Coordinate Independent Formalism for UHFGW detection

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Based on 2404.08572

with Sebastian Schenk and Pedro Schwaller

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WEIZMANN INSTITUTE OF SCIENCE

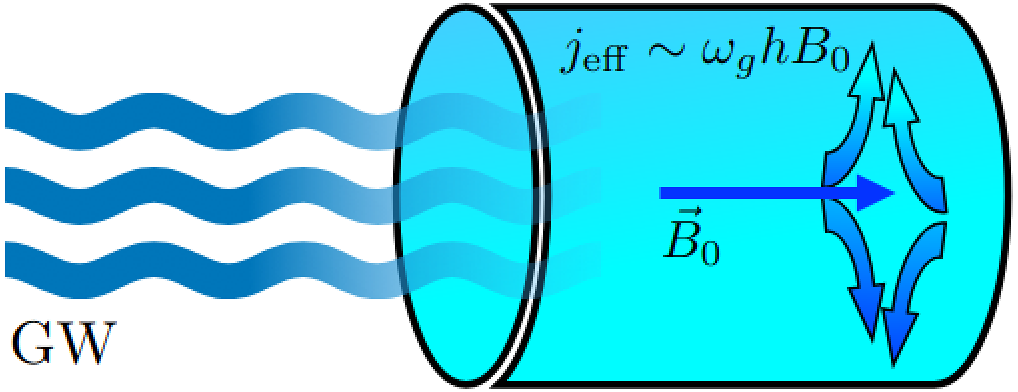
Outline

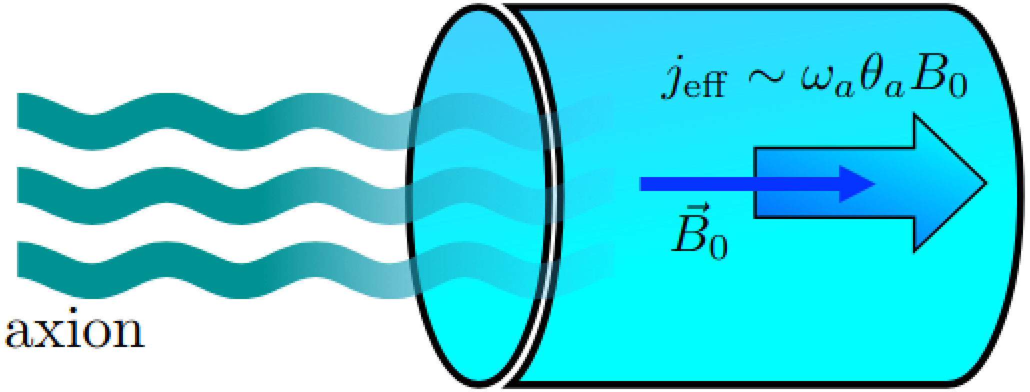
- Axion haloscopes as GW detectors
- What coordinate frame to use when calculating sensitivity?
  - Coordinate independent framework
  - Additional approximations
- Application of our results

Axion haloscopes as detectors

- Conversion of GW in background EM field
- Harness efforts of axion community

Raffelt, Stodolsky '88
A. Berlin et al. '21





$$\sqrt{-g} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \to hFF$$

$$\frac{a}{f} F\tilde{F}$$

Calculation of detector response
What frame to use?

TT gauge

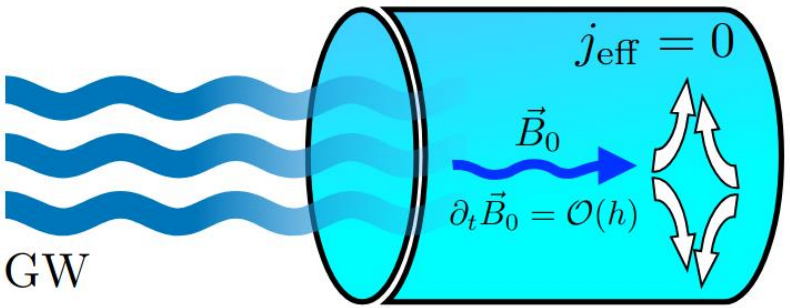
- G. Lupanov '67 …
  N. Herman et al. '20

Proper detector frame

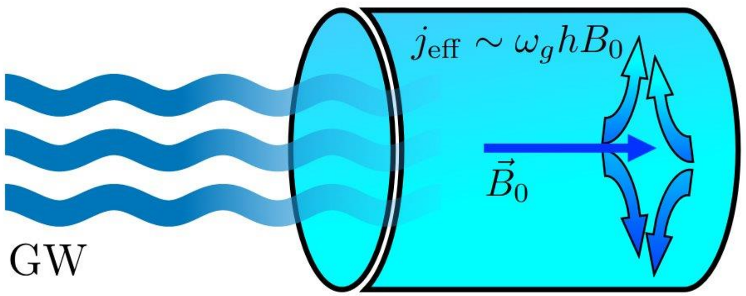
- L. Baroni, et al. '84 …
  A. Berlin et al. '21

Concrete Example: GW comes in parallel to B-field

- There is no signal



- There is a signal



from A. Berlin et al. '21

Calculation of detector response
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More generally: e.g. suppression for large wavelength

- No suppression

- Signal field suppressed $\propto \omega_{\mathrm{GW}} L$

How to frame the question

Description of any GW detector

1. Full theory in GR
   ->coordinate invariant
2. Perturbed theory
   -> inherits gauge invariance
3. Introduce further approximations
   e.g. choice of gauge + dropping terms

Tension

(Old) Literature:

"You have to use TT gauge / proper detector frame!"

Makes Sense!
Applicability / Errors?

