Kinematic Stratifications

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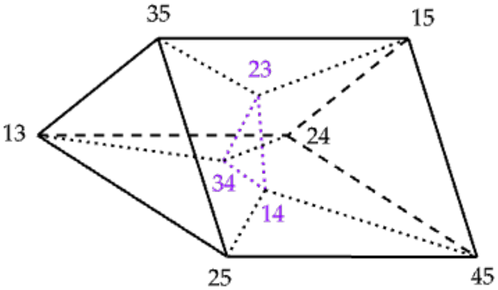


Joint work with Veronica Calvo Cortes and Hadleigh Frost

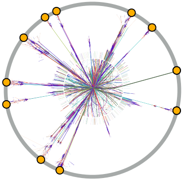
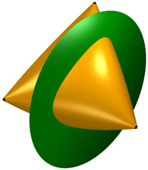
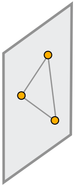
One Particle

In physics, particles are represented by momentum vectors p in *Minkowski space* $\mathbb{R}^{1+d}$, with Lorentzian inner product

$$p \cdot q \;=\; p_0 q_0 - p_1 q_1 - \cdots - p_d q_d .$$

The *universal speed limit* states that $p \cdot p \geq 0$ for each particle.

A particle is *massless* if the equality $p \cdot p = 0$ holds.

Massless: think photon
Massive: think proton

The Lightcone

Several Particles

Consider n particles, with momenta $p^{(1)}, p^{(2)}, \ldots, p^{(n)} \in \mathbb{R}^{1+d}$.

The Lorentz group $\mathrm{SO}(1,d)$ acts on such configurations.
Kinematic data are invariant under this action.

The *Mandelstam invariants* are the entries in the symmetric matrix

$$\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{12} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & \cdots & s_{nn} \end{bmatrix} = \begin{bmatrix} - & p^{(1)} & - \\ - & p^{(2)} & - \\ & \vdots & \\ - & p^{(n)} & - \end{bmatrix} \begin{bmatrix} +1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ (p^{(1)})^T & (p^{(2)})^T & \cdots & (p^{(n)})^T \\ | & | & \cdots & | \end{bmatrix}$$

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The *Mandelstam region* $\mathcal{M}_{n,r}$ is the semi-algebraic set of these matrices, for fixed rank $r \leq 1+d$. It has codimension $\binom{n-r-1}{2}$ in $\mathbb{R}^{\binom{n+1}{2}}$.

We examine the stratification of $\mathcal{M}_{n,r}$ by the signs of the s_{ij}.

Being Lorentzian

The Mandelstam region $\mathcal{M}_{n,r}$ consists of matrices S that satisfy:

- the n diagonal entries s_{ii} of S are non-negative, and
- S has one positive eigenvalue and $r-1$ negative eigenvalues.

The *Lorentzian region* is the subset of nonnegative matrices

$$\mathcal{L}_{n,r} \;=\; \mathcal{M}_{n,r} \,\cap\, (\mathbb{R}_{\geq 0})^{\binom{n+1}{2}}.$$

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Points in $\mathcal{L}_{n,\leq n} = \sqcup_{r=1}^{n} \mathcal{L}_{n,r}$ are *Lorentzian polynomials* of degree two.

Petter Brändén and June Huh: *Lorentzian polynomials*, Ann. Math. (2020).

Bränden proved that $\mathcal{L}_{n,\leq n}$ is a topological ball of dimension $\binom{n+1}{2}$.

P. Brändén: *Spaces of Lorentzian and real stable polynomials are* **Euclidean balls**, Forum Math. Sigma (2021)

Thus, $\mathcal{M}_{n,\leq n} = \sqcup_{r=1}^{n} \mathcal{M}_{n,r}$ is a disjoint union of 2^{n-1} such balls.

