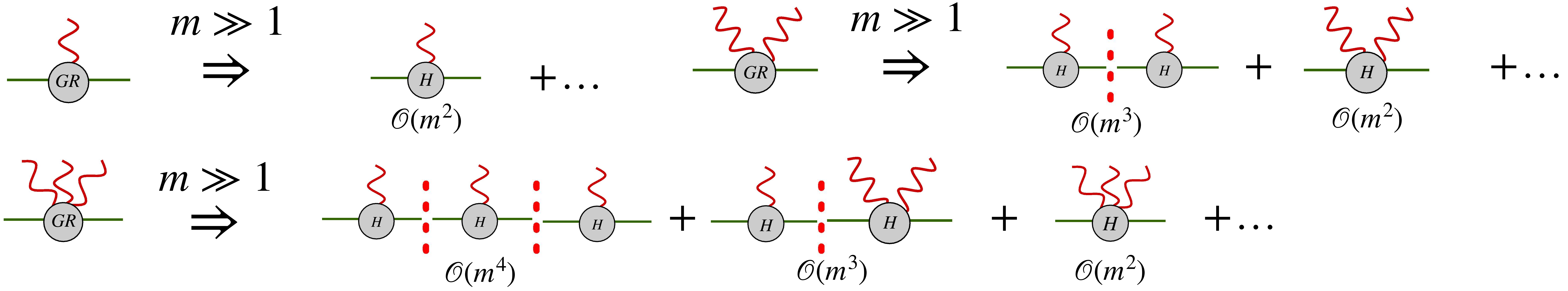


Heavy-mass diagrammatics

Brandhuber, Chen, Travaglini, Wen
Brandhuber, Chen, Johansson, Travaglini, Wen
Brandhuber, Brown, Chen, Gowdy, Travaglini, Wen
Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini



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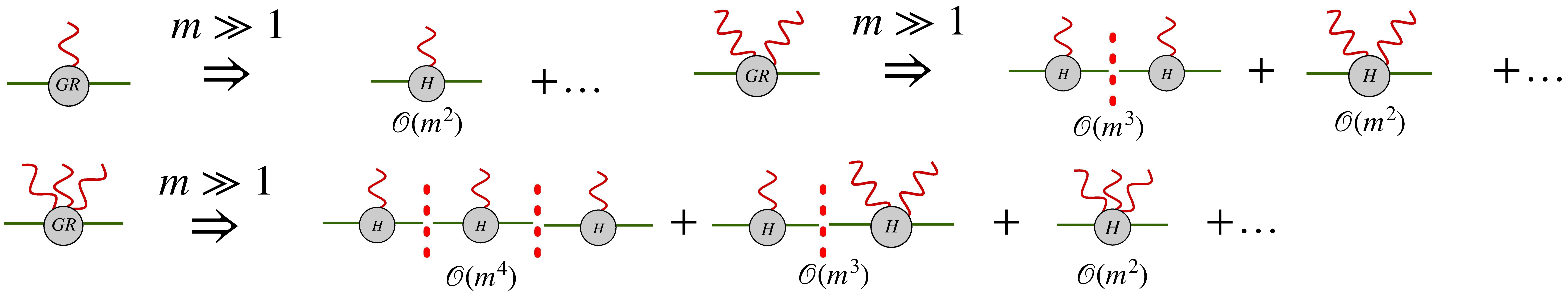

Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

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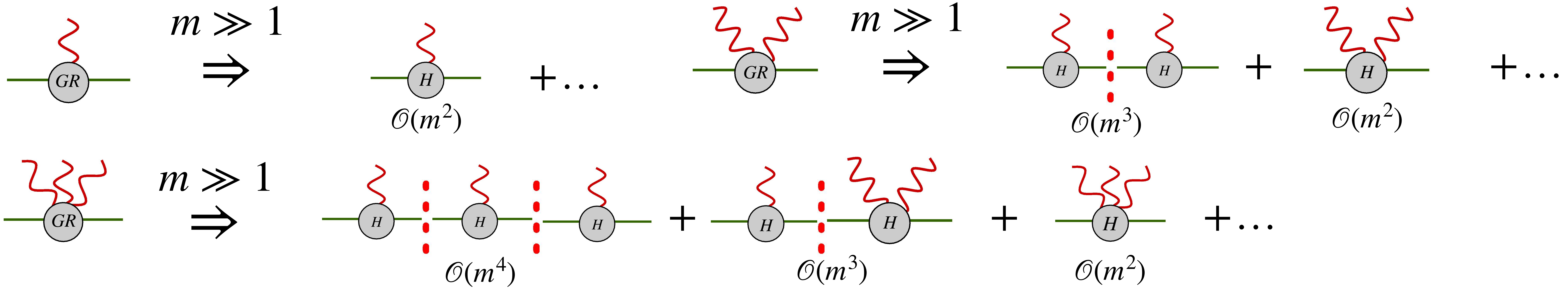
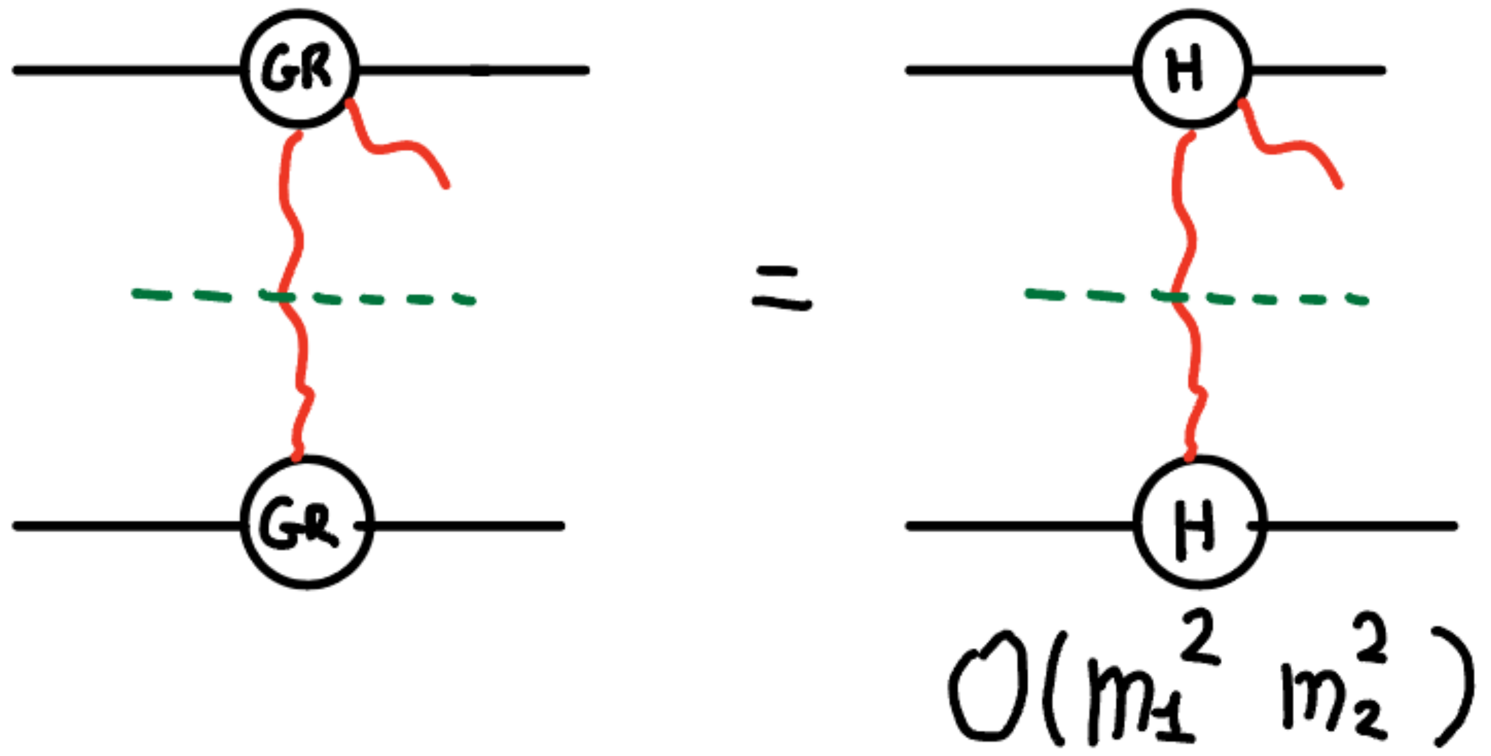


Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

LO



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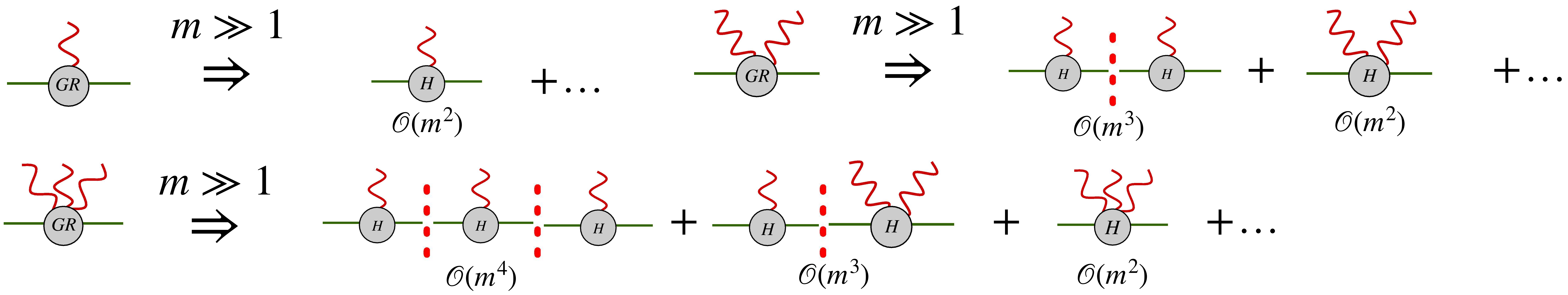

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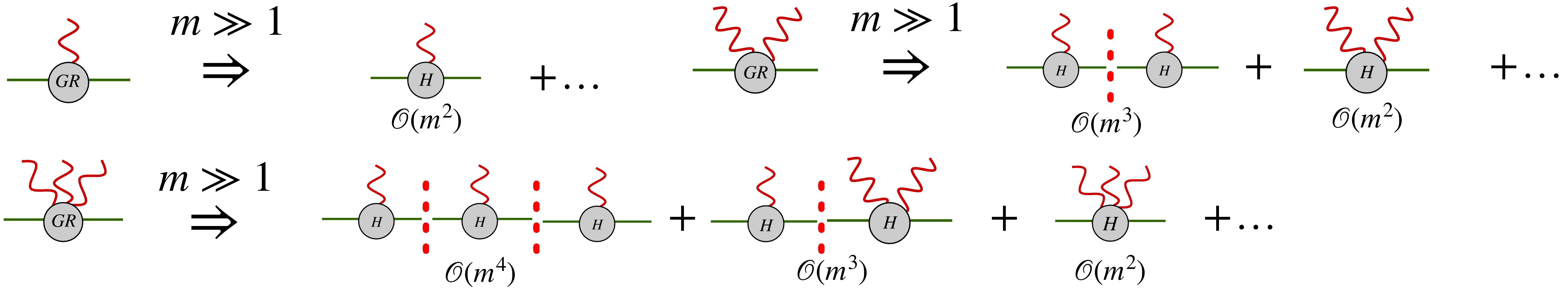
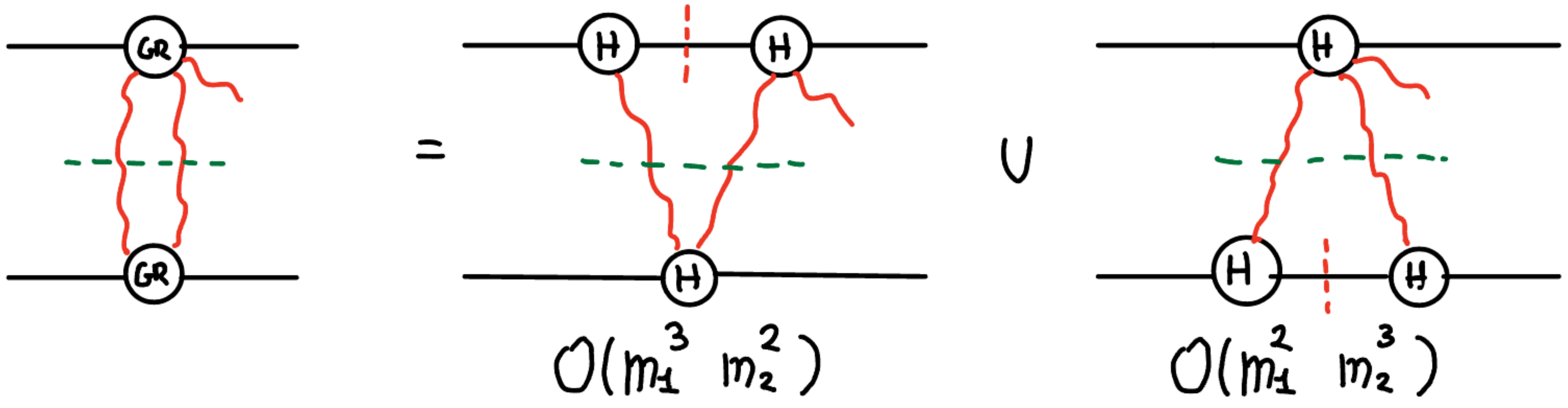


Contributing pieces from HEFT expansion $\mathcal{O}(m^{4+L})$

NLO



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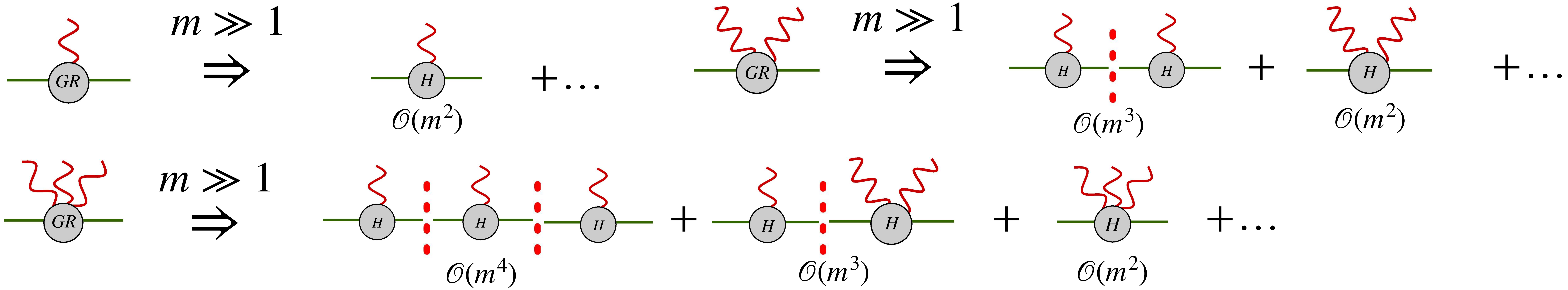

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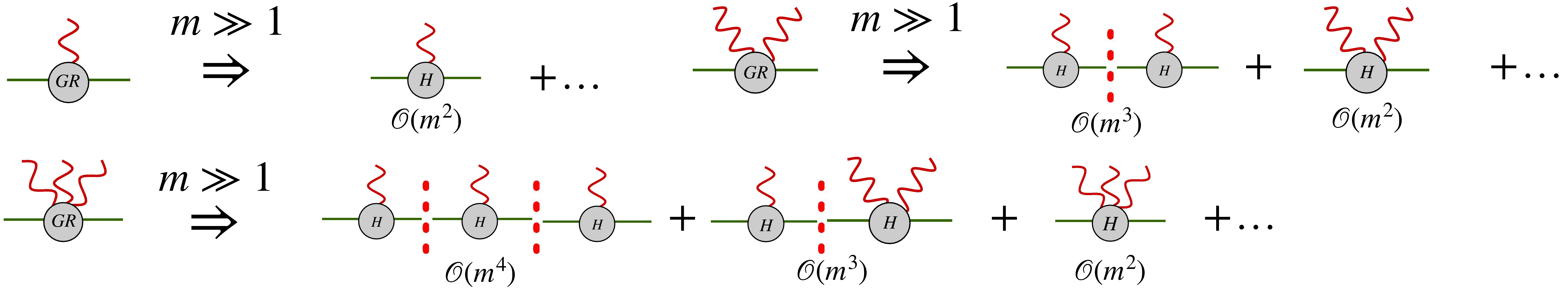

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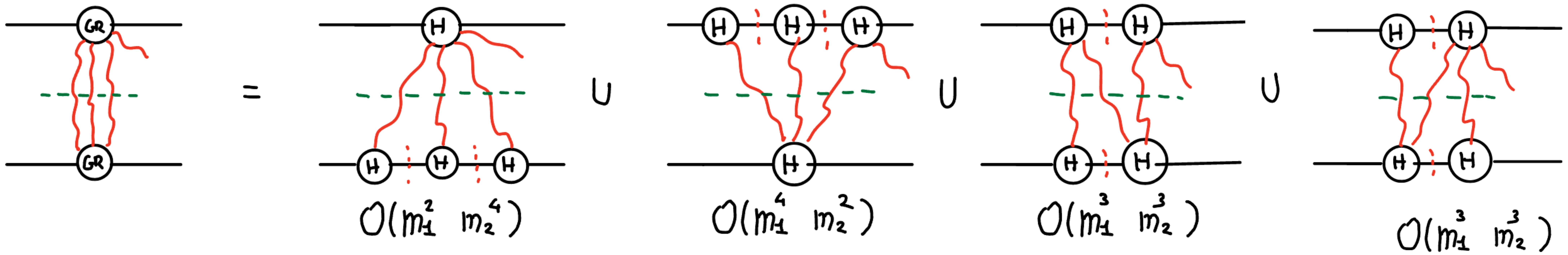
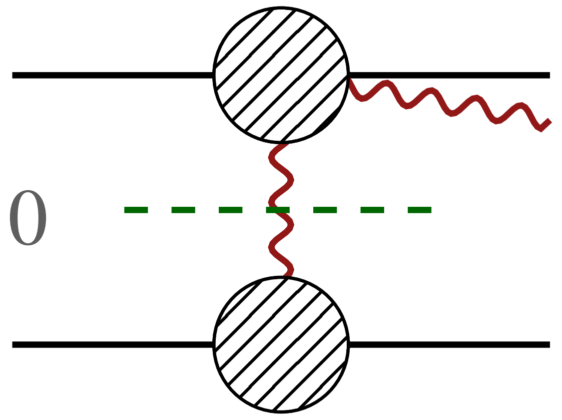


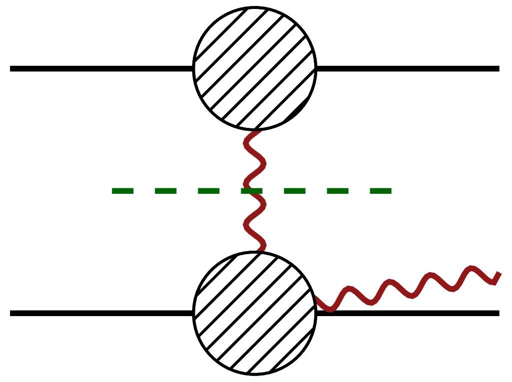
The leading-order waveform

Integrand generation from Generalised Unitarity

$$T^{(0)} = \quad q_2^2 = 0$$



$$\cup$$



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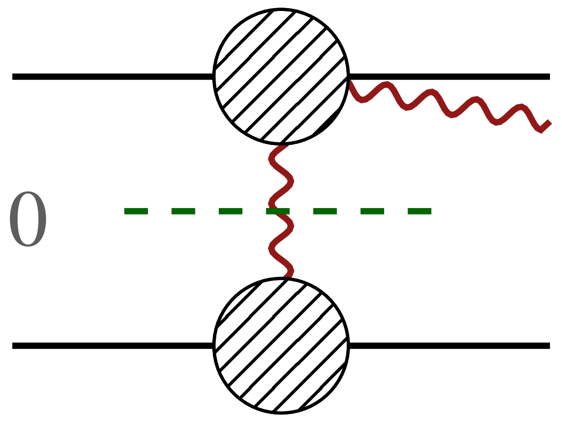
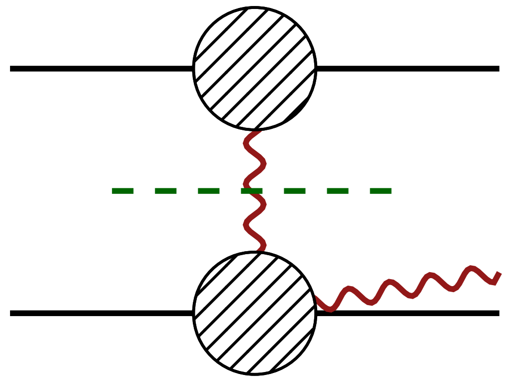
The leading-order waveform