Electro Magnetism in GR

- Field strength $F_{\mu\nu}$ and 4-current j^μ satisfy:

Metric is in here

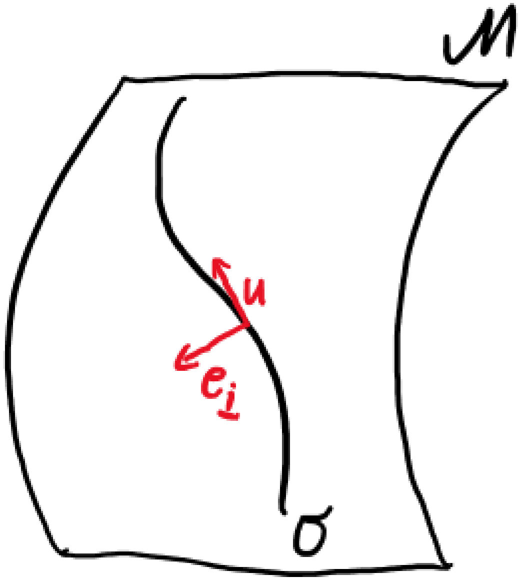
$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \qquad \nabla_\nu F^{\mu\nu} = j^\mu$$

- In going from SR to GR $F_{\mu\nu}$ became a tensor transforming as

$$F'_{\alpha\beta} = F_{\mu\nu} \frac{dx^\mu}{dx'_\alpha} \frac{dx^\nu}{dx'_\beta}$$

-> Can't interpret e.g. F_{i0} as electric field

Electro Magnetism in GR



Observers infinitesimal coord. system, tetrad:

$$g_{\mu\nu} e^{\mu}_{\underline{\mu}} e^{\nu}_{\underline{\nu}} = \eta_{\underline{\mu\nu}} \qquad e^{\mu}_{\underline{0}} = u^{\mu}$$

Obeys:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} e^{\mu}_{\underline{\alpha}} + \Gamma^{\mu}_{\nu\lambda} u^{\nu} e^{\lambda}_{\underline{\alpha}} = \left(a_{\nu} u^{\mu} - a^{\mu} u_{\nu}\right) e^{\nu}_{\underline{\alpha}} + u^{\lambda} \omega^{\rho} \Omega_{\lambda\rho\nu}{}^{\mu} e^{\nu}_{\underline{\alpha}}$$

How is the sensor attached? acceleration,forces rotation,torque

- The observed field:

$$E_{\underline{i}} = F_{\mu\nu}\, e^{\mu}_{\underline{i}} u^{\nu} \qquad B^{\underline{i}} = \frac{1}{2} \epsilon^{\underline{imn}} F_{\mu\nu}\, e^{\mu}_{\underline{m}} e^{\nu}_{\underline{n}}$$

Electro Magnetism in GR

- Boundary conditions on conductor

Consider observer attached to surface of conductor measuring the electric field parallel to surface:

4-velocity of conductor

$$F_{\mu\nu} e^{\mu}_{\underline{1}} u^{\nu} = F_{\mu\nu} e^{\mu}_{\underline{2}} u^{\nu} = 0 \quad \Longleftrightarrow \quad \mathbf{E}_{\parallel} = 0$$

directions parallel to surface of conductor

See D. Rawson-Harris '71

Perturbations around Minkowski

Choose a scheme:

Transition to perturbed quantities:

$$g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$$
$$g^{\mu\nu} \to \eta^{\mu\nu} - h^{\mu\nu}$$
$$F_{\mu\nu} \to \overline{F}_{\mu\nu} + \delta F_{\mu\nu}$$

…

Gauge transformation:

$$x^\mu \to x^\mu + \xi^\mu$$

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$
$$\delta F_{\mu\nu} \to \delta F_{\mu\nu} - \xi^\alpha \partial_\alpha \overline{F}_{\mu\nu} - \overline{F}_{\alpha\nu} \partial_\mu \xi^\alpha - \overline{F}_{\mu\alpha} \partial_\nu \xi^\alpha$$

…

Implies e.g.:

$$F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu} \to \overline{F}^{\mu\nu} + \delta F^{\mu\nu} - h^{\mu\alpha} \overline{F}_\alpha{}^\nu - \overline{F}^\mu{}_\beta h^{\beta\nu}$$

~~$F^{\mu\nu} \to \overline{F}^{\mu\nu} + \delta F^{\mu\nu}$~~ It's a trap!

- **Maxwell's Equations:**

$$0 = \partial_\lambda \delta F_{\mu\nu} + \partial_\mu \delta F_{\nu\lambda} + \partial_\nu \delta F_{\lambda\mu}$$

$$\partial_\nu \delta F^{\mu\nu} = \delta j^\mu + j^\mu_{\text{eff}}$$

$$j^\mu_{\text{eff}} = -\frac{1}{2} \partial_\alpha h \, \overline{F}^{\mu\alpha} + \partial_\nu \left(h^\mu{}_\alpha \overline{F}^{\alpha\nu} + h^\nu{}_\alpha \overline{F}^{\mu\alpha} \right)$$

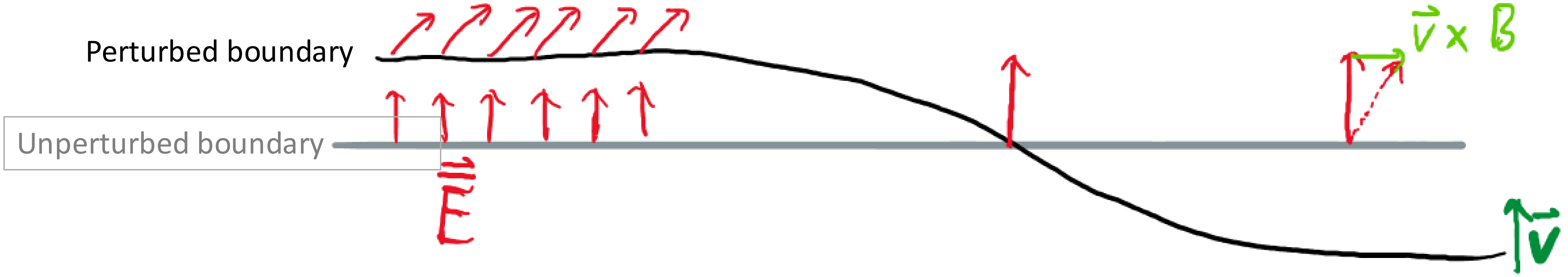
Perturbations around Minkowski

- Perturbed boundary condition: (observed fields work similarly)