Our stratifications match those in

M. Baker, J. Huh, M. Kummer, O. Lorscheid: *Lorentzian polynomials and* **matroids** *over triangular hyperfields*.

Principal Minors

Lemma

$S \in \mathcal{M}_{n,r}$ if and only if the principal minors have alternating signs:

$$(-1)^{|I|-1} \cdot \det(S_I) \;\geq\; 0 \qquad \textit{for all} \;\; I \subseteq [n].$$

For minors of size 2 and 3,

$$s_{ii}s_{jj} \leq s_{ij}^2 \quad \text{and} \quad 2s_{ij}s_{ik}s_{jk} + s_{ii}s_{jj}s_{kk} \geq s_{ii}s_{jk}^2 + s_{jj}s_{ik}^2 + s_{kk}s_{ij}^2.$$

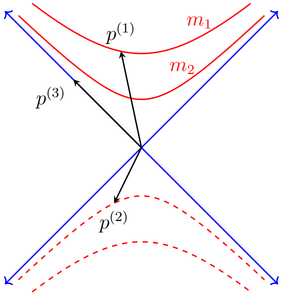
This implies

$$s_{ij}s_{ik}s_{jk} \;\geq\; 0.$$

Proposition

If $S \in \mathcal{M}_{n,r}$ has nonzero entries then there exists $\sigma \in \{-,+\}^n$ such that $\operatorname{sgn}(s_{ij}) = \sigma_i\sigma_j$ for all i, j.

Causality



Corollary

The region $\mathcal{M}_{n,r}$ is the union of the 2^{n-1} signed Mandelstam regions $\mathcal{M}_{n,\sigma,r}$. Their relative interiors are pairwise disjoint:

$$\mathcal{M}_{n,r} \;=\; \bigcup_{\sigma} \mathcal{M}_{n,\sigma,r}.$$

The sign vector σ distinguishes the **future** from the **past**.

The *Lorentzian region* $\mathcal{L}_{n,r}$ is the closure of the region $\mathcal{M}_{n,\sigma,r}$ with $\sigma = (+,+,\ldots,+)$. Everything lies in the future, or in the past.

Massless Particles

From now on, all particles are massless:

$$s_{11} = s_{22} = \cdots = s_{nn} = 0.$$

Principal 4×4 minors of $S \in \mathcal{M}^0_{n,r}$ satisfy $\det(S_{\{i,j,k,l\}}) =$

$$s_{ij}^2 s_{kl}^2 + s_{ik}^2 s_{jl}^2 + s_{il}^2 s_{jk}^2 - 2\cdot\big(s_{ij}s_{ik}s_{jl}s_{kl} + s_{ij}s_{il}s_{jk}s_{kl} + s_{ik}s_{il}s_{jk}s_{jl}\big) \le 0.$$

If we set $p_{ij} = \sqrt{s_{ij}}, \ldots, p_{kl} = \sqrt{s_{kl}}$, then this factors:

$$\begin{aligned}\det(S_{\{i,j,k,l\}}) \; = \; & (p_{ij}p_{kl} + p_{ik}p_{jl} + p_{il}p_{jk})(-p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}) \\ & (-p_{ij}p_{kl} + p_{ik}p_{jl} - p_{il}p_{jk})(p_{ij}p_{kl} - p_{ik}p_{jl} - p_{il}p_{jk}).\end{aligned}$$

Think: Plücker, Schouten, squared Grassmannian, ...

H. Friedman: *Likelihood geometry of the squared Grassmannian*, Proceedings AMS (2025+)

This guides us to **matroids**. In this talk, **all matroids have rank two**.

Matroids of rank two

A *matroid* is a partition $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_m$ of a subset of $[n] = \{1, \ldots, n\}$. The *bases* of P are pairs $\{u, v\}$ where $u \in P_i$ and $v \in P_j$ for $i \neq j$. Elements in $[n] \backslash P$ are *loops*.

The matroid P has $m \geq 2$ parts and $l = n - |P|$ loops.

Example

The *uniform matroid* U_n is the partition of $P = [n]$ into n singletons $P_i = \{i\}$.

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The *uniform matroid* U_n is the partition of $P = [n]$ into n singletons $P_i = \{i\}$.

For $\sigma \in \{-,+\}^P$, the pair (P, σ) is a *signed matroid*.

Definition

The stratum $\mathcal{M}^0_{P,\sigma,r}$ is the subset of the massless Mandelstam region $\mathcal{M}^0_{n,r}$ defined by

$$\mathrm{sign}(s_{ij}) = \sigma_i \sigma_j \ \text{ if } \{i, j\} \text{ is a basis of } P, \text{ and } s_{ij} = 0 \ \text{ otherwise.}$$

Stratification

Theorem

Fix $r \geq 1$. The massless Mandelstam region equals

$$\mathcal{M}^0_{n,r} \;=\; \bigsqcup_{(P,\sigma)} \mathcal{M}^0_{P,\sigma,r}.$$

The disjoint union is over all signed matroids (P,σ) on $[n]$.