Integrand generation from Generalised Unitarity

Building blocks: heavy mass expansion

$T^{(0)}$ = $q_2^2 = 0$  $\cup$  $q_1^2 = 0$

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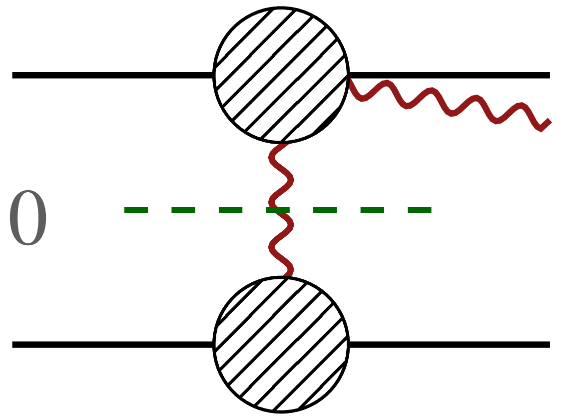
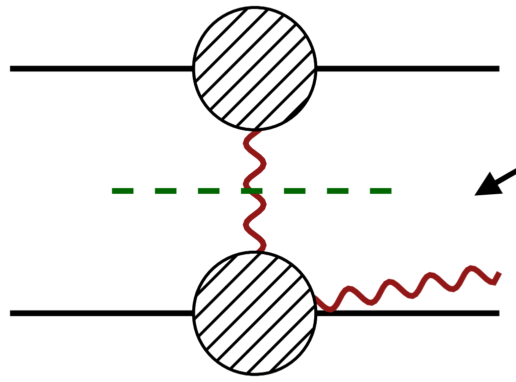
$T^{(0)}$ = $q_2^2 = 0$  $\cup$  $q_1^2 = 0$

Polarisation sum

$$\sum_h \epsilon_{-k}^{\mu_1\nu_1}\epsilon_{k}^{\mu_2\nu_2} = \frac{1}{2}(P^{\mu_1\mu_2}P^{\nu_1\nu_2} + P^{\nu_1\mu_2}P^{\nu_1\mu_2}) - D_0 P^{\mu_1\nu_1}P^{\mu_2\nu_2}$$

$$D_0 = \frac{1}{2} \qquad\qquad D_0 = \frac{2}{D-2}$$

Integrand Generation

The leading-order waveform

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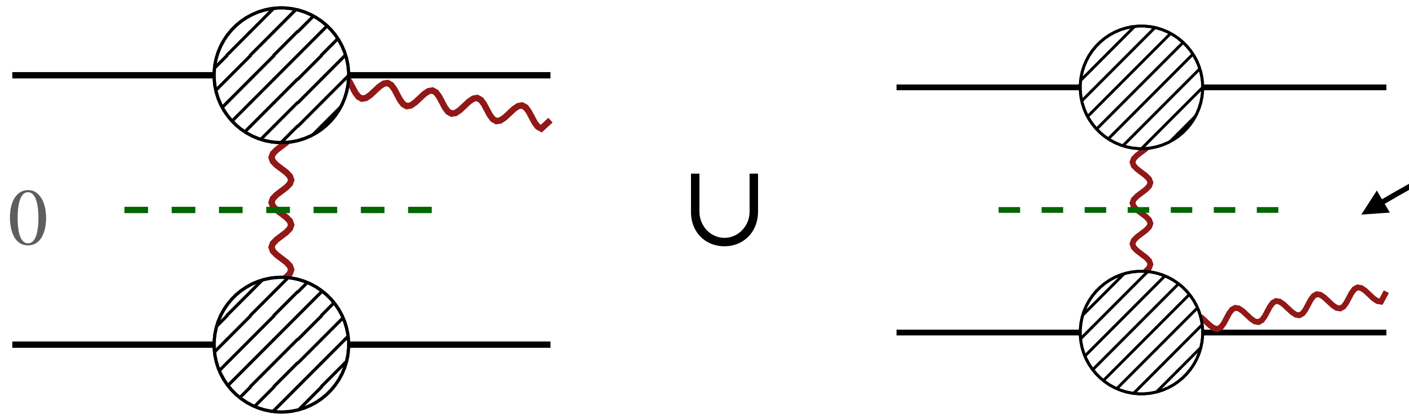
$T^{(0)} = $ $q_2^2 = 0$  $q_1^2 = 0$

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$$D_0 = \frac{1}{2} \qquad D_0 = \frac{2}{D-2}$$

$$T^{(0)} = \frac{m_1^2 m_2^2 \kappa^3}{q_1^2 q_2^2 w_1^2}\Bigg[(q_1\cdot F_k\cdot u_1)^2\Big(y^2 - \frac{1}{D_s-2}\Big) + \frac{q_1\cdot F_k\cdot u_1 u_1\cdot F_k\cdot u_2}{w_2}\left(-y(q_2^2 + 2w_1w_2) + \frac{1}{D_s-2}q_2^2\right)$$
$$+\frac{(u_1\cdot F_k\cdot u_2)^2}{4w_2^2}\left((q_2^2 y + 2w_1w_2)^2 - \frac{1}{D_s-2}q_2^4\right)\Bigg],$$

Integral Reduction

The leading-order waveform

$$\mathcal{W}_h^{(0)} = \frac{e^{i\omega\, n\cdot b_2}}{4\pi r} \int_{\hat{q}} e^{i\, b\cdot q}\, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot (q-k))\; T^{(0)}$$

Fourier transform

Integrand

Tensor structures appearing

$$q \cdot F_k \cdot u_1$$

Integral Reduction

The leading-order waveform

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Decomposition in scalar Form Factors

Anastasiou, Karlen, Vicini

$$P^{\mu\nu} = \sum_{i=1}^{4} \hat{v}_i^\mu v_i^\nu$$

Projector

$$v_i^\mu \in \{u_1^\mu, u_2^\mu, b^\mu, k^\mu\}$$

External vectors

$$v_i^\mu \hat{v}_{j\mu} = \delta_{ij}$$

Dual vectors

$$\hat{v}_i^\mu = G_{ij}^{-1} v_j^\mu\,.$$

Closed form

Integral Reduction

The leading-order waveform

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Closed form

Four dimensional external kinematics: factorisation of tensors

Peraro, Tancredi

$$\eta^{\mu\nu} = P^{\mu\nu}$$

$$T^{\mu_1\ldots\mu_N} = T_{\nu_1\ldots\nu_N}\prod_{i=1}^{N} P^{\mu_i\nu_i}$$

Integral Reduction (II)

The leading-order waveform

$$I_{a_1\,1\,1\,a_4 a_5} = \int_{\hat{q}} e^{i\,b\cdot q}\,\frac{(i\,b\cdot q)^{-a_1}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k))}{(q^2)^{a_4}((q-k)^2)^{a_5}}$$

$$I_{a_1 a_2 a_3 a_4 a_5} = \int_{\hat{q}} e^{D_1}\,\frac{1}{\prod_{i=1}^{5} D_i^{a_i}}$$

Integral family

Reverse Unitarity

$$\begin{aligned} D_1 &= i\; b\cdot q, \\ D_2 &= q^2, \\ D_3 &= (q-k)^2, \\ D_4 &= u_1\cdot q \\ D_5 &= u_2\cdot(k-q) \end{aligned}$$

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Integration-by-parts identities for Fourier integrals

LiteRed

FiniteFlow

$$\int_{\hat{q}} \frac{\partial}{\partial q^\mu}\left(e^{D_1}\,\frac{v^\mu}{\prod_{i=1}^{5} D_i^{a_i}}\right) = 0$$

$$IBP_{a_1,\ldots,a_5}[1-D_1] + IBP_{a_1-1,\ldots,a_5}[1] = 0$$

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The leading-order waveform

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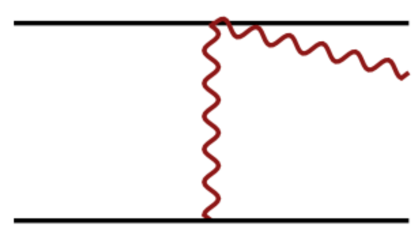
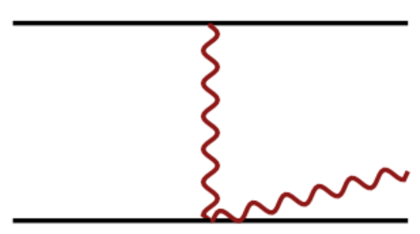
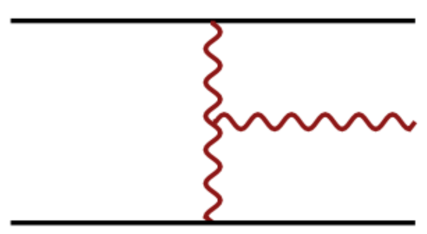
Decomposition into a basis of 6 Master Integrals

$$\delta_b^{(n)} = b^{\mu_1}\cdots b^{\mu_n}\frac{\partial}{\partial b^{\mu_1}}\cdots\frac{\partial}{\partial b^{\mu_n}}$$

$J_{1+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$, $\quad J_{3+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$, $\quad J_{5+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$ $\quad n = 0,1$

Integral Evaluation

Evaluating Fourier Integrals

G.B., S. De Angelis, D. Kosower
In progress

$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{-i\, z_b\sqrt{-b^2}} \int_{-\infty}^{\infty} dz_v f_i(z_b, z_v)$$

$$q^\mu = z_1 u_1^\mu + z_2 u_2^\mu + z_b \hat{b}^\mu + z_v \tilde{v}_\perp^\mu$$

Parametrisation

$$u_1^2 = u_2^2 = \hat{b}^2 = -v_\perp^2 = 1$$
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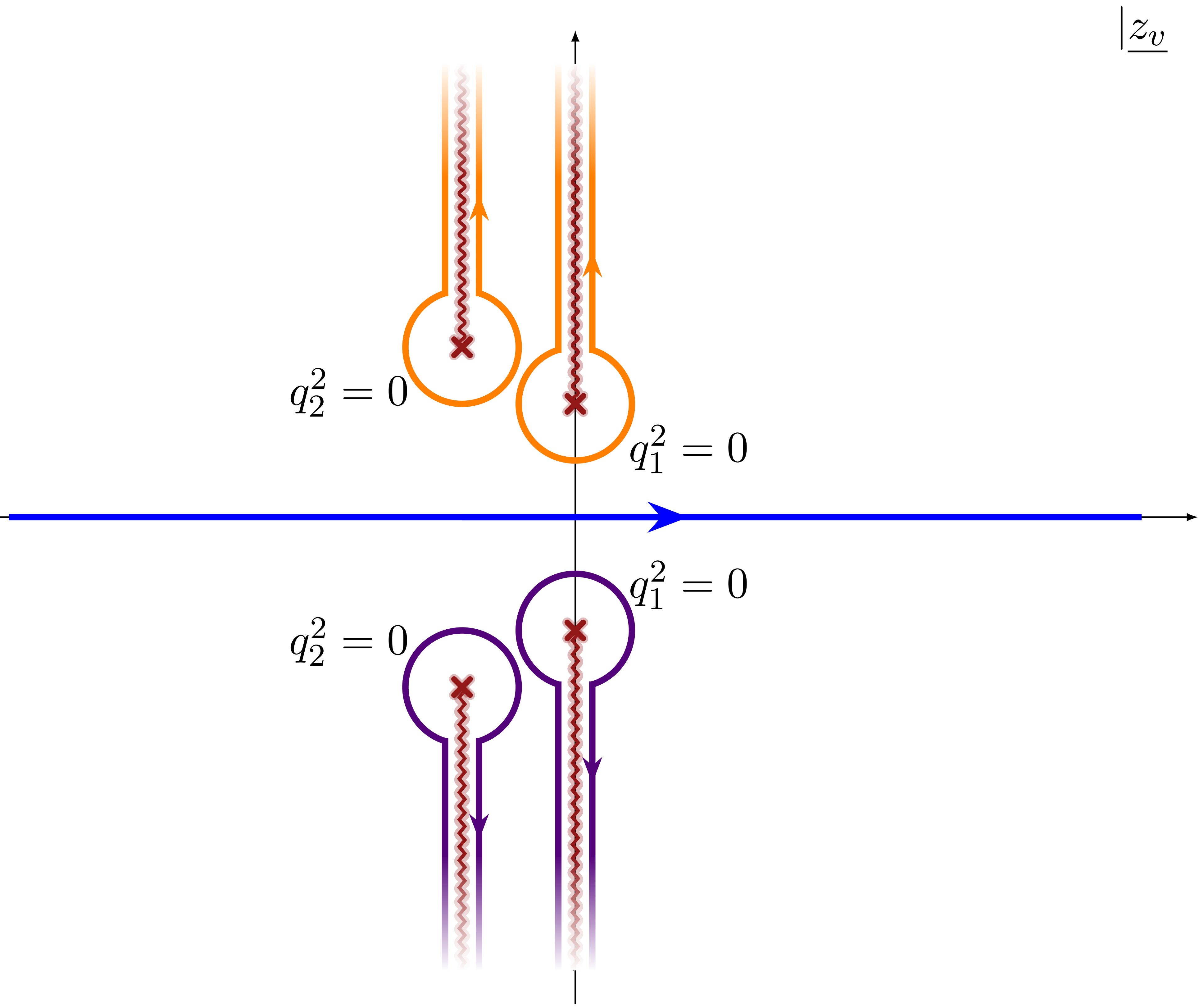

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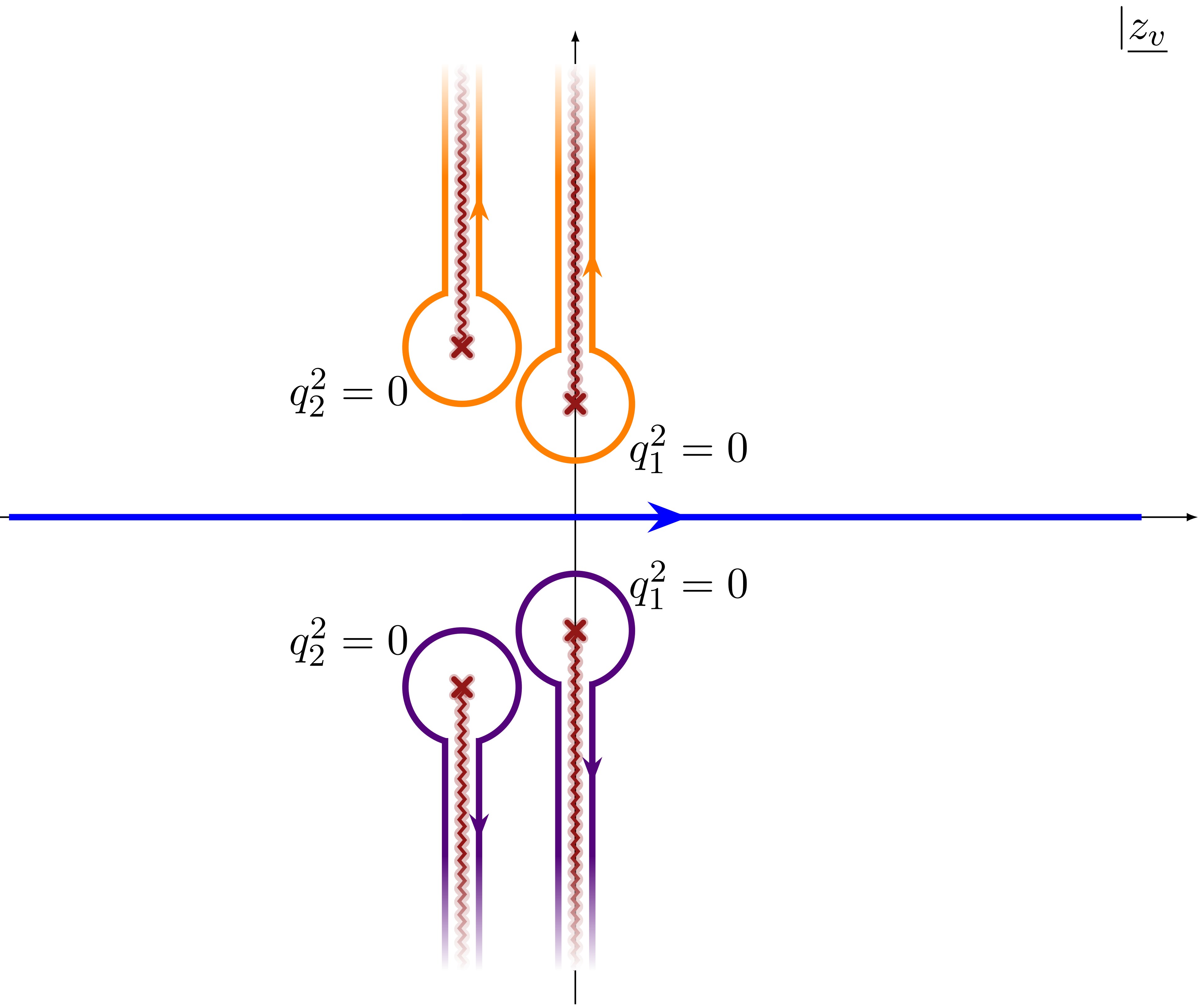
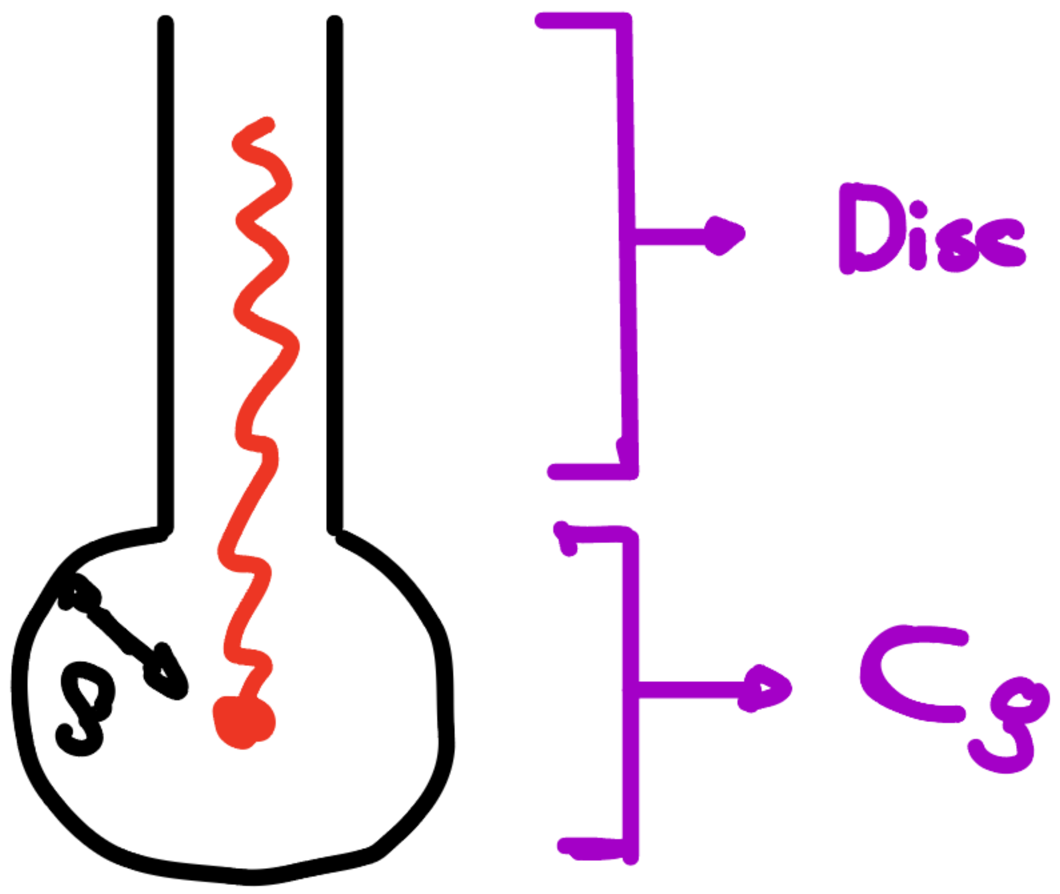
Contour deformation