Most Literature only considers this

Effects related to perceived motion of conductor surface
-> Can drop in frame in which conductor is at rest!!!

$$0 = \delta F_{\mu\nu} \bar{e}^{\mu}_{\underline{1}/\underline{2}} \bar{u}^{\nu} + \delta x^{\lambda} \partial_{\lambda} \bar{F}_{\mu\nu} \bar{e}^{\mu}_{\underline{1}/\underline{2}} \bar{u}^{\nu} + \bar{F}_{\mu\nu} \delta e^{\mu}_{\underline{1}/\underline{2}} \bar{u}^{\nu} + \bar{F}_{\mu\nu} \bar{e}^{\mu}_{\underline{1}/\underline{2}} \delta u^{\nu}$$



Transverse-Traceless (TT) Gauge

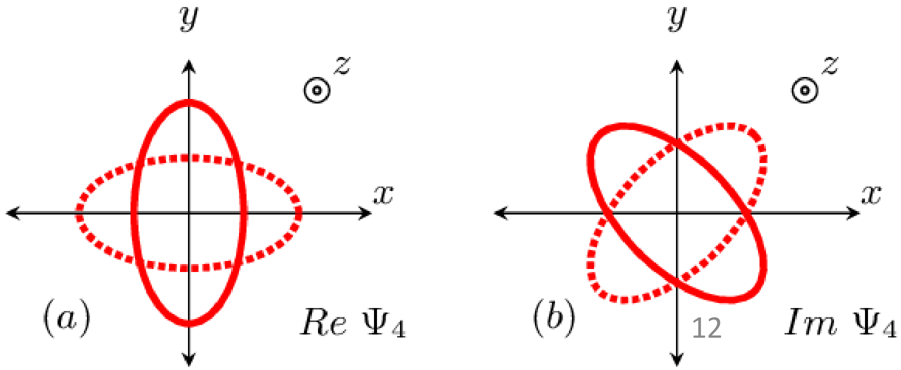
$$h_{0\mu}^{\mathrm{TT}} = 0 \quad h_{ij}^{\mathrm{TT}} = \left(A_{ij}^{+} + A_{ij}^{\times}\right) \mathrm{e}^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$$

e.g. $\mathbf{k} = k\hat{\mathbf{e}}_z,\ A_{ij}^{+} = A^{+}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

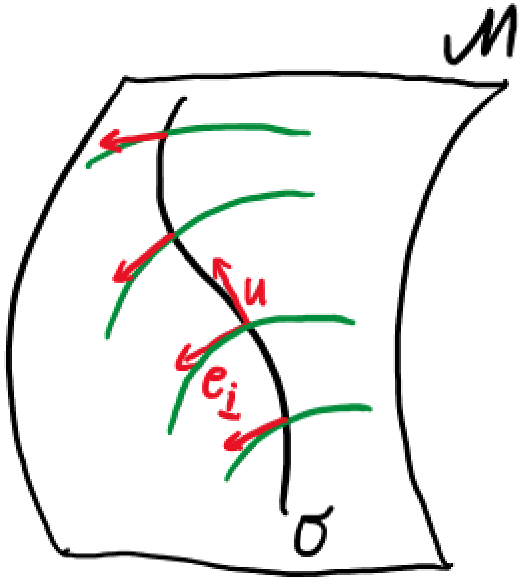
- The synchronous property $h_{0\mu}^{\mathrm{TT}} = 0$ leads to free falling particle staying at rest $\delta x^{\mu} = 0$

  -> Good choice for weakly coupled particles

Change of distance rather than motion:



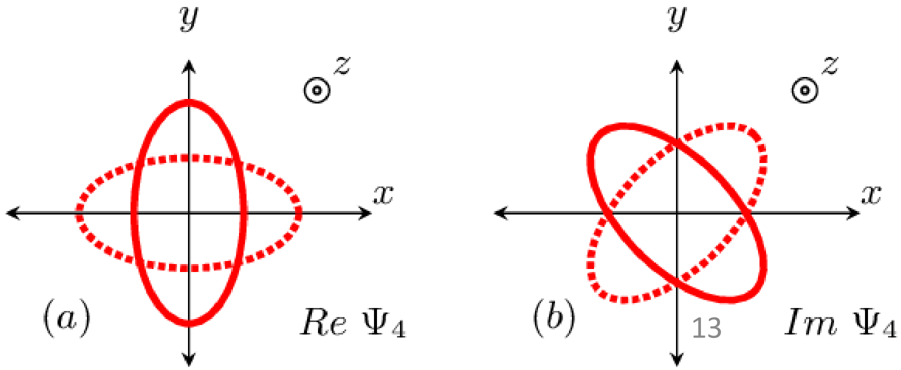
Proper Detector Frame



- Constructed by extending spatial components of tetrad into geodesics
  - Corrections to metric suppressed $h_{\mu\nu} = \mathcal{O}(A\omega_{\mathrm{GW}}^2 L^2)$
- By construction particle at fixed distance stays at rest
  $\delta x^\mu = 0$

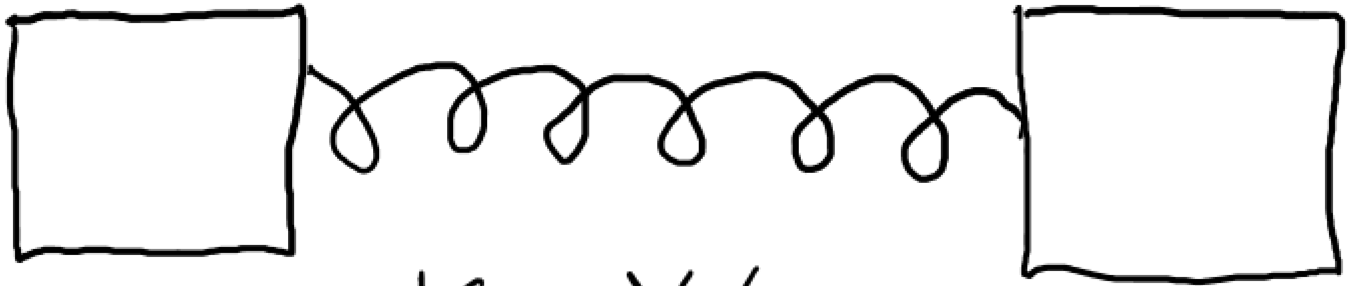
-> Good choice for particles connected by a rigid ruler

Actual motion of free falling particle:



When are particles strongly coupled?

