Modular Invariance in Flavour Physics

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MITP Topical Workshop

Flavour for New Physics at Present and Future Colliders

Mainz, Germany, 18 June 2025

Finanziato dall'Unione europea
NextGenerationEU

Ministero dell'Università e della Ricerca

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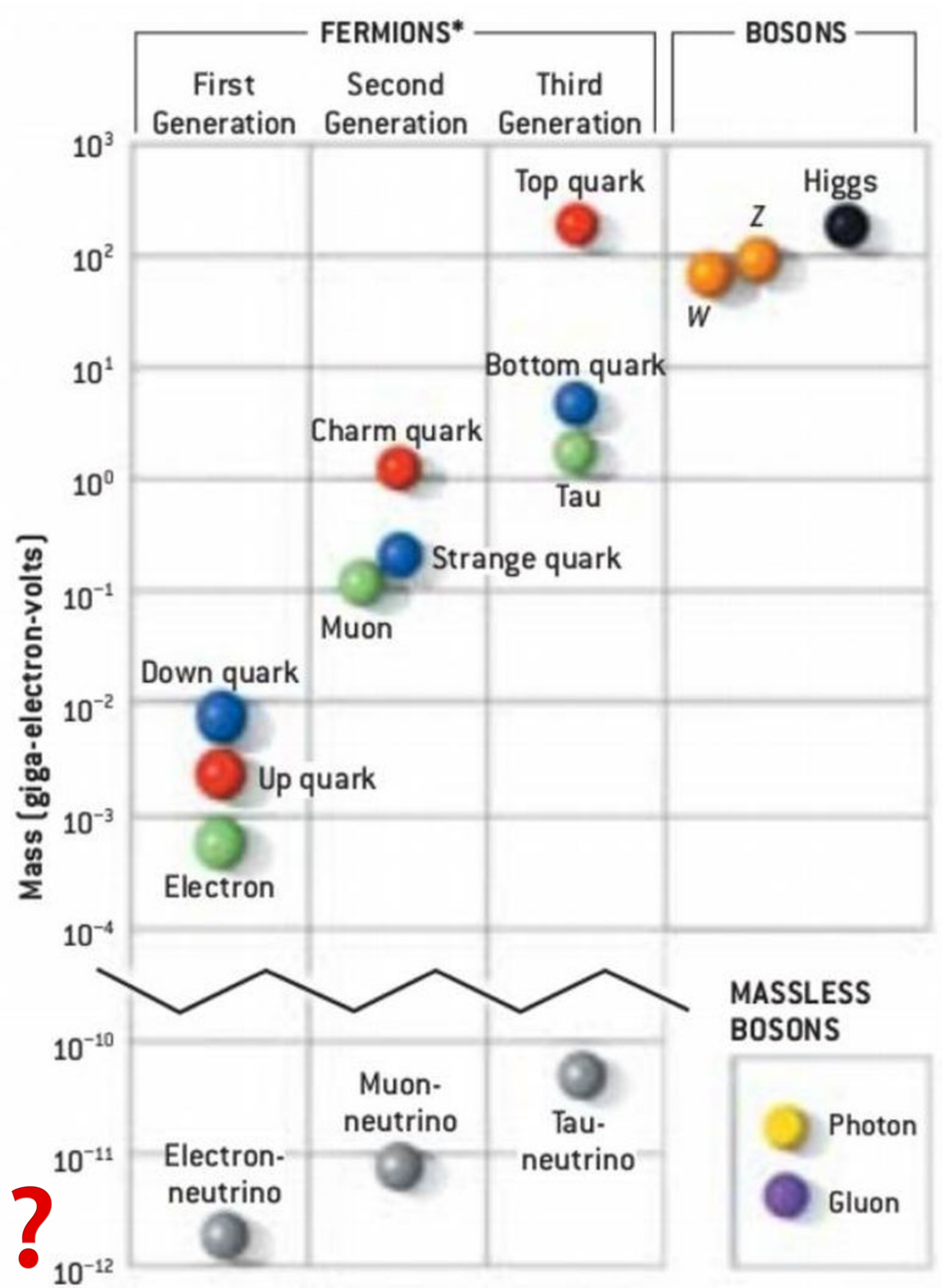
PIANO NAZIONALE DI RIPRESA E RESILIENZA

Outline

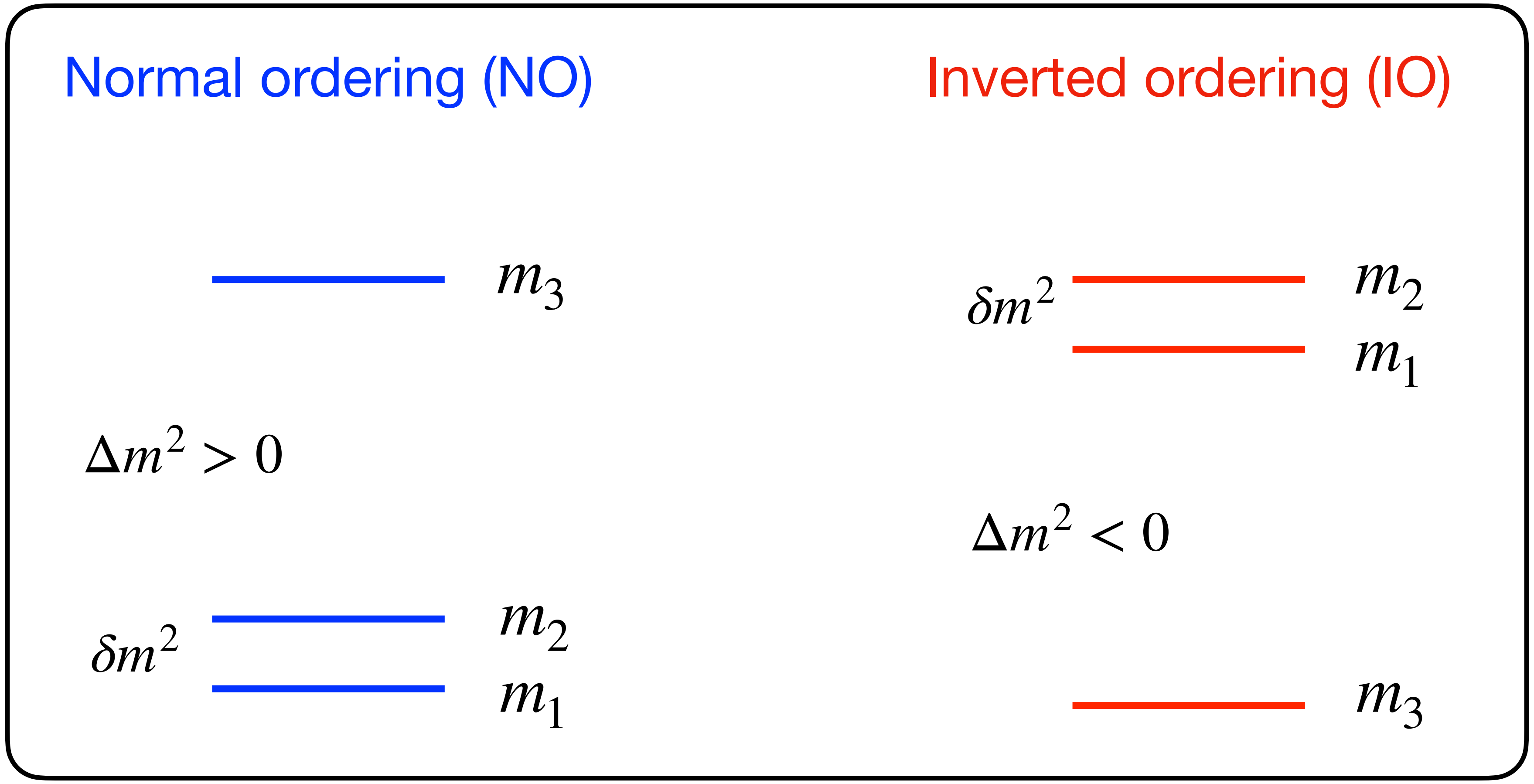
- Flavour problem
- Non-Abelian discrete symmetries
- Modular invariance
  - Modular group
  - Modular forms
  - Modular-invariant supersymmetric theories
- Modular-invariant flavour models
  - Leptons
  - Quarks
- Modular invariance and CP symmetry
  - Strong CP problem

Flavour problem: masses



- Why are there 3 families (generations)?
- Is there any organising principle behind the values of fermion masses?
- What is the value of the lightest neutrino mass?
- Why is the mass of neutrino $\sim 10^7$ times smaller than that of electron?
- What is the mechanism of neutrino mass generation?

Neutrino mass ordering



$$\delta m^2 = m_2^2 - m_1^2 \approx 7.4 \times 10^{-5}\ \text{eV}^2$$

$$\Delta m^2 = m_3^2 - \frac{m_1^2 + m_2^2}{2} \approx \begin{cases} +2.5 \times 10^{-3}\ \text{eV}^2 \ \text{for NO} \\ -2.5 \times 10^{-3}\ \text{eV}^2 \ \text{for IO} \end{cases}$$

$$\frac{\delta m^2}{|\Delta m^2|} \approx 0.03$$

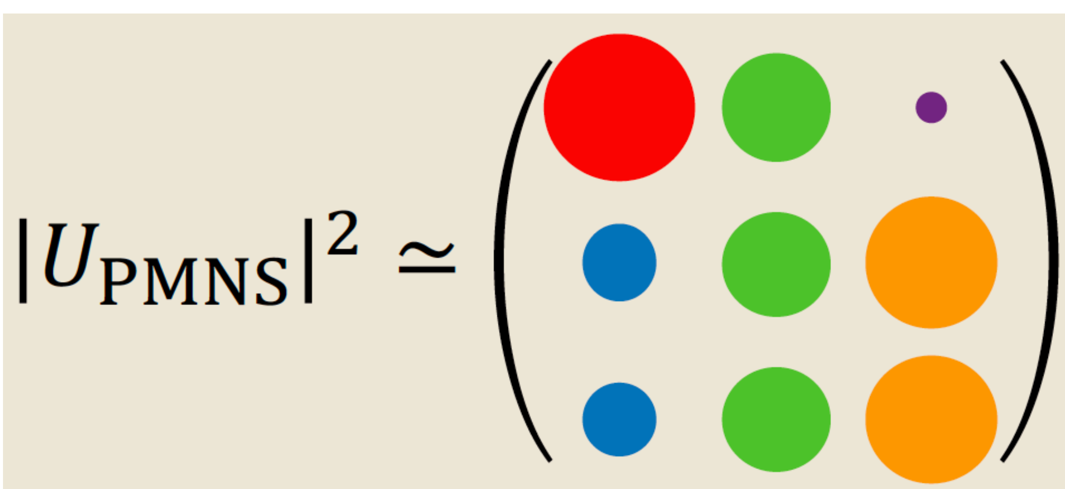
Flavour problem: mixing

Interaction/flavour basis $\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$ Mass basis

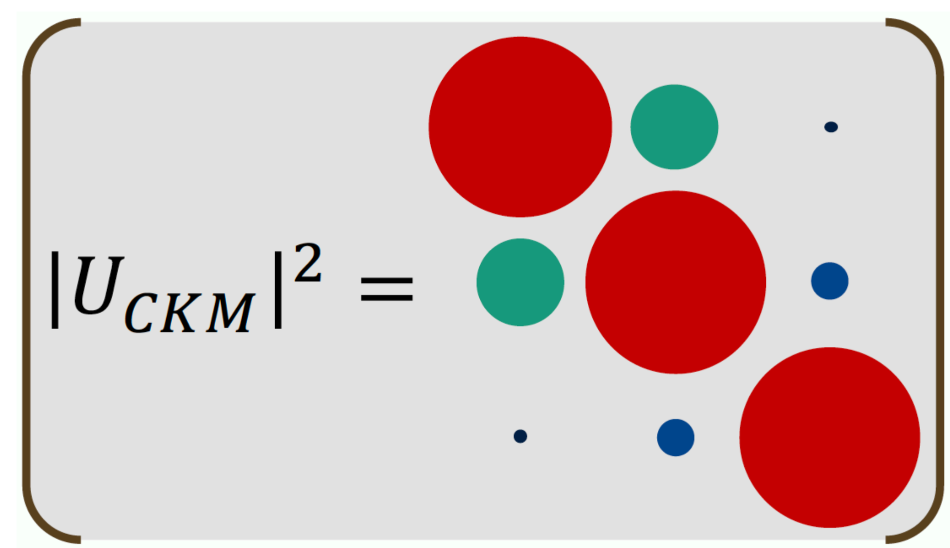
$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}$$

Majorana phases

Leptons



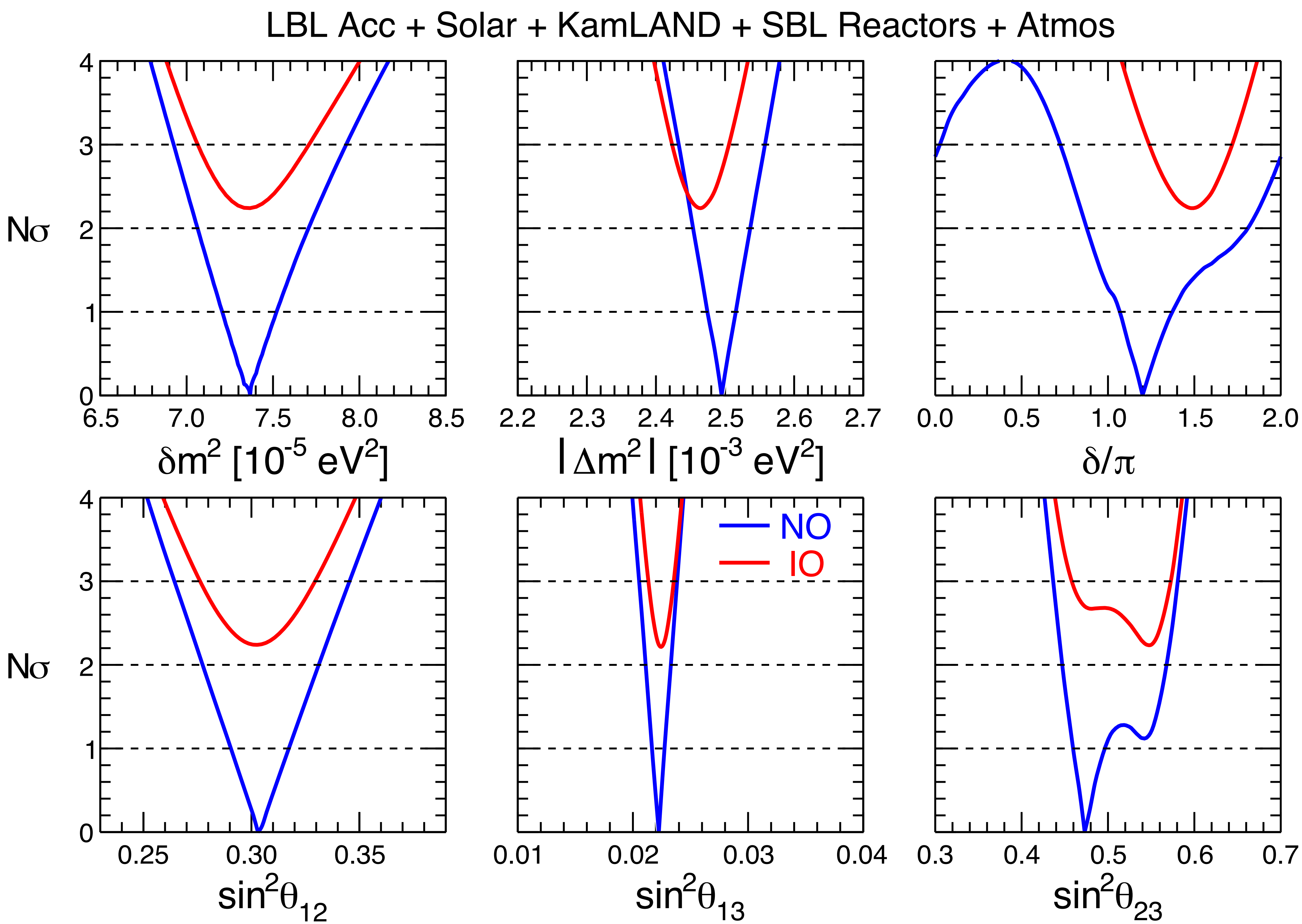
Quarks



$\theta_{12}^{\ell} = 33^\circ$ $\theta_{23}^{\ell} \approx 45^\circ$ $\theta_{13}^{\ell} = 8.6^\circ$ vs $\theta_{12}^{q} = 13^\circ$ $\theta_{23}^{q} = 2.4^\circ$ $\theta_{13}^{q} = 0.21^\circ$

Why are the PMNS and CKM mixing matrices so different?

Neutrino oscillation data



Capozzi et al., 2503.07752; see also Esteban et al., 2410.05380 and de Salas et al., 2006.11237

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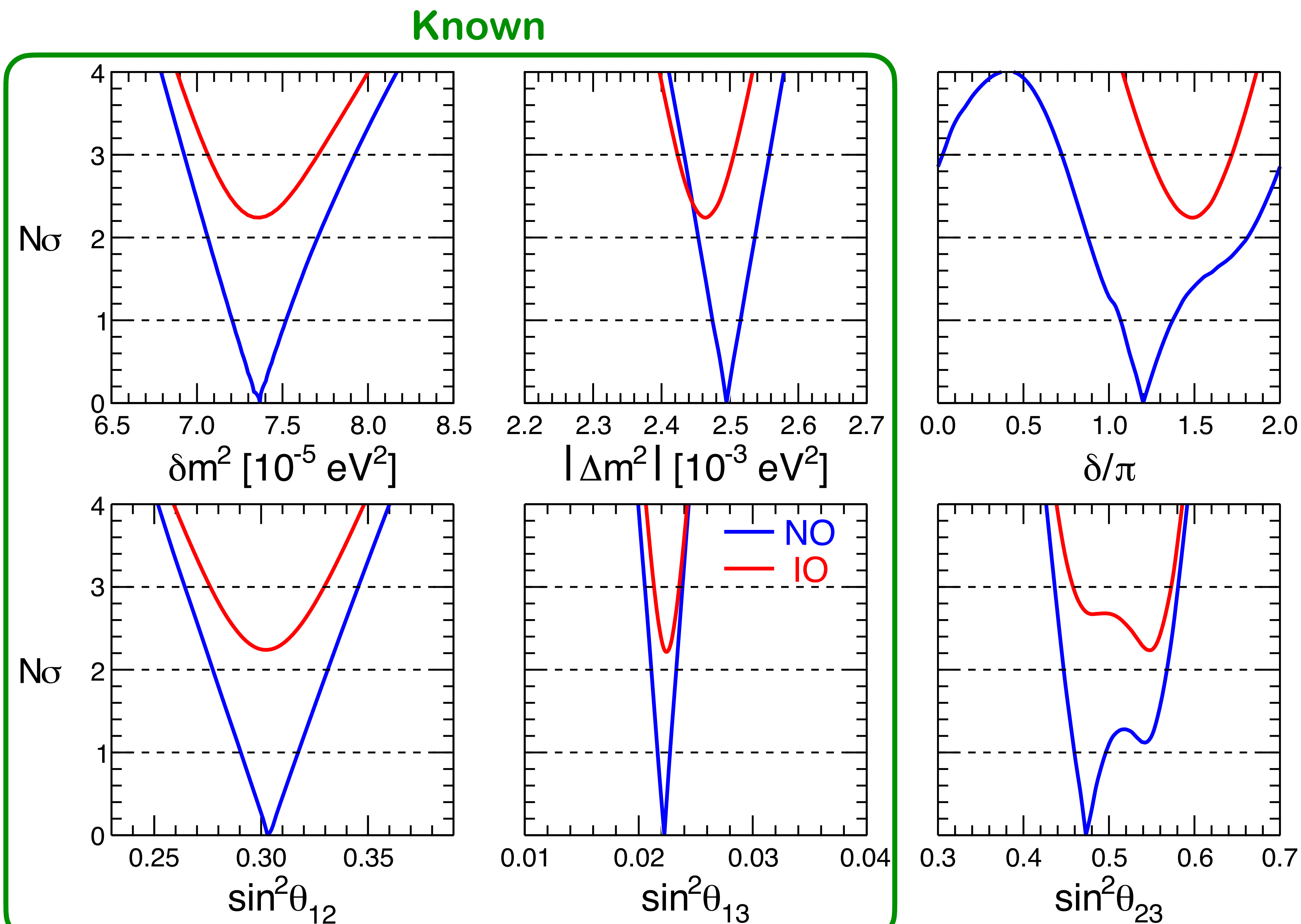

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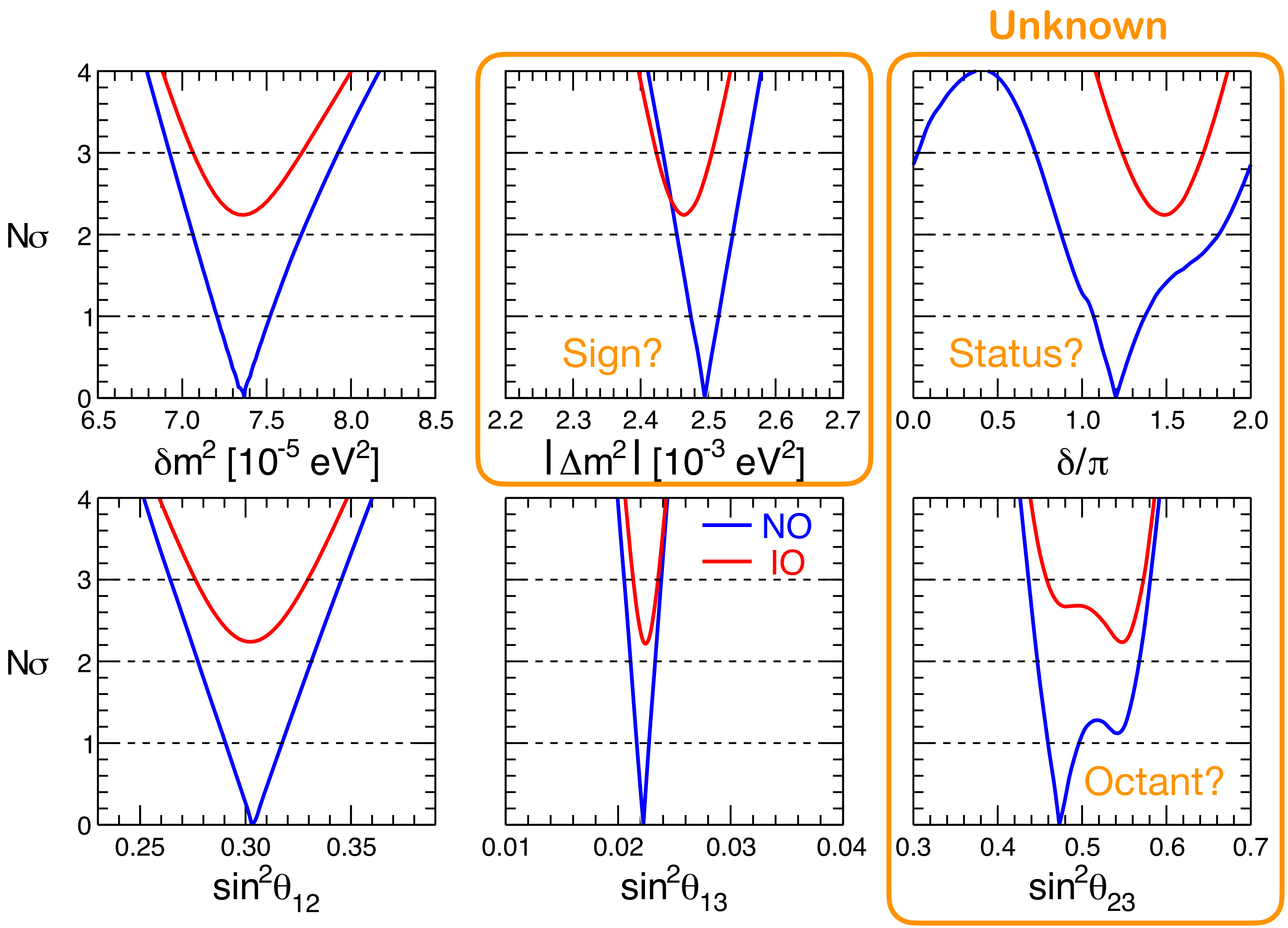