The kinematic stratum $\mathcal{M}^0_{P,\sigma,r}$ is non-empty if and only if $3 \leq r \leq m$ or $r = m = 2$. If this holds, then its dimension is

$$\dim\big(\mathcal{M}^0_{P,\sigma,r}\big) \;=\; m(r-2) + n - l - \binom{r}{2}.$$

Recall: *The matroid P has $m \geq 2$ parts and $l = n - |P|$ loops.*

Enumerative Combinatorics

We write $\left\{{n-l \atop m}\right\}$ for the *Stirling number of second kind*. This is the number of partitions of the set $[n-l]$ into exactly m parts.

Corollary

The number of kinematic strata $\mathcal{M}^0_{P,\sigma,r}$ of dimension d in the Mandelstam region $\mathcal{M}^0_{n,r}$ is given, for a fixed sign vector σ or for all possible sign vectors, respectively, by

$$\sum_{m\geq r}\binom{n}{l}\left\{{n-l \atop m}\right\} \quad \textit{and} \quad \sum_{m\geq r}2^{n-l-1}\binom{n}{l}\left\{{n-l \atop m}\right\}.$$

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d / r	2		3		4	
1	6	12				
2	12	48				
3	7	56	4	16		
4			6	48		
5			1	8		
6					1	8

(a) $n = 4$

d / r	2		3		4		5	
1	10	20						
2	30	120						
3	35	280	10	40				
4	15	240	30	240				
5			30	440				
6			10	160	5	40		
7			1	16	10	160		
8								
9					1	16		
10							1	16

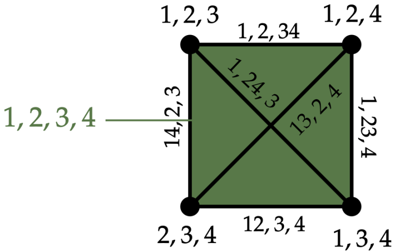
(b) $n = 5$

Posets of Matroids

The strata of $\mathcal{L}^0_{n,r}$ form a poset: $P \leq P'$ if every loop of P' is a loop in P, and the partition P' refines the partition P.

Same as containment of matroid polytopes.

For $n = 4, r = 3$, there are $11 = 1 + 6 + 4$ strata.
The top stratum $\mathcal{L}^0_{U_{4,3}}$ has three connected components:



The strata $\mathcal{M}^0_{P,\sigma,r}$ of $\mathcal{M}^0_{n,r}$ form a poset:
$(P, \sigma) \leq (P', \sigma')$ if $P \leq P'$ and $\sigma = \sigma'$ for all non-loops of P.

Inclusions and Topology

Our kinematic stratifications are nice: if a stratum intersects the closure of another stratum, then containment holds.

But, the topology of strata is quite interesting:

Proposition

$\mathcal{M}^0_{P,\sigma,3}$ *has* $(m-1)!/2$ *connected components.*

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Proposition

$\mathcal{M}^0_{P,\sigma,3}$ has $(m-1)!/2$ connected components.

Theorem

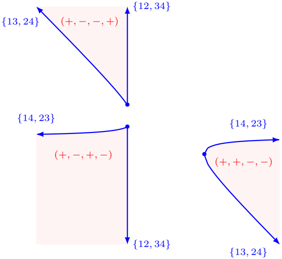
The kinematic stratum $\mathcal{M}^0_{P,\sigma,\leq r}$ is homotopic to the configuration space $F(\mathbb{S}^{r-2}, m)/\mathrm{SO}(r-1)$ for m points on the sphere $\mathbb{S}^{r-2}$.

Corollary

The stratum $\mathcal{M}^0_{P,\sigma,\leq 4}$ is homotopic to the moduli space $M_{0,m}(\mathbb{C})$, and hence to the complement of the affine braid arrangement.

E. Feichtner and G. Ziegler: *The integral cohomology algebras of ordered configuration spaces of spheres* (2000)

What matters for physics?



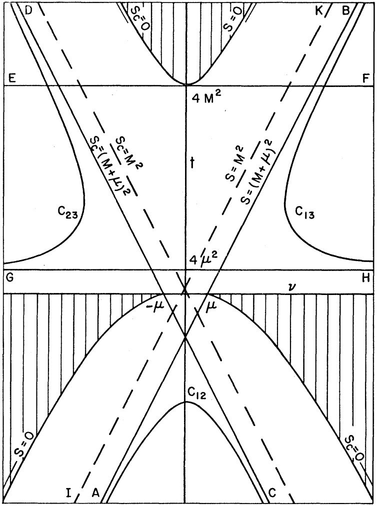


FIG. 1. Kinematics of the reactions I, II, and III.