- Toy model for a stick:



$K \sim Y/L$ $\quad$ $m \sim \rho \cdot L/2$

$$m \ddot{\delta x} = K\,\delta x$$

$$\delta x \sim h \cdot L$$

$$\ddot{\delta x} \sim \omega^2 h \cdot L$$

$$v_s^2 \sim \frac{Y}{\rho} \sim \omega^2 \cdot L^2$$

Comparison of speed of sound to size of detector and wavelength!

See also Bringmann et al. '23

Mechanical Limits

- Transverse traceless gauge
  - -freely falling masses at rest
  - -free falling limit:

$$\delta x^{TT} = h^{TT} L \left(0 \pm \mathcal{O}\left(\frac{v_s}{\omega_{GW} L}\right)\right) \text{ for } \omega_{GW} L \gg v_s$$

- Proper detector frame
  - -bodies with fixed distance at rest
  - -rigid limit:

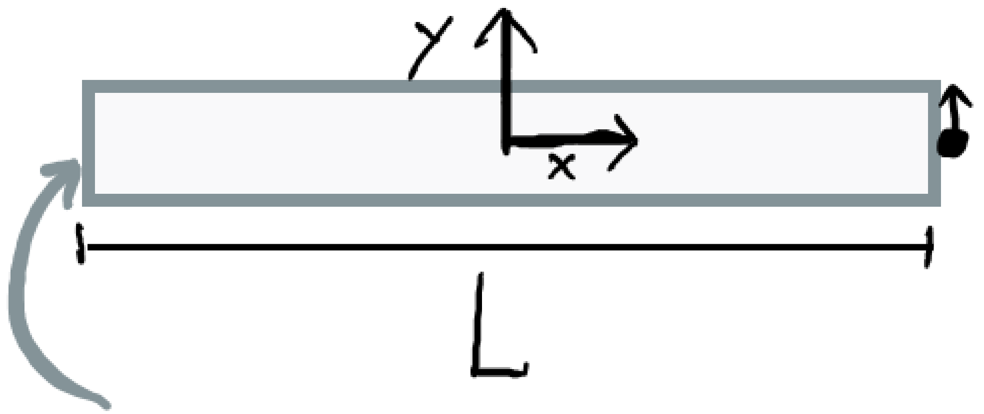
$$\delta x^{PDF} = h^{TT} L \left(0 \pm \mathcal{O}\left(\frac{\omega_{GW}^2 L^2}{v_s^2}\right)\right) \text{ for } \omega_{GW} L \ll v_s$$

Sound velocity in solid
≈ 10^{-5} *c*

  - -corrections to metric suppressed by $\omega_{GW}^2 L^2$
  - -long-wavelength limit (no mechanical approximation):

$$h^{PDF} = h^{TT} \left(\omega_{GW}^2 L^2 \pm \mathcal{O}(\omega_{GW}^3 L^3)\right) \text{ for } \omega_{GW} L \ll 1$$