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Flavour symmetry

At high energies, the theory is invariant under

$$\varphi(x) \to \rho(g)\,\varphi(x)\,, \quad g \in G_f \qquad \text{e.g.} \qquad \varphi = \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix}$$

representation of G_f — flavour symmetry group

$$-\mathscr{L} = \overline{\ell_L} M_e \ell_R + \overline{\nu_L^c} M_\nu \nu_L + \mathrm{h.c.}$$

At low energies, flavour symmetry has to be broken

$$G_f$$

residual symmetries: $G_e \subset G_f$ and $G_\nu \subset G_f$

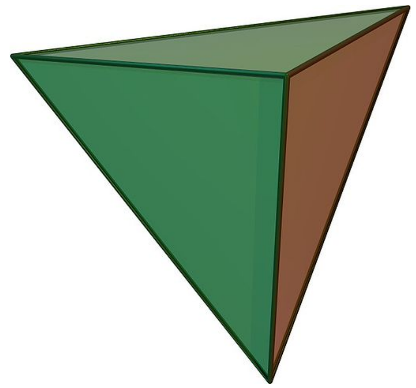
$G_e \subset G_f$	$G_\nu \subset G_f$
$\rho(g_e)^\dagger M_e M_e^\dagger \rho(g_e) = M_e M_e^\dagger$	$\rho(g_\nu)^T M_\nu \rho(g_\nu) = M_\nu$
$U_e^\dagger M_e M_e^\dagger U_e = \mathrm{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right)$	$U_\nu^T M_\nu U_\nu = \mathrm{diag}\left(m_1, m_2, m_3\right)$
$U_e^\dagger \rho(g_e) U_e = \rho(g_e)^{\mathrm{diag}}$	$U_\nu^\dagger \rho(g_\nu) U_\nu = \rho(g_\nu)^{\mathrm{diag}}$

$$U = U_e^\dagger U_\nu$$

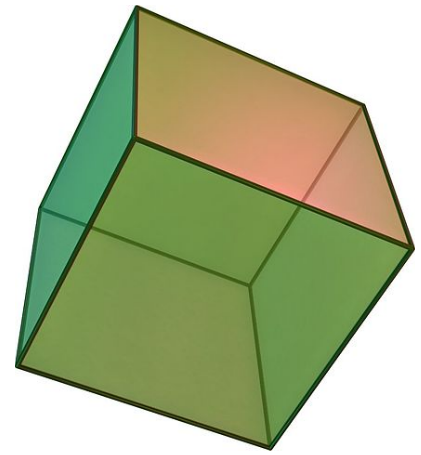
Non-Abelian discrete symmetries

Images: WIKIPEDIA

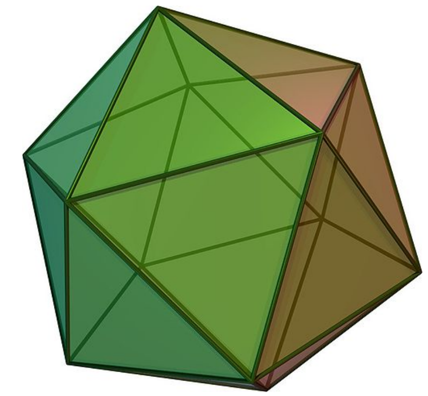
S_3 (6)



A_4 (12)



S_4 (24)



A_5 (60)

Generated by two elements S and T

$$\langle S,\, T \mid S^2 = (ST)^3 = T^N = I \rangle \ , \quad N = 2\,,\,3\,,\,4\,,\,5$$

Another convenient presentation for S_4

$$\langle S,\, T,\, U \mid S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = I \rangle$$

A_4 , S_4 , and A_5 admit a 3-dimensional irrep (unification of families)

Reviews: Altarelli, Feruglio, 1002.0211; Ishimori et al., 1003.3552; King, Luhn, 1301.1340; Petcov, 1711.10806; Feruglio, Romanino, 1912.06028

Discrete flavour symmetry

Pros

✓ Can successfully describe the observed lepton mixing pattern

✓ Unification of three families at high energies: irrep **3**

Cons

☑ Symmetry breaking typically relies on numerous flavons

☑ Elaborated potentials to get desirable vacuum alignment

☑ Higher-dimensional operators can spoil LO predictions

☑ Mainly mixing, and not masses

What is the origin of discrete flavour symmetry?

Discrete flavour symmetry

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- ✓ Can successfully describe the observed lepton mixing pattern
- ✓ Unification of three families at high energies: irrep **3**

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- ☑ Symmetry breaking typically relies on numerous flavons
- ☑ Elaborated potentials to get desirable vacuum alignment
- ☑ Higher-dimensional operators can spoil LO predictions
- ☑ Mainly mixing, and not masses

What is the origin of discrete flavour symmetry?

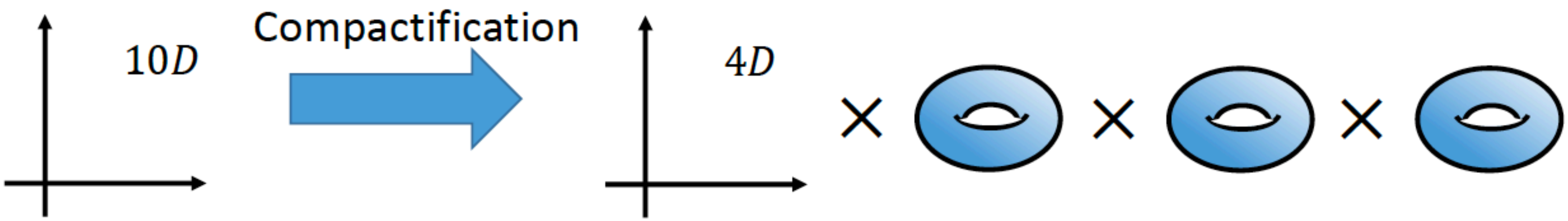
Perhaps modular invariance

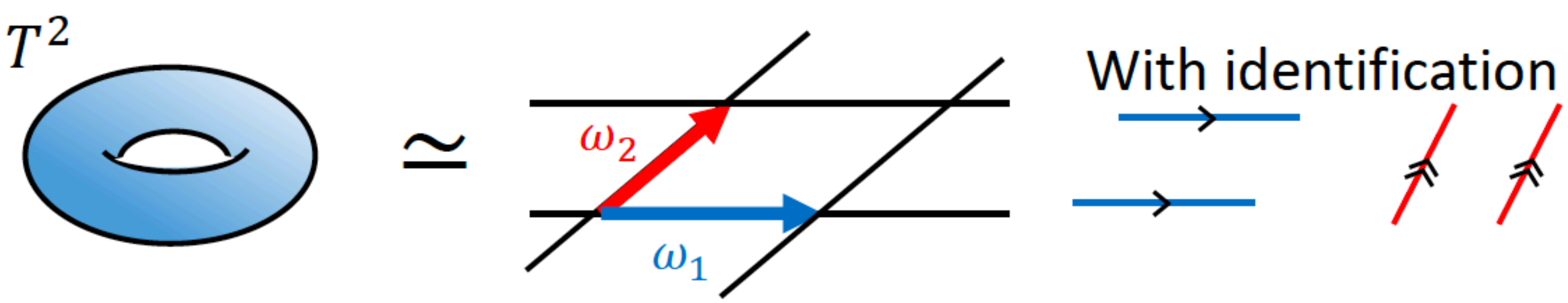
Proposal by Feruglio, in book "From My Vast Repertoire ...: Guido Altarelli's Legacy", 1706.08749

Modular invariance

String theory requires extra dimensions

Images: Takuya H. Tatsuishi





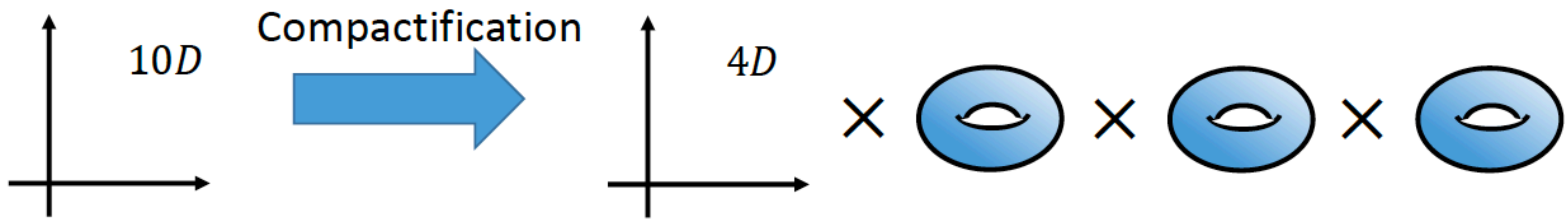
$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}\,(\tau) > 0$$

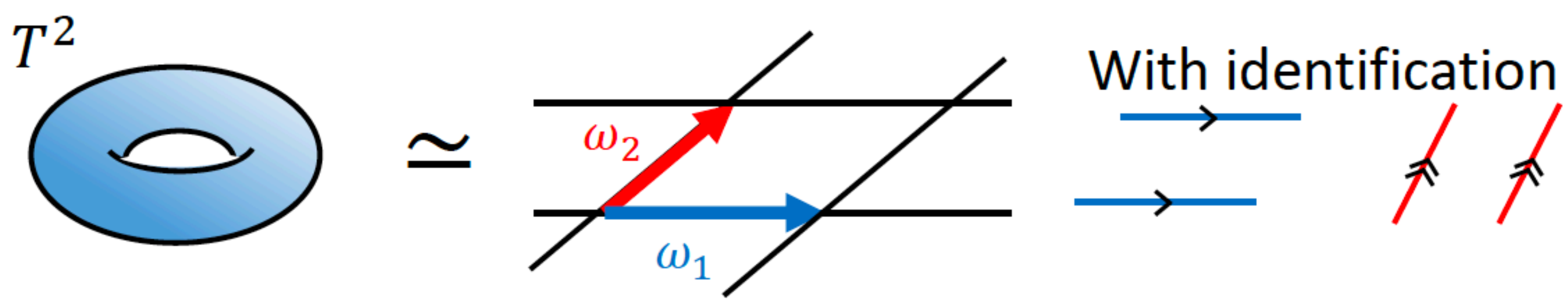
modulus

Modular invariance

String theory requires extra dimensions

Images: Takuya H. Tatsuishi





$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}\,(\tau) > 0$$

modulus

Lattice is invariant under basis transformations

$$\begin{cases} \omega_2 \to \omega_2' = a\omega_2 + b\omega_1 \\ \omega_1 \to \omega_1' = c\omega_2 + d\omega_1 \end{cases} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1$$

$$\boxed{\tau \to \tau' = \frac{a\tau + b}{c\tau + d}}$$

modular transformations

τ and τ' describe the same torus

Modular group

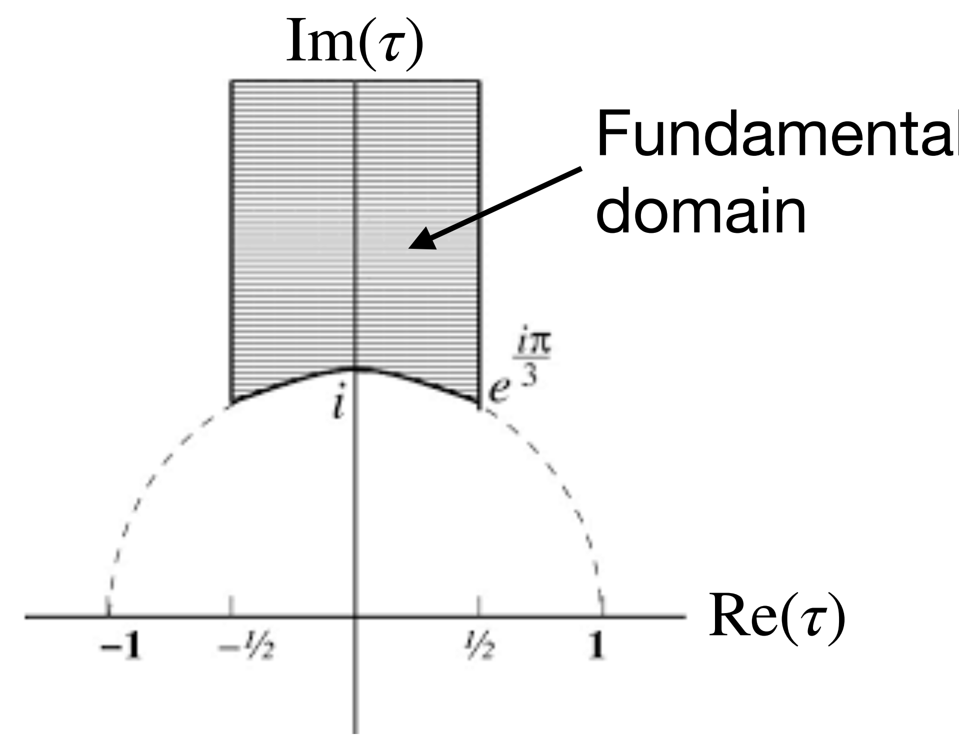
Homogeneous modular group

$$\Gamma = \langle S,\ T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

duality — discrete shift symmetry



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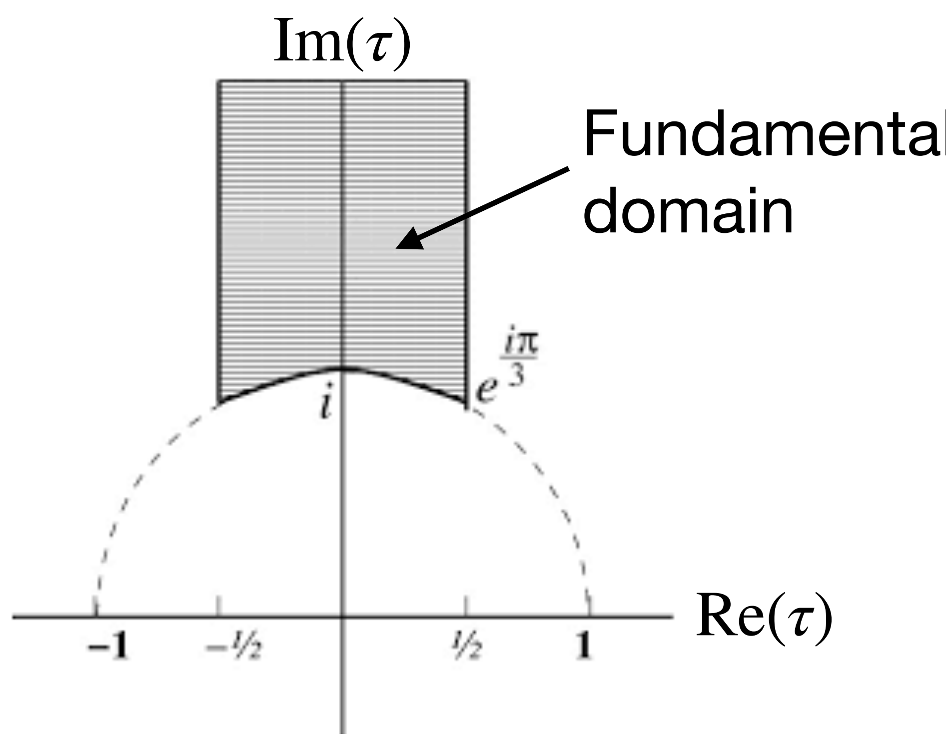
duality   discrete shift symmetry



Inhomogeneous modular group

$$\overline{\Gamma} = \langle S,\ T \mid S^2 = (ST)^3 = I \rangle \cong \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{I, -I\}$$

In other words, $\mathrm{SL}(2,\mathbb{Z})$ matrices γ and $-\gamma$ are identified

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad \tau \xrightarrow{-\gamma} (-\gamma)\,\tau = \frac{-a\tau - b}{-c\tau - d} = \gamma\tau$$

Finite modular groups

Principle congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of level $N = 2\,,\ 3\,,\ 4\,,\ldots$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

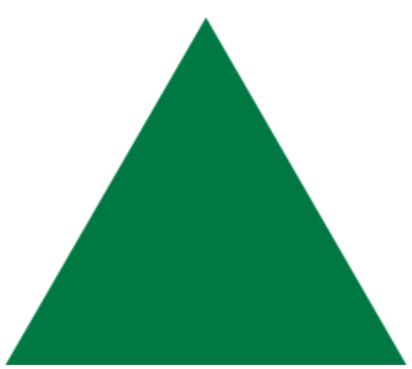
Finite modular groups

$$\overline{\Gamma} = \mathrm{PSL}(2,\mathbb{Z})$$

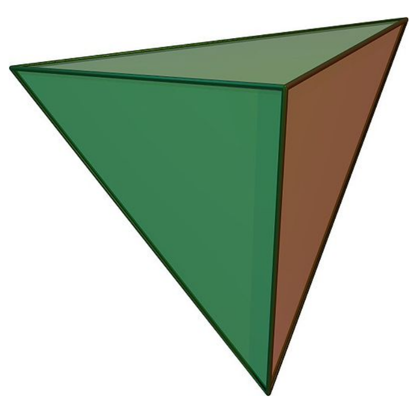
$$\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$$

$$\Gamma_N = \left\langle S,\ T \mid S^2 = (ST)^3 = T^N = I \right\rangle\,, \quad N = 2\,,\ 3\,,\ 4\,,\ 5$$

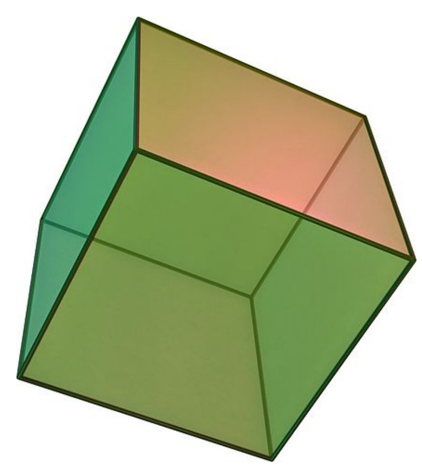
$\Gamma_2 \cong S_3$



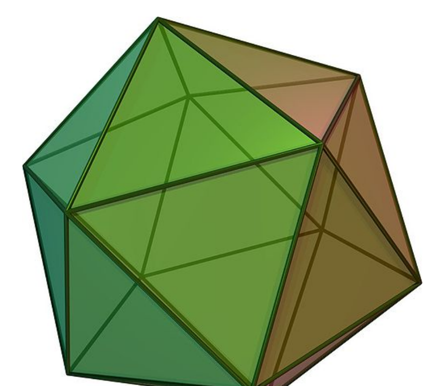
$\Gamma_3 \cong A_4$



$\Gamma_4 \cong S_4$



$\Gamma_5 \cong A_5$



Finite modular groups

Principle congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of level $N = 2\,,\ 3\,,\ 4\,,\ldots$

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Finite modular groups

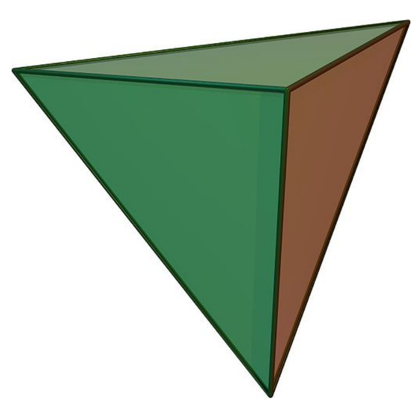
$$\Gamma = \mathrm{SL}(2,\mathbb{Z})$$

$$\Gamma'_N \equiv \Gamma/\Gamma(N)$$

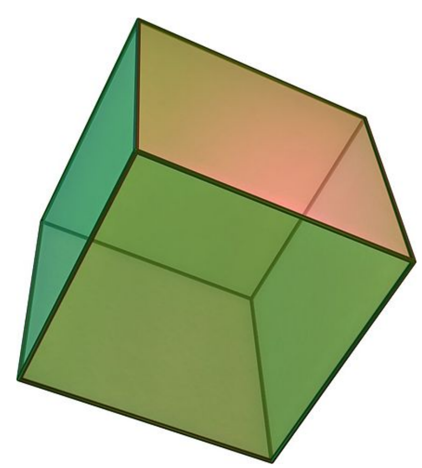
$$\Gamma'_N = \left\langle S,\ T \mid S^4 = (ST)^3 = T^N = I\,,\quad S^2T = TS^2 \right\rangle\,,\quad N = 3\,,\ 4\,,\ 5$$

Double covers

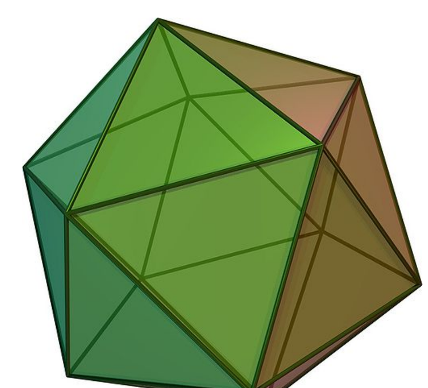
$\Gamma'_3 \cong A'_4 = T'$



$\Gamma'_4 \cong S'_4$



$\Gamma'_5 \cong A'_5$



Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under $\Gamma(N)$ as follows

$$y\left(\gamma\tau\right) = \left(c\tau + d\right)^k y(\tau)\,, \quad \gamma \in \Gamma(N)$$

k is weight
non-negative integer

N is level
natural number

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Modular forms of weight k and level N form a linear space $\mathscr{M}_k\left(\Gamma(N)\right)$ of finite dimension. We can choose a basis in this space s.t. $Y(\tau) \equiv \left(y_1(\tau)\,,\, y_2(\tau)\,,\ldots\right)^T$ transforms as

$$Y\left(\gamma\tau\right) = \left(c\tau + d\right)^k \rho(\gamma)\, Y(\tau)\,, \quad \gamma \in \Gamma$$