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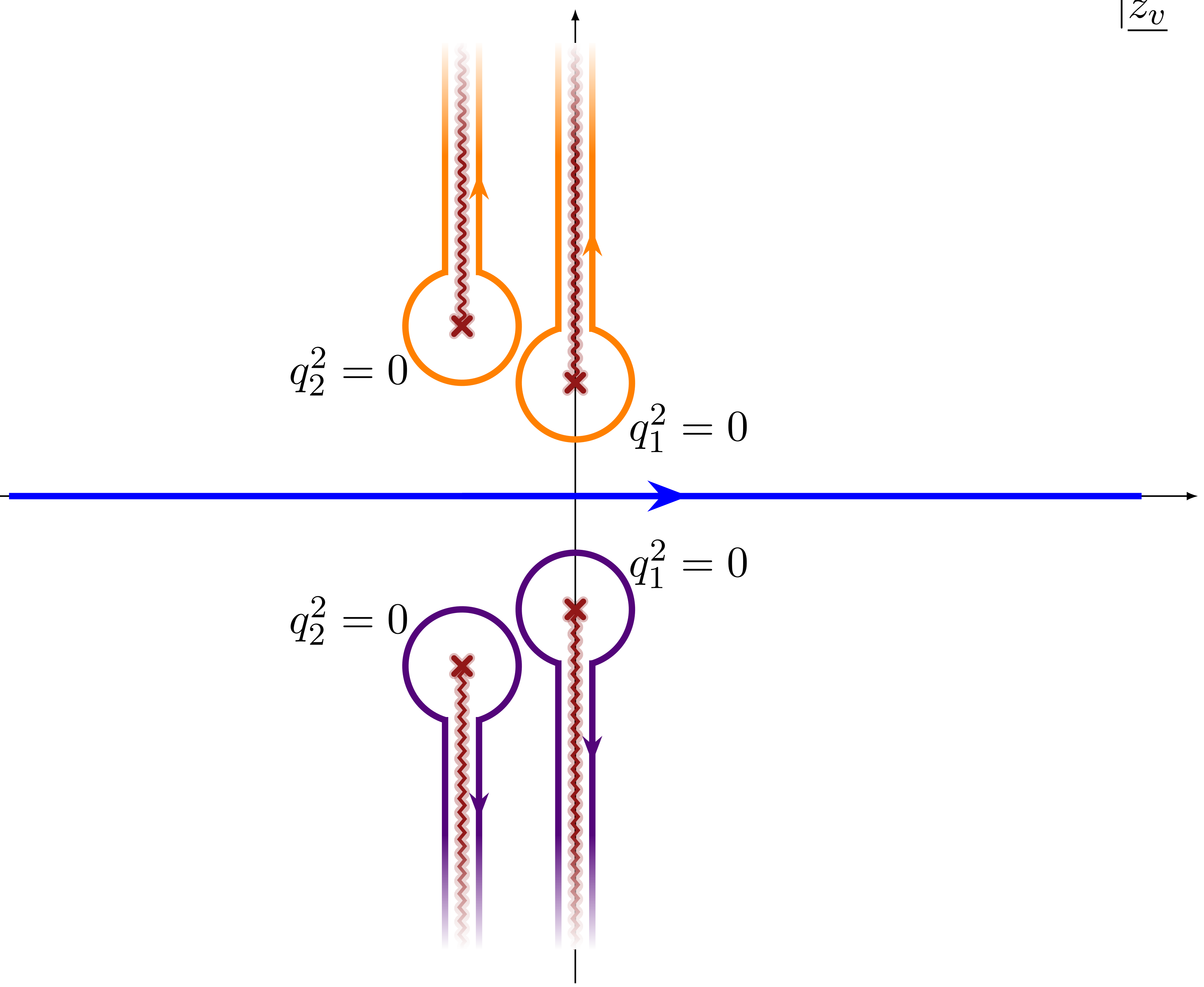
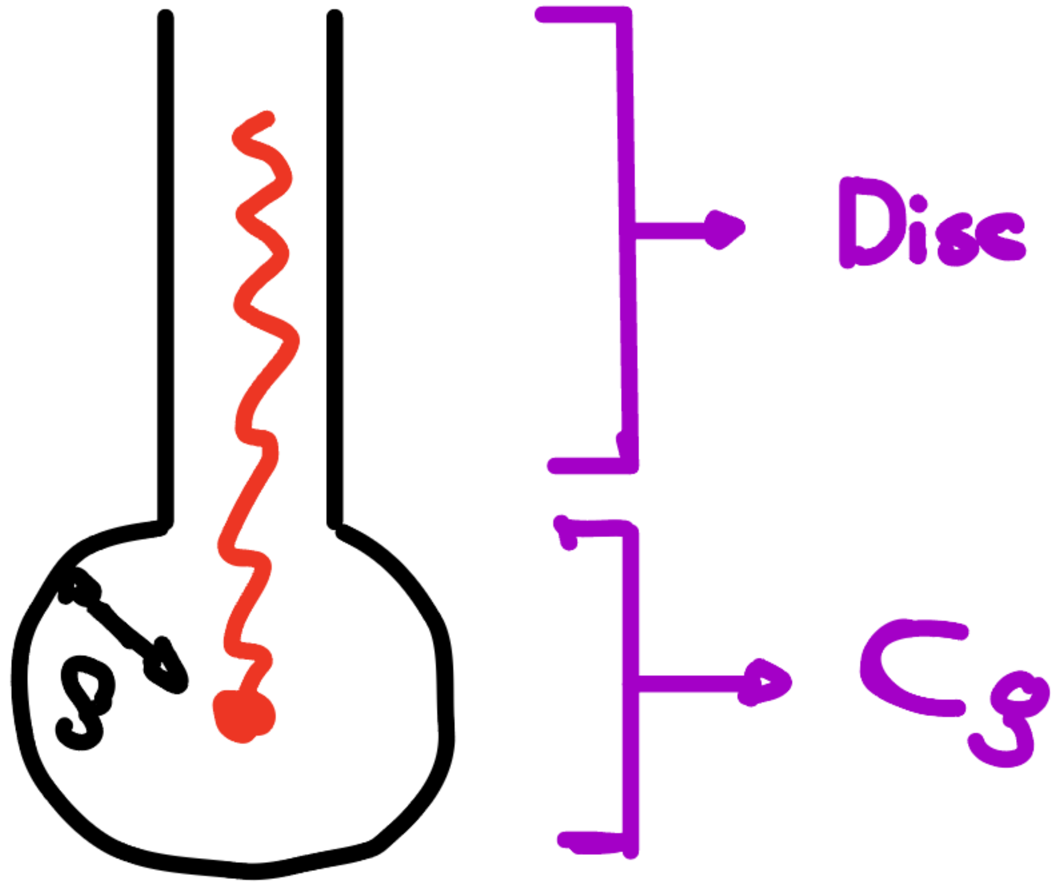


$$F_i(z_b) = \sum_{j=1}^{2} \int_{\hat{z}_j \circlearrowleft \rho} dz_v\ f_i(z_b, z_v) + i \int_\rho^\infty dx\ \mathrm{Disc}_{\hat{z}_j > 0}[f_i(z_b, z_v)]$$



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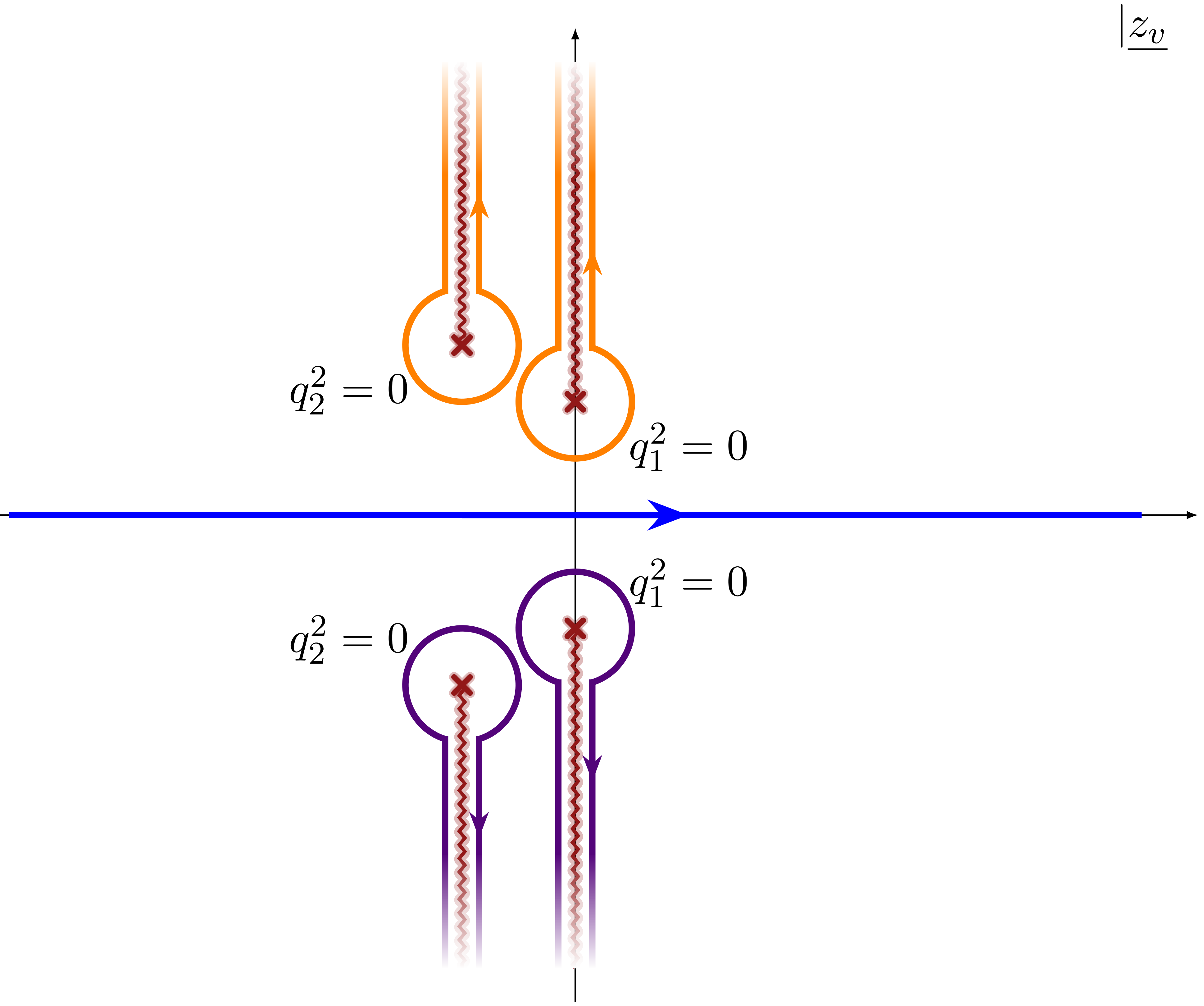

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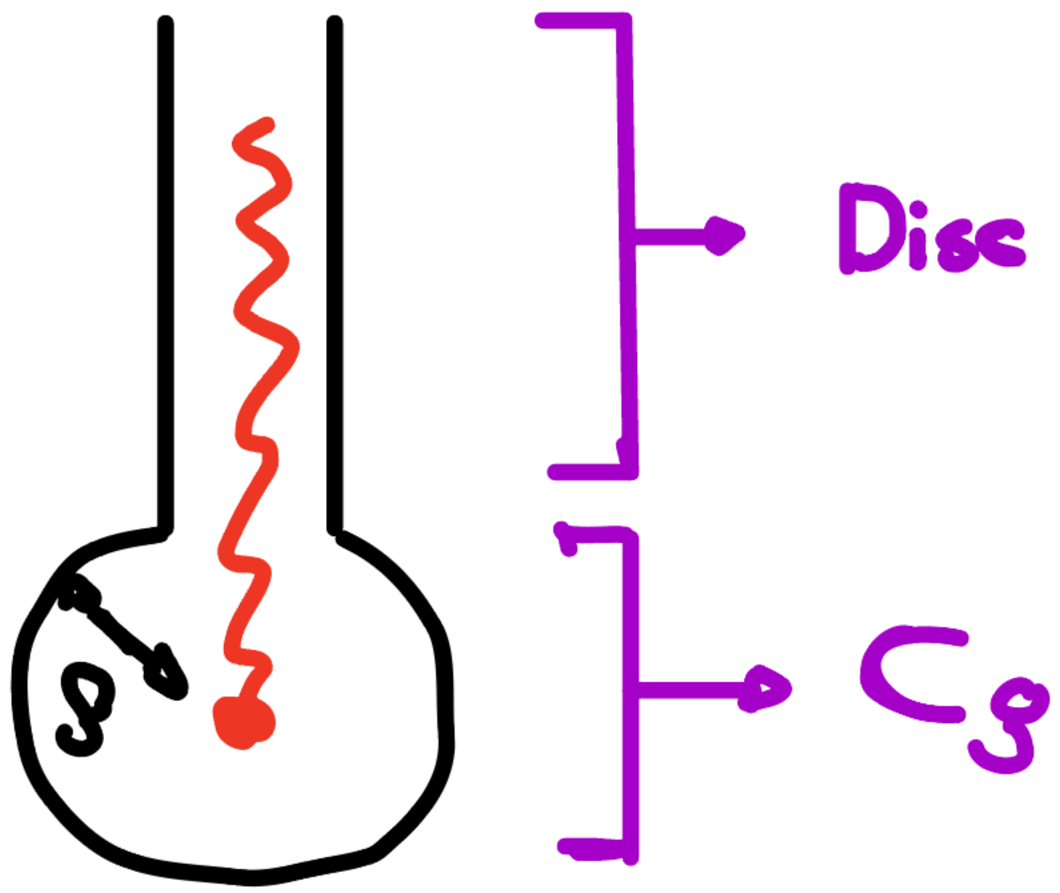

$$\int_a^b dx\ R(x) = -\frac{1}{2\pi i} \int_\gamma dx\ R(x)\ \log\left(\frac{x-a}{x-b}\right)$$
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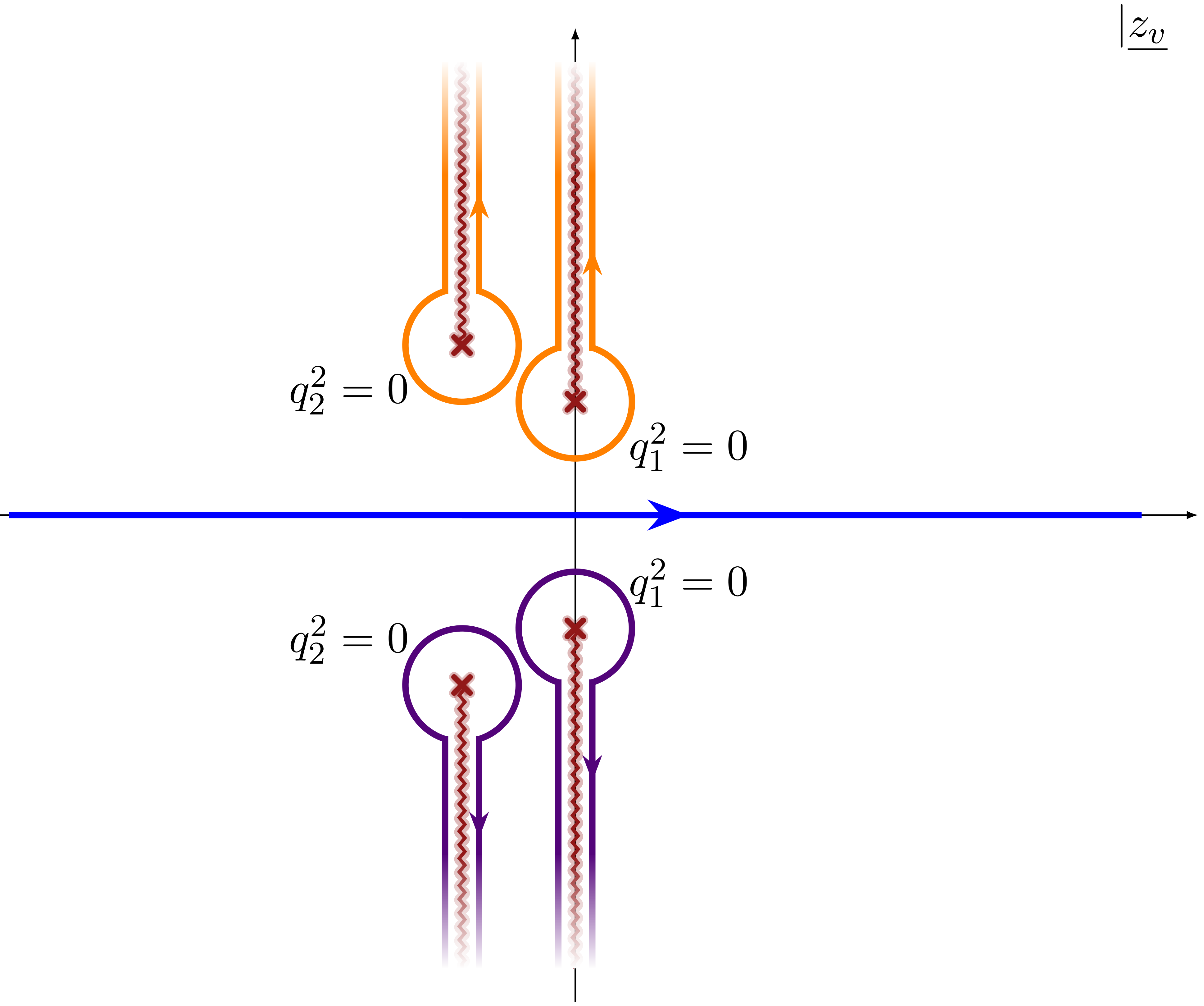