Toy Example modified from A. Berlin et. al. '21



Observer measuring E-field in y-direction

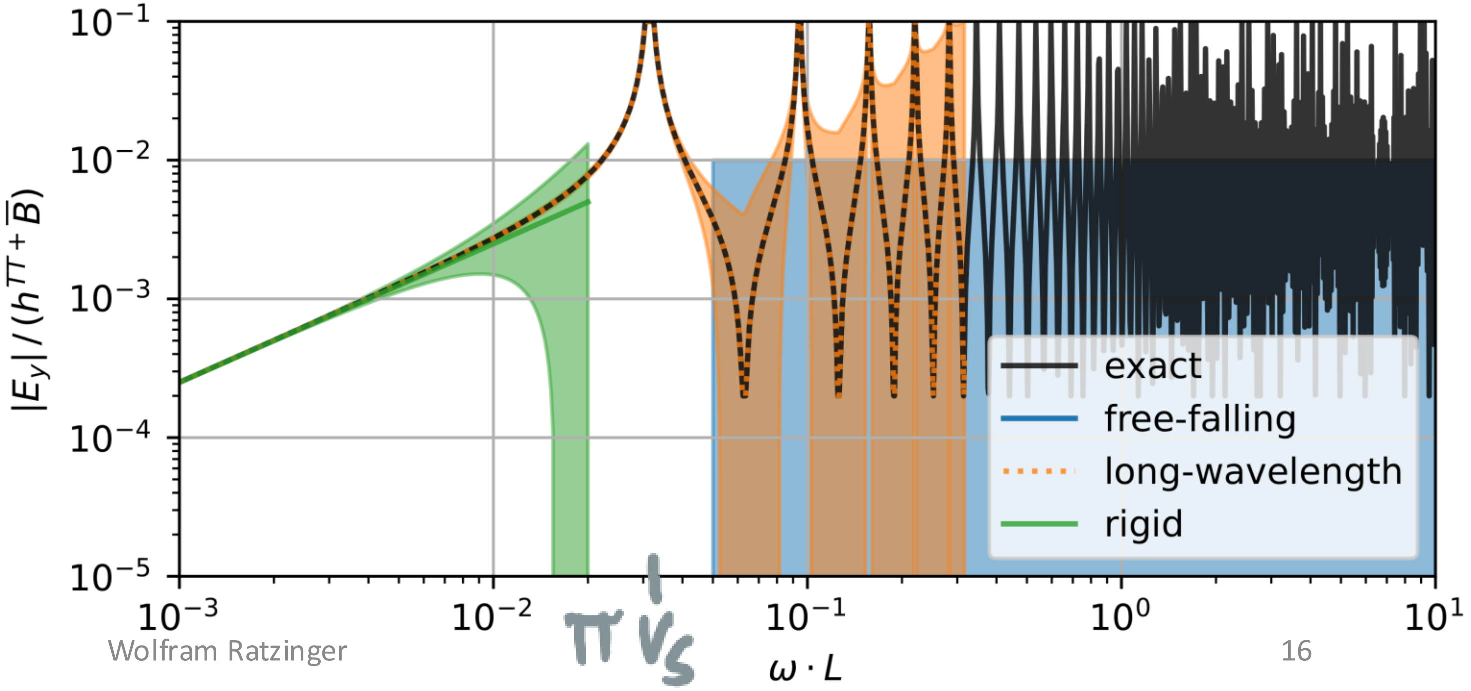
Thin Rod with $v_s = 10^{-2}$

- Homogeneous B-field

$$\vec{\overline{B}} = \overline{B}\hat{e}_z$$

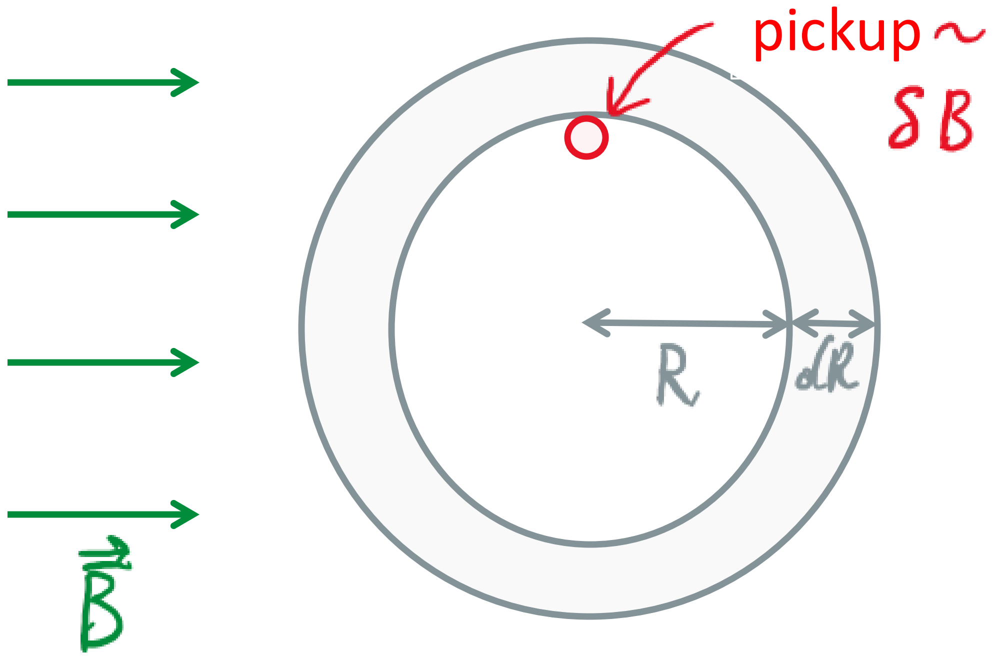
- GW, plus polarized in x-y

$$\vec{k}_{GW} = \omega \hat{e}_z$$



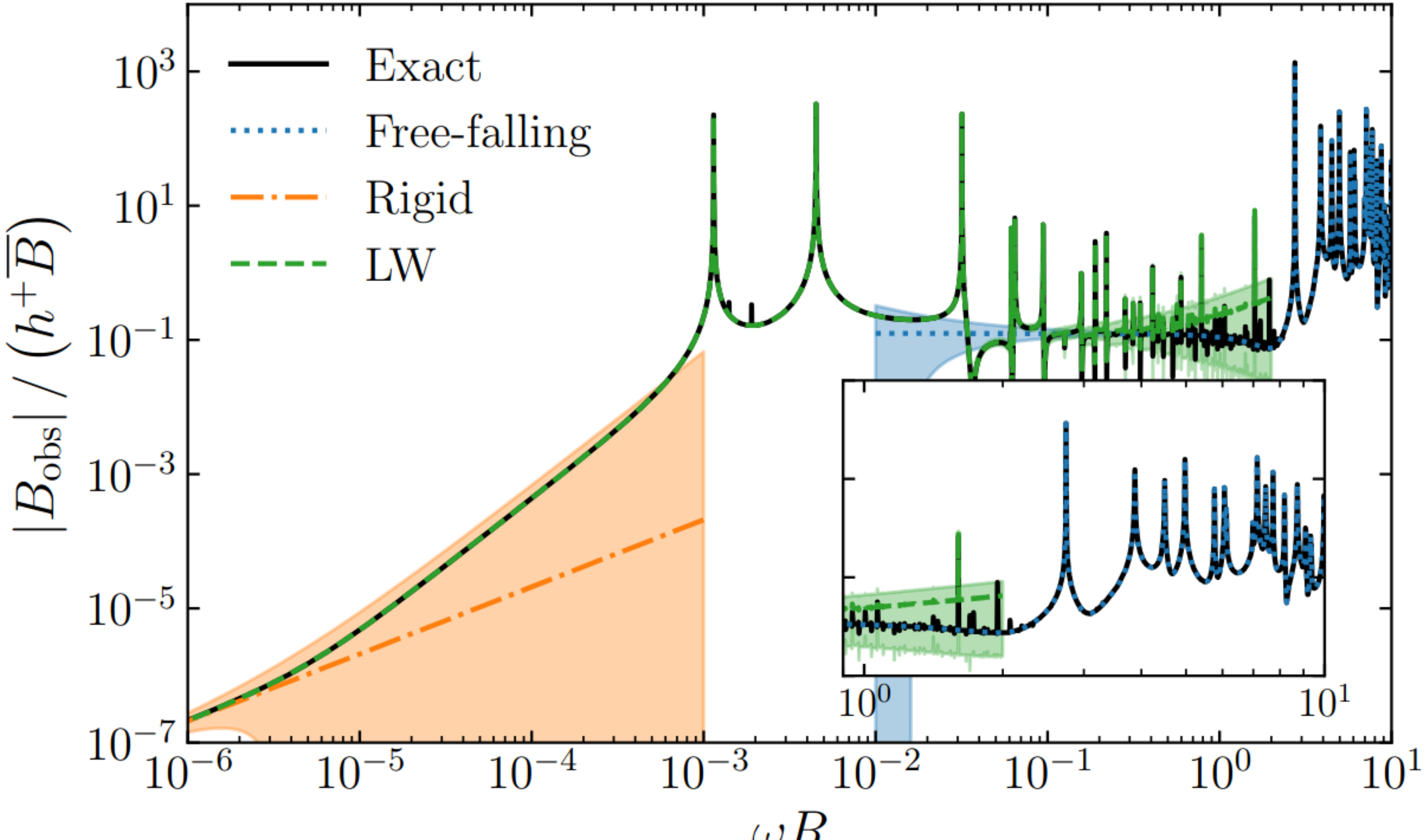