ρ is effectively a representation of finite $\Gamma'_N \equiv \Gamma/\Gamma(N)$

$$\rho\left(\Gamma(N)\right) = 1\,, \quad \rho\left(S\Gamma(N)\right) = \rho(S)\,, \quad \rho\left(T\Gamma(N)\right) = \rho(T)\,, \ldots$$

Feruglio, 1706.08749

Modular forms

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2, \mathbb{Z}_4)$	$A'_5 \equiv SL(2, \mathbb{Z}_5)$
$\dim \mathcal{M}_k(\Gamma(N))$	$k/2+1$	$k+1$	$2k+1$	$5k+1$

A finite set of functions for each k and N

Modular forms

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A finite set of functions for each k and N

Lowest-weight modular forms for each group

Γ_N	$Y^{(k)}_{\mathbf{r}}$	# forms
$\Gamma_2 \cong S_3$	$Y^{(2)}_{\mathbf{2}}$	2
$\Gamma_3 \cong A_4$	$Y^{(2)}_{\mathbf{3}}$	3
$\Gamma_4 \cong S_4$	$Y^{(2)}_{\mathbf{2}}\,,\ Y^{(2)}_{\mathbf{3'}}$	5
$\Gamma_5 \cong A_5$	$Y^{(2)}_{\mathbf{3}}\,,\ Y^{(2)}_{\mathbf{3'}}\,,\ Y^{(2)}_{\mathbf{5}}$	11

See, e.g., Novichkov, Penedo, Petcov, **AT**, 1905.11970 and Ding, King, 2311.09282

Modular forms

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$
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A finite set of functions for each k and N

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$\Gamma_2 \cong S_3$	$Y^{(2)}_{\mathbf{2}}$	2
$\Gamma_3 \cong A_4$	$Y^{(2)}_{\mathbf{3}}$	3
$\Gamma_4 \cong S_4$	$Y^{(2)}_{\mathbf{2}}, Y^{(2)}_{\mathbf{3}'}$	5
$\Gamma_5 \cong A_5$	$Y^{(2)}_{\mathbf{3}}, Y^{(2)}_{\mathbf{3}'}, Y^{(2)}_{\mathbf{5}}$	11

Γ'_N	$Y^{(k)}_{\mathbf{r}}$	# forms
$\Gamma'_3 \cong T'$	$Y^{(1)}_{\hat{\mathbf{2}}}$	2
$\Gamma'_4 \cong S'_4$	$Y^{(1)}_{\hat{\mathbf{3}}}$	3
$\Gamma'_5 \cong A'_5$	$Y^{(1)}_{\hat{\mathbf{6}}}$	6

See, e.g., Novichkov, Penedo, Petcov, **AT**, 1905.11970 and Ding, King, 2311.09282

Modular-invariant SUSY theories

$\mathcal{N} = 1$ global SUSY action

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K\left(\tau, e^{2V}\Phi, \tau^\dagger, \Phi^\dagger\right) + \left[\int \mathrm{d}^2\theta\; W(\tau, \Phi) + \frac{1}{16}\int \mathrm{d}^2\theta\; f_a(\tau)\,\mathscr{W}_a\mathscr{W}_a + \text{h.c.}\right]$$

Kähler potential K
(kinetic terms,
gauge interactions)

Superpotential W
(Yukawa interactions)

Gauge kinetic function f

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\mathrm{QCD}}}{8\pi^2}$$

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Under modular transformations $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \Phi \to \left(c\tau + d\right)^{-k_\Phi} \rho_\Phi(\gamma)\, \Phi \\ V \to V \end{cases}$$

τ is promoted to a (dimensionless) superfield

matter supermultiplets

vector supermultiplets

Modular symmetry acts non-linearly

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

Modular-invariant SUSY theories

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$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\text{QCD}}}{8\pi^2}$$

Modular invariance of the action requires

$$\begin{cases} K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) \to K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) + f_K(\tau, \Phi) + \overline{f_K}\left(\tau^\dagger, \Phi^\dagger\right) \\ W(\tau, \Phi) \to W(\tau, \Phi) \end{cases}$$

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \ln\left(-i\tau + i\tau^\dagger\right) + \sum_\Phi \frac{\Phi^\dagger \Phi}{\left(-i\tau + i\tau^\dagger\right)^{k_\Phi}}$$

Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \ln\left(-i\tau + i\tau^\dagger\right) + \sum_{\Phi} \frac{\Phi^\dagger \Phi}{\left(-i\tau + i\tau^\dagger\right)^{k_\Phi}}$$

Superpotential

$$W = \sum g_{ijk} \left(Y_{ijk}(\tau)\, \Phi_i \Phi_j \Phi_k \right)_{\mathbf{1}}$$

τ-dependent Yukawa couplings

$$Y_{ijk}(\tau) \to (c\tau + d)^{k_{Y_{ijk}}} \rho_{Y_{ijk}}(\gamma)\, Y_{ijk}(\tau) \quad \text{with} \quad \begin{cases} k_{Y_{ijk}} = k_{\Phi_i} + k_{\Phi_j} + k_{\Phi_k} \\ \rho_{Y_{ijk}} \otimes \rho_{\Phi_i} \otimes \rho_{\Phi_j} \otimes \rho_{\Phi_k} \supset \mathbf{1} \end{cases}$$

$Y_{ijk}(\tau)$ are modular forms if $k_{Y_{ijk}} \geq 0$

(SUSY ⇒ holomorphicity)

Modular A4 symmetry

$$\Gamma_3 = \langle S,\, T \mid S^2 = (ST)^3 = T^3 = I \rangle$$

- 12 elements
- 4 irreps: $\mathbf{1}\,,\ \mathbf{1}'\,,\ \mathbf{1}''\,,\ \mathbf{3}$
- Space of the lowest non-trivial weight 2 modular forms has dimension 3
- 3 weight 2 modular forms arrange themselves in a triplet:

$$Y_{\mathbf{3}}^{(2)}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

- $Y_i(\tau)$ are given in terms of the Dedekind eta function

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right) \,, \quad q = e^{2\pi i \tau}$$

- Products of $Y_i(\tau)$ generate modular forms of higher weights, 4, 6, 8, …

Feruglio, 1706.08749

Modular forms of level 3 and weight 2

Feruglio, 1706.08749

$$Y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - 27\frac{\eta'(3\tau)}{\eta(3\tau)}\right] = 1 + 12q + 36q^2 + 12q^3 + \ldots$$

$$Y_2(\tau) = -\frac{i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -6q^{1/3}\left(1 + 7q + 8q^2 + \ldots\right)$$

$$Y_3(\tau) = -\frac{i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -18q^{2/3}\left(1 + 2q + 5q^2 + \ldots\right)$$

Here $\omega = e^{\frac{2\pi i}{3}}$ and $q = e^{2\pi i\tau}$ ($|q| = e^{-2\pi\,\mathrm{Im}\tau} < 1$ since $\mathrm{Im}\,\tau > 0$)

Since modular forms are periodic

$$f(T^N\tau) = f(\tau + N) = (c\tau + d)^k f(\tau) = f(\tau)\,, \qquad T^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \overline{\Gamma}(N)\,,$$

they admit q-expansions:

$$f(\tau) = \sum_{n=0}^{\infty} a_n q_N^n\,, \qquad q_N = e^{\frac{2\pi i\tau}{N}} \qquad (N = 3 \text{ in this example})$$

Feruglio's modular A4 model

Feruglio, 1706.08749

$\Gamma_3 \cong A_4$ (level $N = 3$)

$$\Phi \sim (\mathbf{r},\ k)$$

$$L \sim (\mathbf{3},\ 1) \quad \text{and} \quad H_u \sim (\mathbf{1},\ 0)$$

3 independent modular forms of weight $k = 2$ form a triplet of A_4: $Y^{(2)}_{\mathbf{3}} = \left(Y_1, Y_2, Y_3\right)^T$

$$W_\nu = \frac{1}{\Lambda_L}\Big(\overset{2\,-\,1\,-\,1}{\underset{\mathbf{3}\,\otimes\,\mathbf{3}\,\otimes\,\mathbf{3}}{Y^{(2)}_{\mathbf{3}}(\tau)\,LL}}\Big)_{\mathbf{1}} H_u H_u \qquad \Rightarrow \qquad M_\nu(\tau) = \frac{\mathrm{v}_u^2}{\Lambda_L}\begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

M_ν depends on 3 real parameters: $\mathrm{Re}\,\tau$, $\mathrm{Im}\,\tau$ and the overall scale v_u^2/Λ_L

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$$W_\nu = \frac{1}{\Lambda_L}\left(\overset{2-1-1}{\underset{\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}}{Y^{(2)}_{\mathbf{3}}(\tau)\, LL}}\right)_{\mathbf{1}} H_u H_u \qquad \Rightarrow \qquad M_\nu(\tau) = \frac{\mathrm{v}_u^2}{\Lambda_L}\begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

M_ν depends on 3 real parameters: $\mathrm{Re}\,\tau$, $\mathrm{Im}\,\tau$ and the overall scale v_u^2/Λ_L

$$\langle\tau\rangle = 0.0111 + 0.9946\, i$$

$\sin^2\theta_{12} = 0.295$ $\quad$ $\sin^2\theta_{13} = 0.0447$ $\quad$ $\sin^2\theta_{23} = 0.651$

$\delta/\pi = 1.55$ $\quad$ $\alpha_{21}/\pi = 0.22$ $\quad$ $\alpha_{31}/\pi = 1.80$

$m_1 = 0.0500$ eV $\quad$ $m_2 = 0.0507$ eV $\quad$ $m_3 = 0.0007$ eV (IO)

Minimal modular S4 seesaw model

Novichkov, Penedo, Petcov, **AT**, 1811.04933

$L \sim (\mathbf{3},\, 2)$ $\quad E^c \sim (\mathbf{1}',\, 0) \oplus (\mathbf{1},\, 2) \oplus (\mathbf{1}',\, 2)$ $\quad N^c \sim (\mathbf{3}',\, 0)$ and $H_{u,d} \sim (\mathbf{1},\, 0)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d$$
$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}$$

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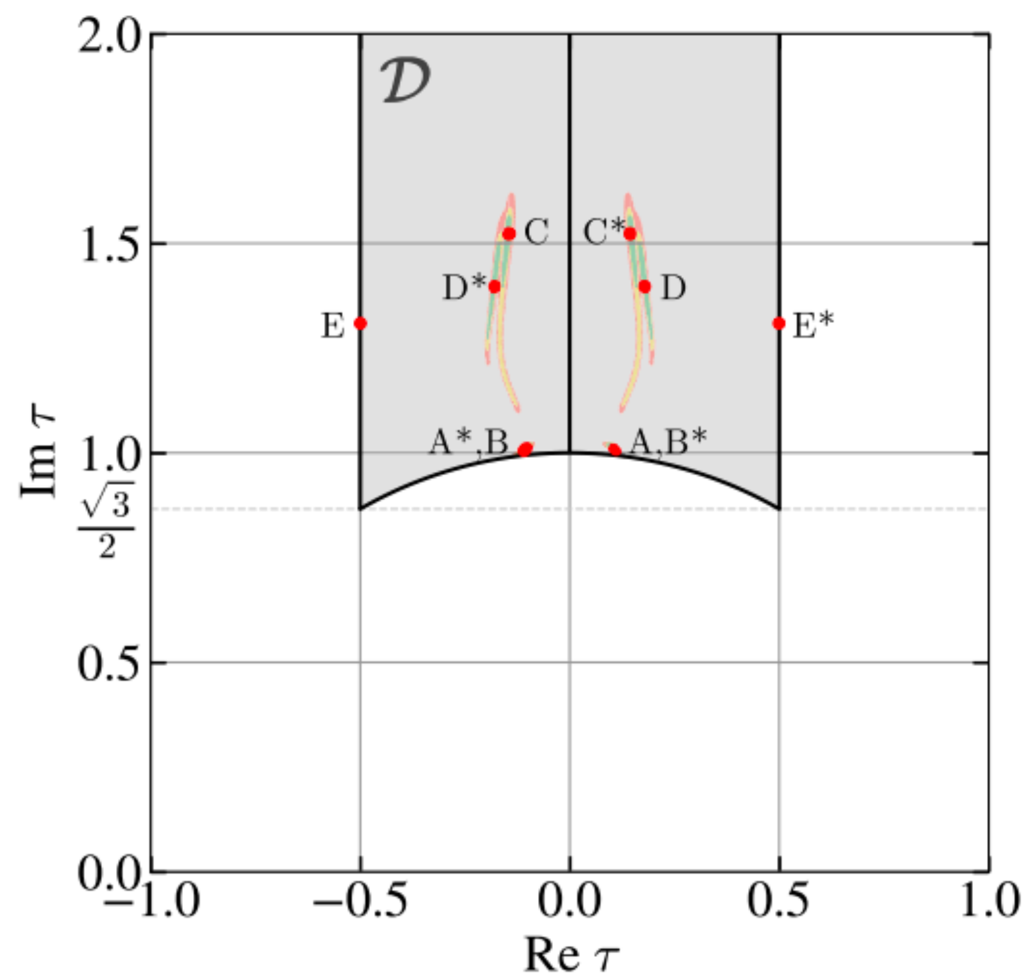
Solutions A and A*

Input parameters		Observables		Predictions	
$\mathrm{Re}\,\tau$	± 0.1045	m_e/m_μ	0.0048	m_1 [eV]	0.017
$\mathrm{Im}\,\tau$	1.0100	m_μ/m_τ	0.0565	m_2 [eV]	0.019
β/α	9.465	r	0.0299	m_3 [eV]	0.053
γ/α	0.0022	$\sin^2\theta_{12}$	0.305	δ/π	± 1.31
$\mathrm{Re}\,(g'/g)$	0.2330	$\sin^2\theta_{13}$	0.0213	α_{21}/π	± 0.30
$\mathrm{Im}\,(g'/g)$	± 0.4924	$\sin^2\theta_{23}$	0.551	α_{31}/π	± 0.87
$v_d\alpha$ [MeV]	53.19	δm^2 $[10^{-5}\ \mathrm{eV}^2]$	7.34	$\lvert m_{ee}\rvert$ [eV]	0.017
$v_u^2 g_1^2/\Lambda$ [eV]	0.0093	$\lvert\Delta m^2\rvert$ $[10^{-3}\ \mathrm{eV}^2]$	2.455	$\sum_i m_i$ [eV]	0.090
		$N\sigma$	0.02	Ordering	NO

8 (5) parameters vs 12 (9) observables



Minimal modular S4 seesaw model

Novichkov, Penedo, Petcov, **AT**, 1811.04933

$L \sim (\mathbf{3}, 2)$ $E^c \sim (\mathbf{1}', 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1}', 2)$ $N^c \sim (\mathbf{3}', 0)$ and $H_{u,d} \sim (\mathbf{1}, 0)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d$$

No CP

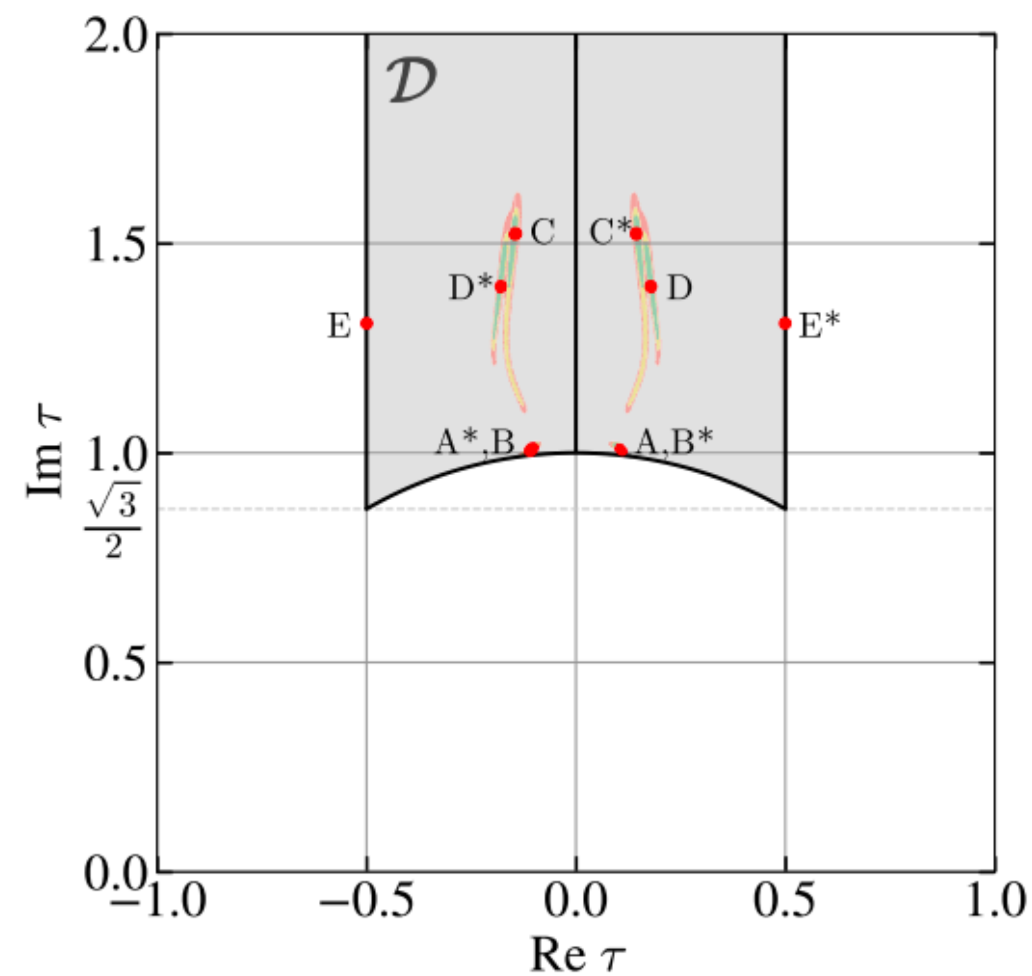
$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}$$

complex

Solutions A and A*

Input parameters		Observables		Predictions	
$\mathrm{Re}\,\tau$	± 0.1045	m_e/m_μ	0.0048	m_1 [eV]	0.017
$\mathrm{Im}\,\tau$	1.0100	m_μ/m_τ	0.0565	m_2 [eV]	0.019
β/α	9.465	r	0.0299	m_3 [eV]	0.053
γ/α	0.0022	$\sin^2\theta_{12}$	0.305	δ/π	± 1.31
$\mathrm{Re}\,(g'/g)$	0.2330	$\sin^2\theta_{13}$	0.0213	α_{21}/π	± 0.30
$\mathrm{Im}\,(g'/g)$	± 0.4924	$\sin^2\theta_{23}$	0.551	α_{31}/π	± 0.87
$v_d\alpha$ [MeV]	53.19	δm^2 [10^{-5} eV^2]	7.34	$\lvert m_{ee}\rvert$ [eV]	0.017
$v_u^2 g_1^2/\Lambda$ [eV]	0.0093	$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.455	$\sum_i m_i$ [eV]	0.090
		$N\sigma$	0.02	Ordering	NO

8 (5) parameters vs 12 (9) observables



Minimal modular S4 seesaw model

Novichkov, Penedo, Petcov, **AT**, 1905.11970

$L \sim (\mathbf{3}, 2)$ $E^c \sim (\mathbf{1}', 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1}', 2)$ $N^c \sim (\mathbf{3}', 0)$ and $H_{u,d} \sim (\mathbf{1}, 0)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d$$

With CP

$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}$$

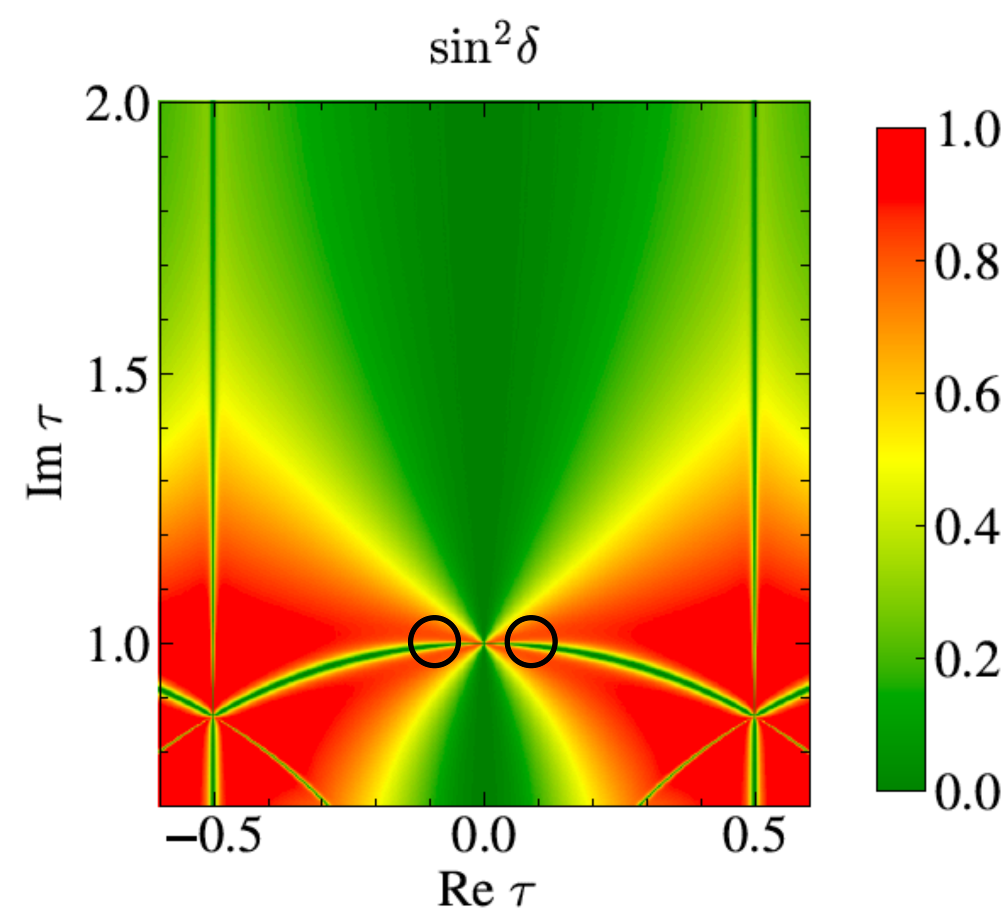
real

Solutions A and A*

Input parameters	
$\mathrm{Re}\,\tau$	± 0.0992
$\mathrm{Im}\,\tau$	1.0160
β/α	9.348
γ/α	0.0022
g'/g	-0.0209
$v_d\alpha$ [MeV]	53.61
$v_u^2 g_1^2/\Lambda$ [eV]	0.0135