Stanley Mandelstam: *Determination of the pion-nucleon scattering amplitude from dispersion relations and unitarity. General theory*, Physical Review (1958).

Momentum Conservation

The *massless momentum conserving (MMC) region* $\mathcal{C}^0_{n,r}$ is

$$\mathcal{C}^0_{n,r} \;=\; \bigsqcup_{(P,\sigma)} \mathcal{C}^0_{P,\sigma,r},$$

where $\mathcal{C}^0_{P,\sigma,r}$ is the intersection of $\mathcal{M}^0_{P,\sigma,r}$ with the subspace

$$\mathbb{R}^{n(n-3)/2} = \{S \,:\, s_{i1}+s_{i2}+\cdots+s_{in} \;=\; 0 \quad \text{for } i = 1,2,\ldots,n\}.$$

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Theorem

The stratum $\mathcal{C}^0_{P,\sigma,r}$ is non-empty if and only if

1. *For $3 \leq r < m$: there exist i, j, k, l in $[n]$, with $\sigma_i = \sigma_j = +$ and $\sigma_k = \sigma_l = -$, such that the restriction of the matroid P to $\{i, j, k, l\}$ is either U_4 or $\{ik, jl\}$,* **and**
2. *for $2 \leq r = m$: each part of P has elements with opposite signs.*

In this case, $\dim(\mathcal{C}^0_{P,\sigma,r}) \;=\; (m-1)(r-1) - \binom{r}{2} + (n-l-m) - 1.$

Four Particles

Use the rank 3 matrix

$$S = \begin{bmatrix} 0 & x & -x-y & y \\ x & 0 & y & -x-y \\ -x-y & y & 0 & x \\ y & -x-y & x & 0 \end{bmatrix}.$$

The principal 3×3 minors are $\det(S_{ijk}) = -2xy(x+y) \geq 0$.

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The MMC region $\mathcal{C}^0_{4,\leq 3} = \mathcal{C}^0_{4,3} \cup \mathcal{C}^0_{4,2}$ has nine strata:

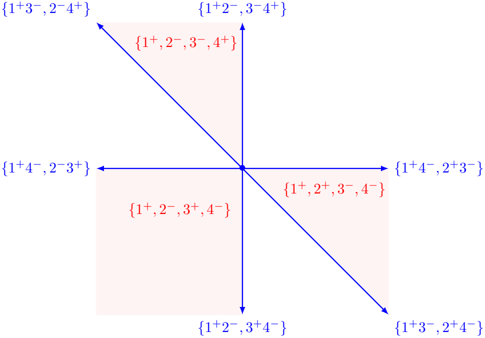

Figure 3: The $3 + 6$ MMC strata for $n = 4$.

On-Shell

For $n = 4$ particles, fix masses $\mathbf{m} = (\mu, \mu, m, m)$ with $m > \mu > 0$.
We studied the MMC region $\mathcal{C}^{\mathbf{m}}_{4,3}$ by modifying our matrix:

$$S = \begin{bmatrix} \mu^2 & x & -x-y-\mu^2 & y \\ x & \mu^2 & y & -x-y-\mu^2 \\ -x-y-\mu^2 & y & m^2 & \mu^2-m^2+x \\ y & -x-y-\mu^2 & \mu^2-m^2+x & m^2 \end{bmatrix}.$$