Spherical cavity in B field

- Hollow sphere with radius R
  and thickness dR=0.1 R
  -speed of sound $v_s=10^{-3}$
- In homogeneous magnetic field
- Small pickup-loop (rigid) + freely rotating
  -> Measures oscillating B field orthogonal to loop



Result

Mechanical Resonances

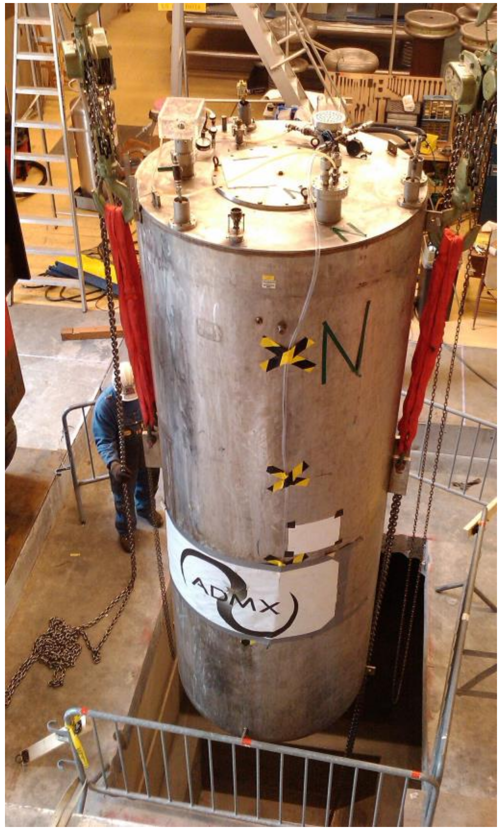


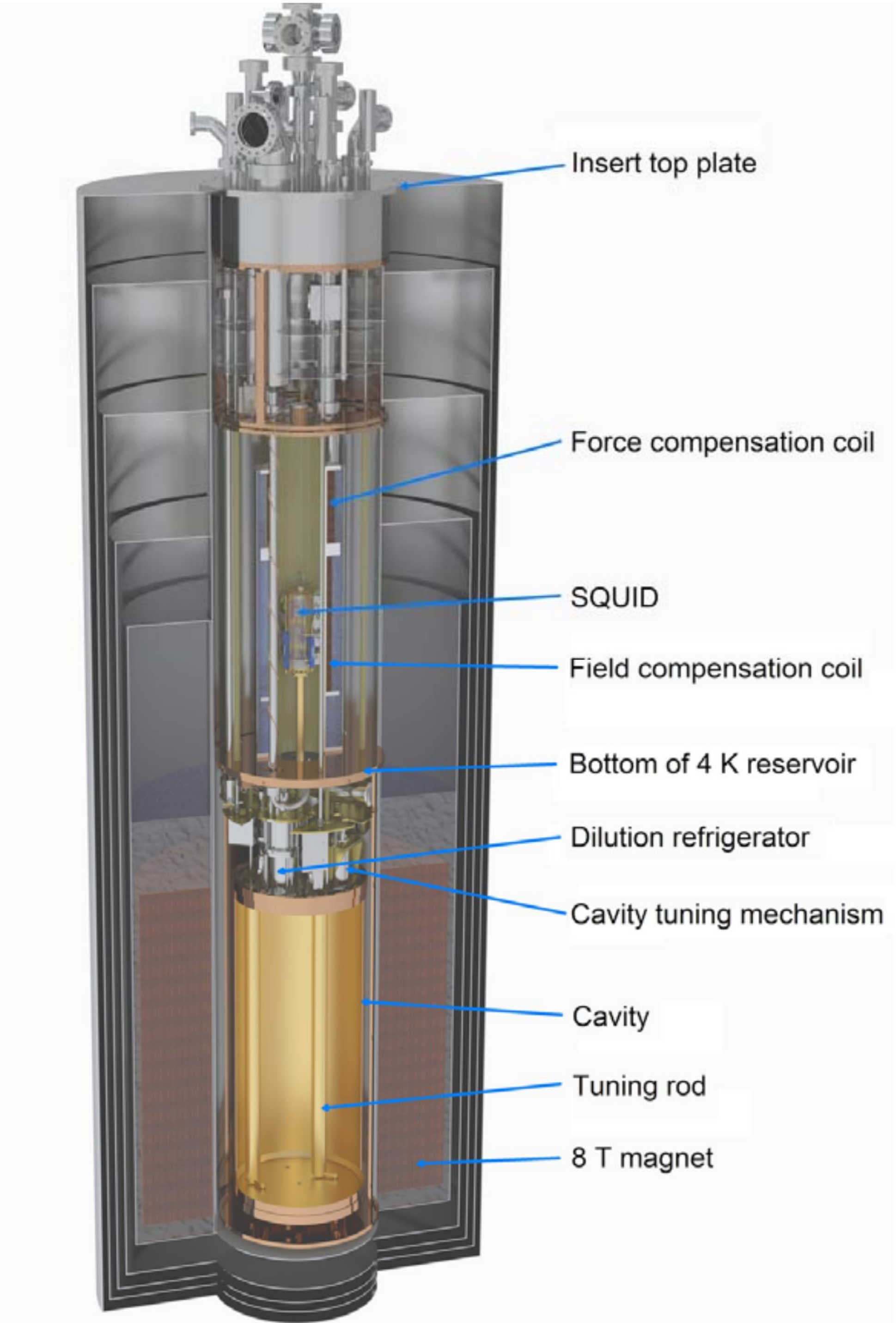
EM Resonances

How to use this result?