Observables	
m_e/m_μ	0.0048
m_μ/m_τ	0.0576
r	0.0298
$\sin^2\theta_{12}$	0.305
$\sin^2\theta_{13}$	0.0214
$\sin^2\theta_{23}$	0.486
δm^2 [10^{-5} eV^2]	7.33
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.457
$N\sigma$	1.01

Predictions	
m_1 [eV]	0.012
m_2 [eV]	0.015
m_3 [eV]	0.051
δ/π	± 1.64
α_{21}/π	± 0.35
α_{31}/π	± 1.25
$\lvert m_{ee}\rvert$ [eV]	0.012
$\sum_i m_i$ [eV]	0.078
Ordering	NO



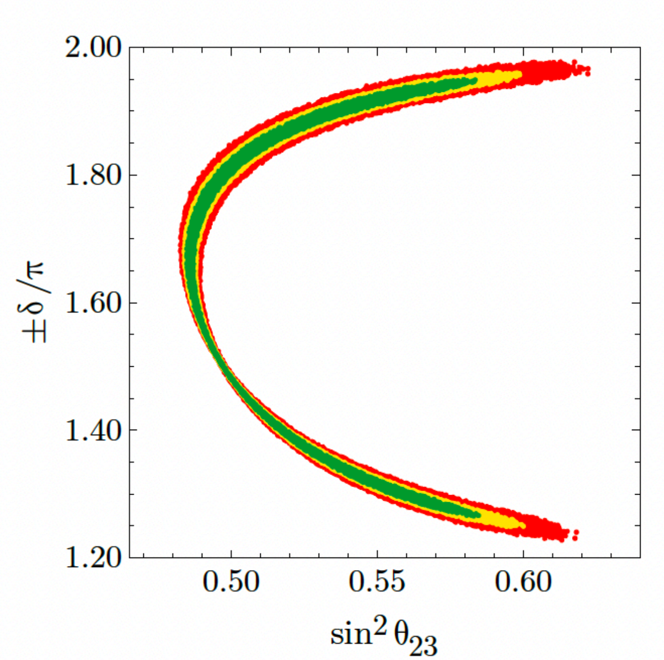
7 (4) parameters vs 12 (9) observables

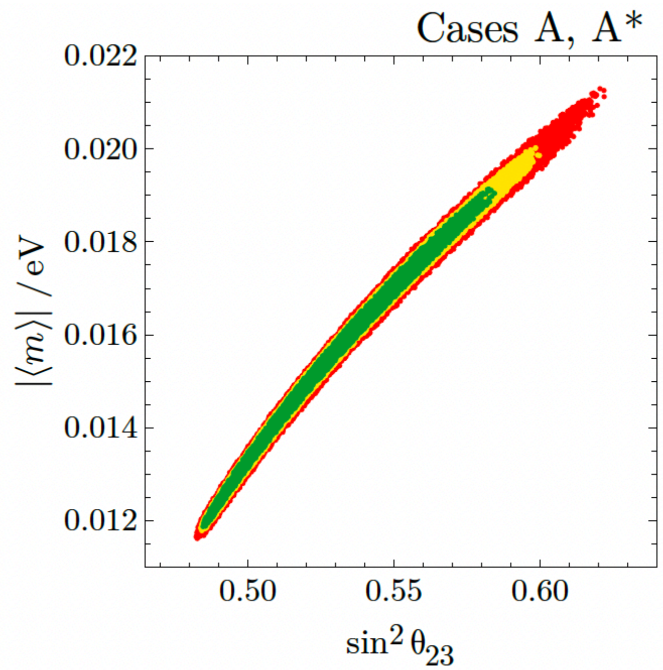
Minimal modular S4 seesaw model

Novichkov, Penedo, Petcov, **AT**, 1811.04933

Correlations between observables

No CP





$< 2\sigma$ $2-3\sigma$ $3-5\sigma$

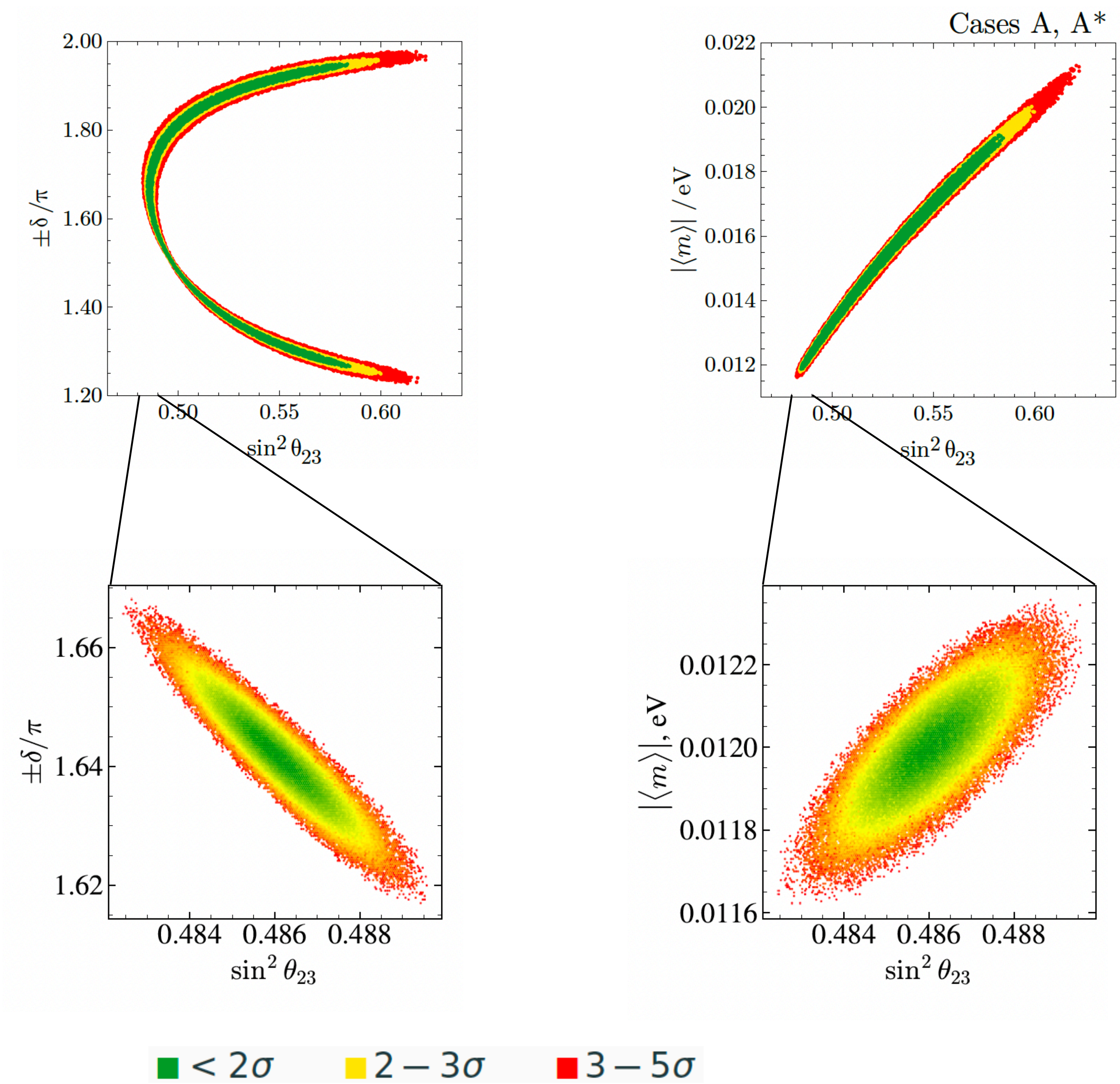
Minimal modular S4 seesaw model

Novichkov, Penedo, Petcov, **AT**, 1811.04933
1905.11970

Correlations between observables

No CP

With CP



A modular A4 model of quarks

Yao, Lu, Ding, 2012.13390

$Q \sim (\mathbf{3},\, 2)$ $U^c \sim (\mathbf{1},\, 0) \oplus (\mathbf{1}',\, 0) \oplus (\mathbf{1}'',\, 0)$ $D^c \sim (\mathbf{1}'',\, 0) \oplus (\mathbf{1}',\, 2) \oplus (\mathbf{1}'',\, 4)$ and $H_{u,d} \sim (\mathbf{1},\, 0)$

$$\begin{aligned}
\mathcal{W}_u &= \alpha_u u^c_{\mathbf{1}}(Q_L Y^{(2)}_{\mathbf{3}})_{\mathbf{1}} H_u + \beta_u c^c_{\mathbf{1}'}(Q_L Y^{(2)}_{\mathbf{3}})_{\mathbf{1}''} H_u + \gamma_u t^c_{\mathbf{1}''}(Q_L Y^{(2)}_{\mathbf{3}'})_{\mathbf{1}'} H_u\,, \\
\mathcal{W}_d &= \alpha_d d^c_{\mathbf{1}''}(Q_L Y^{(2)}_{\mathbf{3}})_{\mathbf{1}'} H_d + \beta_d s^c_{\mathbf{1}'}(Q_L Y^{(4)}_{\mathbf{3}})_{\mathbf{1}''} H_d + \gamma_{d,1} b^c_{\mathbf{1}''}(Q_L Y^{(6)}_{\mathbf{3}I})_{\mathbf{1}'} H_d \\
&\quad + \gamma_{d,2} b^c_{\mathbf{1}''}(Q_L Y^{(6)}_{\mathbf{3}II})_{\mathbf{1}'} H_d\,.
\end{aligned}$$

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$$\mathcal{W}_u = \alpha_u u^c_{\mathbf{1}}(Q_L Y^{(2)}_{\mathbf{3}})_{\mathbf{1}} H_u + \beta_u c^c_{\mathbf{1}'}(Q_L Y^{(2)}_{\mathbf{3}})_{\mathbf{1}''} H_u + \gamma_u t^c_{\mathbf{1}''}(Q_L Y^{(2)}_{\mathbf{3}'})_{\mathbf{1}'} H_u \,,$$

$$\mathcal{W}_d = \alpha_d d^c_{\mathbf{1}''}(Q_L Y^{(2)}_{\mathbf{3}})_{\mathbf{1}'} H_d + \beta_d s^c_{\mathbf{1}'}(Q_L Y^{(4)}_{\mathbf{3}})_{\mathbf{1}''} H_d + \gamma_{d,1} b^c_{\mathbf{1}''}(Q_L Y^{(6)}_{\mathbf{3}I})_{\mathbf{1}'} H_d + \gamma_{d,2} b^c_{\mathbf{1}''}(Q_L Y^{(6)}_{\mathbf{3}II})_{\mathbf{1}'} H_d \,.$$

Parameters

$$\langle\tau\rangle = 0.49175 + 0.88563i \,, \quad \beta_u/\alpha_u = 518.22933 \,, \quad \gamma_u/\alpha_u = 1.83596 \times 10^5 \,,$$
$$\beta_d/\alpha_d = 9.39751 \,, \quad \gamma_{d,1}/\alpha_d = 32.46046 \,, \quad \gamma_{d,2}/\alpha_d = -0.02697 \,,$$
$$\alpha_u v_u = 0.00034\,\text{GeV}, \quad \alpha_d v_d = 0.05081\,\text{GeV} \,.$$

Observables

$$\theta^q_{12} = 0.22734 \,, \quad \theta^q_{13} = 0.00332 \,, \quad \theta^q_{23} = 0.05708 \,, \quad \delta^q_{CP} = 0.39532\,\pi \,,$$
$$m_u/m_c = 0.00193 \,, \quad m_c/m_t = 0.00282 \,, \quad m_d/m_s = 0.05055 \,, \quad m_s/m_b = 0.01815 \,.$$
$$m_t = 87.46\ \text{GeV} \,, \quad m_b = 0.968\ \text{GeV} \,.$$

9 parameters vs 10 observables 😢

Modular invariance and CP

Fields

$$\tau \xrightarrow{\text{CP}} -\tau^\dagger \qquad \text{and} \qquad \Phi \xrightarrow{\text{CP}} \Phi^\dagger$$

Modular forms

$$Y(\tau) \xrightarrow{\text{CP}} Y(-\tau^*) = Y(\tau)^*$$

Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

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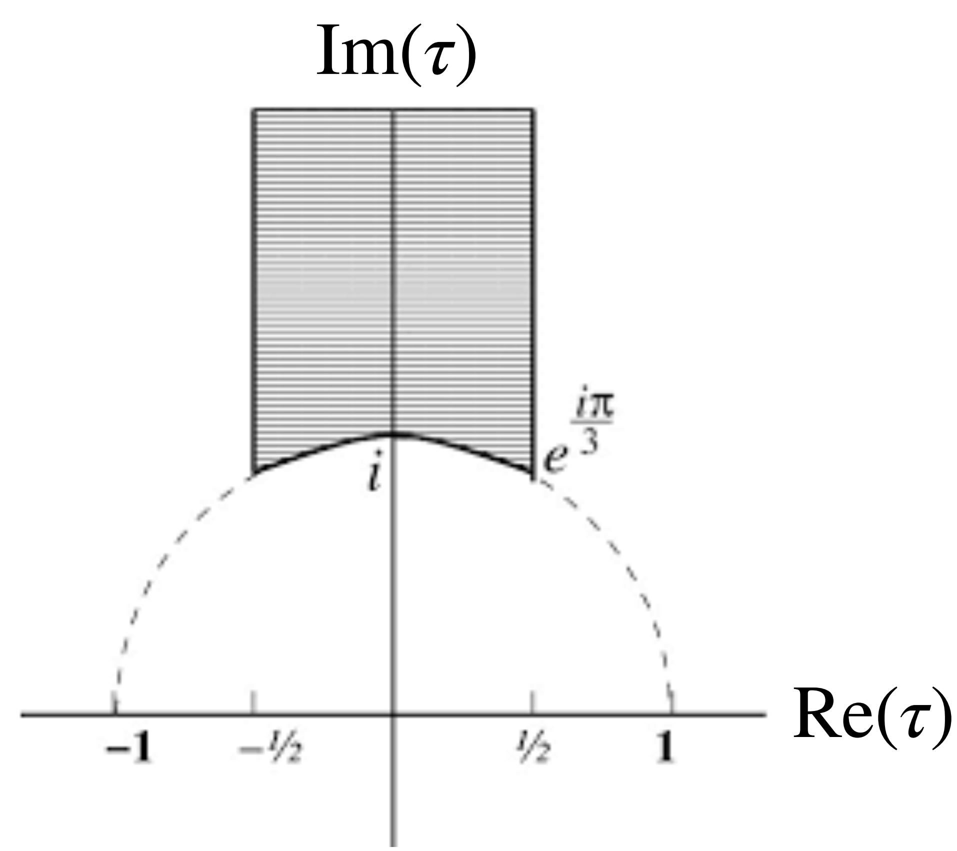
Modular forms

$$Y(\tau) \xrightarrow{\text{CP}} Y(-\tau^*) = Y(\tau)^*$$

CP-conserving values of τ

$\tau \xrightarrow{\text{CP}} -\tau^* = \gamma\tau$ (goes to itself up to γ)

1. $\tau = iy \xrightarrow{\text{CP}} iy$
2. $\tau = -\frac{1}{2} + iy \xrightarrow{\text{CP}} \frac{1}{2} + iy = T\tau$
3. $\tau = e^{i\varphi} \xrightarrow{\text{CP}} -e^{-i\varphi} = S\tau$



Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

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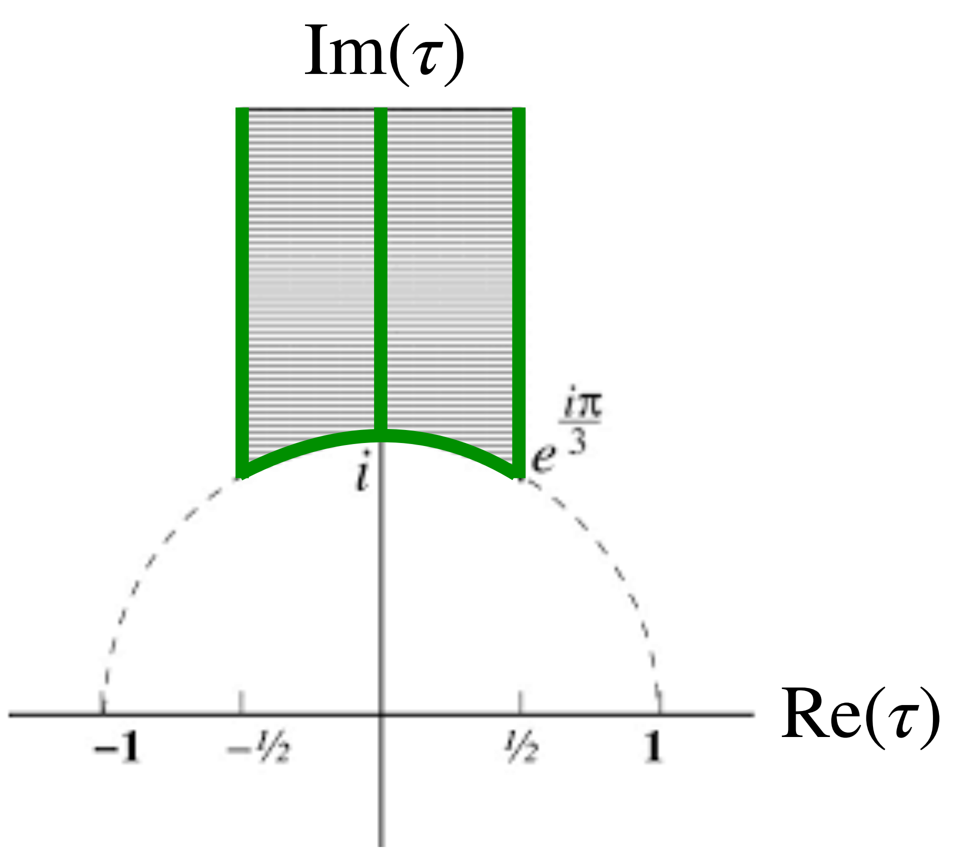


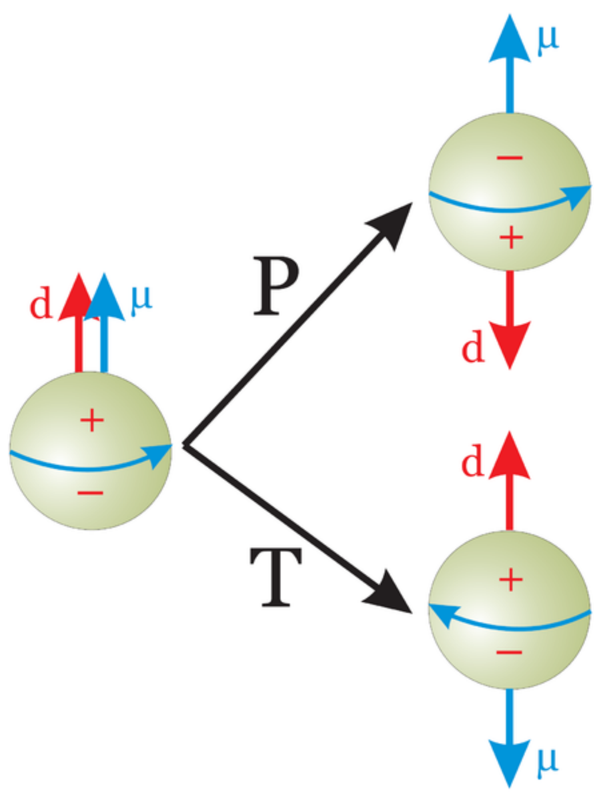
Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

The strong CP problem

$$\mathcal{L}_{\rm QCD} = \overline{q}\left(i\not{D} - M_q\right)q - \frac{1}{4g_3^2}G^a_{\mu\nu}G^{a,\mu\nu} + \frac{\theta_{\rm QCD}}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$$

$\bar{\theta} = \theta_{\rm QCD} + \arg\det M_q$ CPV parameter

Neutron EDM d



$d = 2.4 \times 10^{-16}\ \bar{\theta}\ e \cdot$ cm Pospelov, Ritz, hep-ph/9908508v4

$|d| \leq 1.8 \times 10^{-26}\ e \cdot$ cm (90% C.L.) Abel et al., 2001.11966

$|\bar{\theta}| \lesssim 10^{-10}$

Why so small???