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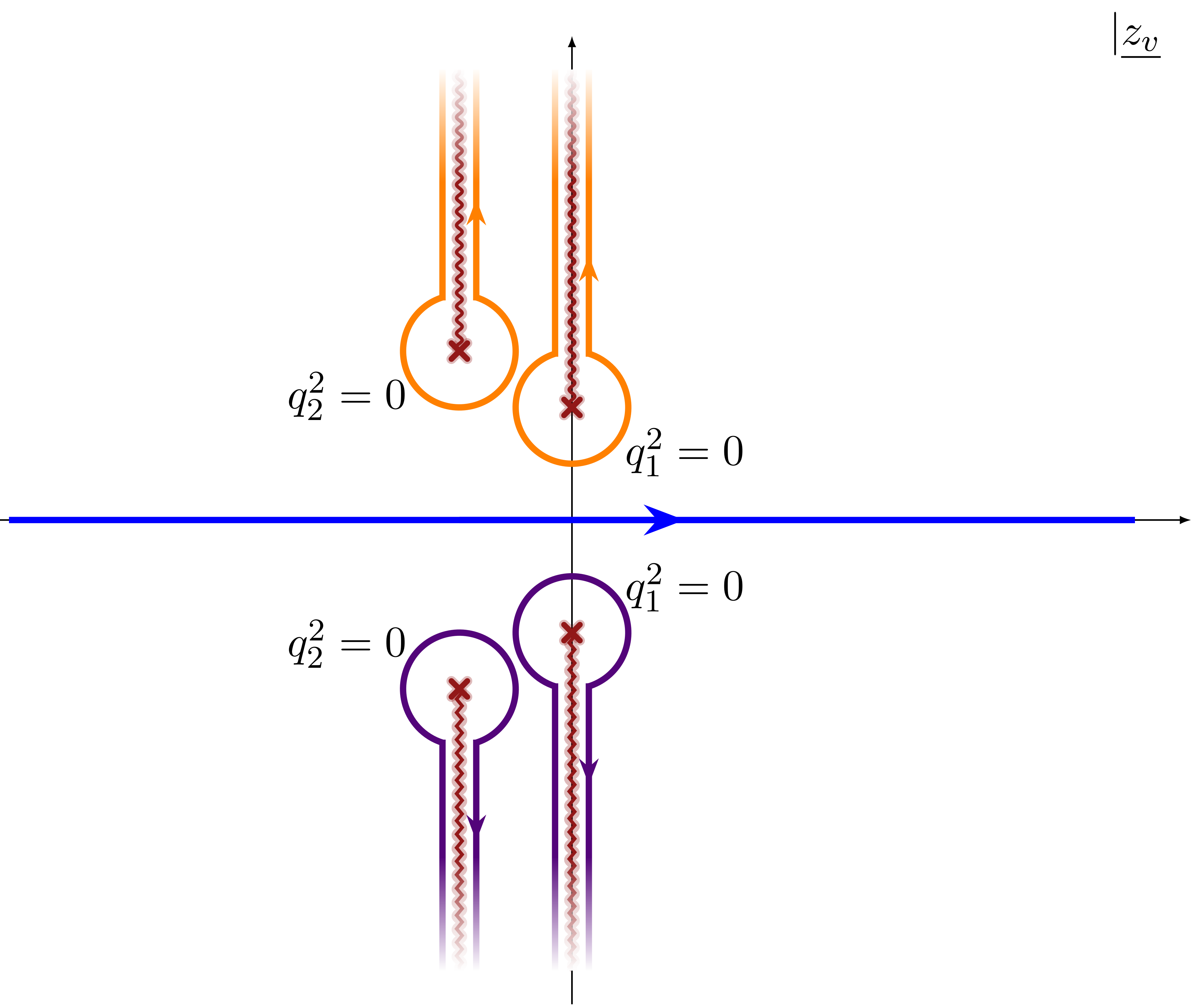
Contour deformation



$$F_i(z_b) = \sum_{j=1}^{2} \int_{\hat{z}_j} dz_v \; f_i(z_b, z_v) + i \int_\rho^\infty dx \; \underset{\hat{z}_j > 0}{\mathrm{Disc}}[f_i(z_b, z_v)]$$

Integral Evaluation

Evaluating Fourier Integrals

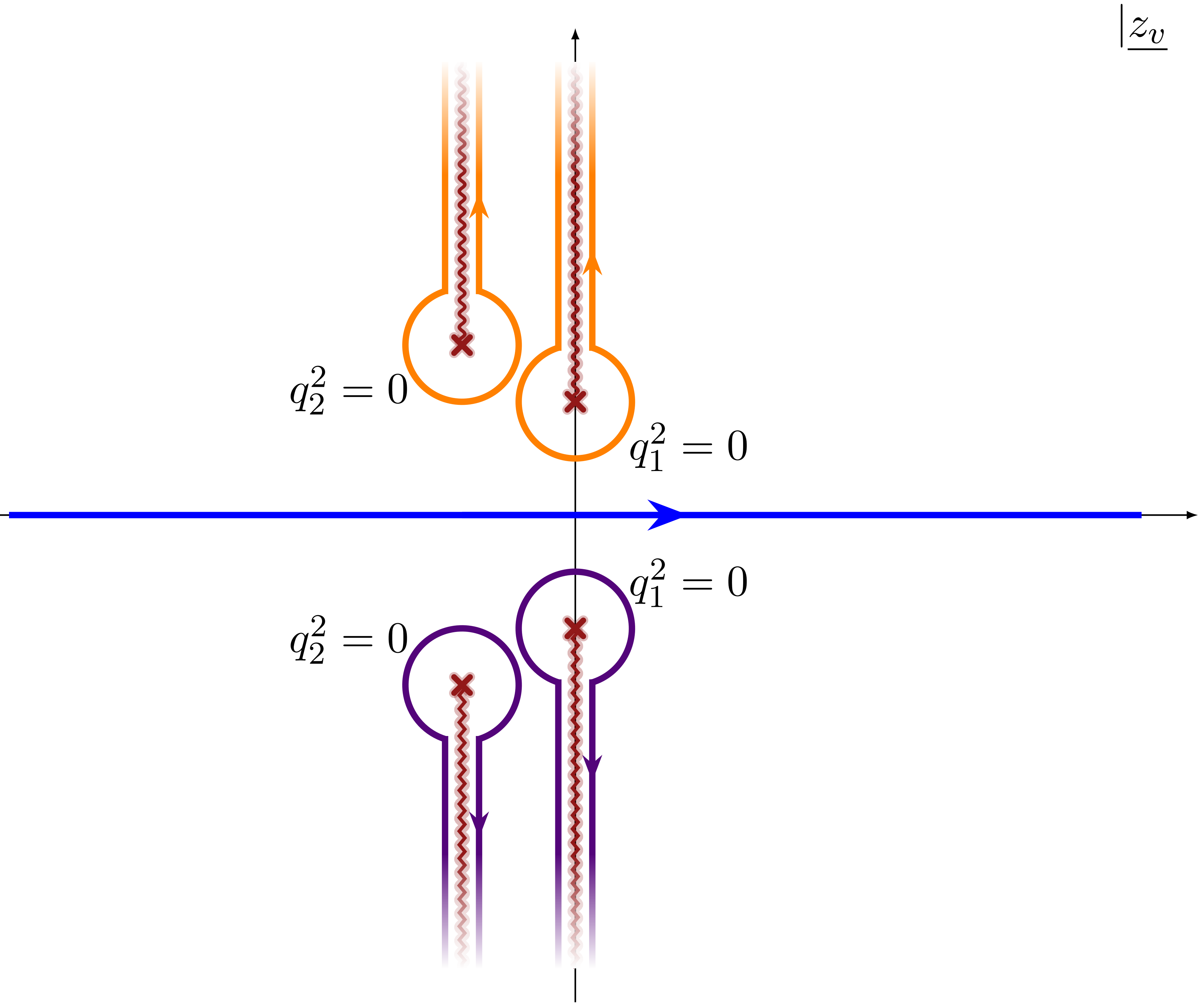
G.B., S. De Angelis, D. Kosower
In progress

$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{-i\, z_b\sqrt{-b^2}} \int_{-\infty}^{\infty} dz_v f_i(z_b, z_v)$$

$$q^\mu = z_1 u_1^\mu + z_2 u_2^\mu + z_b \hat{b}^\mu + z_v \tilde{v}_\perp^\mu$$

Parametrisation

$$u_1^2 = u_2^2 = \hat{b}^2 = -v_\perp^2 = 1$$
$$v_\perp \cdot u_1 = v_\perp \cdot u_2 = v_\perp \cdot \hat{b} = 1$$



Contour deformation

$$F_i(z_b) = \sum_{j=1}^{2} \oint_{\hat{z}_j} dz_v\ f_i(z_b, z_v) + i \int_\rho^\infty dx\ \underset{\hat{z}_j>0}{\mathrm{Disc}}[f_i(z_b, z_v)]$$

Numerically fast-convergent one-fold integrals

$$J_i \sim \int_{-\infty}^{+\infty} e^{-i\, z_b\sqrt{-b^2}}\ F_i(z_b)$$

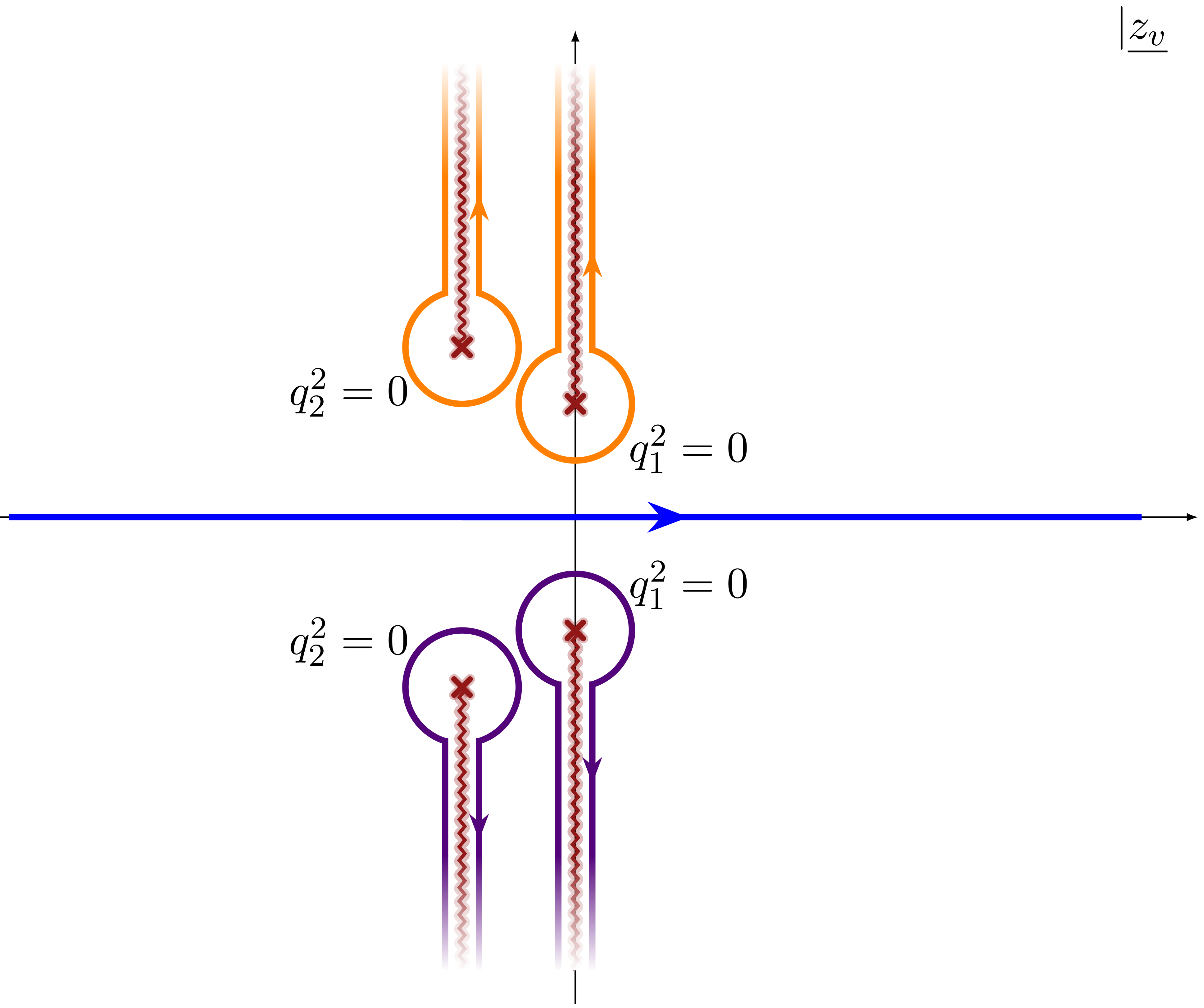
Integral Evaluation

A couple of examples

G.B., S. De Angelis, D. Kosower
In progress

$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{-i\, z_b \sqrt{-b^2}} \int_{-\infty}^{\infty} dz_v f_i(z_b, z_v)$$

$$f_1 = \frac{\log \frac{-q^2}{\mu^2}}{-(k-q)^2} \qquad f_2 = \frac{\log \frac{-q^2}{\mu^2}}{-q^2}$$



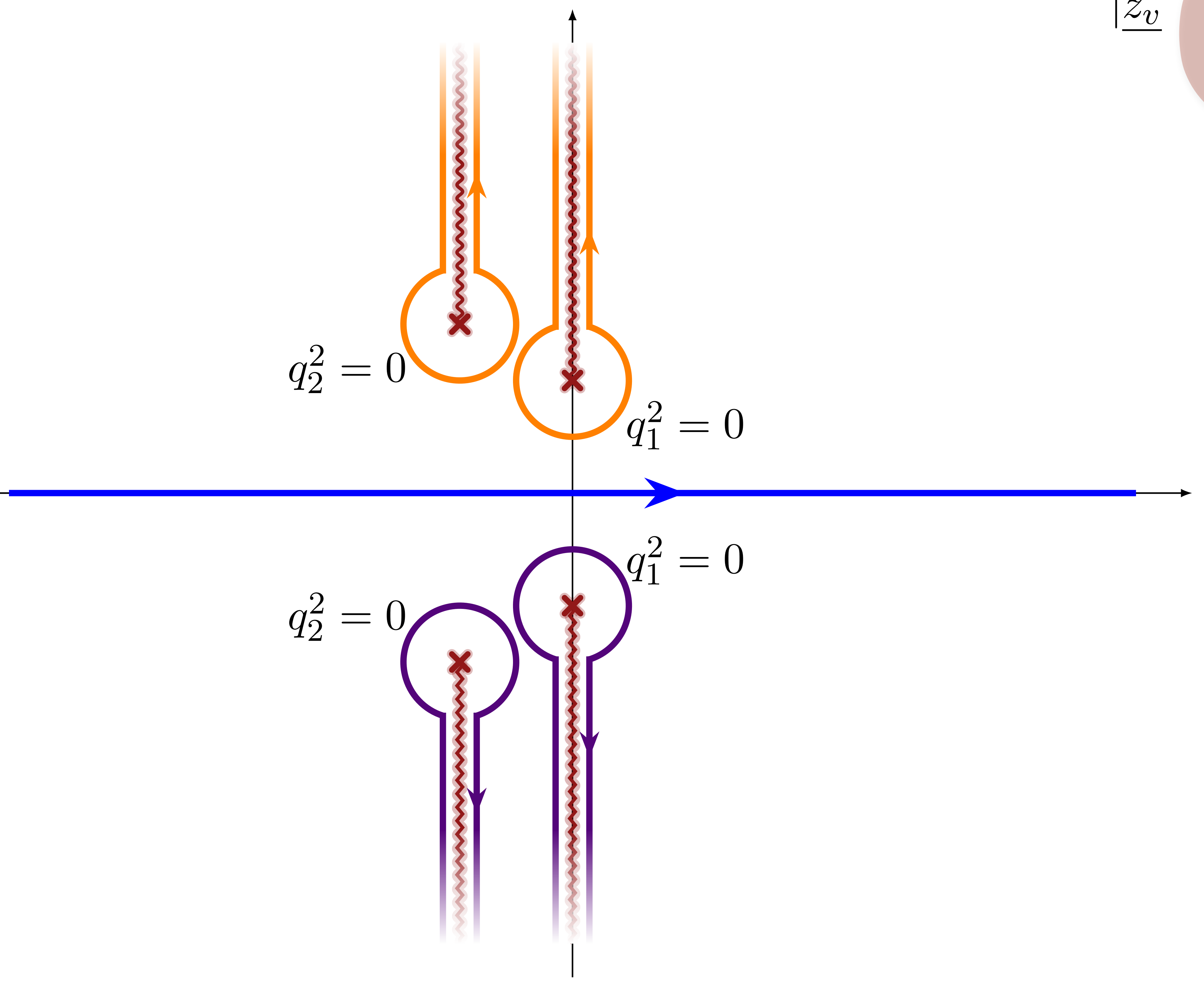
Integral Evaluation

A couple of examples

G.B., S. De Angelis, D. Kosower
In progress

$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{-i\, z_b\sqrt{-b^2}} \int_{-\infty}^{\infty} dz_v f_i(z_b, z_v)$$

$$f_1 = \frac{\log \frac{-q^2}{\mu^2}}{-(k-q)^2} \qquad f_2 = \frac{\log \frac{-q^2}{\mu^2}}{-q^2}$$



$$\underline{|z_v} \quad F_1(z_b) = \int_{\hat{z}_2} dz_v\, f_1(z_b, z_v) + i \int_0^{\infty} dx\, \underset{\hat{z}_1>0}{\mathrm{Disc}}[f_1(z_b, z_v)]$$

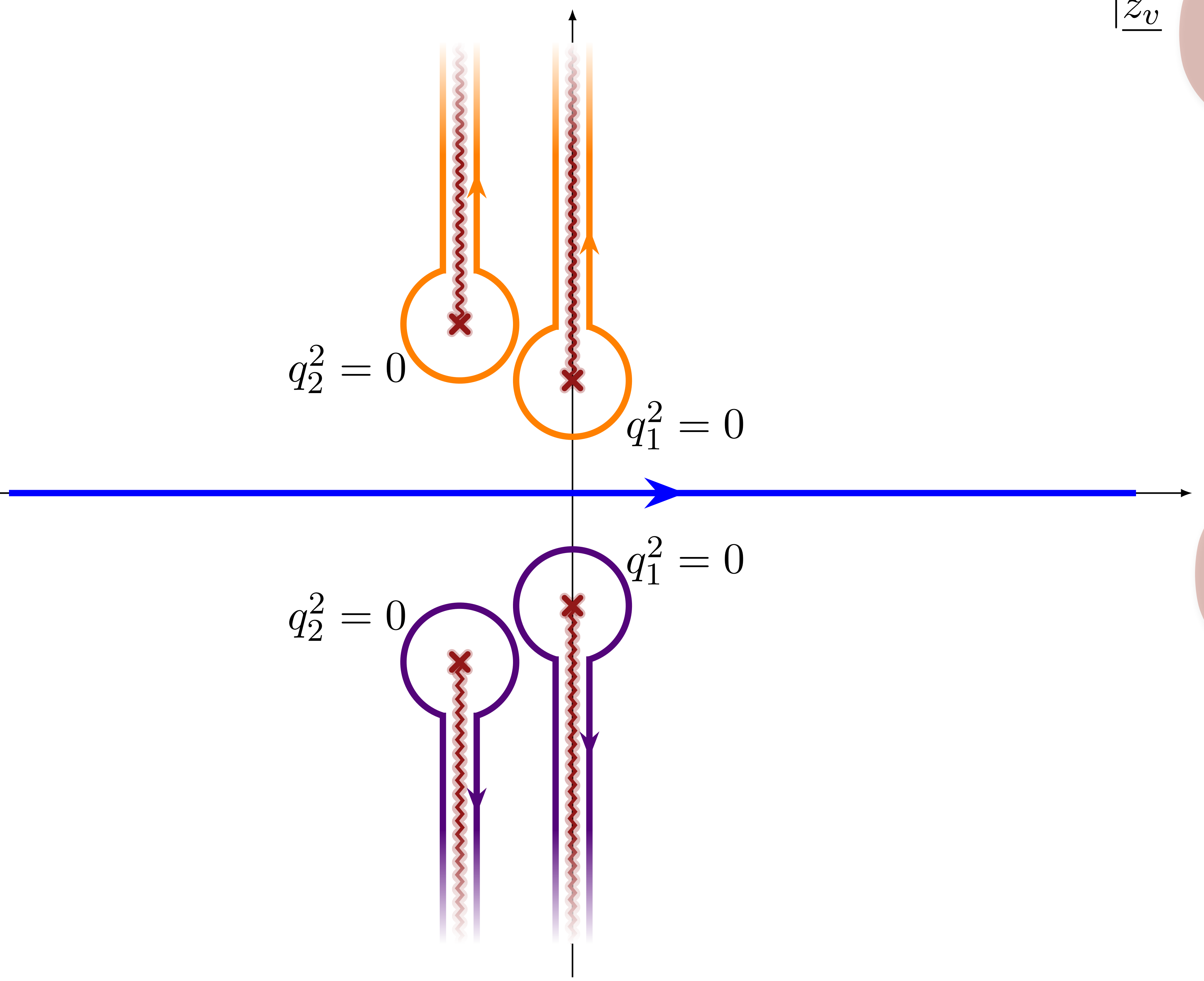
$$= \frac{\pi}{\sqrt{(\hat{b}\cdot k + z_b)^2 + w_1^2}} \log\left[\left(\sqrt{(\hat{b}\cdot k + z_b)^2 + \hat{w}_1^2} + \sqrt{\hat{w}_2^2 + z_b^2}\right)^2 + \hat{b}\cdot k^2\right]$$