1st Example:

Want to compute response of ADMX

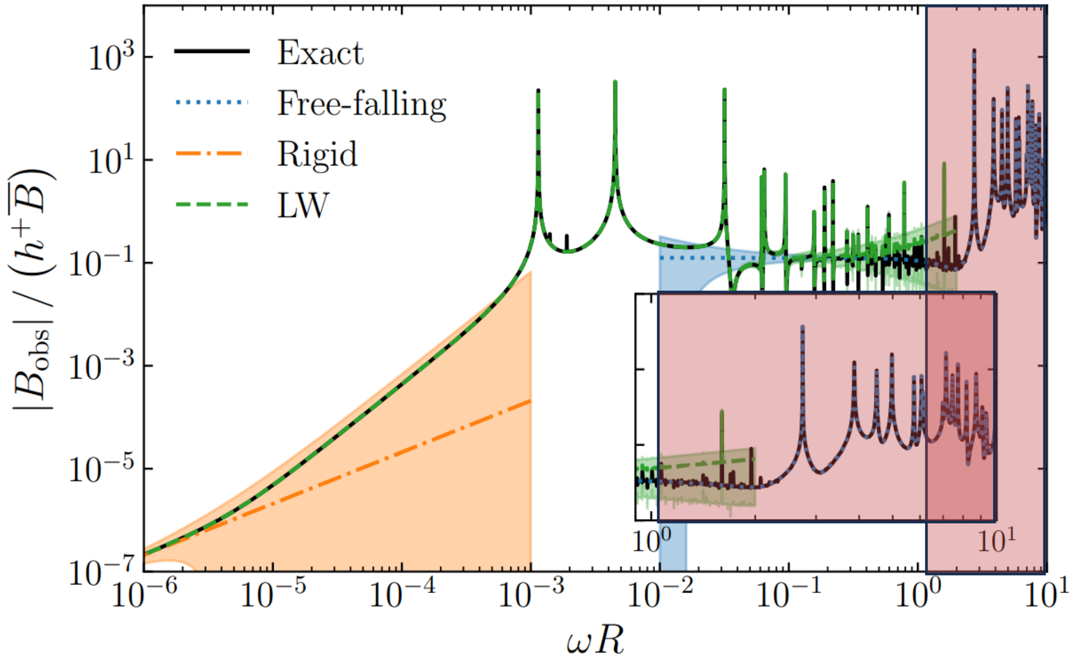




How to use this result?

1st Example: Want to compute response of ADMX

-ADMX relies on EM resonances



Lies in regime $\omega_{GW} L \gg v_s$
-> Free-falling approximation good
-> Don't need to model mechanics
Just use effective current in TT

Going further:
-> Only $\delta F^{\mu\nu}$ encodes EM resonances
-> Neglect tetrad

How to use this result?

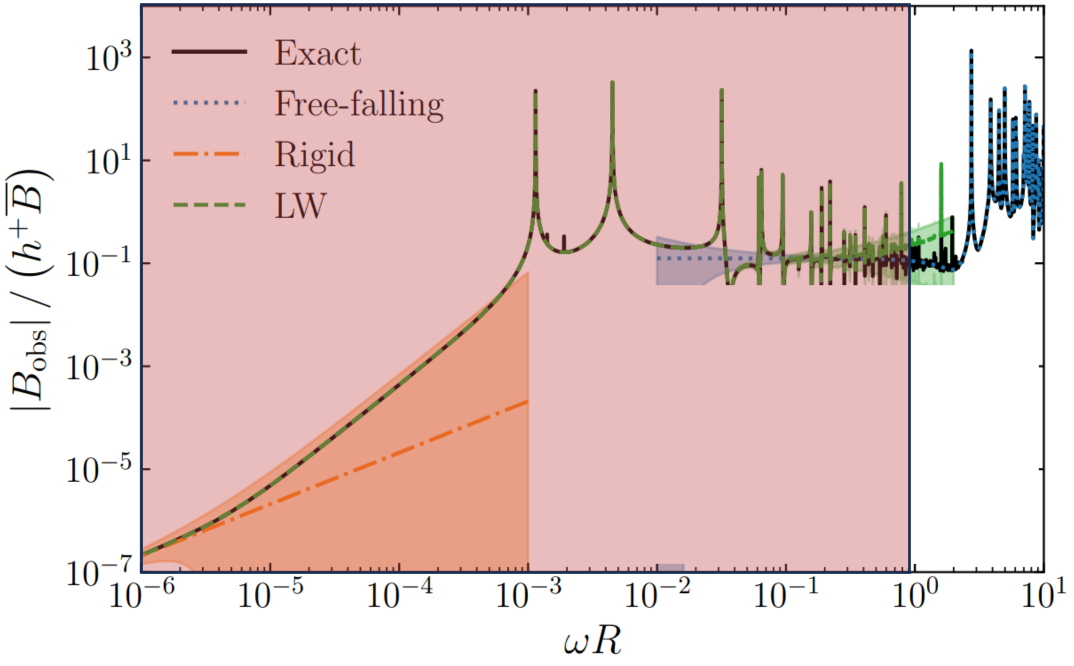
2nd Example:

Want to compute response of MAGO

How to use this result?