… and the CPV phase in the CKM matrix $\delta_{\rm CKM} \approx 1.2$

Our solution: CP + modular invariance

Feruglio, Strumia, **AT**, 2305.08908

1. CP is a symmetry => $\theta_{\text{QCD}} = 0$ (and real Lagrangian couplings)

2. Modular invariance/anomaly cancellation => $\arg\det M_q = 0$

3. CP is broken spontaneously by the VEV of a single complex scalar field, the modulus τ => $\delta_{\text{CKM}} = \mathcal{O}(1)$

4. Quark mass hierarchies and mixing angles are reproduced by $\mathcal{O}(1)$ parameters

5. Corrections to $\bar{\theta} = 0$ are small under certain assumptions on SUSY breaking

Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \ln\left(-i\tau + i\tau^\dagger\right) + \sum_M \frac{\Phi^\dagger e^{2V}\Phi}{\left(-i\tau + i\tau^\dagger\right)^{k_M}}$$

Superpotential

$$W = Y^u_{ij}(\tau)\, U^c_i\, Q_j\, H_u + Y^d_{ij}(\tau)\, D^c_i\, Q_j\, H_d$$

τ-dependent Yukawa couplings

$$Y^q_{ij}(\tau) \to (c\tau + d)^{k^q_{ij}}\, Y^q_{ij}(\tau) \qquad \text{with} \qquad k^{u(d)}_{ij} = k_{U^c_i(D^c_i)} + k_{Q_j} + k_{H_{u(d)}}$$

are modular forms!

$$Y^q_{ij}(\tau) = c^q_{ij}\, Z_{k^q_{ij}}(\tau) \qquad \text{with} \qquad c^q_{ij} \in \mathbb{R} \qquad \text{because of CP}$$

Gauge kinetic function

$$f = \frac{1}{g_3^2} \qquad \theta_{\rm QCD} = 0 \ \ \text{because of CP}$$

Determinant of quark mass matrix

$$M_u = \mathrm{v}_u Y^u \qquad M_d = \mathrm{v}_d Y^d$$

$$\det M_q = \det M_u \det M_d \propto \det Y^u \det Y^d$$

$$Y^q(\tau) = \begin{pmatrix} Z_{k^q_{11}} & Z_{k^q_{12}} & Z_{k^q_{13}} \\ Z_{k^q_{21}} & Z_{k^q_{22}} & Z_{k^q_{23}} \\ Z_{k^q_{31}} & Z_{k^q_{32}} & Z_{k^q_{33}} \end{pmatrix} \quad \Rightarrow$$

$\det Y^q(\tau)$ is a modular form of weight k^q_{det}

$$k^u_{\mathrm{det}} = k^u_{11} + k^u_{22} + k^u_{33} = \ldots = \sum_{i=1}^{3} \left(k_{U^c_i} + k_{Q_i} \right) + 3k_{H_u}$$

And $\det Y^u(\tau) \det Y^d(\tau)$ is a modular form of weight k_{det}

$$k_{\mathrm{det}} = k^u_{\mathrm{det}} + k^d_{\mathrm{det}} = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U^c_i} + k_{D^c_i} \right) + 3\left(k_{H_u} + k_{H_d} \right)$$

$$k_{\mathrm{det}} = 0 \quad \Rightarrow \quad \det Y^u(\tau) \det Y^d(\tau) = \text{(real) constant}$$

Simplest example: quarks

Simplest non-trivial example giving $k_{\rm det} = 0$

$$k_Q = k_{U^c} = k_{D^c} = (-6, 0, 6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Yukawa matrices

$$Y^q = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23} E_6 \\ c^q_{31} & c^q_{32} E_6 & c^q_{33} E_4^3 + c'^q_{33} E_6^2 \end{pmatrix} \qquad \text{with} \qquad \det Y^q = -\, c^q_{13} c^q_{22} c^q_{31} \in \mathbb{R}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^u_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.56 \\ 0 & -1.86 & 0.87 \\ 1.29 & 4.14 & 3.51,\ 1.40 \end{pmatrix} \qquad c^d_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.55 \\ 0 & -2.59 & 4.59 \\ 0.378 & 0.710 & 0.734,\ 1.76 \end{pmatrix}$$

reproduce the quark masses, mixing angles and $\delta_{\rm CKM}$ at the GUT scale of 2×10^{16} GeV

Simplest example: leptons

$k_L = k_{E^c} = (-6, 0, 6)$

Weinberg operator $\mathscr{C}^{\nu}_{ij}(L_i H_u)(L_j H_u)$ for neutrino masses

Charged lepton Yukawa matrix and coefficient of the Weinberg operator

$$Y^e = \begin{pmatrix} 0 & 0 & c^e_{13} \\ 0 & c^e_{22} & c^e_{23}E_6 \\ c^e_{31} & c^e_{32}E_6 & c^e_{33}E_4^3 + c'^e_{33}E_6^2 \end{pmatrix} \qquad \mathscr{C}^{\nu} = \begin{pmatrix} 0 & 0 & c^{\nu}_{13} \\ 0 & c^{\nu}_{22} & c^{\nu}_{23}E_6 \\ c^{\nu}_{13} & c^{\nu}_{23}E_6 & c^{\nu}_{33}E_4^3 + c'^{\nu}_{33}E_6^2 \end{pmatrix}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^e_{ij} = 10^{-3}\begin{pmatrix} 0 & 0 & 1.29 \\ 0 & 5.95 & 0.35 \\ -2.56 & 1.47 & 1.01,\ 1.32 \end{pmatrix} \qquad c^{\nu}_{ij} = \frac{1}{10^{16}\ \mathrm{GeV}}\begin{pmatrix} 0 & 0 & 3.4 \\ 0 & 7.1 & 1.2 \\ 3.4 & 1.2 & 0.19,\ 0.95 \end{pmatrix}$$

reproduce the lepton masses and mixing angles

Generalisations

- With SM quarks only, finite modular groups Γ_N do not help to reduce the number of parameters while keeping $\bar{\theta} = 0$ and accommodating m_q and V_{CKM}

  Penedo, Petcov, 2404.08032

- Unlike in the Nelson—Barr models, heavy vector-like quarks are not needed, but can help to lower modular weights favoured in string compactifications

  Feruglio, Parriciatu, Strumia, **AT**, 2406.01689

- In string compactifications, gauge kinetic function is usually a complex, non-trivial function of τ

$$f(\tau) = \frac{1}{g_3^2} + i\frac{\theta_{\text{QCD}}}{8\pi^2}$$

  The solution still works (under certain assumptions)

  Feruglio, Marrone, Strumia, **AT**, 2505.20395

I have not discussed...

- Fermion mass hierarchies from modular symmetry along

  Novichkov, Penedo, Petcov, 2102.07488; Feruglio, Gherardi, Romanino, **AT,** 2101.08718
  Okada, Tanimoto, 2009.14242

- Dynamical selection of the vacuum/scalar potential

  potential for the modulus recently revisited in
  Novichkov, Penedo, Petcov, 2201.02020; Leedom, Righi, Westphal, 2212.03876

- Control over the Kähler potential

  top-down concept of eclectic flavour symmetries helps
  Nilles, Ramos-Sánchez, Vaudrevange, 2001.01736, 2004.05200, 2006.03059, 2010.13798

- Non-SUSY/non-holomorphic version of the construction

  Qu, Ding, 2406.02527

- Modular-invariant inflation

  Ding, Jiang, Zhao, 2405.06497, 2411.18603 (+Xu); King, Wang, 2405.08924

- Modular-invariant baryogenesis

  Duch, Strumia, **AT**, 2504.03506

Conclusions

- Modular invariance as flavour symmetry
  - can be implemented in a bottom-up approach
  - Yukawa couplings are functions of a modulus τ
  - both lepton masses and mixing are constrained
- Minimal modular-invariant flavour models
  - no flavons
  - lightest neutrino mass, mass ordering and CPV phases are predicted
  - quarks are challenging…
- Modular invariance and CP
  - can be consistently combined
  - real couplings => smaller number of free parameters
  - τ is the only source of CPV
  - provide an alternative solution to the strong CP problem

Backup slides

3-neutrino mixing

Charged current weak interactions

$$-\mathscr{L}_{\rm CC} = \frac{g}{\sqrt{2}} \sum_{\ell=e,\mu,\tau} \overline{\ell_L}(x)\,\gamma_\alpha\,\nu_{\ell L}(x)\,W^{\alpha\dagger}(x) + \text{h.c.}$$

Mismatch between the interaction and mass eigenstates

$$\nu_{\ell L}(x) = \sum_{j=1}^{3} U_{\ell j}\,\nu_{jL}(x)$$

U is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix

The standard parameterisation (adopted by the PDG)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}$$

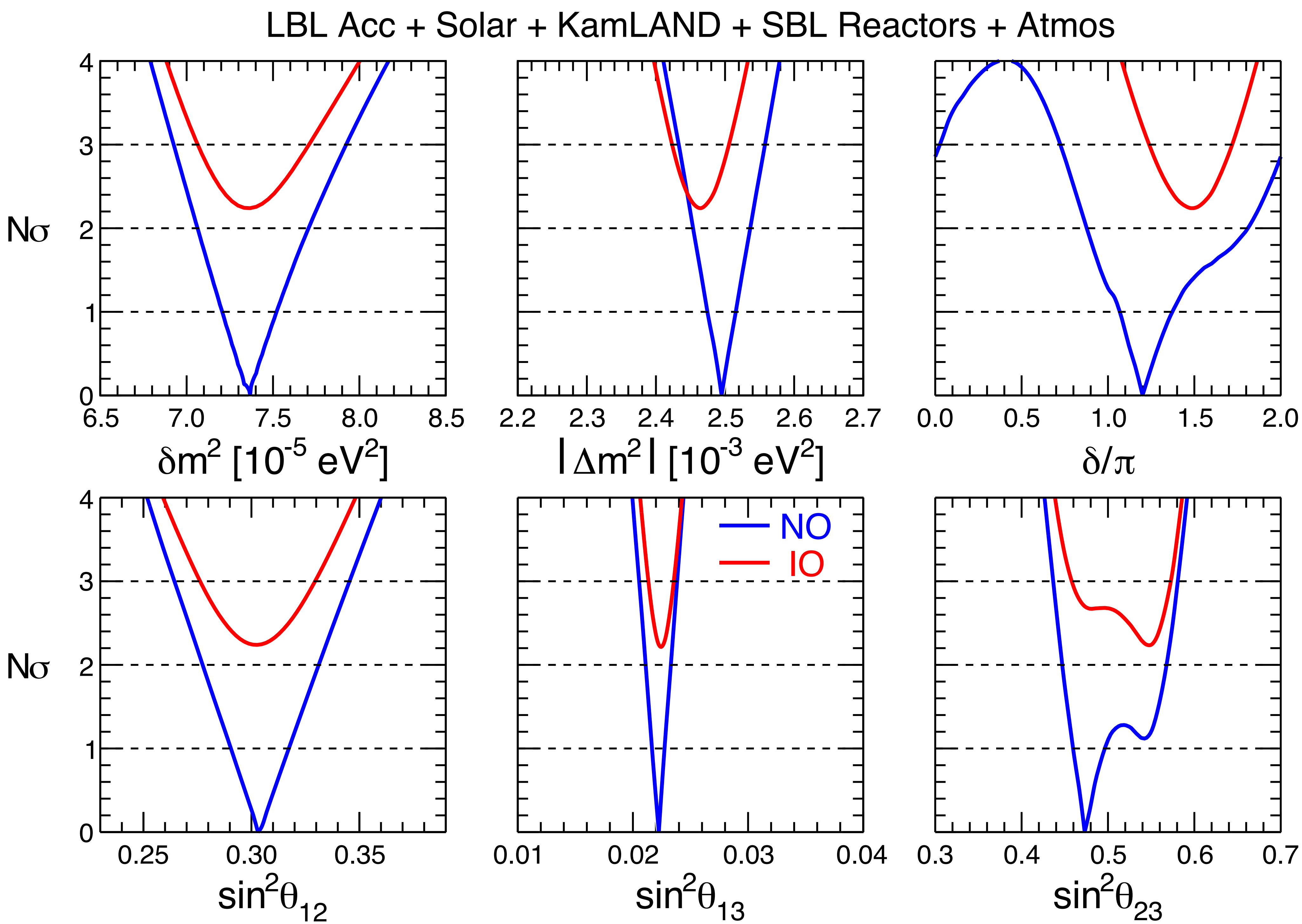
Atmospheric angle θ_{23}

Reactor angle θ_{13}
Dirac phase δ

Solar angle θ_{12}

Majorana phases
α_{21} and α_{31}
(if ν are Majorana)

Neutrino oscillation data



Capozzi et al., 2503.07752; see also Esteban et al., 2410.05380 and de Salas et al., 2006.11237

Neutrino oscillation data

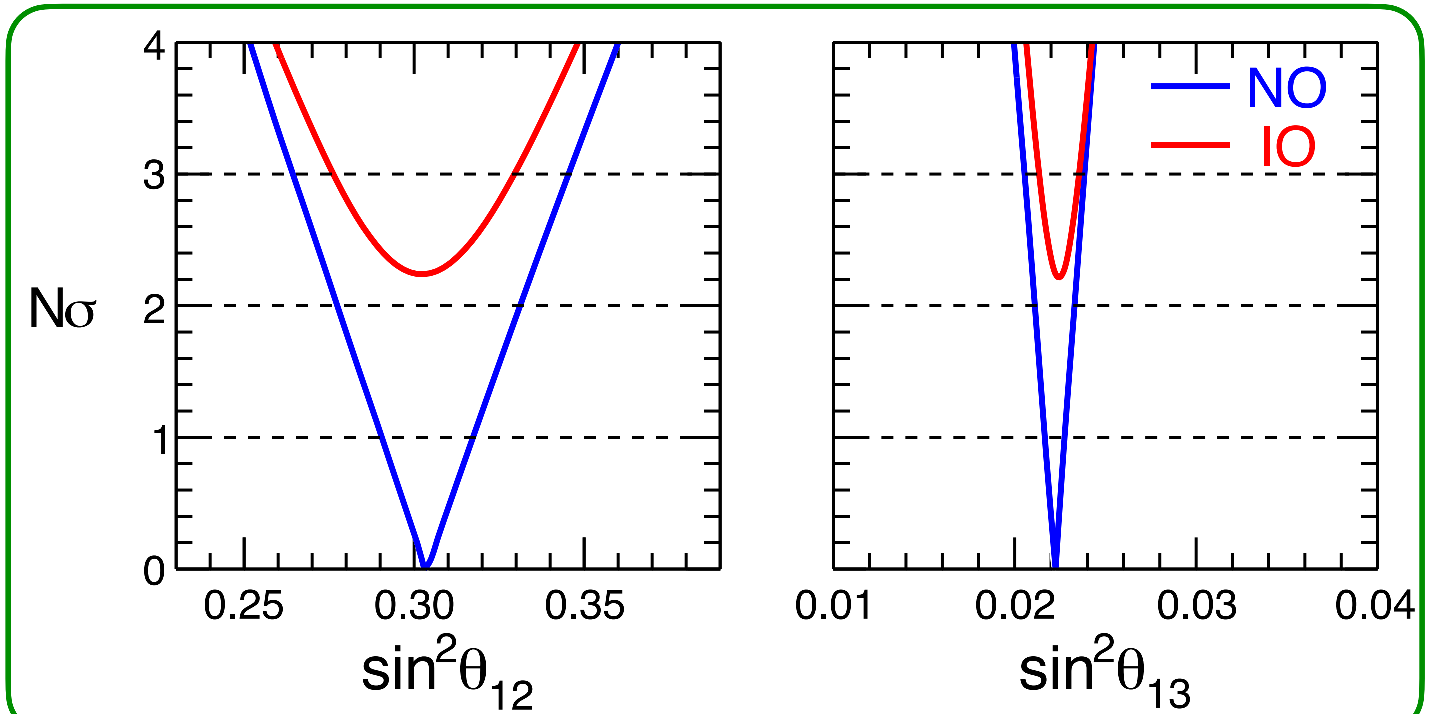
$\sin^2\theta_{12} = 0.303\ (4.5\%)$

$\sin^2\theta_{13} = 0.0223\ (2.4\%)$

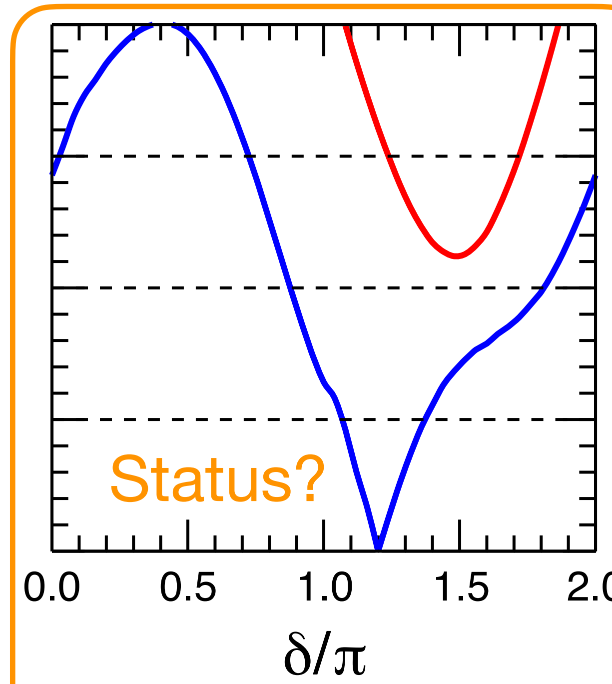
$\sin^2\theta_{23} =$ $0.473\ (5.1\%)$ or $0.545\ (4.3\%)$

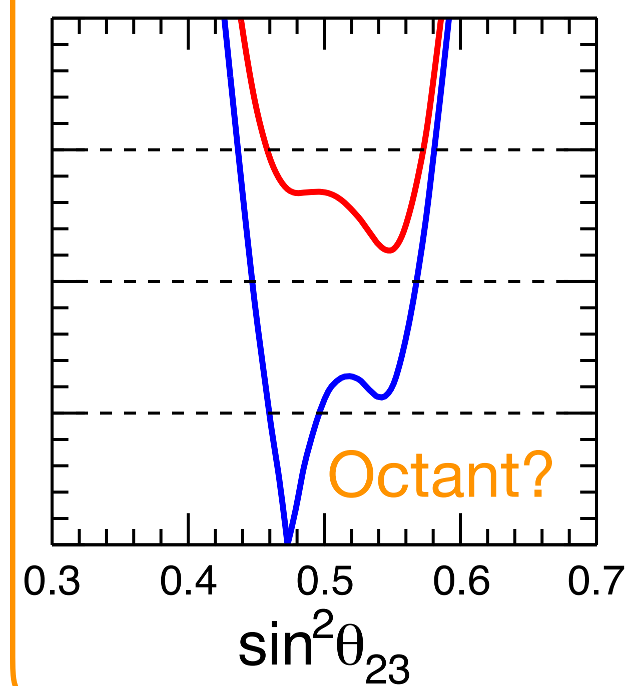
$\delta/\pi =$ $1.20\ (18\%)$ or $1.48\ (8\%)$

Known



Unknown





Capozzi et al., 2503.07752; see also Esteban et al., 2410.05380 and de Salas et al., 2006.11237

Lepton masses and mixings

NuFIT 5.2 (2022)

with SK atmospheric data		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
	$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
	$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
	$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
	$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
	$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
	$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
	$\delta_{\rm CP}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \rm eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \rm eV^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$

Esteban et al., 2007.14792 and www.nu-fit.org

$$m_e/m_\mu = 0.0048 \pm 0.0002$$

$$m_\mu/m_\tau = 0.0565 \pm 0.0045$$

Quark masses and mixings

At the GUT scale of 2×10^{16} GeV,
assuming MSSM with $\tan\beta = 10$ and SUSY breaking scale of 10 TeV

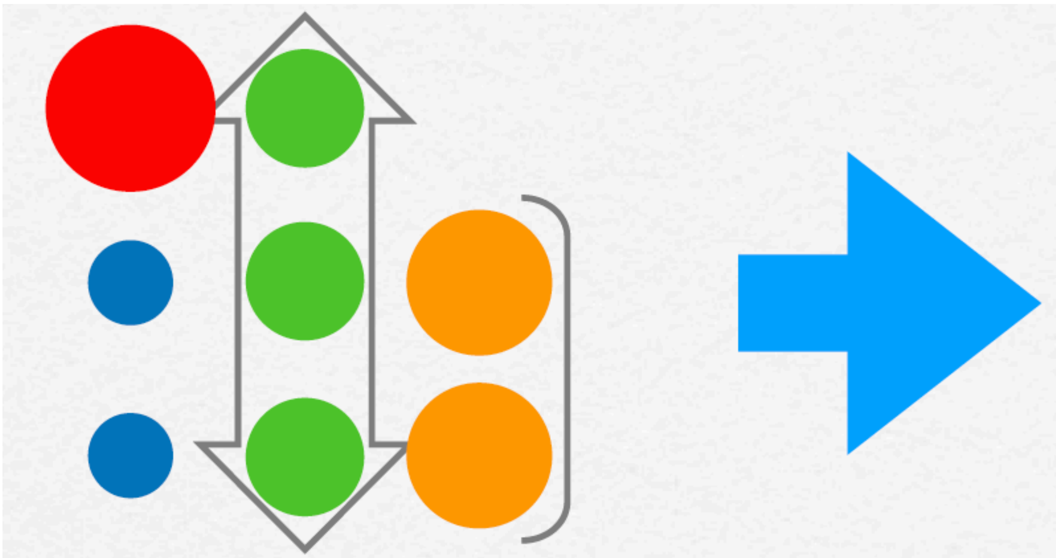
m_u/m_c	$(1.93 \pm 0.60) \times 10^{-3}$
m_c/m_t	$(2.82 \pm 0.12) \times 10^{-3}$
m_d/m_s	$(5.05 \pm 0.62) \times 10^{-2}$
m_s/m_b	$(1.82 \pm 0.10) \times 10^{-2}$
$\sin^2\theta_{12}$	$(5.08 \pm 0.03) \times 10^{-2}$
$\sin^2\theta_{13}$	$(1.22 \pm 0.09) \times 10^{-5}$
$\sin^2\theta_{23}$	$(1.61 \pm 0.05) \times 10^{-3}$
δ/π	0.385 ± 0.017

$m_t = 87.46$ GeV
$m_b = 0.9682$ GeV

Antusch, Maurer, 1306.6879
Yao, Lu, Ding, 2012.13390

Tri-bimaximal (TBM) mixing

Harrison, Perkins, Scott, hep-ph/0202074



$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sin^2\theta_{23} = \frac{1}{2} \qquad \sin^2\theta_{13} = 0 \qquad \sin^2\theta_{12} = \frac{1}{3}$$

Allowed at 2σ | Excluded at many σ | Allowed at 2σ

TBM mixing from S4

Example

$$G_f = S_4$$

$$G_e = Z_3^T \qquad\qquad G_\nu = Z_2^S \times Z_2^U$$

$$\rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad \rho(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \omega = e^{2\pi i/3}$$

$U_e = \mathbb{1}$

diagonalised by $U_\nu = U_{\text{TBM}}$

$$U_{\text{PMNS}} = U_e^\dagger U_\nu = U_{\text{TBM}}$$

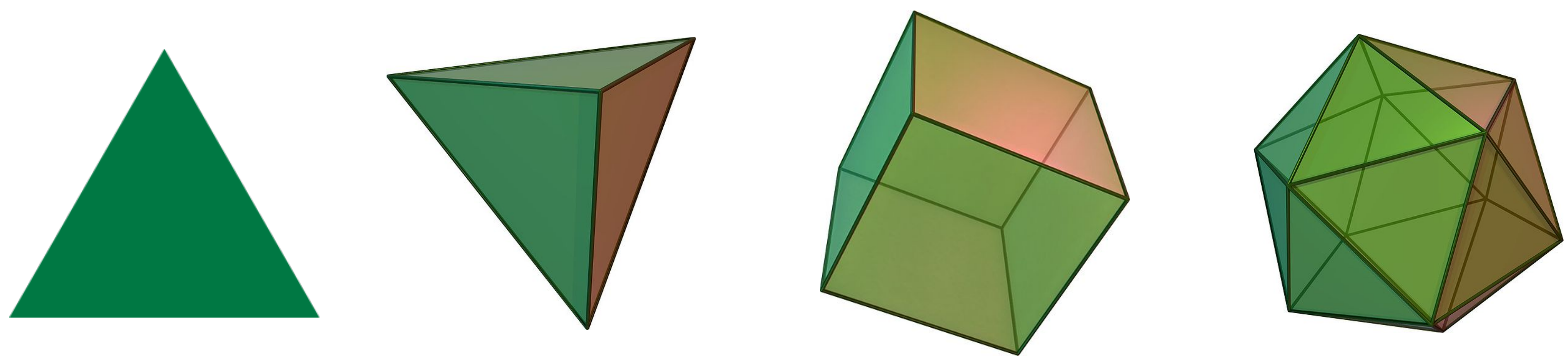
In concrete models, flavour symmetry breaking occurs spontaneously when flavons (scalar fields not charged under the SM) acquire VEVs

$\langle \phi^e \rangle \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ preserves T and $\langle \phi^\nu \rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ preserves S and U

Finite modular groups

$$\Gamma_N = \left\langle S,\, T \mid S^2 = (ST)^3 = T^N = I \right\rangle\,, \quad N = 2\,,\, 3\,,\, 4\,,\, 5$$

$\Gamma_2 \cong S_3$ $\Gamma_3 \cong A_4$ $\Gamma_4 \cong S_4$ $\Gamma_5 \cong A_5$