A couple of examples

G.B., S. De Angelis, D. Kosower
In progress

$$J_i \sim \int_{-\infty}^{+\infty} dz_b e^{-i\, z_b \sqrt{-b^2}} \int_{-\infty}^{\infty} dz_v f_i(z_b, z_v)$$

$$f_1 = \frac{\log \frac{-q^2}{\mu^2}}{-(k-q)^2} \qquad f_2 = \frac{\log \frac{-q^2}{\mu^2}}{-q^2}$$



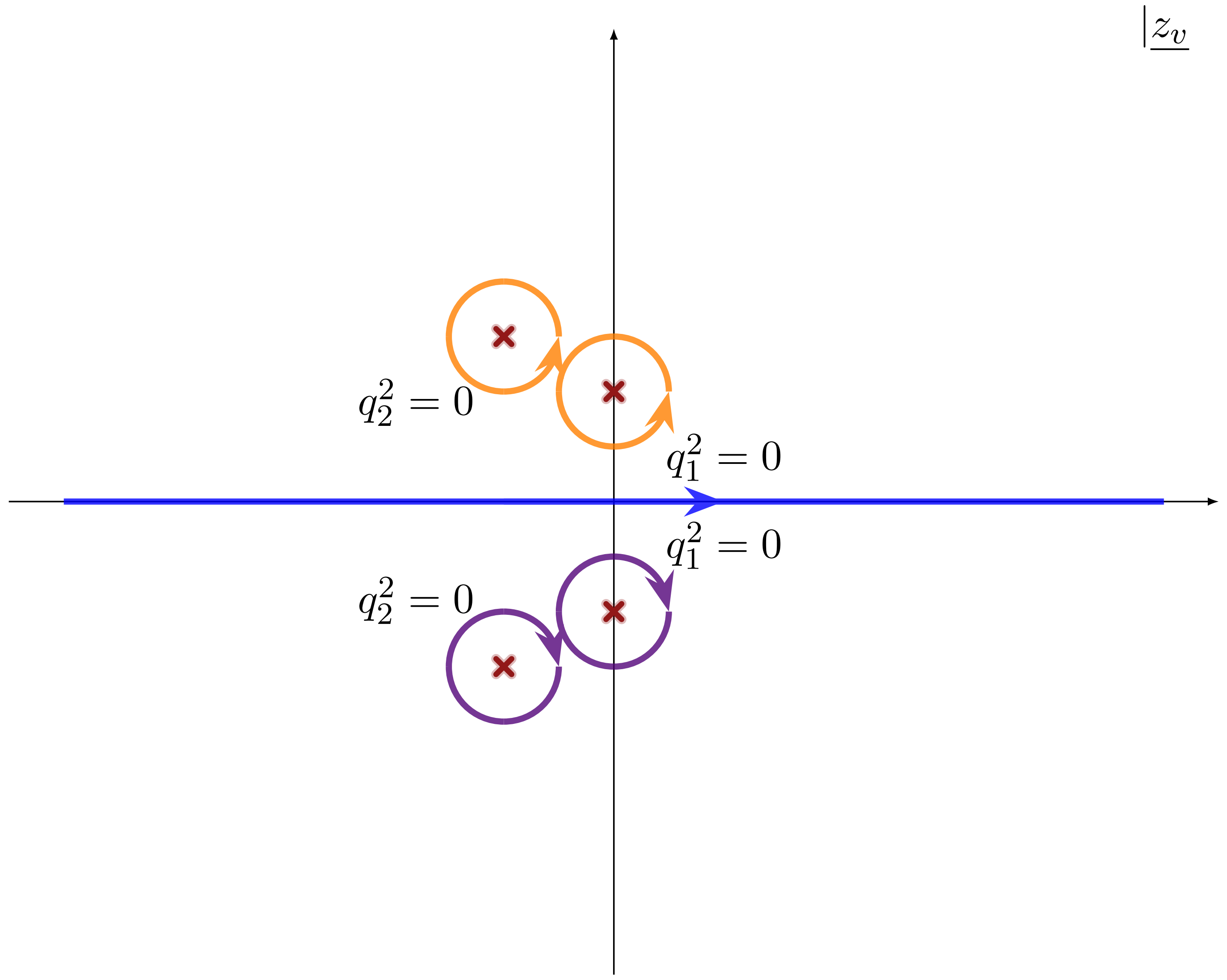
$$\Big|_{z_v} F_1(z_b) = \int_{\hat{z}_2 \circlearrowleft} dz_v\ f_1(z_b, z_v) + i \int_0^\infty dx\ \underset{\hat{z}_1 > 0}{\mathrm{Disc}}[f_1(z_b, z_v)]$$

$$= \frac{\pi}{\sqrt{(\hat{b}\cdot k + z_b)^2 + w_1^2}} \log\left[\left(\sqrt{(\hat{b}\cdot k + z_b)^2 + \hat{w}_1^2} + \sqrt{\hat{w}_2^2 + z_b^2}\right)^2 + \hat{b}\cdot k^2\right]$$

$$F_2(z_b) = \int_{\hat{z}_1 \circlearrowleft} dz_v\ f_2(z_b, z_v) + i \int_0^\infty dx\ \underset{\hat{z}_1 = 0}{\mathrm{Disc}}[f_2(z_b, z_v)]$$

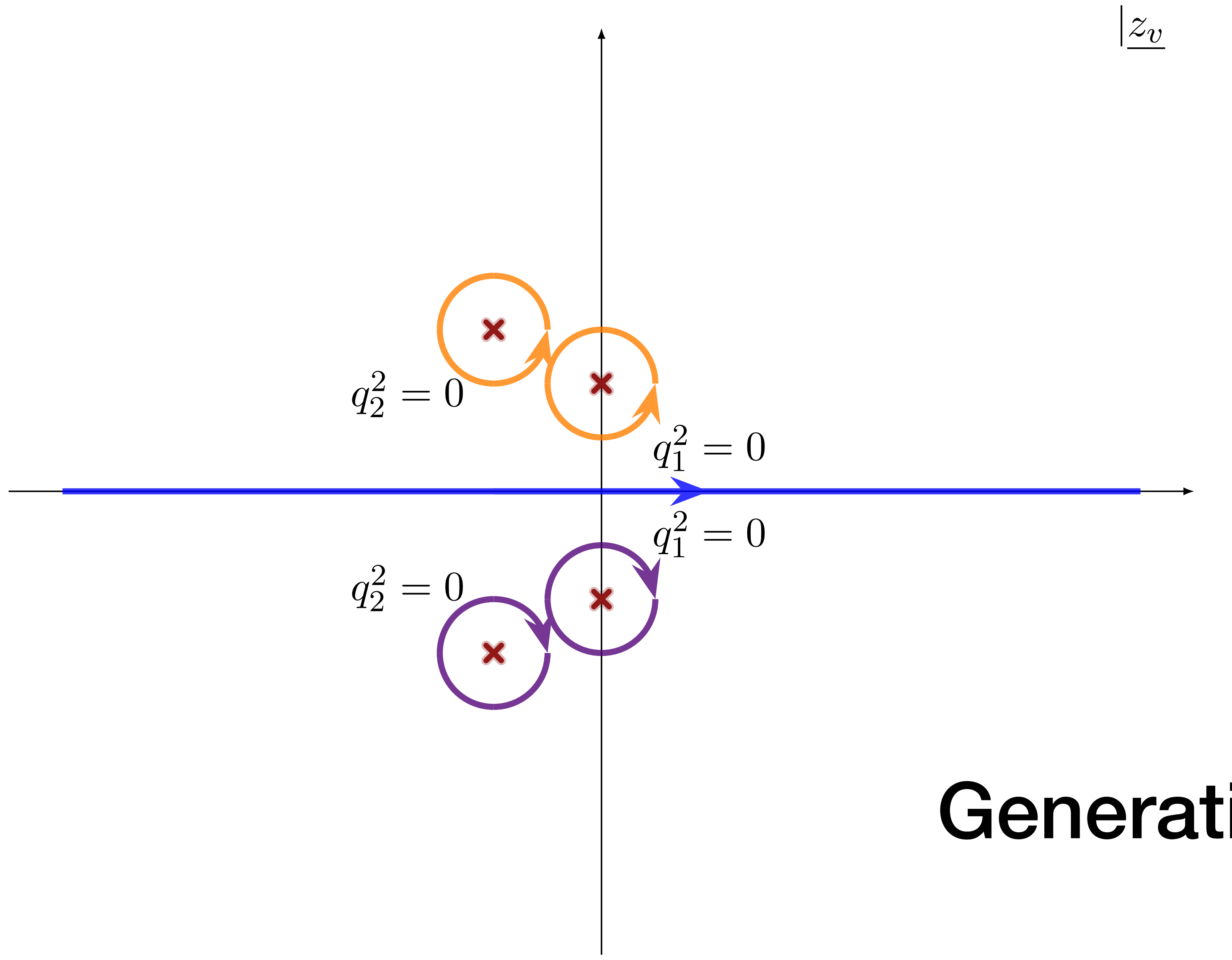
$$= \frac{2\pi \log\left(2\sqrt{\hat{w}_2^2 + z_b^2}\right)}{\sqrt{\hat{w}_2^2 + z_b^2}}$$

Overlapping pole and branch cut!



Results at LO

$$\mathscr{W}_h^{(0)}(\omega, \mathbf{n}) = \sum_{i=1}^{6} c_i \, J_i$$

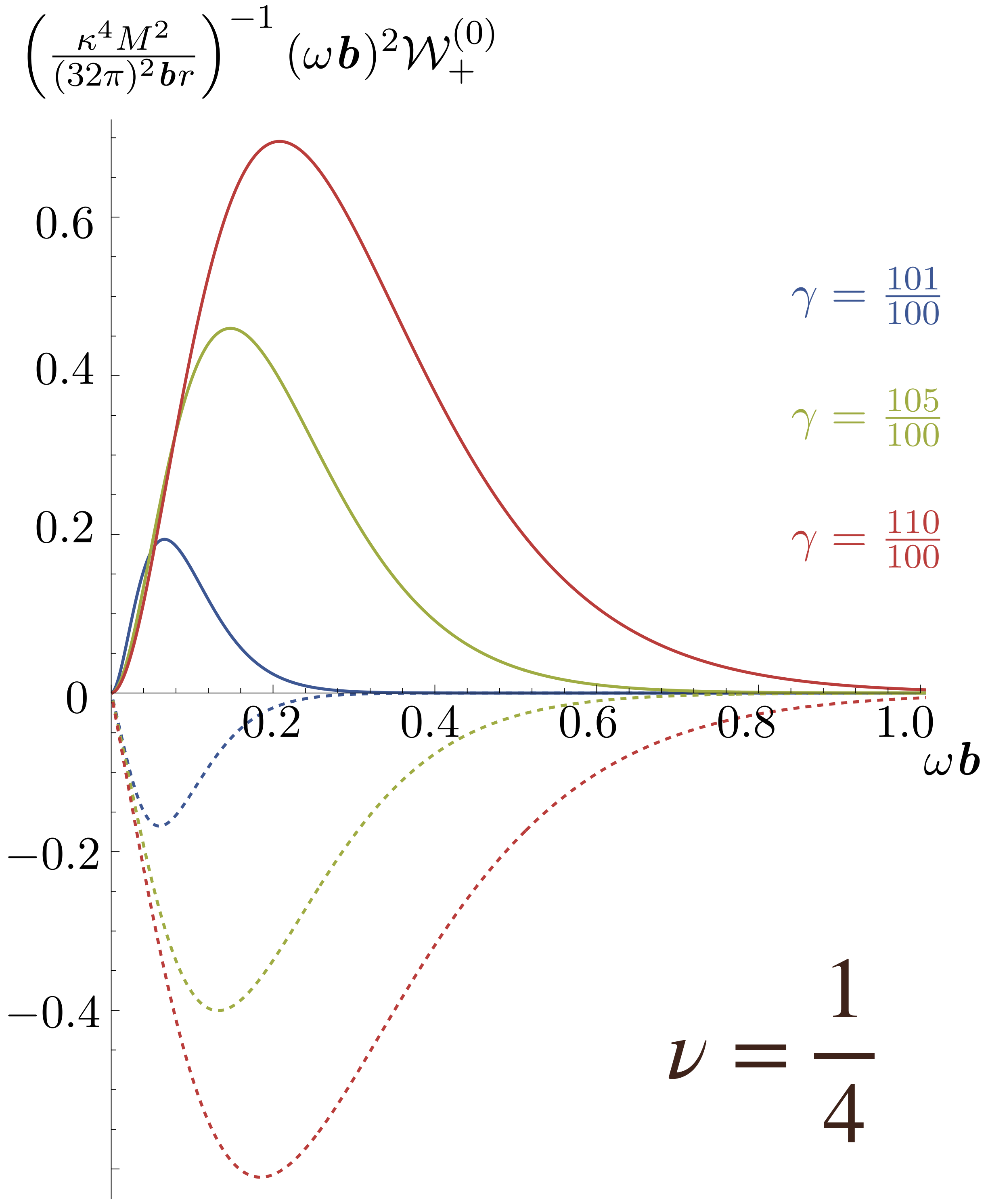


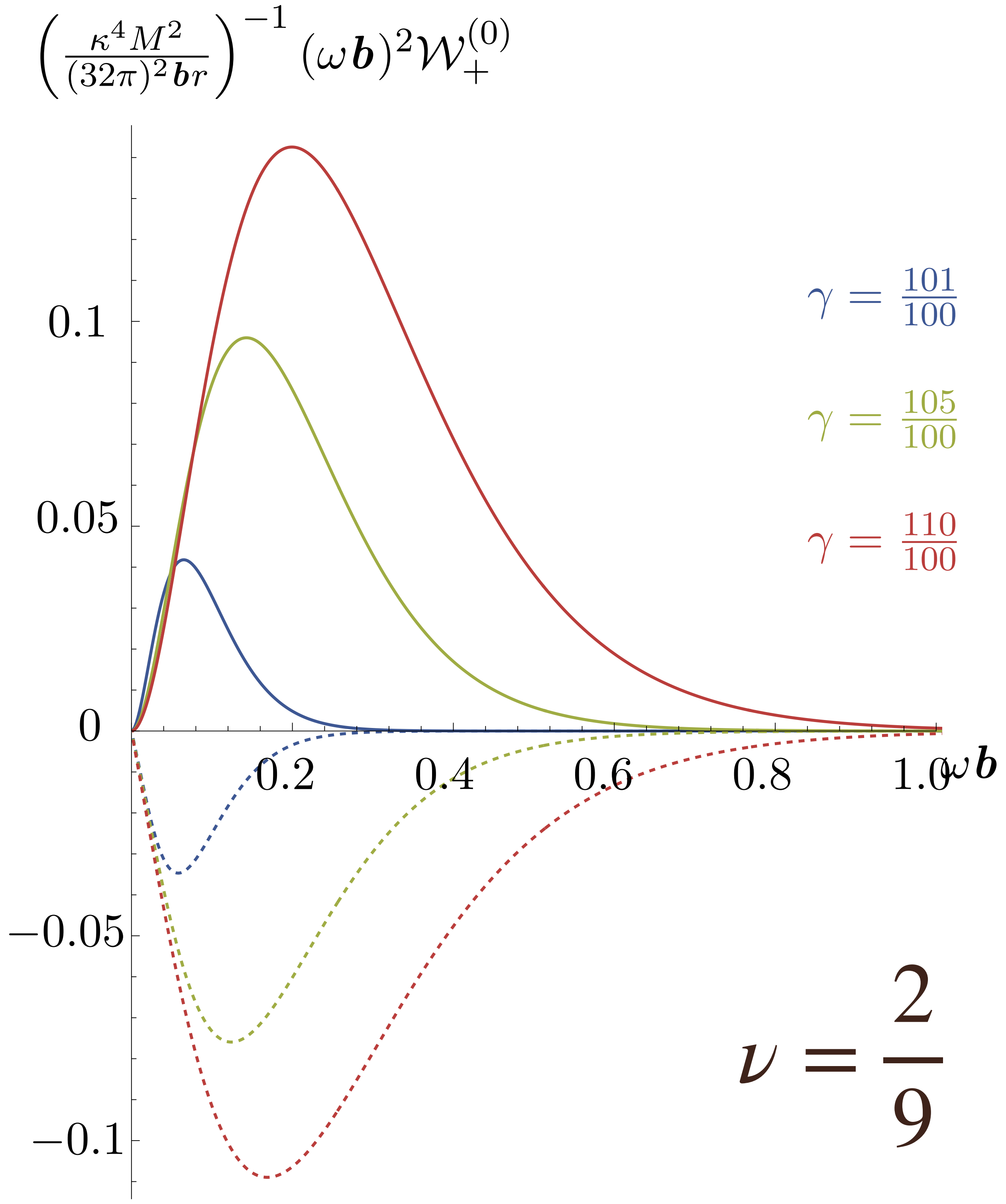
Results at LO

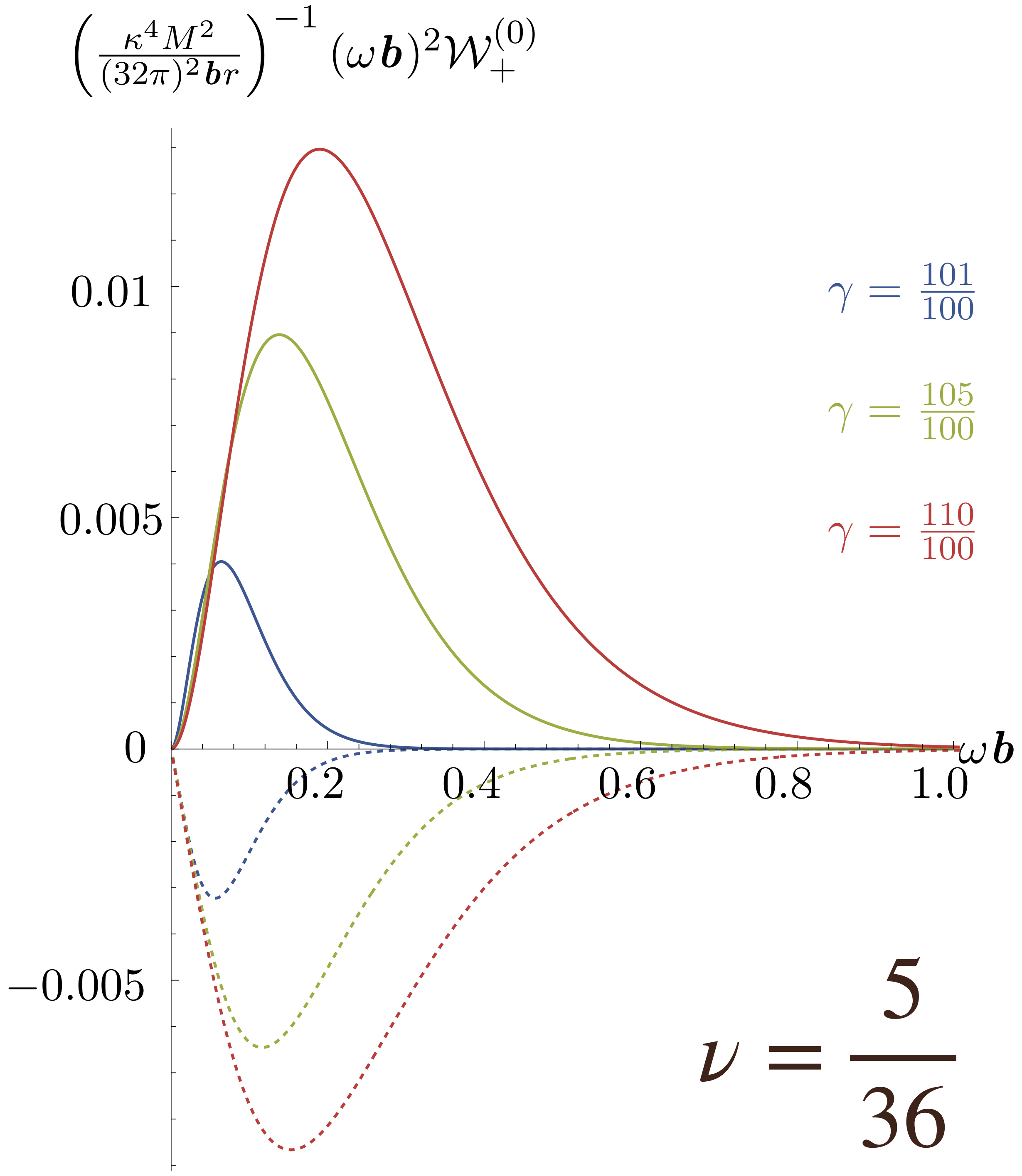
$$\mathscr{W}_h^{(0)}(\omega, \mathbf{n}) = \sum_{i=1}^{6} c_i\, J_i$$