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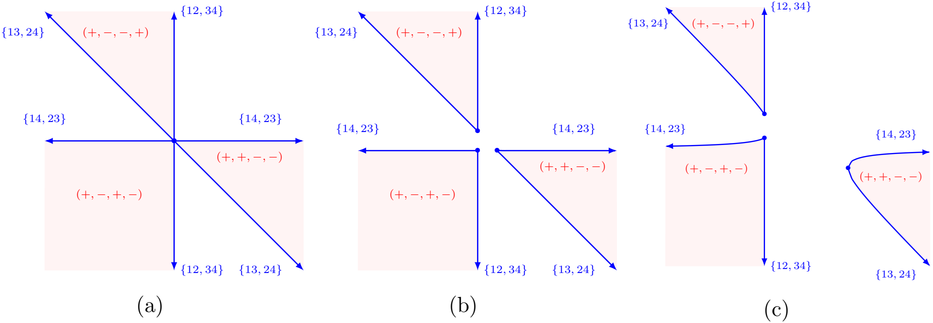
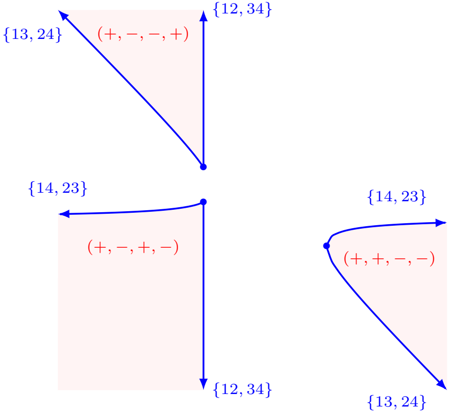
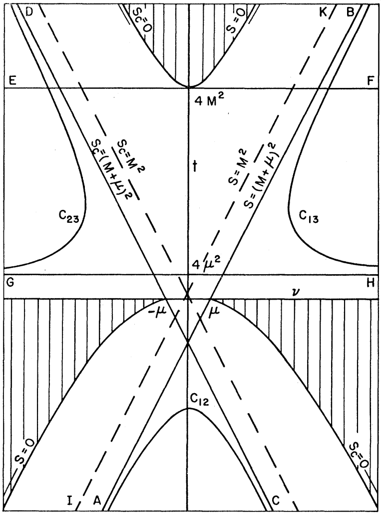


Figure 4: Regions for (a) massless, (b) equal masses, and (c) two unequal masses.

Back to 1958





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Stanley Mandelstam: *Determination of the pion-nucleon scattering amplitude from dispersion relations and unitarity. General theory*, Physical Review (1958).

Five Particles

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1	6	30				
2	6	60	3	15		
3			9	90		
4			1	10		
5					1	10

10 **cyclic polytopes** $C(4,6)$
$f = (6, 15, 18, 9)$

$$S = \begin{bmatrix} 0 & a & -a-b+d & b-d-e & e \\ a & 0 & b & -b-c+e & -a+c-e \\ -a-b+d & b & 0 & c & a-c-d \\ b-d-e & -b-c+e & c & 0 & d \\ e & -a+c-e & a-c-d & d & 0 \end{bmatrix}.$$

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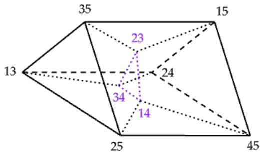
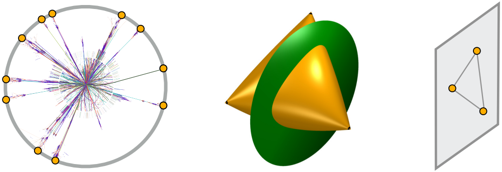
Igusa quartic

$$\begin{aligned} & a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + a^2e^2 \\ & + 2abcd + 2abce + 2abde + 2acde + 2bcde \\ & - 2ab^2c - 2bc^2d - 2cd^2e - 2ade^2 - 2a^2be \;<\; 0. \end{aligned}$$



σ	s_{12}	s_{13}	s_{14}	s_{15}	s_{23}	s_{24}	s_{25}	s_{34}	s_{35}	s_{45}
$(-,-,+,+,+)$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$(-,+,-,+,+)$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$(-,+,+,-,+)$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$(-,+,+,+,-)$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$+$	$-$	$-$
$(+,-,-,+,+)$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$
$(+,-,+,-,+)$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$-$
$(+,-,-,+,+)$	$-$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$
$(+,+,-,-,+)$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
$(+,+,-,+,-)$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$-$
$(+,+,+,-,-)$	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$+$

Conclusion



- We are part of the ERC Synergy project UNIVERSE+, titled

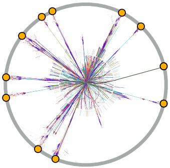
  Positive Geometry in Particle Physics and Cosmology.

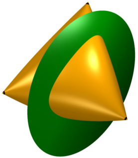
- This talk discussed a collaboration between the nodes in Leipzig and Princeton. The other two are Amsterdam and Munich.

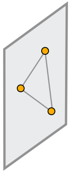
- Our paper was submitted to the journal

  Discrete and Computational Geometry

Conclusion







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- **Thanks for listening!**