Images: WIKIPEDIA

For $N > 5$ additional relations $f(S, T) = I$ needed to render Γ_N finite

de Adelhart Toorop, Feruglio, Hagedorn, 1112.1340

Vacuum selection

In the considered bottom-up approach the VEV of τ is a free parameter

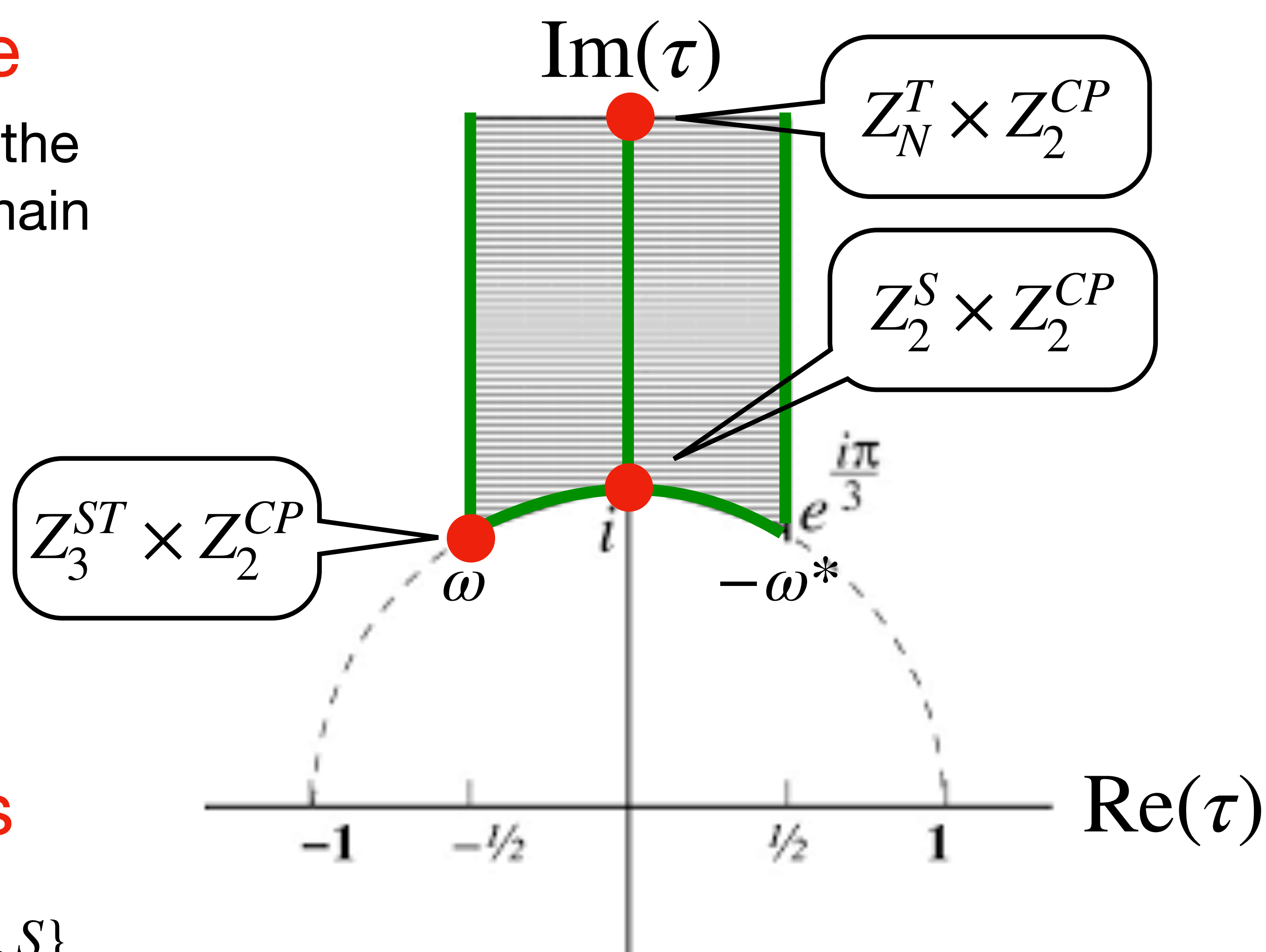
Top-down conjecture

All extrema of the potential lie on the boundary of the fundamental domain and on the imaginary axis

M. Cvetic et al., NPB **361** (1991) 194

Recent studies find new, CP-violating minima

Novichkov, Penedo, Petcov, 2201.02020
Leedom, Righi, Westphal, 2212.03876



Residual symmetries

$$\tau = i: \quad i \xrightarrow{S} -\frac{1}{i} = i \quad \Rightarrow \quad Z_2^S = \{I, S\}$$

$$\tau = \omega \equiv e^{\frac{2\pi i}{3}}: \quad \omega \xrightarrow{ST} -\frac{1}{\omega+1} = \omega \quad \Rightarrow \quad Z_3^{ST} = \{I, ST, (ST)^2\}$$

$$\tau = i\infty: \quad i\infty \xrightarrow{T} i\infty + 1 = i\infty \quad \Rightarrow \quad Z_N^T = \{I, T, T^2, \ldots, T^N\}$$

Matter fields and canonical normalisation

Gauge quantum numbers

	Q	U^c	D^c	L	E^c	H_u	H_d
$\mathrm{SU}(3)_c$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{SU}(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$
$\mathrm{U}(1)_Y$	$\frac{1}{6}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$

Canonical normalisation

$$K \supset \frac{\Phi^\dagger \Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}} = \Phi^\dagger_{\mathrm{can}} \Phi_{\mathrm{can}} \qquad \Phi_{\mathrm{can}} = \{\phi_{\mathrm{can}}, \psi_{\mathrm{can}}\}$$

$$\psi_{\mathrm{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{-\frac{k_\Phi}{2}} \psi_{\mathrm{can}} = e^{-ik_\Phi \alpha(\tau)} \psi_{\mathrm{can}} \qquad \alpha(\tau) = \arg(c\tau + d)$$

Modular symmetry acts on canonically normalised fields
as a τ-dependent phase rotation (with $\tau = \tau(x)$)

Modular-SM anomalies

Conditions for modular-gauge anomaly cancellation

$$\mathrm{SU}(3)_c: \quad A \equiv \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) = 0$$

$$\mathrm{SU}(2)_L: \quad \sum_{i=1}^{3} \left(3k_{Q_i} + k_{L_i}\right) + k_{H_u} + k_{H_d} = 0$$

$$\mathrm{U}(1)_Y: \quad \sum_{i=1}^{3} \left(k_{Q_i} + 8k_{U_i^c} + 2k_{D_i^c} + 3k_{L_i} + 6k_{E_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Simplest solution

$$k_Q = k_{U^c} = k_{D^c} = k_L = k_{E^c} = (-k, 0, k) \qquad \text{and} \qquad k_{H_u} + k_{H_d} = 0$$

Cancellation of modular-QCD anomaly along with $k_{H_u} + k_{H_d} = 0$ implies

$$k_{\det} = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

More on modular-gauge anomalies

$\Phi \to \Phi' = \Lambda(\gamma, \tau)^{-k_\Phi}\,\Phi$

Jacobian J: $\mathscr{D}\,\Phi' = J\,\mathscr{D}\,\Phi$

Arkani-Hamed, Murayama, hep-th/9707133

$$\log J = -\frac{i}{64\pi^2}\int d^4x\, d^2\theta\,\left[\sum_\Phi T(\Phi)\,k_\Phi\right] W^a W^a\;\ln\Lambda$$

$T(\Phi)$ is the Dynkin index of the rep of Φ: $\mathrm{tr}\left(t_a t_b\right) = T(\Phi)\,\delta_{ab}$

$$\sum_\Phi T(\Phi)\,k_\Phi = 0$$

$$\mathrm{SU}(3)_c: \quad \sum_i \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) = 0$$

$$\mathrm{SU}(2)_L: \quad \sum_i \left(3k_{Q_i} + k_{L_i}\right) + k_{H_u} + k_{H_d} = 0$$

$$\mathrm{U}(1)_Y: \quad \sum_i \left(k_{Q_i} + 8k_{U_i^c} + 2k_{D_i^c} + 3k_{L_i} + 6k_{E_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Simplest example: quarks

Simplest non-trivial example giving $k_{\det} = 0$ and $A = 0$

$$k_Q = k_{U^c} = k_{D^c} = (-6, 0, 6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Yukawa matrices

$$Y^q = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23} E_6 \\ c^q_{31} & c^q_{32} E_6 & c^q_{33} E_4^3 + c'^q_{33} E_6^2 \end{pmatrix} \quad \Rightarrow \quad Y^q|_{\text{can}} = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23}\,(2\text{Im}\tau)^3\,E_6 \\ c^q_{31} & c^q_{32}\,(2\text{Im}\tau)^3\,E_6 & (2\text{Im}\tau)^6 \left[c^q_{33} E_4^3 + c'^q_{33} E_6^2\right] \end{pmatrix}$$

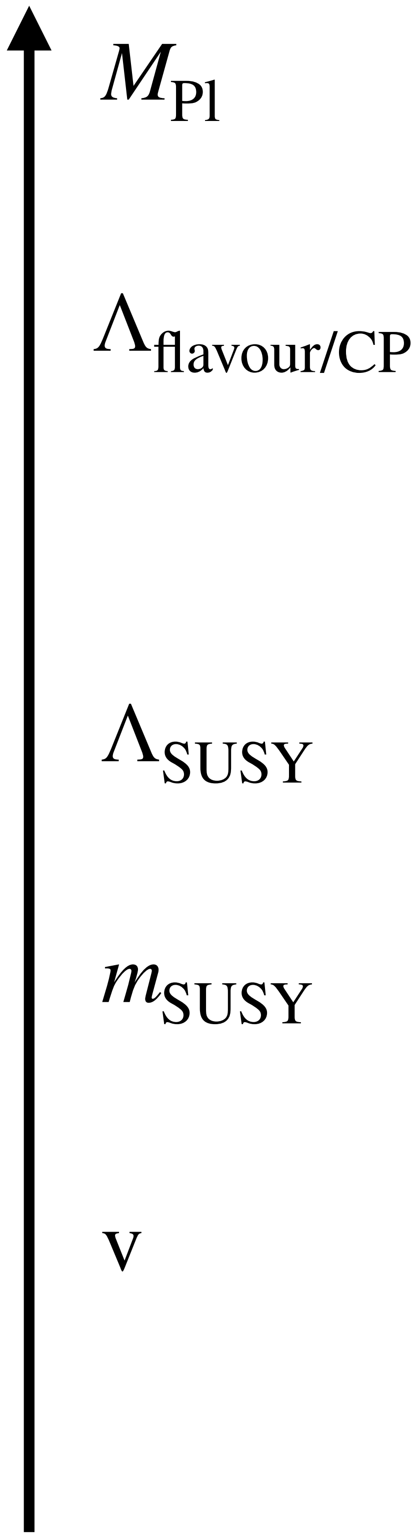
$$\det Y^q|_{\text{can}} = -\,c^q_{13} c^q_{22} c^q_{31} \in \mathbb{R}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^u_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.56 \\ 0 & -1.86 & 0.87 \\ 1.29 & 4.14 & 3.51,\ 1.40 \end{pmatrix} \qquad c^d_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.55 \\ 0 & -2.59 & 4.59 \\ 0.378 & 0.710 & 0.734,\ 1.76 \end{pmatrix}$$

reproduce the quark masses, mixing angles and δ_{CKM} at the GUT scale of 2×10^{16} GeV

Corrections to $\bar{\theta} = 0$



Corrections to $\bar{\theta} = 0$

$M_{\rm Pl}$

$\Lambda_{\rm flavour/CP}$

$\Lambda_{\rm SUSY}$

$m_{\rm SUSY}$

v

SUSY unbroken

- Modular invariance determines completely (up to real couplings) the functional dependence $W(\tau)$
- It is not the case for K, but $\bar{\theta}$ is insensitive to K
- No-renormalisation theorems Ellis, Ferrara, Nanopoulos, PLB **114** (1982) 231

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$\Lambda_{\rm SUSY}$

SUSY breaking corrections

- In general, can be large

$m_{\rm SUSY}$

- Small if $\Lambda_{\rm flavour/CP} \gg \Lambda_{\rm SUSY}$ (as e.g. in gauge mediation) and soft ~~SUSY~~ terms respect the flavour structure of the SM

v

$$\bar{\theta} \lesssim \frac{M_t^4 M_b^4 M_c^2 M_s^2}{{\rm v}^{12}} J_{\rm CP} \tan^6\beta \sim 10^{-28} \tan^6\beta$$

Corrections to $\bar{\theta} = 0$

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SM corrections are negligible

- $\bar{\theta} \sim 10^{-18}$ at four loops Khriplovich, PLB **173** (1986) 193
Ellis, Gaillard, NPB **150** (1979) 141

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

$$\gamma = -I \quad \Rightarrow \quad f(\tau) = (-1)^k f(\tau) \quad \Rightarrow \quad k \text{ is even}$$

Modular forms are periodic and admit q-expansions

$$\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad f(\tau+1) = f(\tau) \quad \Rightarrow \quad f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,, \quad q = e^{2\pi i \tau}$$

Modular forms of weight k form a linear space $\mathscr{M}_k$ of finite dimension

$$\dim \mathscr{M}_k = \begin{cases} 0 & \text{if } k \text{ is negative or odd} \\ \lfloor k/12 \rfloor & \text{if } k = 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \neq 2 \pmod{12} \end{cases}$$

Modular forms of level 1: E4 and E6

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$Z\left(\gamma\tau\right) = \left(c\tau + d\right)^{k} Z(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

Normalised Eisenstein series

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k}$$

Each modular form of weight k can be written as a polynomial in E_4 and E_6

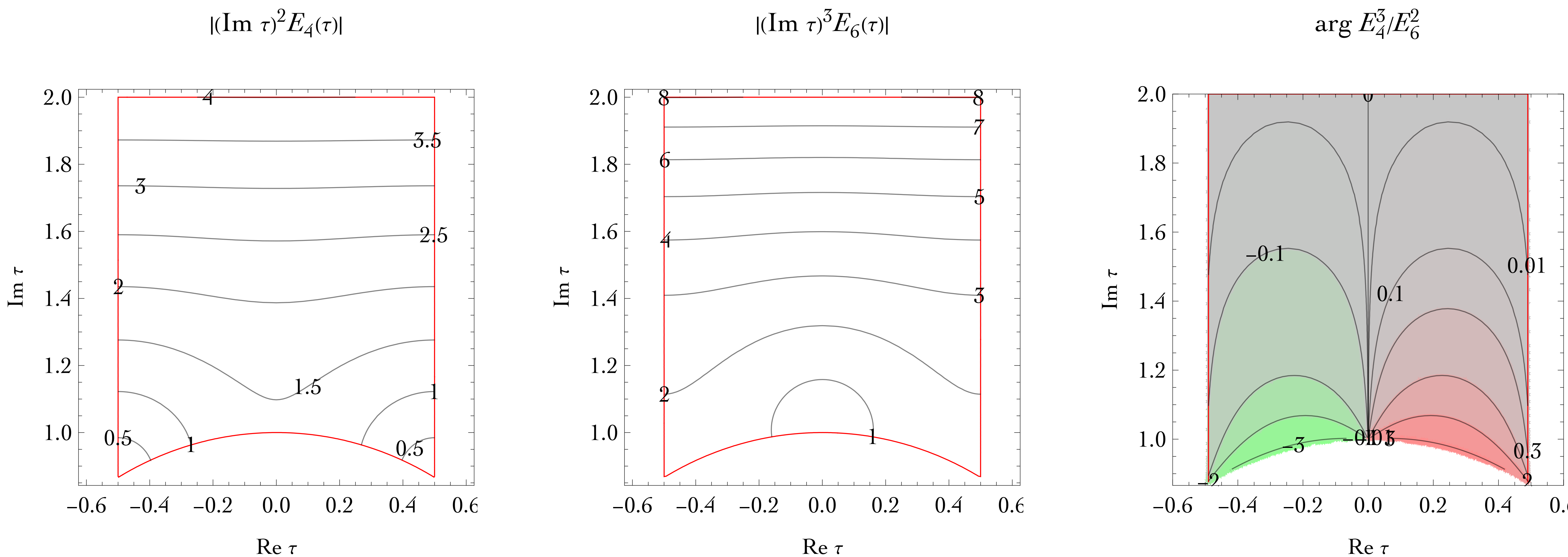
$$Z(\tau) = \sum_{a,b\geq 0} c_{ab}\, E_4^a(\tau)\, E_6^b(\tau) \qquad \text{with} \qquad 4a + 6b = k$$

Modular weight k	0	2	4	6	8	10	12	14
Modular forms	1	–	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	E_4^3, E_6^2	$E_{14} = E_4^2 E_6$

Modular forms of level 1: E4 and E6

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}(q^5)$$

$$E_6(\tau) = 1 - 540 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \mathcal{O}(q^5)$$



Modular forms of level 2

Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^n\right) \qquad q \equiv e^{2\pi i \tau}$$

$$Z_1^{(2)} = \frac{2i}{\pi}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - 8\frac{\eta'(2\tau)}{\eta(2\tau)}\right] = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + \mathcal{O}\left(q^5\right)$$

$$Z_2^{(2)} = \frac{2\sqrt{3}i}{\pi}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right] = 8\sqrt{3}q^{1/2}\left(1 + 4q + 6q^2 + 8q^3 + \mathcal{O}\left(q^4\right)\right)$$

$$\begin{pmatrix} Z_1^{(2)} \\ Z_2^{(2)} \end{pmatrix} \sim \mathbf{2} \quad \text{of} \quad \Gamma_2 \cong S_3$$

$$\left\{Z_1^{(4)}, Z_2^{(4)}, Z_3^{(4)}\right\} = \left\{{Z_2^{(2)}}^2 - {Z_1^{(2)}}^2, 2Z_1^{(2)}Z_2^{(2)}, {Z_1^{(2)}}^2 + {Z_2^{(2)}}^2\right\}$$

$$\begin{pmatrix} Z_1^{(4)} \\ Z_2^{(4)} \end{pmatrix} \sim \mathbf{2} \qquad Z_3^{(4)} \sim \mathbf{1}_0 \quad \text{of} \quad \Gamma_2 \cong S_3$$

Group properties of $\Gamma_2 \cong S_3$

$$\Gamma_2 = \left\langle S,\, T \mid S^2 = (ST)^3 = T^2 = I \right\rangle$$

$$\mathscr{S}_{\mathbf{2}} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \qquad \mathscr{T}_{\mathbf{2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

S_3	$\mathbf{1}_0$	$\mathbf{1}_1$	$\mathbf{2}$
$\mathscr{S}$	1	−1	$\mathscr{S}_{\mathbf{2}}$
$\mathscr{T}$	1	−1	$\mathscr{T}_{\mathbf{2}}$

Tensor products

$$\mathbf{1}_1 \otimes \mathbf{1}_1 = \mathbf{1}_0 \qquad \mathbf{1}_1 \otimes \mathbf{2} = \mathbf{2} \qquad \mathbf{2} \otimes \mathbf{2} = \mathbf{1_0} \oplus \mathbf{1_1} \oplus \mathbf{2}$$

Clebsch-Gordan coefficients

$$\left(\gamma_{\mathbf{1}_1} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{2}} = \left(-\gamma\beta_2\,,\, \gamma\beta_1\right)^T$$

$$\left(\alpha_{\mathbf{2}} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{1}_0} = \alpha_1\beta_1 + \alpha_2\beta_2$$

$$\left(\alpha_{\mathbf{2}} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{1}_1} = \alpha_1\beta_2 - \alpha_2\beta_1$$

$$\left(\alpha_{\mathbf{2}} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{2}} = \left(\alpha_2\beta_2 - \alpha_1\beta_1\,,\, \alpha_1\beta_2 + \alpha_2\beta_1\right)^T$$

Modular S4 symmetry

$$\Gamma_4 = \left\langle S,\, T \mid S^2 = (ST)^3 = T^4 = I \right\rangle$$

- 24 elements
- 5 irreps: $\mathbf{1}\,,\ \mathbf{1}'\,,\ \mathbf{2}\,,\ \mathbf{3}\,,\ \mathbf{3}'$
- Space of the lowest non-trivial weight 2 modular forms has dimension 5
- 5 weight 2 modular forms arrange themselves in a doublet and a triplet:

$$Y_{\mathbf{2}}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix} \qquad Y_{\mathbf{3}'}(\tau) \equiv \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}$$

- $Y_i(\tau)$ are given in terms of the Dedekind eta function

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right)\,, \quad q = e^{2\pi i \tau}$$

- Products of $Y_i(\tau)$ generate modular forms of higher weights, 4, 6, 8, …

Penedo, Petcov, 1806.11040

Minimal modular S4 seesaw models

Novichkov, Penedo, Petcov, **AT**, 1811.04933

Seesaw type I models with no flavons

	E_1^c	E_2^c	E_3^c	N^c	L	H_d	H_u
$SU(2)_L \times U(1)_Y$	$(\mathbf{1},+1)$	$(\mathbf{1},+1)$	$(\mathbf{1},+1)$	$(\mathbf{1},0)$	$(\mathbf{2},-1/2)$	$(\mathbf{2},-1/2)$	$(\mathbf{2},+1/2)$
$\Gamma_4 \cong S_4$	$\mathbf{1}$ or $\mathbf{1}'$	$\mathbf{1}$ or $\mathbf{1}'$	$\mathbf{1}$ or $\mathbf{1}'$	$\mathbf{3}$ or $\mathbf{3}'$	$\mathbf{3}$ or $\mathbf{3}'$	$\mathbf{1}$	$\mathbf{1}$
k_I	k_1	k_2	k_3	k_N	k_L	0	0

$$W = \sum_{i=1}^{3} \alpha_i \left(E_i^c\, L\, F_{E_i}(\tau)\right)_{\mathbf{1}} H_d + g\left(N^c\, L\, F_N(\tau)\right)_{\mathbf{1}} H_u + \Lambda\left(N^c\, N^c\, F_M(\tau)\right)_{\mathbf{1}}$$

Modular invariance imposes the following constraints on the weights:

$$\begin{cases} k_{\alpha_i} = k_i + k_L \\ k_g = k_N + k_L \\ k_\Lambda = 2\,k_N \end{cases} \quad \Leftrightarrow \quad \begin{cases} k_i = k_{\alpha_i} - k_g + k_\Lambda/2 \\ k_L = k_g - k_\Lambda/2 \\ k_N = k_\Lambda/2 \end{cases}$$

$$W = \lambda_{ij}(\tau)\, E_i^c\, L_j\, H_d + \mathcal{Y}_{ij}(\tau)\, N_i^c\, L_j\, H_u + \frac{1}{2} M_{ij}(\tau)\, N_i^c\, N_j^c$$

After integrating out heavy neutrinos and after EWSB

$$M_e = \mathrm{v}_d\, \lambda^\dagger \qquad M_\nu = -\,\mathrm{v}_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}$$