Generation of templates for gravitational waves analysis

$\phi = \frac{7\pi}{10}\,,\ \theta = \frac{7\pi}{5}\,,\ m_1 = m_2\,,\ b = 1$





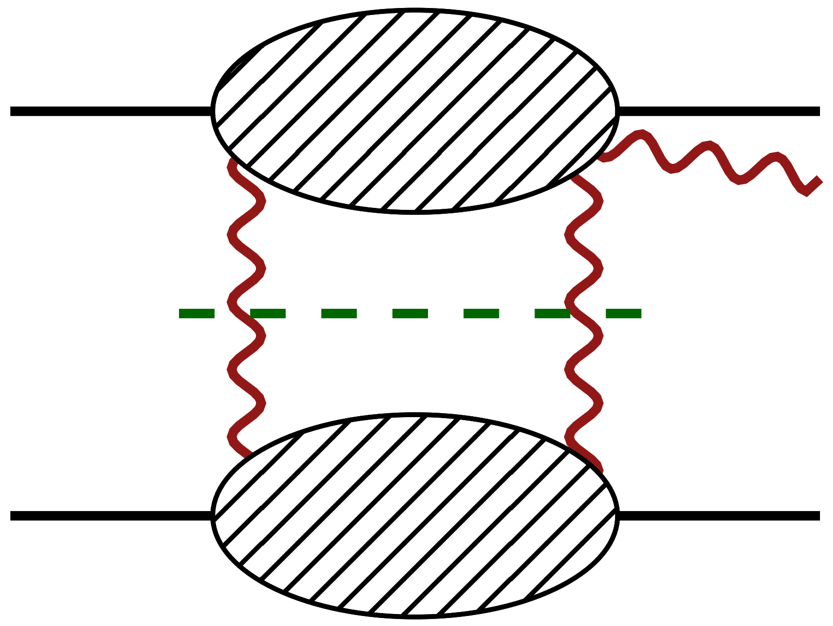
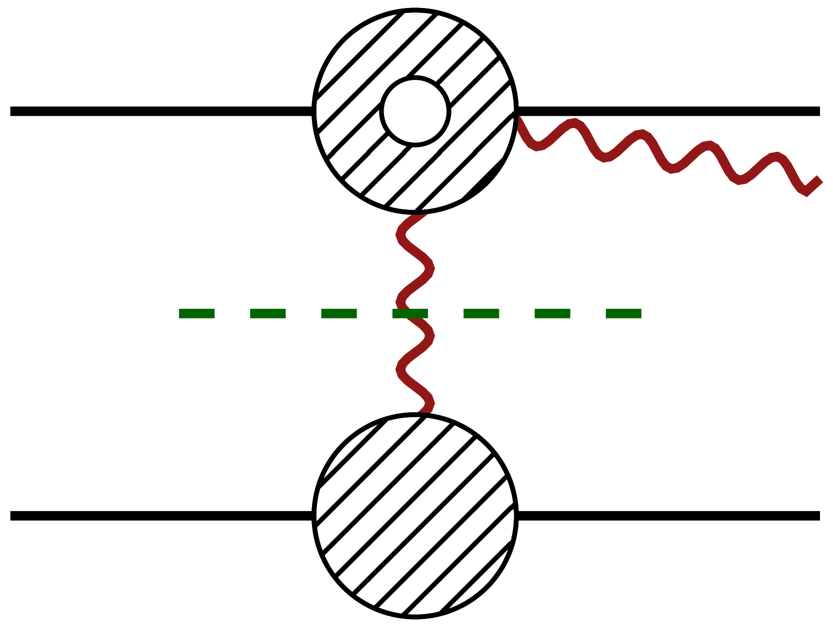


Integrand Generation

NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

Integrand generation from double and single graviton exchange

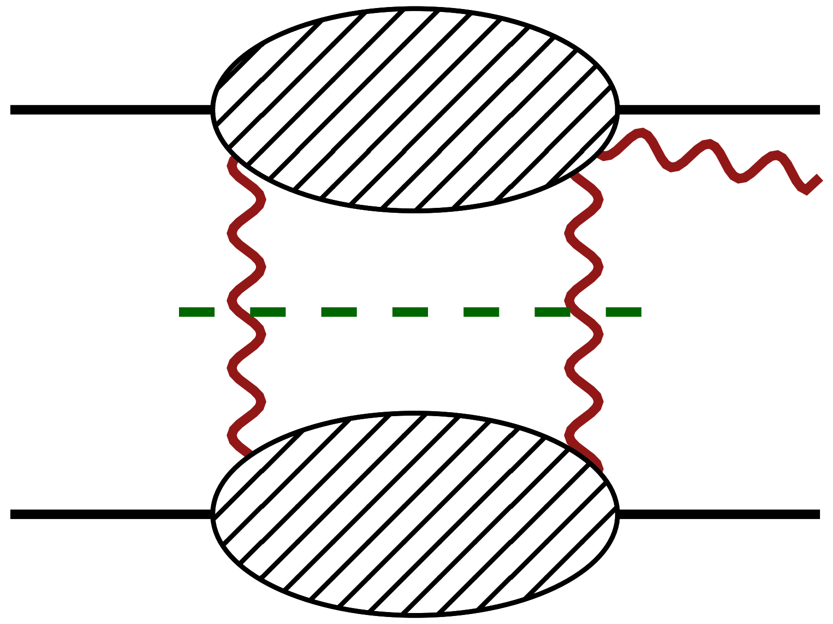
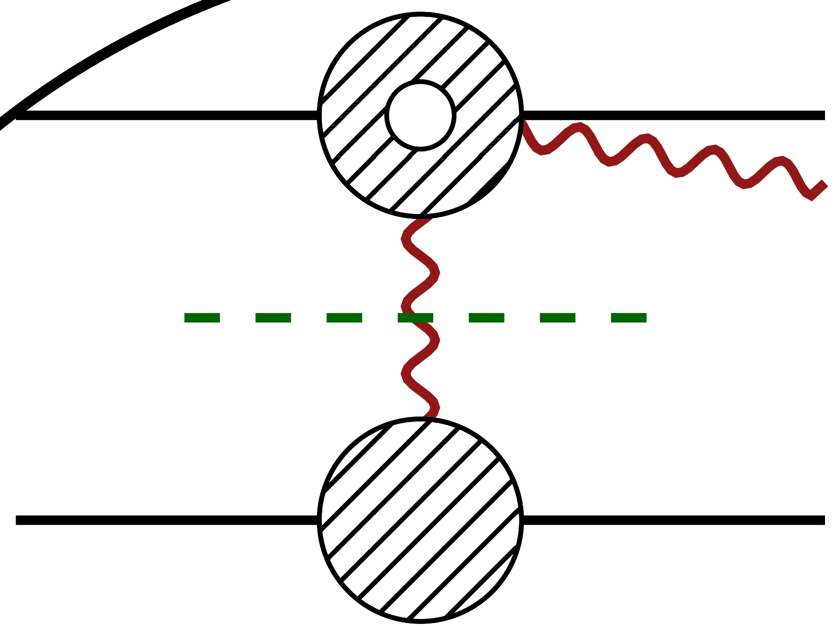
$T^{(1)} =$  $\cup$ 

Integrand Generation

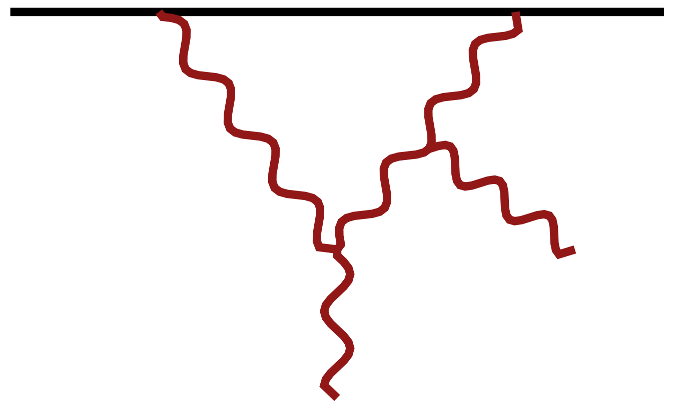
NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

Integrand generation from double and single graviton exchange

$T^{(1)} =$  $\cup$ 

- Cut merging

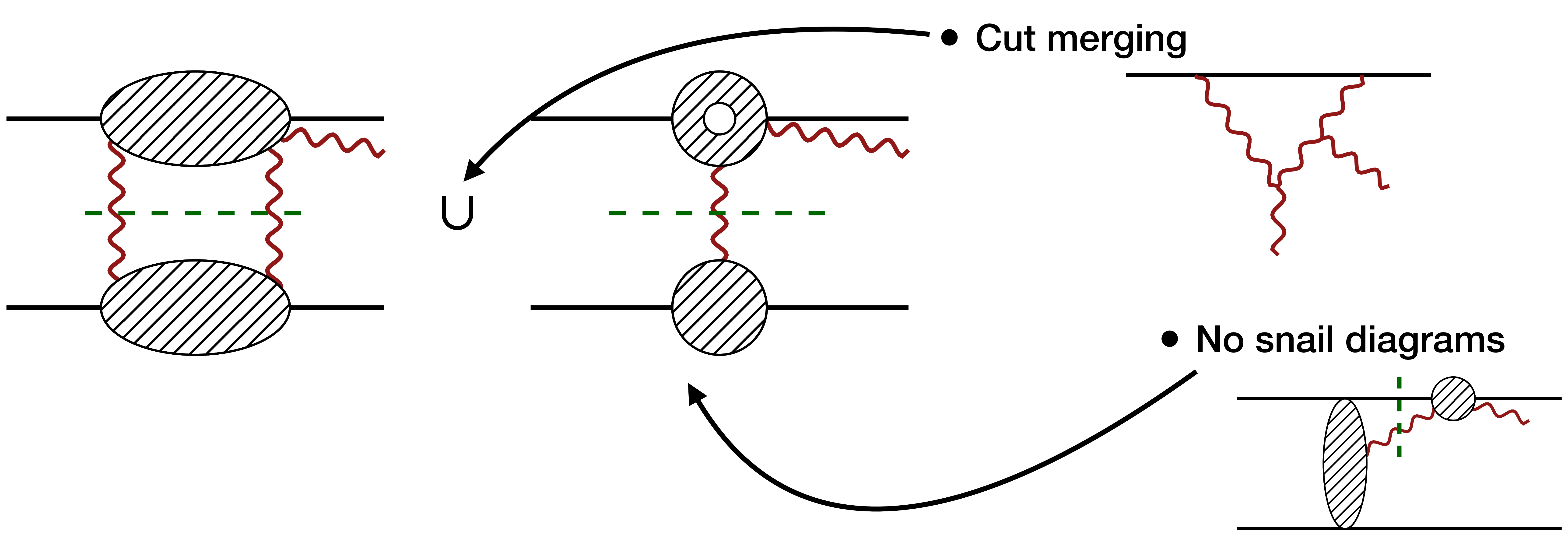


NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

Integrand generation from double and single graviton exchange

$T^{(1)} =$



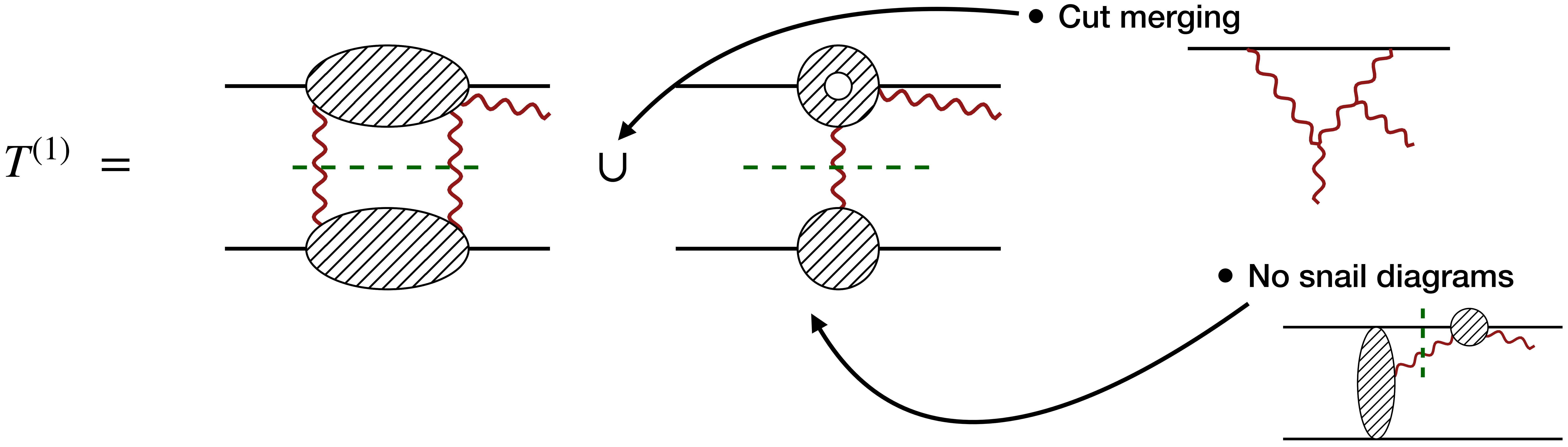
Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini
Herderschee, Roiban, Teng

Integrand Generation

NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

Integrand generation from double and single graviton exchange



Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini
Herderschee, Roiban, Teng

$$\mathscr{W}_h^{(1)} = \frac{e^{i\omega\, n\cdot b_2}}{4\pi r} \int_{\hat{q}} e^{i\, b\cdot q}\, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot (q-k))\ T^{(1)}$$

Fourier transform — Integrand

Integral Reduction

NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

$$I^{(1)}_{a_1a_2a_3111a_7a_8a_9a_{10}a_{11}} = \int_{\hat{q},\hat{\ell}} \frac{e^{D_1}D_1^{-a_1}D_{11}^{-a_{11}}\hat{\delta}(D_4)\hat{\delta}(D_5)\hat{\delta}(D_6)}{D_2^{a_2}D_3^{a_3}D_7^{a_7}D_8^{a_8}D_9^{a_9}D_{10}^{a_{10}}}$$

$D_1 = i\ b \cdot q,\ D_2 = q^2,\ D_3 = (q-k)^2,$

$D_4 = u_1 \cdot q,\ D_5 = u_2 \cdot (k-q)$

$D_6 = u_1 \cdot \ell\ ,\ \ D_7 = u_2 \cdot \ell,\ \ D_8 = \ell^2,$

$D_9 = (\ell - q_2)^2,\ \ D_{10} = (\ell + q_1)^2,\ D_{11} = i\ b \cdot \ell$

Integral Reduction

NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

$$I^{(1)}_{a_1 a_2 a_3 1 1 1 a_7 a_8 a_9 a_{10} a_{11}} = \int_{\hat{q},\hat{\ell}} \frac{e^{D_1} D_1^{-a_1} D_{11}^{-a_{11}} \hat{\delta}(D_4)\hat{\delta}(D_5)\hat{\delta}(D_6)}{D_2^{a_2} D_3^{a_3} D_7^{a_7} D_8^{a_8} D_9^{a_9} D_{10}^{a_{10}}}$$

$$D_1 = i\, b\cdot q,\ D_2 = q^2,\ D_3 = (q-k)^2,$$
$$D_4 = u_1\cdot q,\ D_5 = u_2\cdot(k-q)$$
$$D_6 = u_1\cdot\ell\ ,\ \ D_7 = u_2\cdot\ell,\ \ D_8 = \ell^2,$$
$$D_9 = (\ell - q_2)^2,\ \ D_{10} = (\ell + q_1)^2,\ D_{11} = i\, b\cdot\ell$$

Integration-by-parts identities for Fourier-Loop integrals

$$\int_{\hat{q},\hat{\ell}} \frac{\partial}{\partial s^\mu}\left(e^{D_1}\frac{v^\mu}{\prod_{i=1}^{11} D_i^{a_i}}\right) = 0 \qquad s^\mu \in \{q^\mu, \ell^\mu\} \qquad IBP_{a_1,\ldots,a_{11}}[1-D_1] + IBP_{a_1-1,\ldots,a_{11}} = 0$$

Integral Reduction

NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

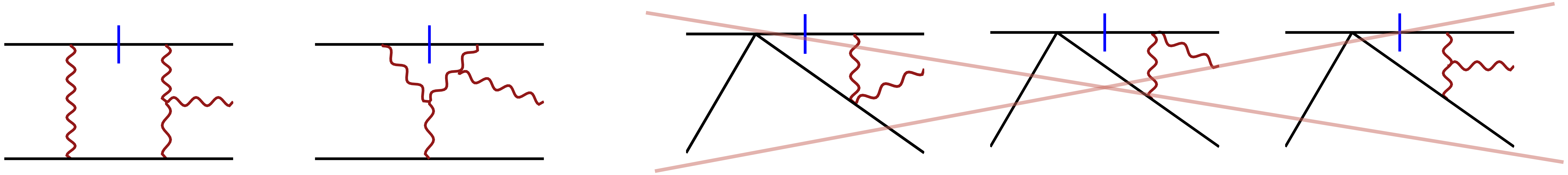
$$I^{(1)}_{a_1a_2a_3111a_7a_8a_9a_{10}a_{11}} = \int_{\hat{q},\hat{\ell}} \frac{e^{D_1}D_1^{-a_1}D_{11}^{-a_{11}}\hat{\delta}(D_4)\hat{\delta}(D_5)\hat{\delta}(D_6)}{D_2^{a_2}D_3^{a_3}D_7^{a_7}D_8^{a_8}D_9^{a_9}D_{10}^{a_{10}}}$$

$$D_1 = i\, b\cdot q,\ D_2 = q^2,\ D_3 = (q-k)^2,$$
$$D_4 = u_1\cdot q,\ D_5 = u_2\cdot(k-q)$$
$$D_6 = u_1\cdot \ell\ ,\ \ D_7 = u_2\cdot\ell,\ \ D_8 = \ell^2,$$
$$D_9 = (\ell-q_2)^2,\ \ D_{10} = (\ell+q_1)^2,\ D_{11} = i\, b\cdot \ell$$

Integration-by-parts identities for Fourier-Loop integrals

$$\int_{\hat{q},\hat{\ell}} \frac{\partial}{\partial s^\mu}\left(e^{D_1}\frac{v^\mu}{\prod_{i=1}^{11} D_i^{a_i}}\right) = 0 \qquad s^\mu \in \{q^\mu, \ell^\mu\} \qquad IBP_{a_1,\ldots,a_{11}}[1-D_1] + IBP_{a_1-1,\ldots,a_{11}} = 0$$