2nd Example: Want to compute response of MAGO

-MAGO relies on EM resonances but $\Delta\omega_{\rm EM} \sim \omega_{\rm mech} \ll \omega_{\rm EM}$



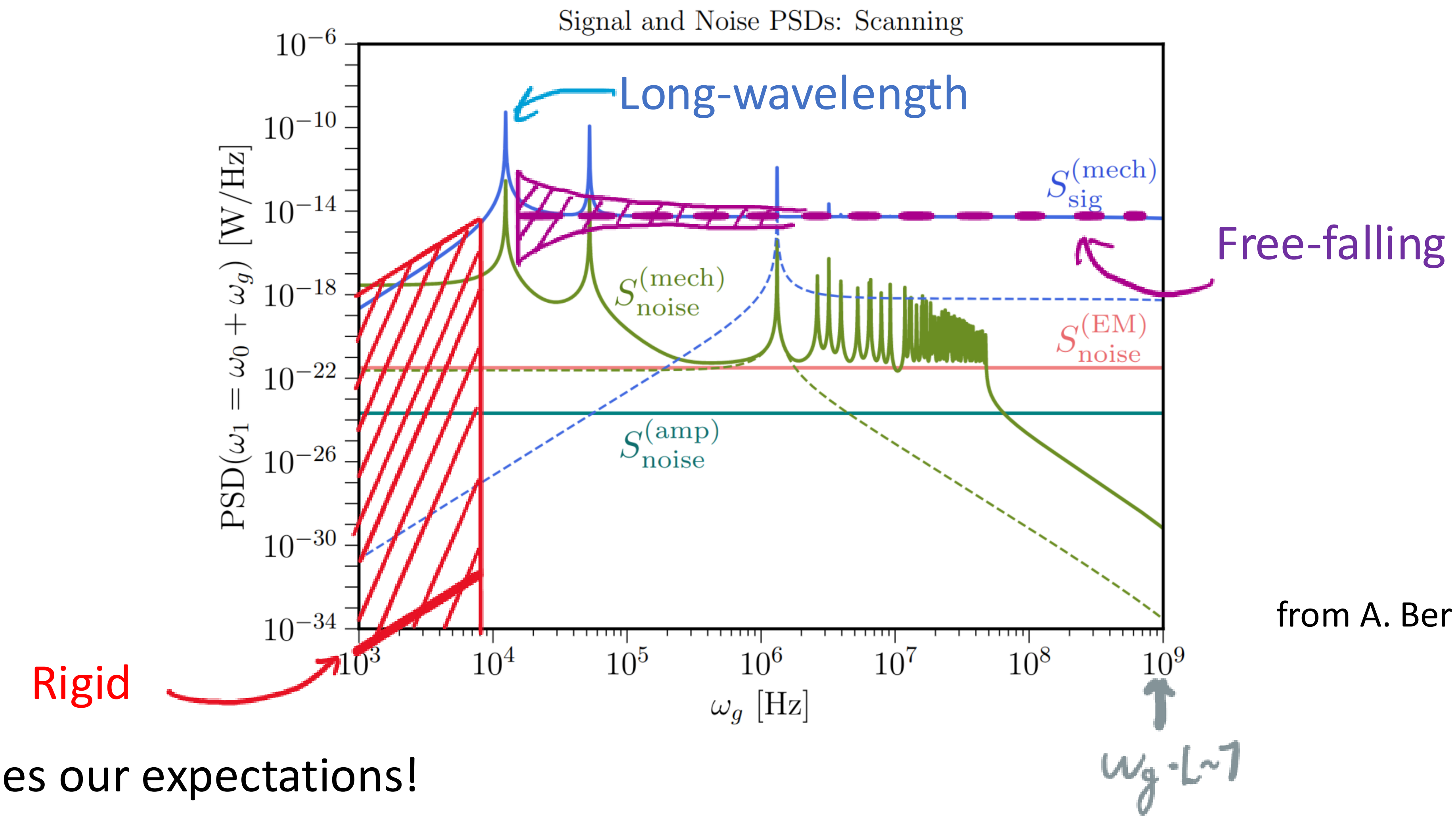
Lies in regime $\omega_{GW} L \ll 1$

-> Free-falling approximation for $\omega_{GW} L \gg v_s$
-> Effect of mechanical resonances neglected

-> Rigid limit for $\omega_{GW} L \gg v_s$
-> Mechanical effect might still be large

-> LW approximation (good but tedious)

Comparison with prediction for MAGO



from A. Berlin et al. '23

Result matches our expectations!

Discussion: Sensitivity of ABRACADABRA

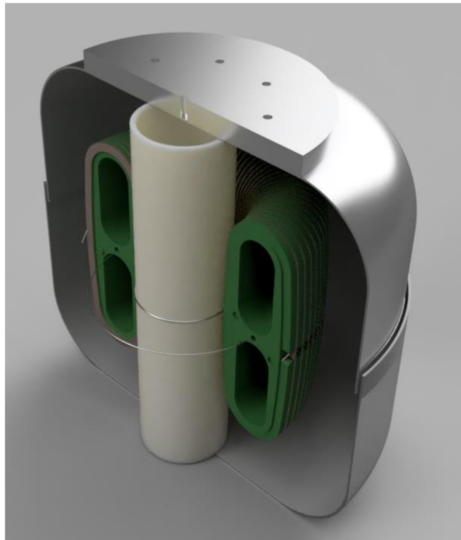
- Lies in regime $\omega_{GW} L \ll 1$

- Pappas et al. '25 uses rigid approximation even though $\omega_{GW} L \gg v_s$

- On the other hand Domcke '24 et al. finds mechanics dominated signal

  (as expected)

-> Probably more work to be done ;)



Conclusion

- Detector development requires theoretical and experimental effort
- Bulk equations + boundary conditions + observables must be coordinate invariant
- Choice of gauge + neglecting motion, is approximation
  -> Make sure that one is in the right limit + introduce errors

Thanks