Minimal modular S4 seesaw models

Systematic exploration of low weights k_{α_i}, k_g, k_N

Higher weights => more free parameters in the superpotential

Majorana mass term for N^c

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$$

$k_\Lambda = 0 \;\Rightarrow\; F_M = \text{const}:$ $\quad \left(N^c N^c\right)_{\mathbf{1}} = N_1^c N_1^c + N_2^c N_3^c + N_3^c N_2^c \qquad M = 2\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$k_\Lambda = 2 \;\Rightarrow\; F_M = Y_{\mathbf{2}},\, Y_{\mathbf{3}'}:$ $\quad \Lambda \left(N^c N^c Y_{\mathbf{2}}\right)_{\mathbf{1}} + \Lambda' \left(N^c N^c Y_{\mathbf{3}'}\right)_{\mathbf{1}} \qquad M = 2\Lambda \begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix}$

$k_\Lambda = 4 \;\Rightarrow\; F_M = Y^{(4)}_{\mathbf{1}},\, Y^{(4)}_{\mathbf{2}},\, Y^{(4)}_{\mathbf{3}},\, Y^{(4)}_{\mathbf{3}'}:$

$$\Lambda \left(N^c N^c Y^{(4)}_{\mathbf{1}}\right)_{\mathbf{1}} + \Lambda' \left(N^c N^c Y^{(4)}_{\mathbf{2}}\right)_{\mathbf{1}} + \Lambda'' \left(N^c N^c Y^{(4)}_{\mathbf{3}}\right)_{\mathbf{1}} + \Lambda''' \left(N^c N^c Y^{(4)}_{\mathbf{3}'}\right)_{\mathbf{1}}$$

Minimal modular S4 seesaw models

Charged-lepton Yukawa matrix

$$\left(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_3}\right) = (2, 4, 4) \;\Rightarrow\; \left(F_{E_1}, F_{E_2}, F_{E_3}\right) = \left(Y_{\mathbf{3}'}, Y^{(4)}_{\mathbf{3}}, Y^{(4)}_{\mathbf{3}'}\right):$$

$$\lambda = \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\left(Y_1 Y_4 - Y_2 Y_5\right) & \beta\left(Y_1 Y_3 - Y_2 Y_4\right) & \beta\left(Y_1 Y_5 - Y_2 Y_3\right) \\ \gamma\left(Y_1 Y_4 + Y_2 Y_5\right) & \gamma\left(Y_1 Y_3 + Y_2 Y_4\right) & \gamma\left(Y_1 Y_5 + Y_2 Y_3\right) \end{pmatrix}$$

Number of free real parameters in M_e and M_ν
($\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$ + coupling constants in the superpotential)

k_g \ k_Λ	0	2	4
0	6	6	10
2	8	8	12
4, $\rho_N \neq \rho_L$	10	10	14
4, $\rho_N = \rho_L$	12	12	16

We aim to describe/predict
12 observables:

m_e, m_μ, m_τ

m_1, m_2, m_3 or m_3, m_1, m_2

$\sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}$

$\delta, \alpha_{21}, \alpha_{31}$

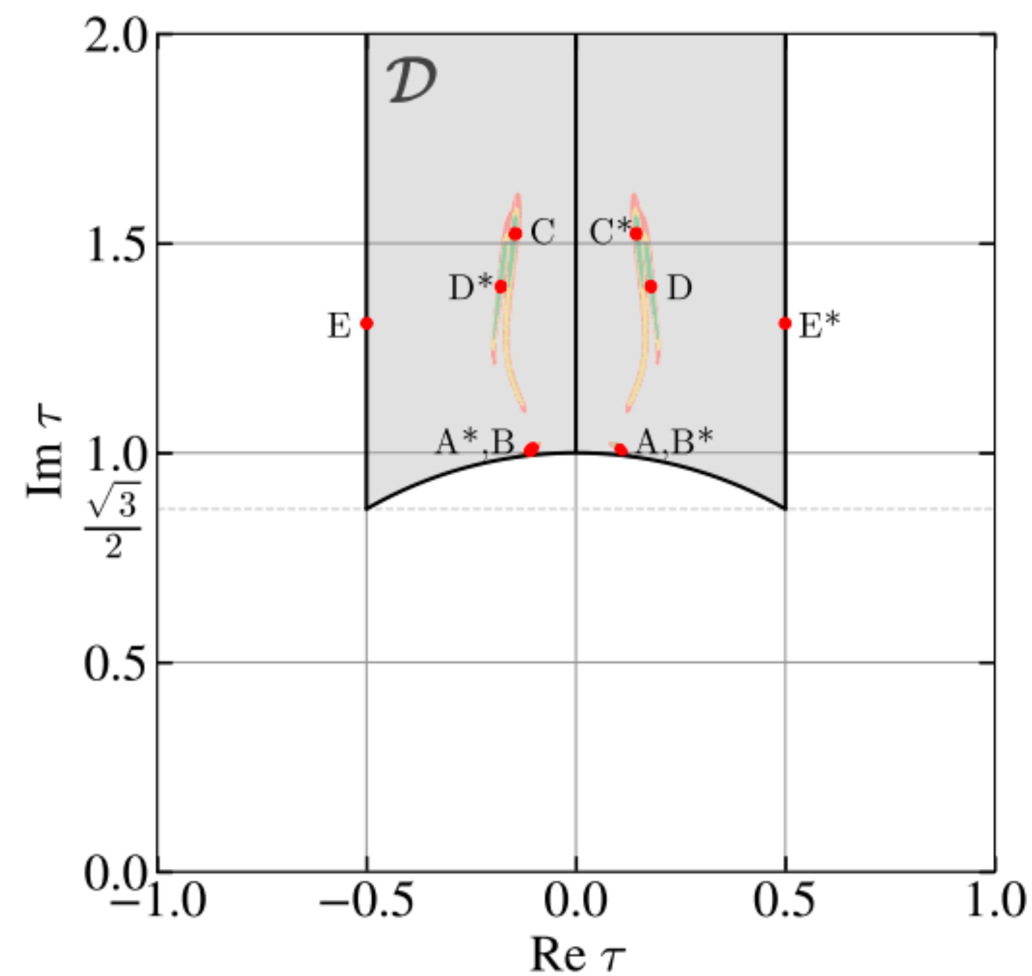
Minimal modular S4 seesaw models

Charged leptons: $(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_3}) = (2, 4, 4)$ Neutrinos: $(k_\Lambda, k_g) = (0, 2)$

$$W = \alpha\left(E_1^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_d + \beta\left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma\left(E_3^c L\, Y_{\mathbf{3'}}^{(4)}\right)_{\mathbf{1}} H_d$$
$$+ g\left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g'\left(N^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda\left(N^c N^c\right)_{\mathbf{1}}$$

Solutions A and A*

Input parameters		Observables		Predictions	
$\mathrm{Re}\,\tau$	± 0.1045	m_e/m_μ	0.0048	m_1 [eV]	0.017
$\mathrm{Im}\,\tau$	1.0100	m_μ/m_τ	0.0565	m_2 [eV]	0.019
β/α	9.465	r	0.0299	m_3 [eV]	0.053
γ/α	0.0022	$\sin^2\theta_{12}$	0.305	δ/π	± 1.31
$\mathrm{Re}\,(g'/g)$	0.2330	$\sin^2\theta_{13}$	0.0213	α_{21}/π	± 0.30
$\mathrm{Im}\,(g'/g)$	± 0.4924	$\sin^2\theta_{23}$	0.551	α_{31}/π	± 0.87
$v_d\alpha$ [MeV]	53.19	δm^2 [10^{-5} eV^2]	7.34	$\|m_{ee}\|$ [eV]	0.017
$v_u^2 g_1^2/\Lambda$ [eV]	0.0093	$\|\Delta m^2\|$ [10^{-3} eV^2]	2.455	$\sum_i m_i$ [eV]	0.090
		$N\sigma$	0.02	Ordering	NO



8 (5) parameters vs 12 (9) observables

Modular invariance and CP

Novichkov, Penedo, Petcov, **AT**, 1905.11970

- $\tau \xrightarrow{CP} -\tau^*$

- $\chi(x) \xrightarrow{CP} X\,\overline{\chi}(x_P)\,, \quad x_P = (t, -\mathbf{x})$

  In the symmetric basis where $\rho(S)^T = \rho(S)$ and $\rho(T)^T = \rho(T)$,
  $X = \mathbb{1}$ (canonical CP basis)

- $Y(\tau) \xrightarrow{CP} Y(-\tau^*) = X\,Y^*(\tau) = Y^*(\tau)$ in the symmetric basis

Extended modular group

$CP \to \gamma \to CP^{-1}$ on the modulus

$$\tau \xrightarrow{CP} -\tau^* \xrightarrow{\gamma} -\frac{a\tau^* + b}{c\tau^* + d} \xrightarrow{CP^{-1}} \frac{a\tau - b}{-c\tau + d}$$

Outer automorphism of $\overline{\Gamma}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to u(\gamma) \equiv CP\,\gamma\,CP^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$$u(S) = S \qquad u(T) = T^{-1}$$

Extended modular group

$$\overline{\Gamma}^* = \left\langle \tau \xrightarrow{S} -1/\tau\,,\ \ \tau \xrightarrow{T} \tau + 1\,,\ \ \tau \xrightarrow{CP} -\tau^* \right\rangle \simeq \overline{\Gamma} \rtimes Z_2^{CP}$$

$$\overline{\Gamma}^* \simeq PGL(2,\mathbb{Z}) \ \text{ with } \ CP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{if} \quad ad - bc = 1 \qquad \text{and} \qquad \tau \to \frac{a\tau^* + b}{c\tau^* + d} \quad \text{if} \quad ad - bc = -1$$

CP-conserving values of the modulus

τ and $\gamma\tau$ are physically equivalent, thus CP is preserved for

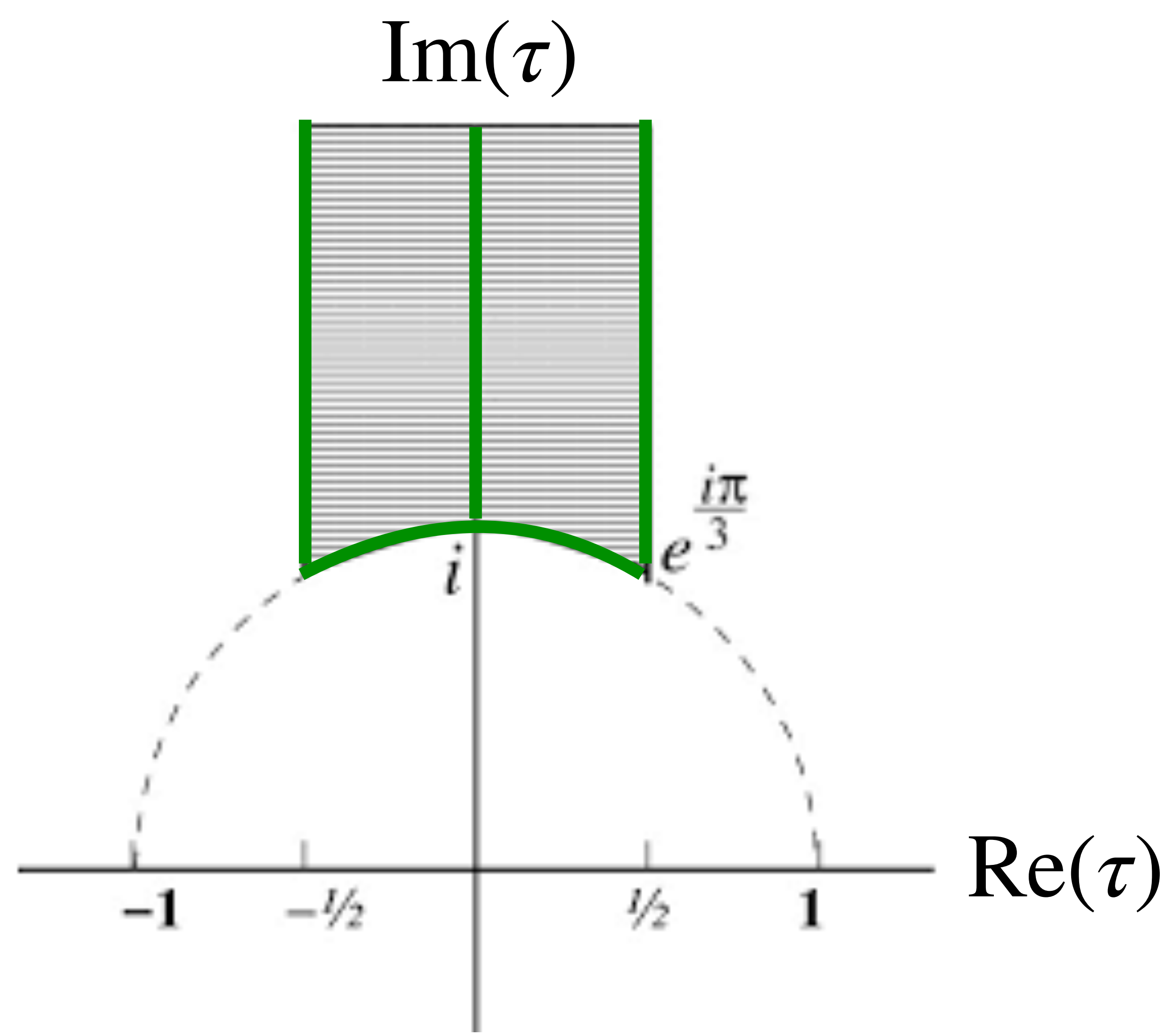
$$\tau \xrightarrow{CP} -\tau^* = \gamma\tau$$

CP is violated in the fundamental domain, except for:

1) $\tau = iy \xrightarrow{CP} iy$

2) $\tau = -\frac{1}{2} + i\,y \xrightarrow{CP} \frac{1}{2} + i\,y = T\tau$

3) $\tau = e^{i\varphi} \xrightarrow{CP} -e^{-i\varphi} = S\tau$



Implication of CP for couplings

$$W \supset \sum_s g_s \left(Y_s(\tau)\chi_1 \ldots \chi_n\right)_{\mathbf{1},s} \qquad \overline{W} \supset \sum_s g_s^* \overline{\left(Y_s(\tau)\chi_1 \ldots \chi_n\right)_{\mathbf{1},s}}$$

In a symmetric basis ($X = \mathbb{1}$)

$$g_s \left(Y_s(\tau)\chi_1 \ldots \chi_n\right)_{\mathbf{1},s} \xrightarrow{CP} g_s \left(Y_s^*(\tau)\overline{\chi}_1 \ldots \overline{\chi}_n\right)_{\mathbf{1},s} = g_s \overline{\left(Y_s(\tau)\chi_1 \ldots \chi_n\right)_{\mathbf{1},s}}$$

reality of Clebsch-Gordan coefficients (holds for $N \leq 5$)

$$g_s = g_s^*$$

Couplings must be real

Sources of corrections

SUSY breaking

Can be made negligible via separation of
SUSY-breaking scale and messenger scale

Renormalisation group running

Small for $\tan\beta \lesssim 10\ (25)$ dependent on the model

Criado, Feruglio, 1807.01125

Kähler potential

This is a problem in the bottom-up approach, since many terms allowed by modular invariance can be present in K

Feruglio, 1706.08749

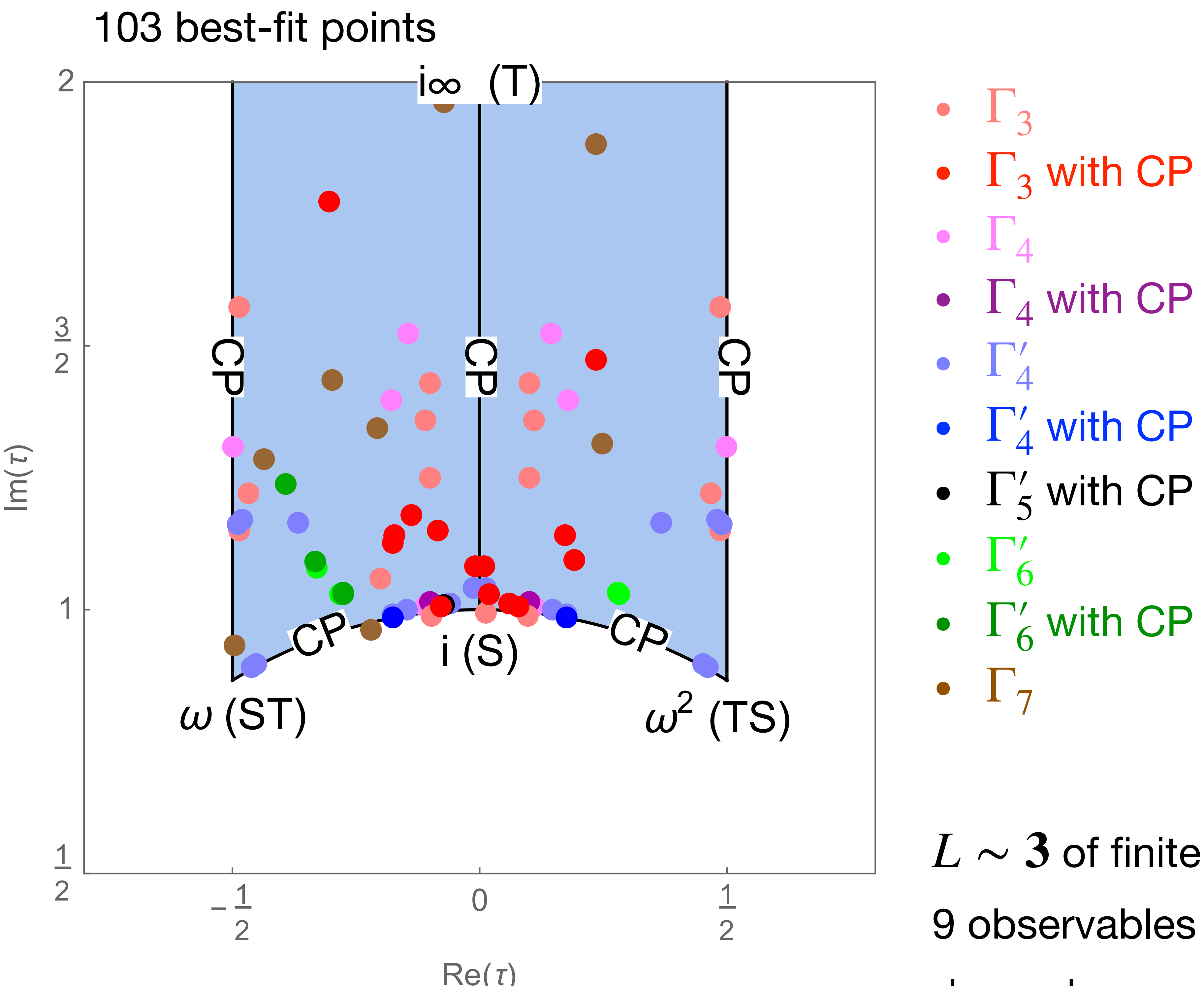
Chen, Ramos-Sánchez, Ratz 1909.06910, + et al. 2108.02240

Feruglio, Gherardi, Romanino, **AT**, 2101.08718

Eclectic flavour symmetries: Nilles, Ramos-Sánchez, Vaudrevange, 2001.01736, 2004.05200

Selection of models

Feruglio, 2211.00659



$L \sim \mathbf{3}$ of finite modular group $\Gamma_N^{(\prime)}$

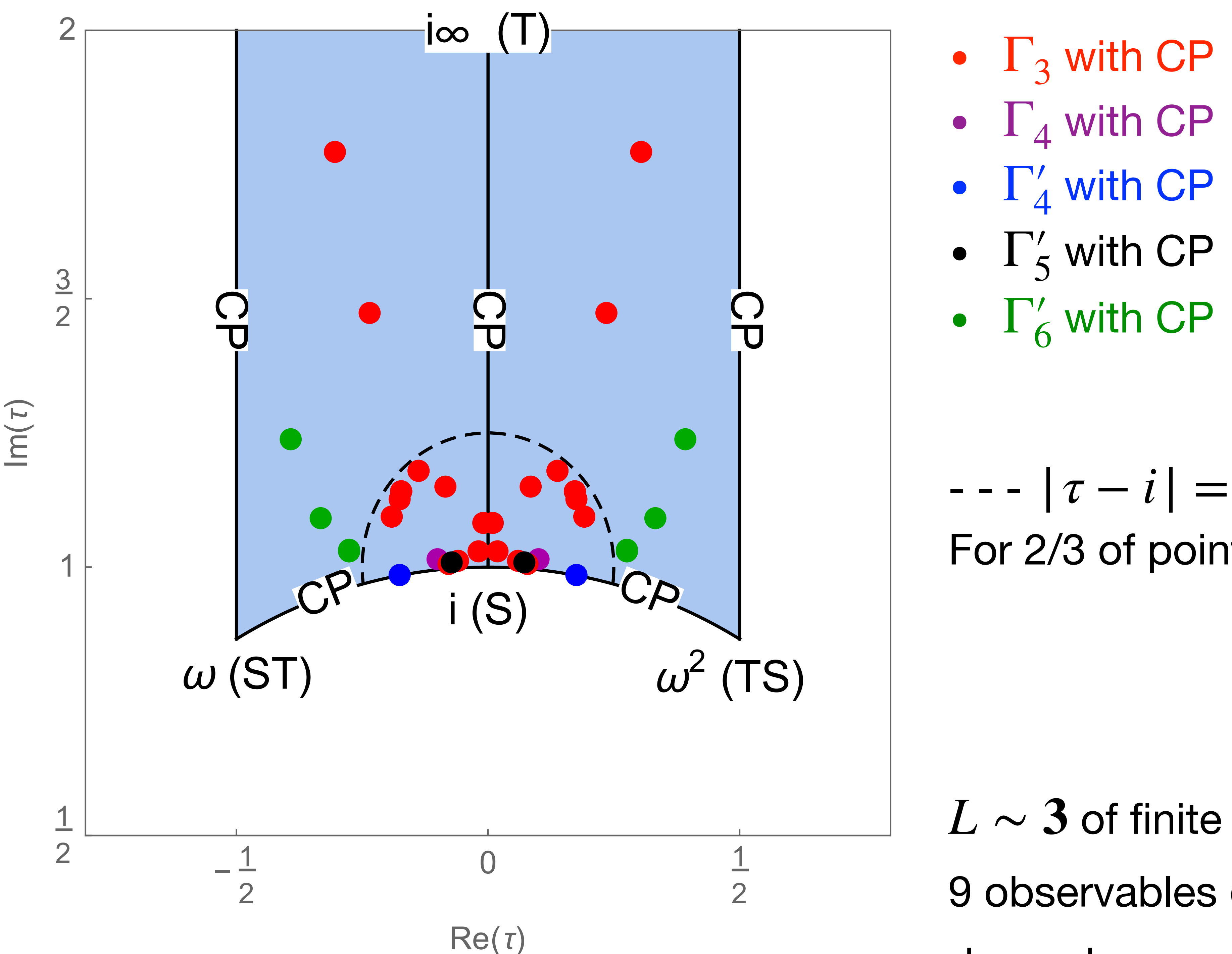
9 observables $(m_i\,,\ \theta_{ij}\,,\ \delta\,,\ \alpha_{ij})$