76 MIs, 28 appearing. Master integrals belonging to two top topologies, no snail diagrams!



Integral Evaluation

NLO Waveform

G.B., S. De Angelis, D. Kosower
In progress

Loop-by-loop approach for Master Integrals Evaluation

$$J_i = \int_{\hat{q}} e^{ib\cdot q}\hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot (q-k)) \int_{\hat{\ell}} \mathcal{I}_i(q,\ell)$$

Step2: Complex analysis

Step1: Differential Equations

Integral Evaluation

NLO Waveform

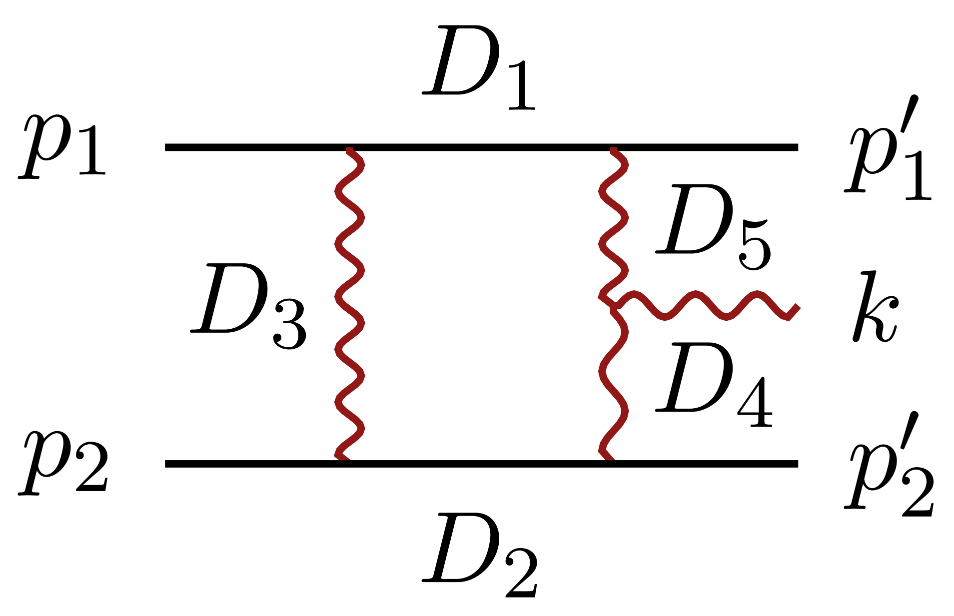
G.B., S. De Angelis, D. Kosower
In progress

Loop-by-loop approach for Master Integrals Evaluation

$$J_i = \int_{\hat{q}} e^{ib\cdot q}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k)) \int_{\hat{\ell}} \mathcal{I}_i(q,\ell)$$

Step2: Complex analysis **Step1: Differential Equations**

Caron-Huot, Giroux, Hannesdottir, Mizera
Bohnenblust, Ita, Kraus, Schlenk
G.B., S. De Angelis



Loop integrals via canonical differential equations

$$I_{1,a_2,a_3,a_4,a_5} := -\, e^{\gamma_E \epsilon} \int_{\hat{\ell}} \frac{i\pi\delta(D_1)}{(D_2 + i\alpha\,\varepsilon)^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5}}\,,$$

Distribution identity

$$\frac{1}{D_2 + i\alpha\,\varepsilon} = \mathscr{P}\left(\frac{1}{D_2}\right) - i\alpha\,\pi\delta(D_2)$$

$$D_1 = u_1\cdot\ell\,, \qquad D_2 = u_2\cdot\ell\,, \qquad D_3 = \ell^2\,, \qquad D_4 = (\ell+q_1)^2\,, \qquad D_5 = (\ell-q_2)^2\,.$$

At $\mathcal{O}[\epsilon^0]$ we obtain an expression in terms of logarithms

Integral Evaluation

NLO Waveform

Loop-by-loop approach for Master Integrals Evaluation

$$J_i = \int_{\hat{q}} e^{ib\cdot q}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k)) \int_{\hat{\ell}} \mathcal{I}_i(q,\ell)$$

Step2: Complex analysis

Step1: Differential Equations

NLO Waveform

Loop-by-loop approach for Master Integrals Evaluation

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Step2: Complex analysis

Step1: Differential Equations

Fourier integration via contour deformation

$$J_i = \frac{1}{\epsilon}\mathcal{F}\left[J_i^{(-1)}\right] + \mathcal{F}\left[J_i^{(0)}\right]$$

Integral Evaluation

NLO Waveform

Loop-by-loop approach for Master Integrals Evaluation

$$J_i = \int_{\hat{q}} e^{ib\cdot q}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k))\int_{\hat{\ell}}\mathcal{I}_i(q,\ell)$$

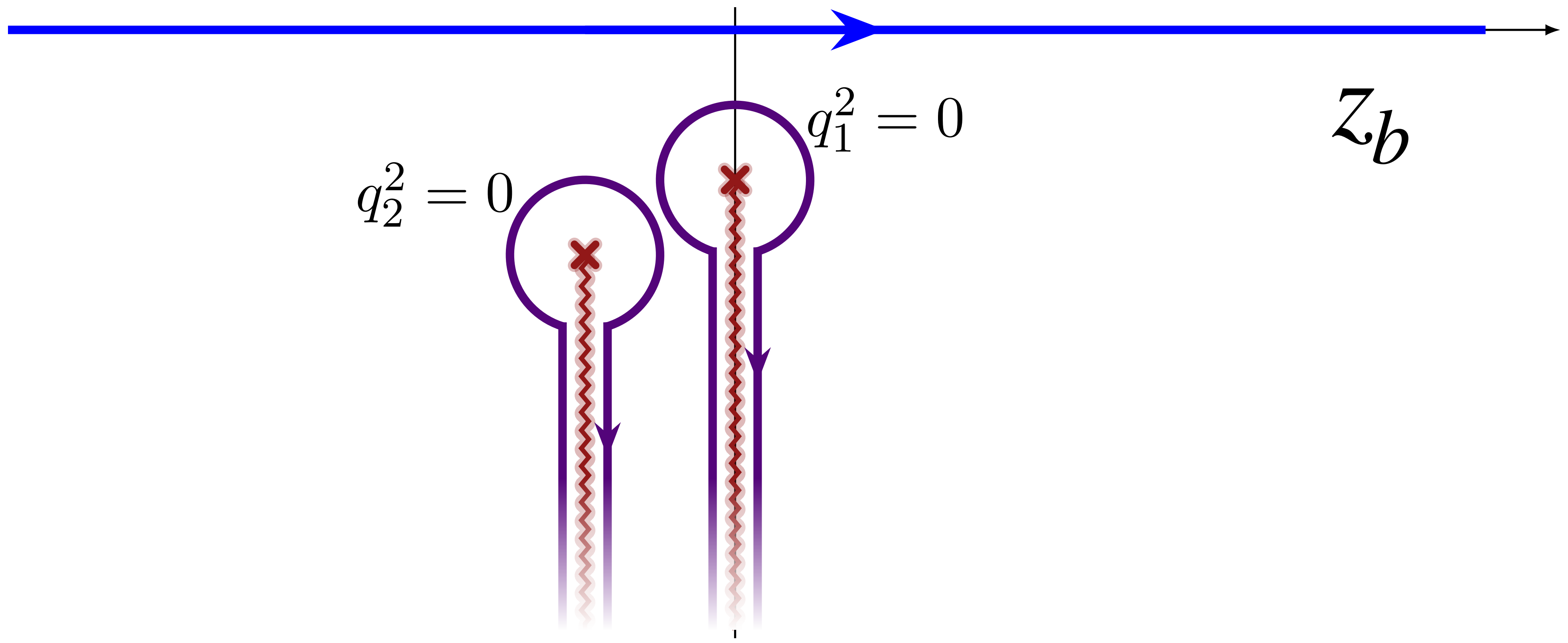
Step2: Complex analysis **Step1: Differential Equations**

Fourier integration via contour deformation

$$J_i = \frac{1}{\epsilon}\mathcal{F}\left[J_i^{(-1)}\right] + \mathcal{F}\left[J_i^{(0)}\right]$$

Fast numerically convergent one-fold integrals

$$J_i = \int_{-\infty}^{\infty} dz_b\, e^{-iz_b\sqrt{-b^2}}\, f(z_b)$$



Results at NLO

G.B., S. De Angelis, D. Kosower
In progress

$$\mathscr{W}_h^{(1)} = \mathscr{W}_{h,\text{IR}}^{(1)} + \mathscr{W}_{h,\text{tail}}^{(1)} + \mathscr{W}_{h,\text{fin}}^{(1)}$$

Results at NLO

G.B., S. De Angelis, D. Kosower
In progress

$$\mathscr{W}_h^{(1)} = \mathscr{W}_{h,\text{IR}}^{(1)} + \mathscr{W}_{h,\text{tail}}^{(1)} + \mathscr{W}_{h,\text{fin}}^{(1)}$$

Weinberg

IR divergent term

$$\mathscr{W}^{(1)}h,\text{IR} = -\frac{i\kappa^2(m_1 w_1 + m_2 w_2)}{32\pi\epsilon}\Gamma_\alpha \mathscr{W}_h^{(0)}$$

$$\Gamma_\alpha = 1 - \alpha\frac{\gamma\left(\gamma^2 - \frac{3}{2}\right)}{\left(\gamma^2 - 1\right)^{3/2}} \ .$$

Results at NLO

G.B., S. De Angelis, D. Kosower
In progress

$$\mathscr{W}_h^{(1)} = \mathscr{W}_{h,\text{IR}}^{(1)} + \mathscr{W}_{h,\text{tail}}^{(1)} + \mathscr{W}_{h,\text{fin}}^{(1)}$$

Weinberg **IR divergent term**

$$\mathscr{W}^{(1)}h,\text{IR} = -\frac{i\kappa^2(m_1 w_1 + m_2 w_2)}{32\pi\,\epsilon}\Gamma_\alpha \mathscr{W}_h^{(0)} \qquad \Gamma_\alpha = 1 - \alpha\frac{\gamma\left(\gamma^2 - \frac{3}{2}\right)}{\left(\gamma^2 - 1\right)^{3/2}}\ .$$

Tail term

$$\mathscr{W}_{h,\text{tail}}^{(1)} = \frac{i\,\kappa^2\,(m_1 w_1 + m_2 w_2)}{32\pi}\Gamma_\alpha \log\left(\frac{\omega^2}{4\pi\,\mu^2}\right)\mathscr{W}_h^{(0)},$$

Results at NLO

G.B., S. De Angelis, D. Kosower
In progress

$$\mathcal{W}^{(1)}_{h} = \mathcal{W}^{(1)}_{h,\mathrm{IR}} + \mathcal{W}^{(1)}_{h,\mathrm{tail}} + \mathcal{W}^{(1)}_{h,\mathrm{fin}}$$

Weinberg **IR divergent term**

$$\mathcal{W}^{(1)}h,\mathrm{IR} = -\frac{i\kappa^2(m_1 w_1 + m_2 w_2)}{32\pi\epsilon}\Gamma_\alpha \mathcal{W}^{(0)}_h \qquad \Gamma_\alpha = 1 - \alpha\frac{\gamma\left(\gamma^2 - \frac{3}{2}\right)}{\left(\gamma^2-1\right)^{3/2}}\,.$$

Tail term

$$\mathcal{W}^{(1)}_{h,\mathrm{tail}} = \frac{i\,\kappa^2\,(m_1 w_1 + m_2 w_2)}{32\pi}\Gamma_\alpha \log\left(\frac{\omega^2}{4\pi\,\mu^2}\right)\mathcal{W}^{(0)}_h\,,$$

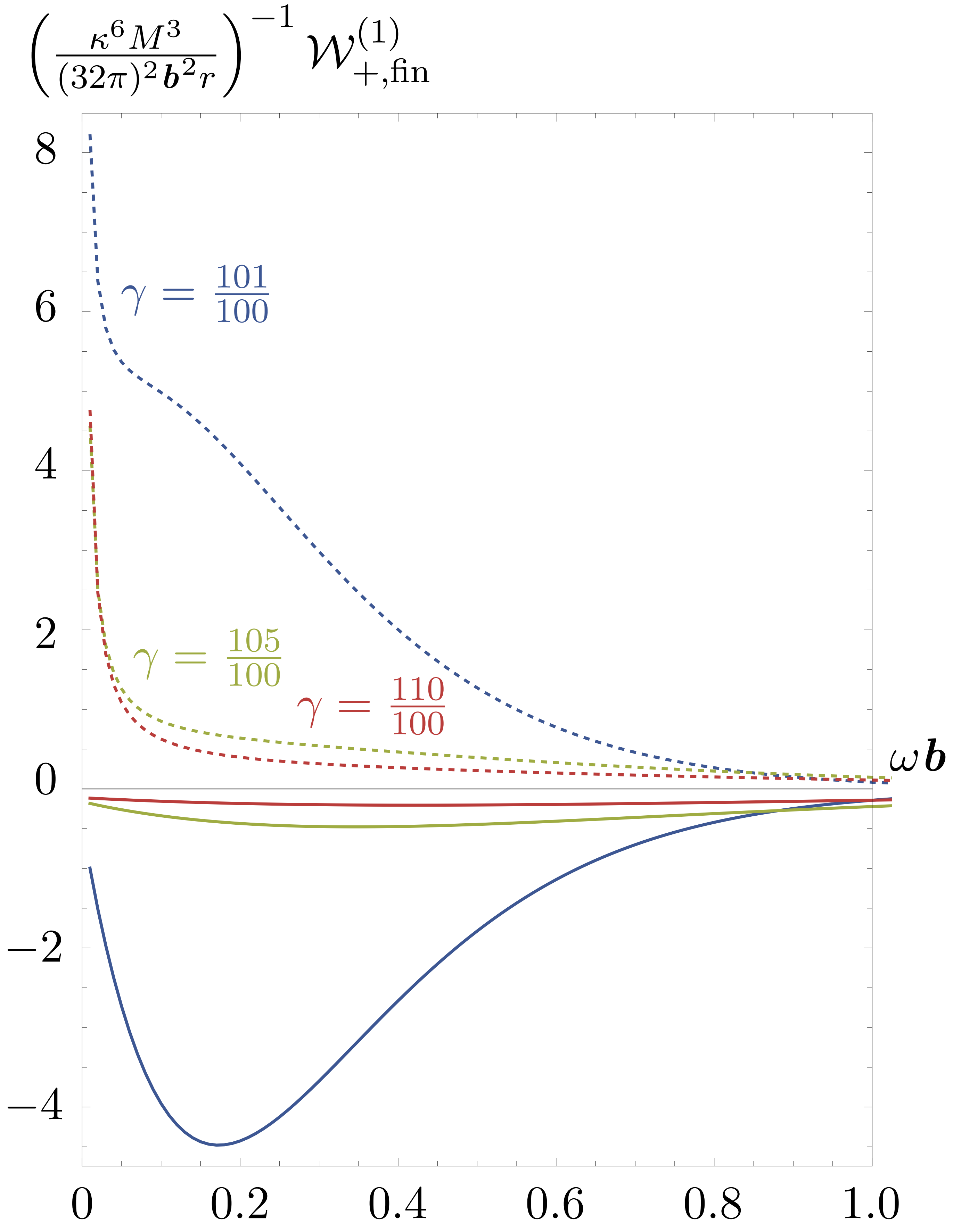
Finite term

$$\mathcal{W}^{(1)}_{h,\mathrm{fin}}$$

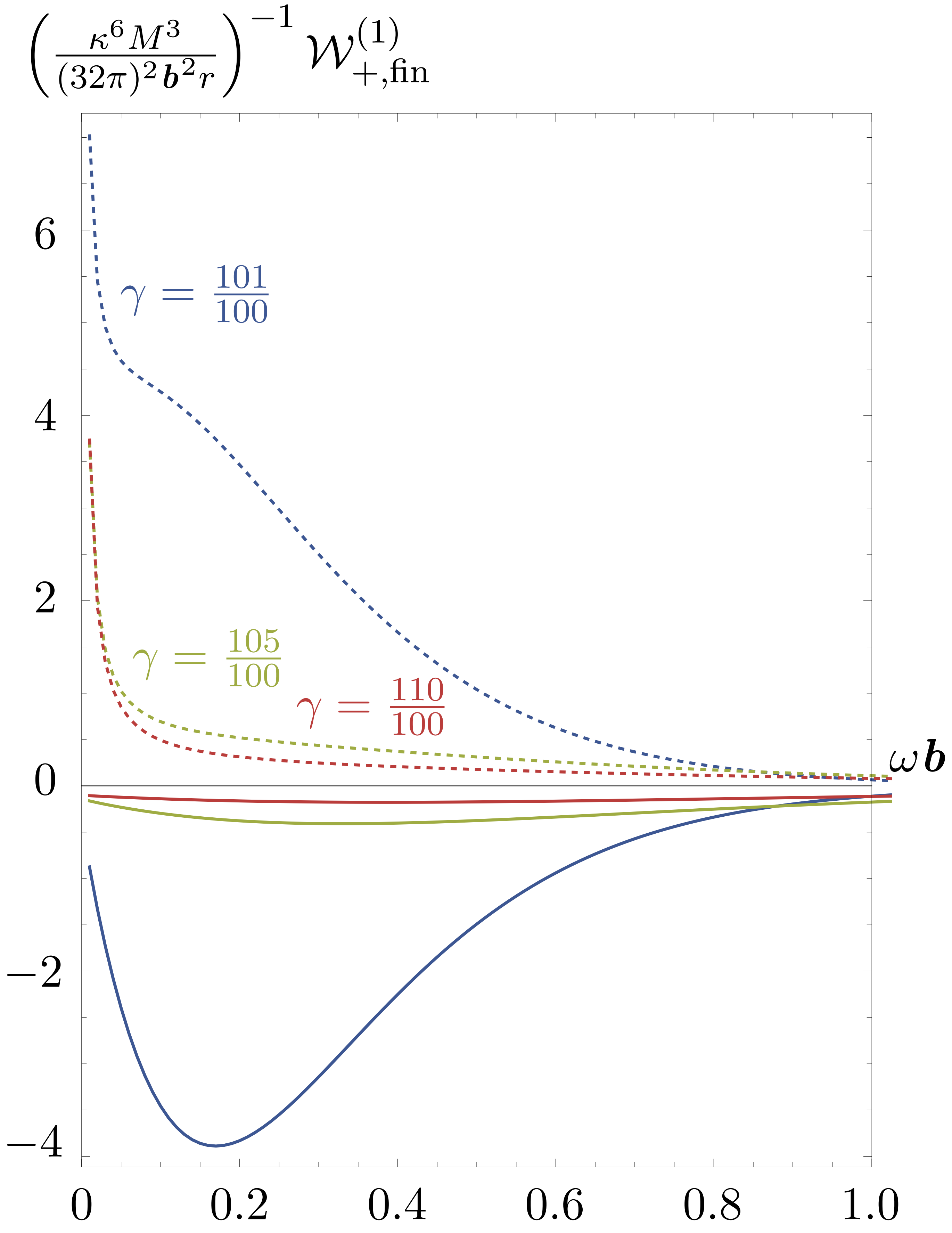
Plots at NLO

$\phi = \frac{7\pi}{10}\,,\ \theta = \frac{7\pi}{5}\,,\ m_1 = m_2\,,\ b = 1$

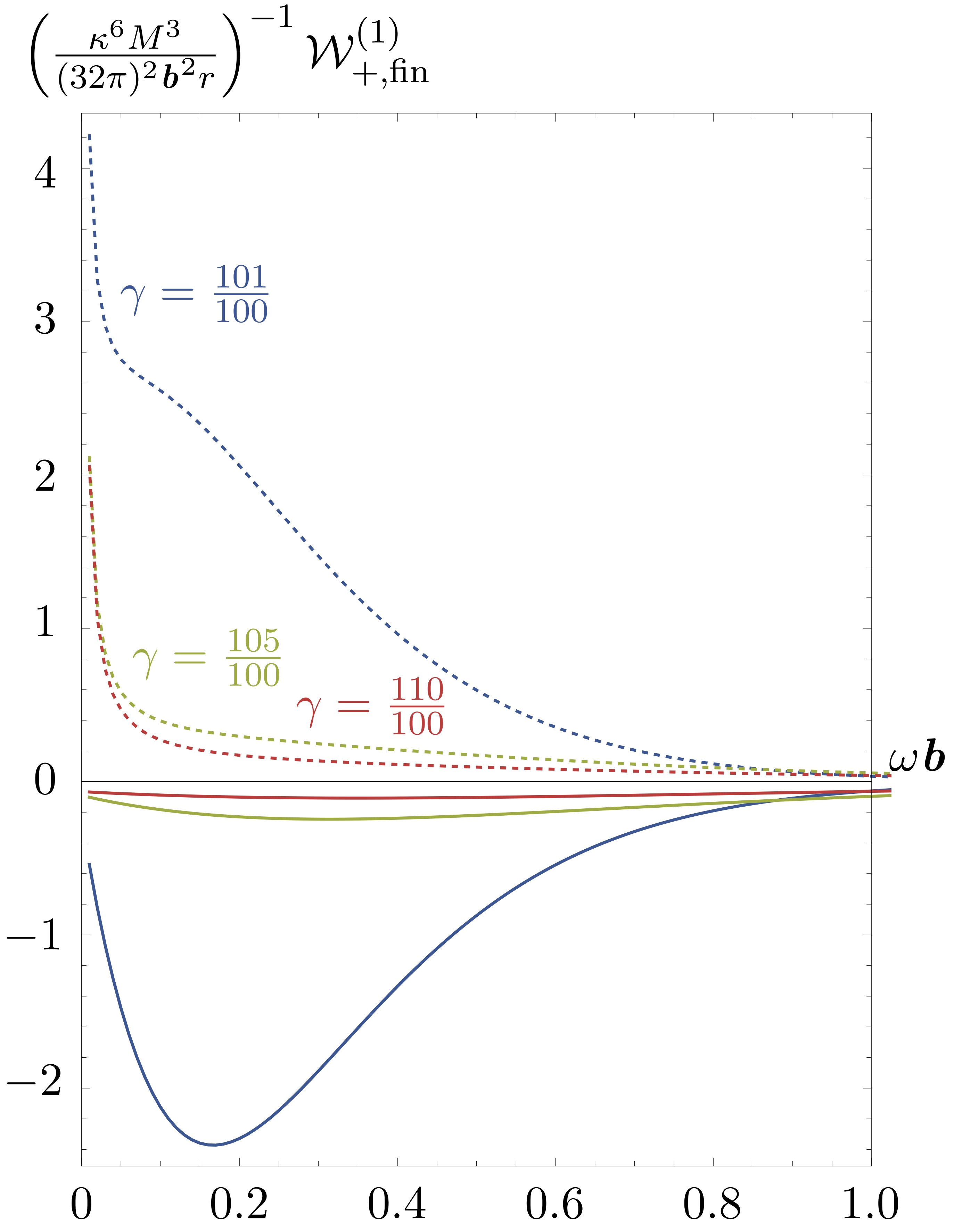
G.B., S. De Angelis, D. Kosower
In progress



$\nu = \frac{1}{4}$



$\nu = \frac{2}{9}$



$\nu = \frac{5}{36}$

Gravitational Power Spectrum

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \frac{2}{\kappa^2}\int \mathrm{d}\Omega \sum_h \left|\, \omega\, \mathscr{W}_h(\omega,\Omega) \,\right|^2$$

Gravitational Power Spectrum

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \frac{2}{\kappa^2}\int \mathrm{d}\Omega \sum_h \left| \,\omega\,\mathscr{W}_h(\omega,\Omega)\, \right|^2$$

Numerical evaluation via **Lebedev Quadrature**

$$\int \mathrm{d}\Omega\, f(\theta,\varphi) \simeq \ (4\pi) \sum_{\theta_i\varphi_i \in L_N} W_i^{(N)}(\theta_i,\varphi_i) f(\theta_i,\varphi_i)$$

$$L_N = \{(\theta_i,\varphi_i)\}_{i=1}^{P}$$

Grid of points

$$\{W_i^{(N)}\}_{i=1}^{P}$$

Weights

Gravitational Power Spectrum

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \frac{2}{\kappa^2}\int \mathrm{d}\Omega \sum_h \left| \, \omega \mathscr{W}_h(\omega, \Omega) \, \right|^2$$

Numerical evaluation via **Lebedev Quadrature**

$$\int \mathrm{d}\Omega \, f(\theta, \varphi) \simeq \ (4\pi) \sum_{\theta_i \varphi_i \in L_N} W_i^{(N)}(\theta_i, \varphi_i) f(\theta_i, \varphi_i)$$

$$L_N = \{(\theta_i, \varphi_i)\}_{i=1}^P$$

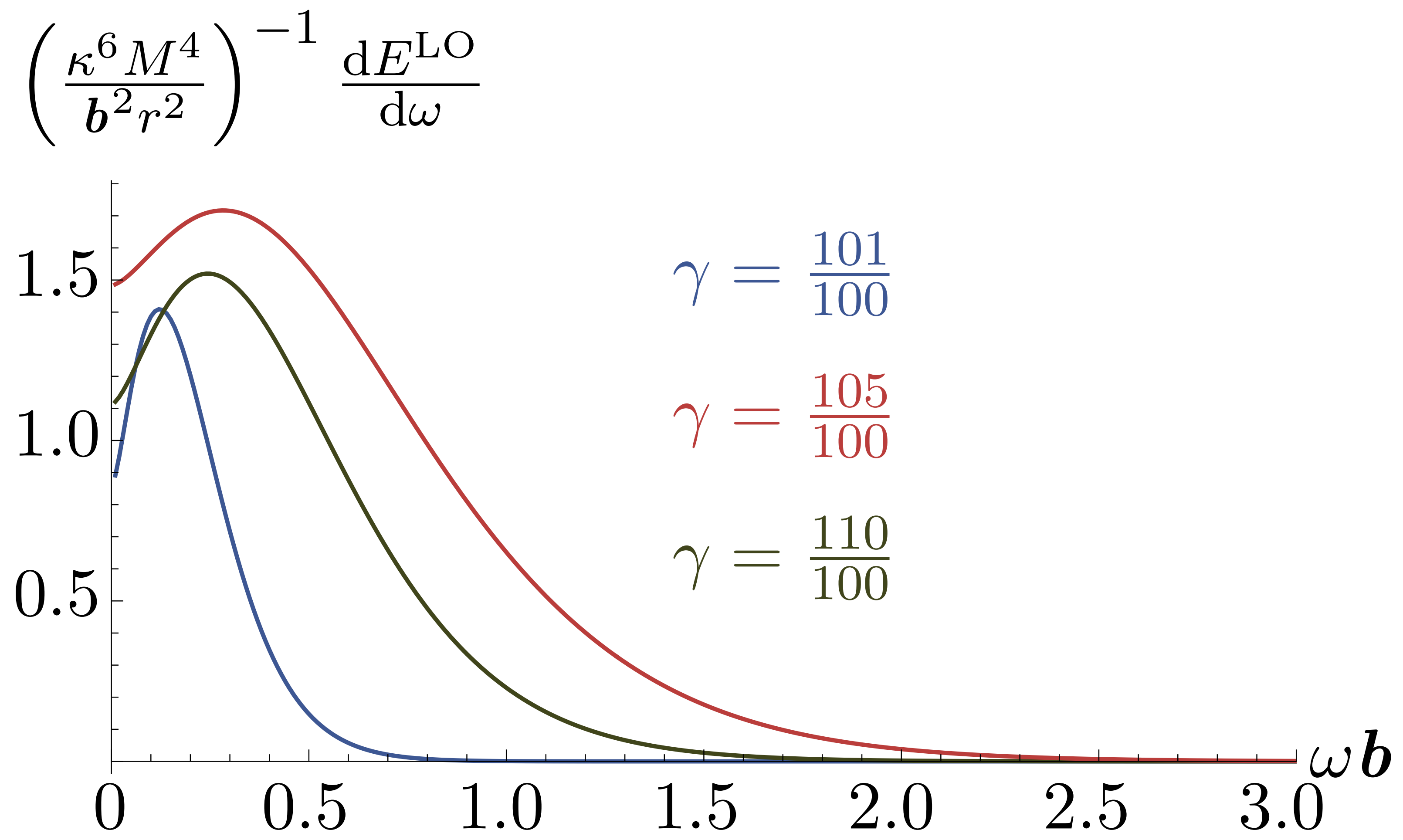
Grid of points

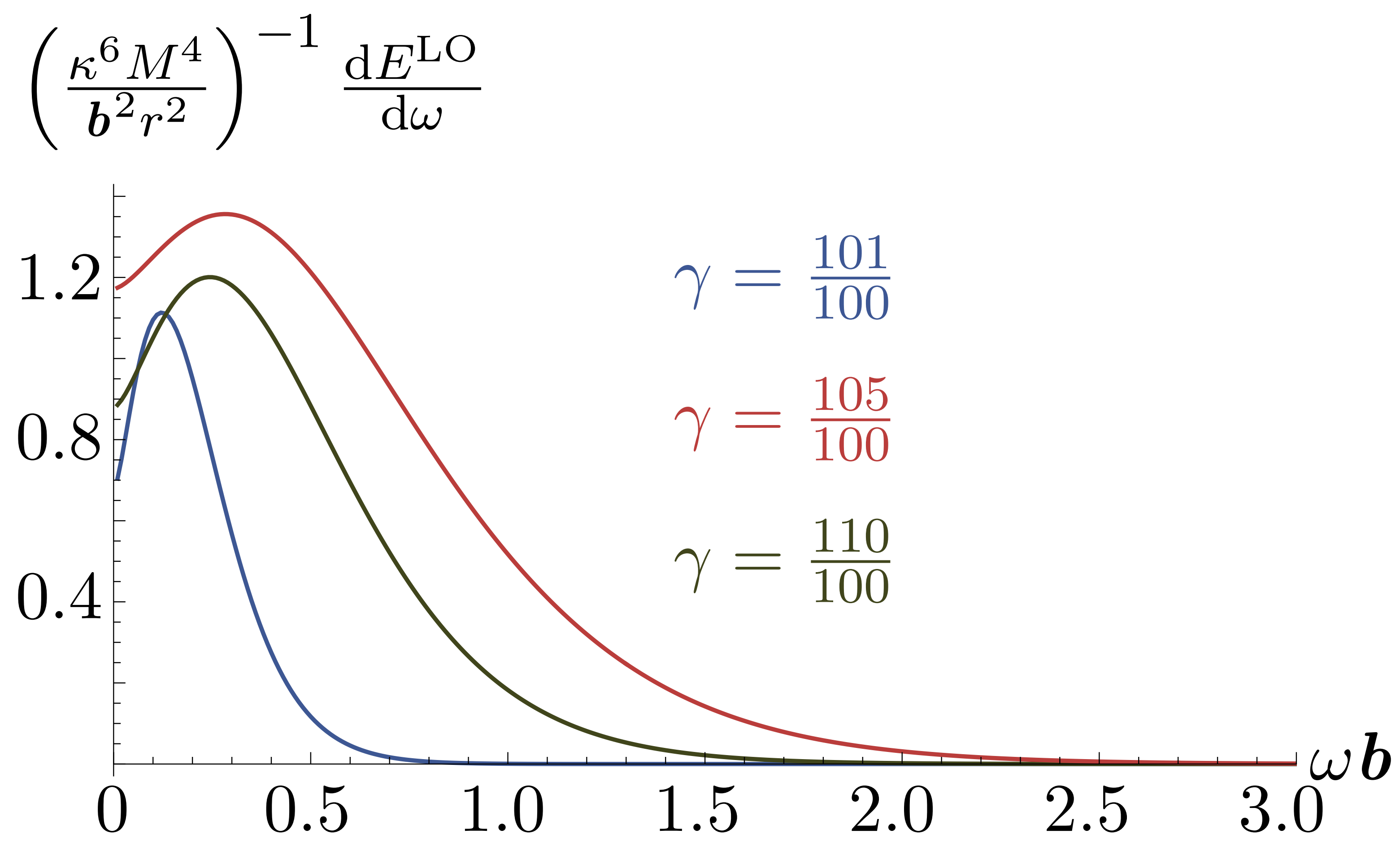
$$\{W_i^{(N)}\}_{i=1}^P$$

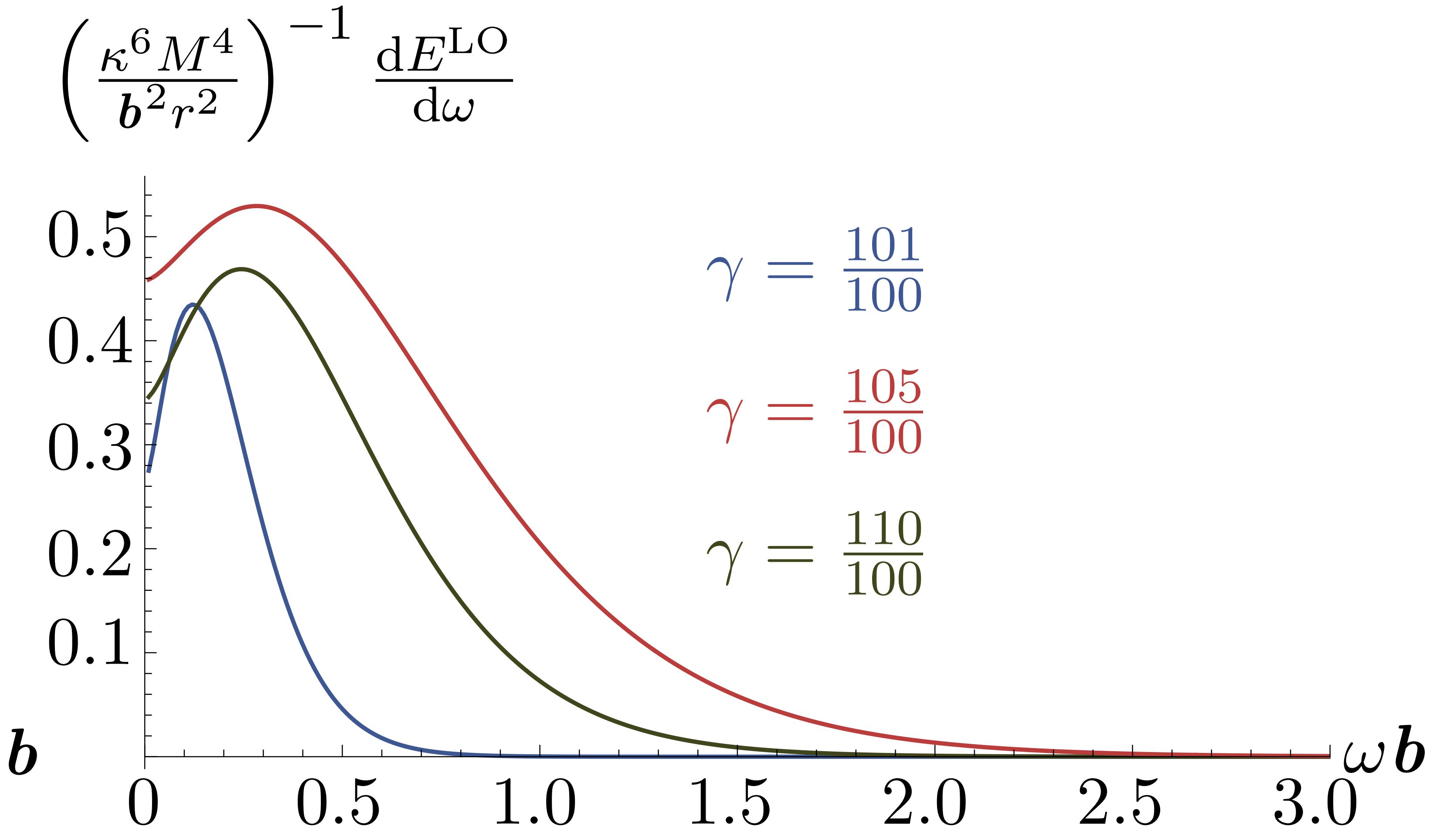
Weights

Leading Order Power Spectrum

Jakobsen, Mogull, Plefka, Steinhoff

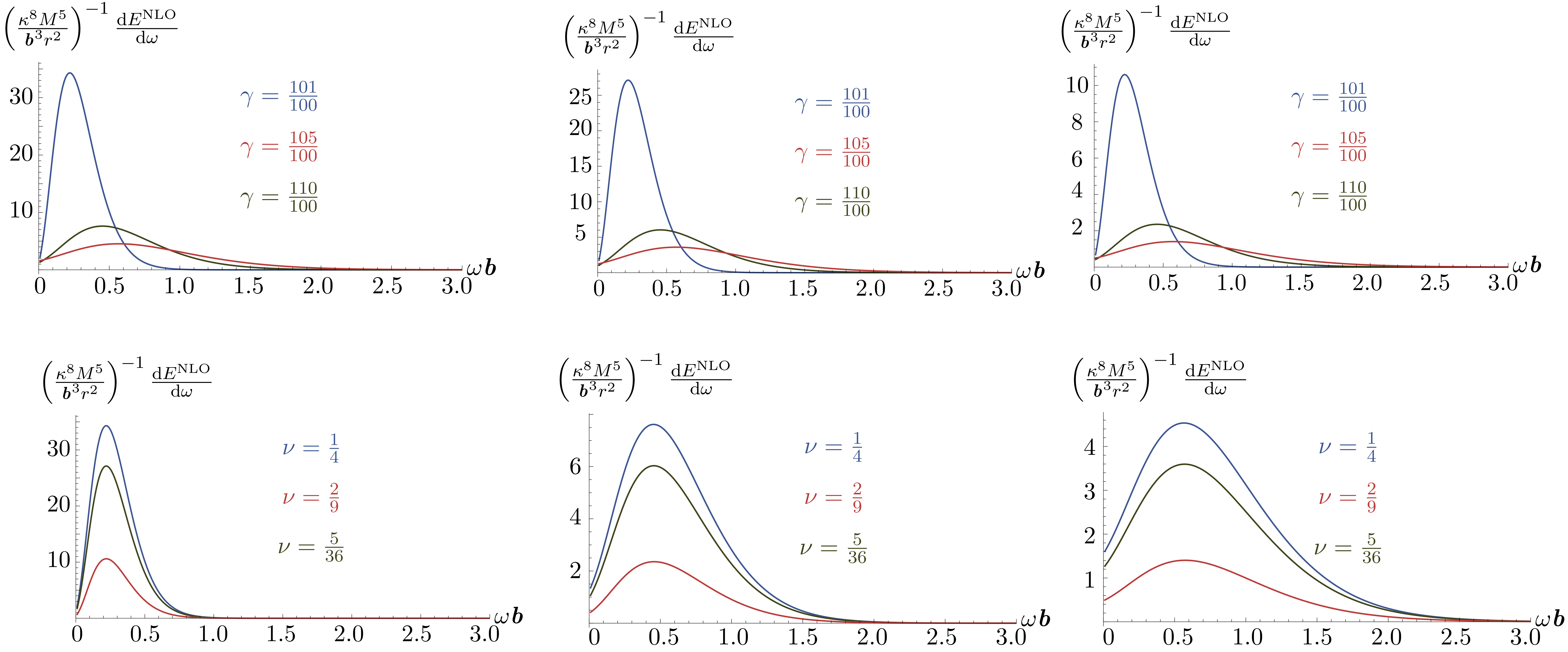






NLO Power Spectrum

G.B., De Angelis, Kosower



Outlooks

Gravitational Waveforms from scattering amplitudes

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Gravitational Waveforms from scattering amplitudes

Combined approach: Scattering amplitudes techniques in frequency space

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Analytic One-loop gravitational waveform in frequency domain

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Analytic One-loop gravitational waveform in frequency domain

Differential equations in frequency domain? [G.B., Chestnov, Crisanti, Giroux, Smith, in progress]

Thank you!