depend on τ and 2 or 3 additional Lagrangian parameters

Selection of models with CP

Feruglio, 2211.00659

27 pairs of best-fit points



- - - $|\tau - i| = 0.25$

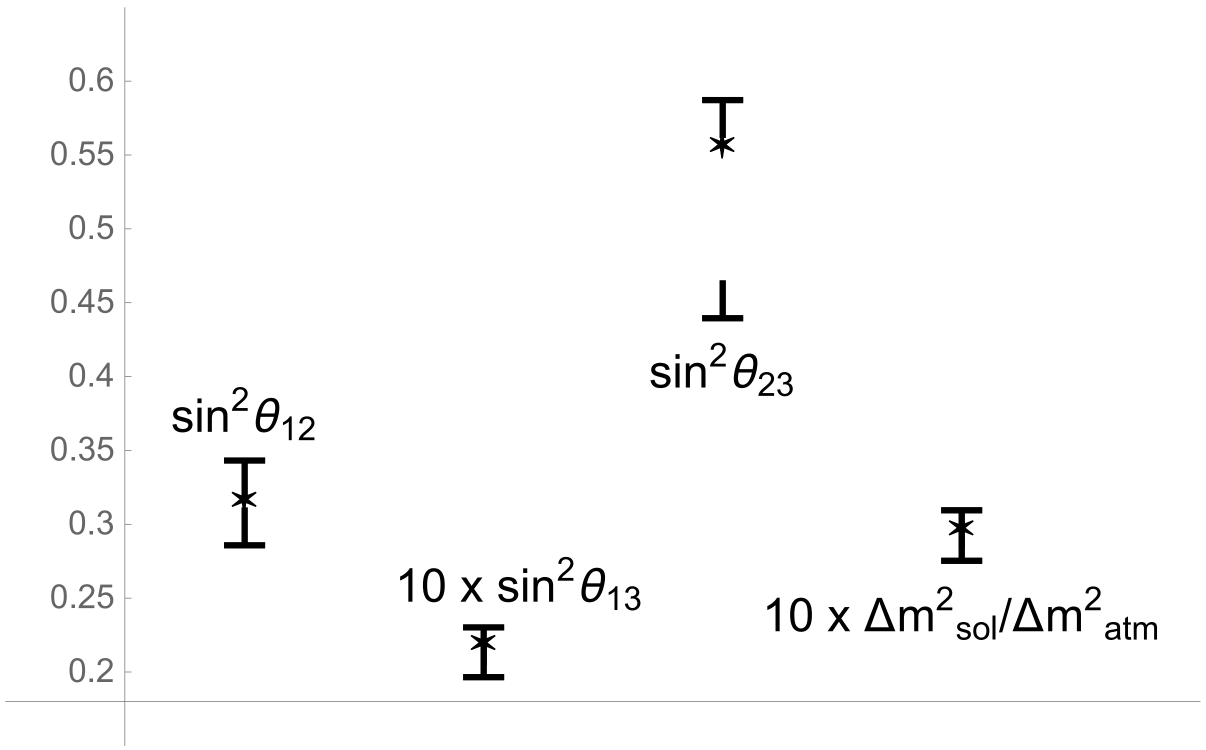
For 2/3 of points $|\tau - i| < 0.25$

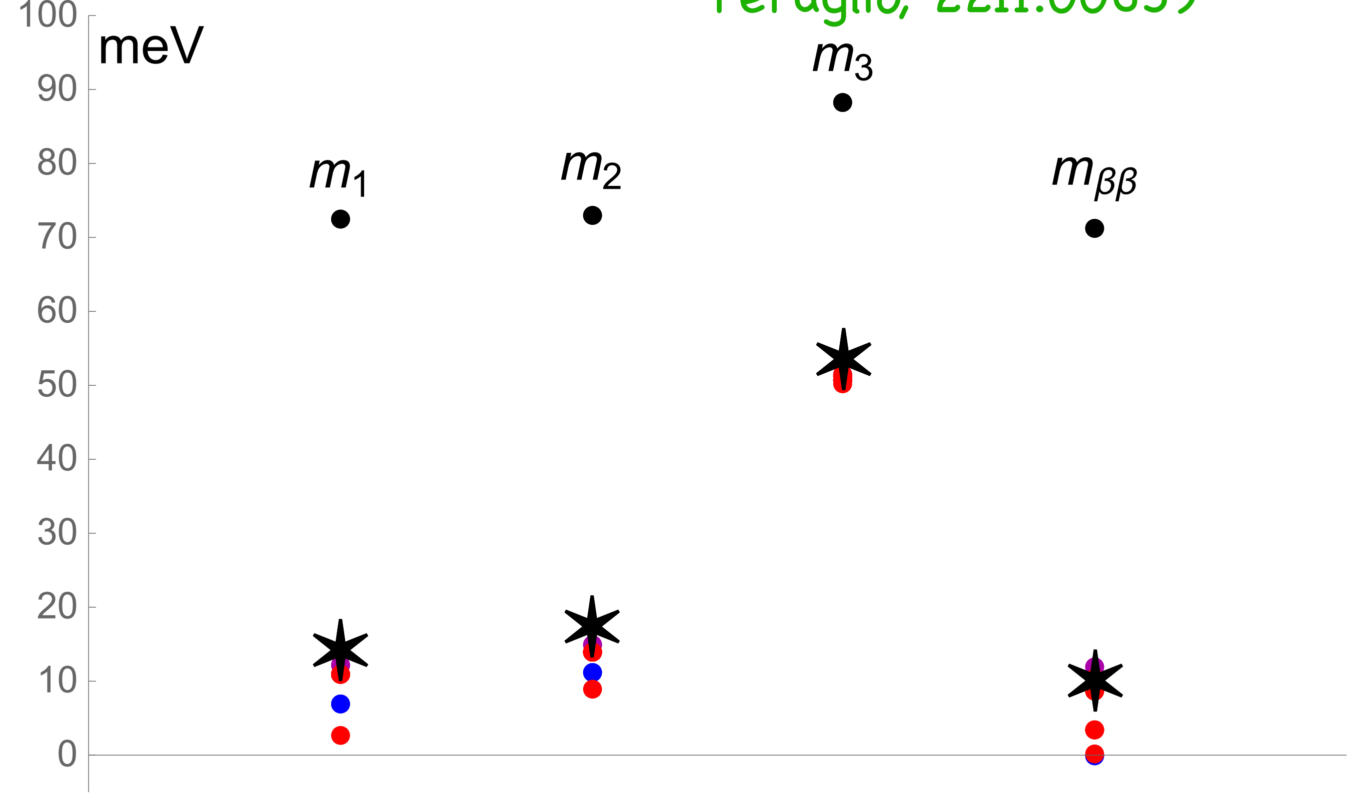
$L \sim \mathbf{3}$ of finite modular group $\Gamma_N^{(\prime)}$

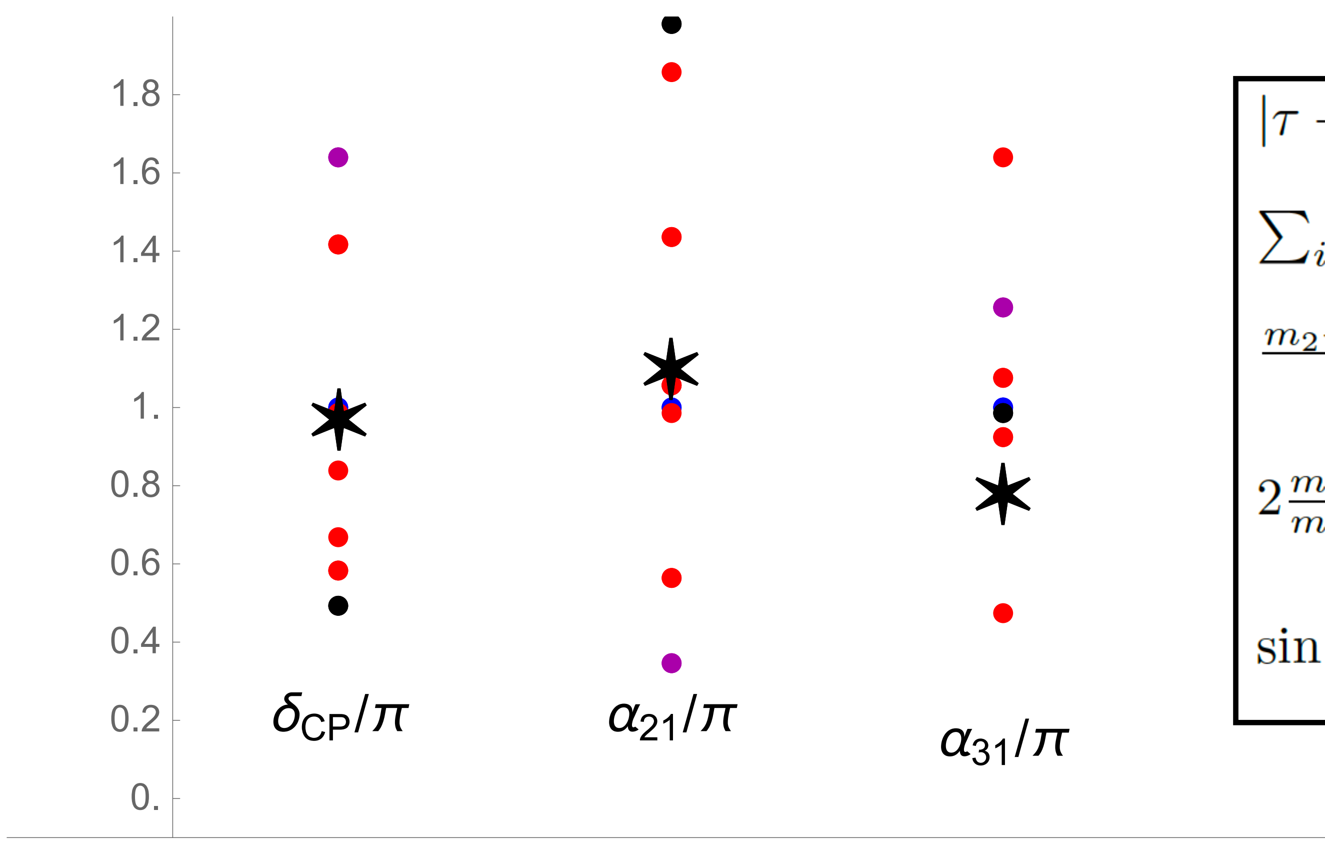
9 observables (m_i, θ_{ij}, δ, α_{ij}) depend on τ and 2 or 3 additional Lagrangian parameters

Models with $|\tau - i| < 0.25$ leading to NO

Feruglio, 2211.00659







$|\tau - i| = 0.20 \pm 0.04$ $\quad |u| = \left|\frac{\tau - i}{\tau + i}\right| = 0.095 \pm 0.015$

$\sum_i m_i = 73.9 \pm 4.6\ \text{meV}$ $\quad m_{\beta\beta} = 5.5 \pm 4.2\ \text{meV}$

$\frac{m_2 + m_1}{2} = 11.5 \pm 2.2\ \text{meV}$ $\quad \frac{m_2 + m_1}{2 m_3} = 0.23 \pm 0.04$

$2\frac{m_2 - m_1}{m_2 + m_1} = 0.34 \pm 0.25$ $\quad \left(\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}\right)^{1/3} = 0.310 \pm 0.001$

$\sin\theta_{13} = 0.148 \pm 0.001$ $\quad \left|\frac{1}{\sqrt{2}} - \sin\theta_{12}\right| = 0.14 \pm 0.01$

Linearly realised symmetries

Change of variables:

Feruglio, Gherardi, Romanino, **AT**, 2101.08718
Novichkov, Penedo, Petcov, 2102.07488

$$u = \frac{\tau - i}{\tau + i} \qquad \Phi = (1-u)^{k_\varphi}\varphi$$

$$u \xrightarrow{S} -u \qquad \Phi \xrightarrow{S} \Omega_\varphi(S)\,\Phi \qquad \Omega_\varphi(S) = i^{k_\varphi}\rho_\varphi(S)$$

$$u \xrightarrow{CP} \overline{u} \qquad \Phi \xrightarrow{CP} \overline{\Phi}$$

After canonical normalisation of the kinetic terms:

Feruglio, 2211.00659

$$\Omega(S)^\dagger\, m_e^2(-u,-\overline{u})\,\Omega(S) = m_e^2(u,\overline{u}) \qquad \Omega(S)^T\, m_\nu(-u,-\overline{u})\,\Omega(S) = m_\nu(u,\overline{u})$$

$$[m_e^2(\overline{u},u)]^* = m_e^2(u,\overline{u}) \qquad m_\nu(\overline{u},u)^* = m_\nu(u,\overline{u})$$

$$\Omega(S) \equiv \Omega_{H_u}(S)\,\Omega_L(S) = i^{k_{H_u}+k_L}\rho_{H_u}(S)\,\rho_L(S)$$

Under the assumption that ρ_L is irreducible, $\Omega(S)$ is fixed (up to an overall phase)

$$\Omega(S) = i^{k_S}\,\mathrm{diag}\,(1,-1,-1) \qquad \forall\ N$$

Universal scaling

The case of k_S odd and $m_\nu(0,0)$ singular (originating from seesaw)

$$m_\nu^{-1} = m_{0\nu}^{-1} \begin{pmatrix} x_{11}\, x & x_{12}^0 & x_{13}^0 \\ \cdot & x_{22}\, x & x_{23}\, x \\ \cdot & \cdot & x_{33}\, x \end{pmatrix} + \mathcal{O}(x^2)$$

$$u = xe^{i\theta}$$

$$x \approx 0.1$$

$$m_{0\nu}/h = 11.5 \text{ meV}$$

$$m_{1,2} = \frac{m_{0\nu}}{h}\left(1 \mp \frac{s}{2h}x\right) \qquad m_3 = \frac{m_{0\nu}}{|q|x}$$

$$\sin^2\theta_{12} = \frac{1}{2}\left(1 + \frac{l\bar{k} + \bar{l}k}{hs}x\right) \qquad \sin^2\theta_{13} = 2\frac{|n|^2}{h^2}x^2$$

$$\sin^2\theta_{23} = \frac{(x_{13}^0)^2}{h^2}(1 + \mathcal{O}(x))$$

$$\delta_{CP} = \pi - \arg\left[\frac{(c_\nu - is_\nu)^2 x_{12}^0 x_{13}^0}{n}\right] + \mathcal{O}(x^2)$$

$$\alpha_{21} = \pi + \mathcal{O}(x)$$

$$\alpha_{31} = \arg(q) - \arg\left[(c_\nu - is_\nu)^2\right] + \mathcal{O}(x) \ .$$

$$\frac{|q|}{h} \approx 2.3 \qquad \frac{s}{h} \approx 3.4 \qquad \frac{\sqrt{2}|n|}{h} \approx 1.5$$

$$\frac{|l\bar{k} + \bar{l}k|}{2\sqrt{2}hs} \approx 1.4 \qquad \frac{|x_{23}^2 - x_{22}x_{33}|}{2h|q|} \approx 4.8$$

$\mathcal{O}(1)$ constants
to get the averages

Dimension of linear space of modular forms

N	g	$d_{2k}(\Gamma(N))$	μ_N	Γ_N
2	0	$k+1$	6	S_3
3	0	$2k+1$	12	A_4
4	0	$4k+1$	24	S_4
5	0	$10k+1$	60	A_5
6	1	$12k$	72	
7	3	$28k-2$	168	

$k(\text{this presentation}) \equiv 2\,k(\text{this table})$

F. Feruglio, in book "From My Vast Repertoire ...: Guido Altarelli's Legacy", 1706.08749

Modular forms of weight 2 and level 2

Level *N* = 2 $\left(\Gamma_2 \simeq S_3 : \quad S^2 = (ST)^3 = T^2 = I\right)$

N\k	0	2	4	6
2	1	**2**	3	4

S

(up to multiplicative factors)

Seed functions: $\eta\left(\frac{\tau}{2}\right), \quad \eta\left(\frac{\tau+1}{2}\right), \quad \eta\left(2\tau\right)$

T

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right) , \quad q = e^{2\pi i \tau},$$ is the Dedekind eta function

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau) \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, PRD **98** (2018) 016004

Modular forms of weight 2 and level 2

Level *N* = 2 $\left(\Gamma_2 \simeq S_3 : \quad S^2 = (ST)^3 = T^2 = I\right)$

N\k	0	2	4	6
2	1	**2**	3	4

$$Y(a_1, a_2, a_3 \,|\, \tau) \equiv \sum_{i=1}^{3} a_i \frac{\mathrm{d}}{\mathrm{d}\tau} \log \eta_i(\tau) \,, \qquad \sum_{i=1}^{3} a_i = 0$$

$$Y_{\mathbf{2}}(-1/\tau) = \tau^2 \rho(S)\ Y_{\mathbf{2}}(\tau) \qquad Y_{\mathbf{2}}(\tau+1) = \rho(T)\ Y_{\mathbf{2}}(\tau)$$

$$\rho(S) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y_{\mathbf{2}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix} \equiv c \begin{pmatrix} Y(1,1,-2 \,|\, \tau) \\ Y(\sqrt{3}, -\sqrt{3}, 0 \,|\, \tau) \end{pmatrix}$$

S_3 doublet of weight 2 modular forms

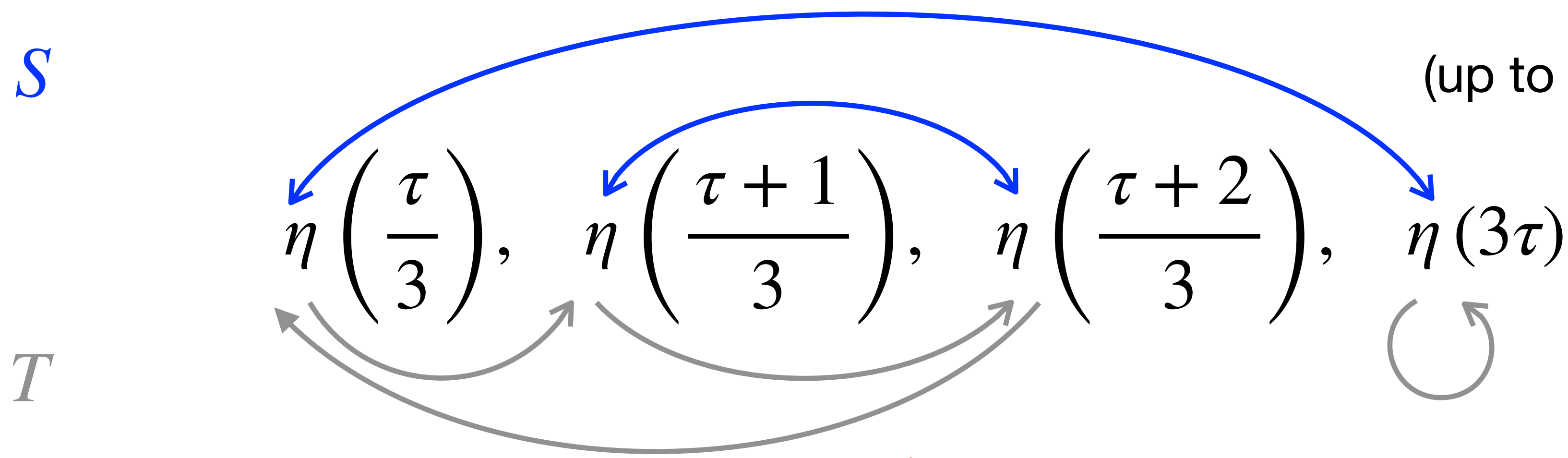
T. Kobayashi, K. Tanaka, T.H. Tatsuishi, PRD **98** (2018) 016004

Modular forms of weight 2 and levels 3 and 4

Level *N* = 3 $\left(\Gamma_3 \simeq A_4 : \quad S^2 = (ST)^3 = T^3 = I\right)$

N\k	0	2	4	6
3	1	**3**	5	7

(up to multiplicative factors)

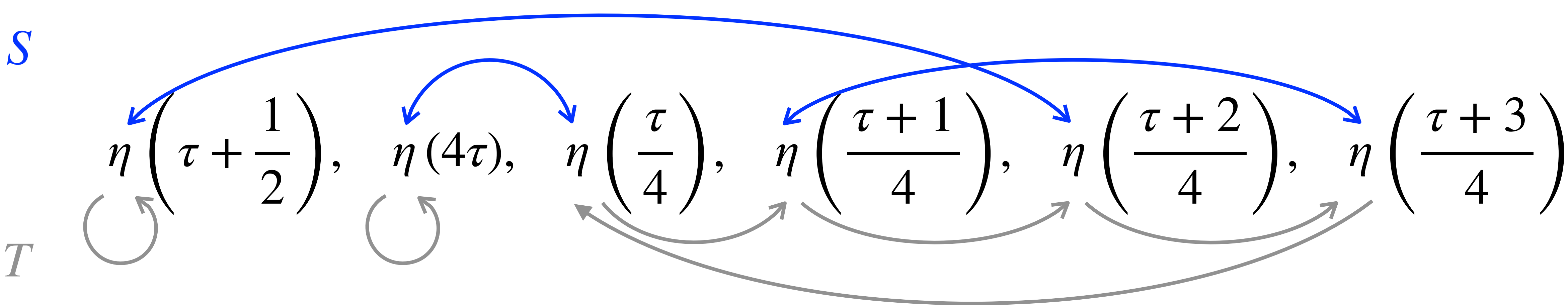


A_4 triplet of weight 2 modular forms

F. Feruglio, in book "From My Vast Repertoire ...: Guido Altarelli's Legacy", 1706.08749

Level *N* = 4 $\left(\Gamma_4 \simeq S_4 : \quad S^2 = (ST)^3 = T^4 = I\right)$

N\k	0	2	4	6
4	1	**5**	9	13



S_4 doublet and triplet ($\mathbf{3}'$) of weight 2 modular forms

J.T. Penedo, S.T. Petcov, NPB **939** (2019) 292

Modular forms of weight 2 and level 4

Building lowest-weight forms

$$Y(a_1,\ldots,a_6|\tau) \equiv \frac{d}{d\tau}\left(\sum_{i=1}^{6} a_i \log \eta_i(\tau)\right)$$

$$\sum_i a_i = 0$$

$$S:\; Y(a_1,\ldots,a_6|\tau) \to Y(a_1,a_2,a_3,a_4,a_5,a_6|-1/\tau) = \tau^2\, Y(a_5,a_3,a_2,a_6,a_1,a_4|\tau)$$

$$T:\; Y(a_1,\ldots,a_6|\tau) \to Y(a_1,a_2,a_3,a_4,a_5,a_6|\tau+1) = Y(a_1,a_2,a_6,a_3,a_4,a_5|\tau)$$

$Y(\tau) \to (c\tau+d)^{2k_Y}\,\rho_Y(\gamma)\,Y(\tau)$ $\Longrightarrow$ **Modular forms of weight 2**

From J.T. Penedo's talk at DISCRETE 2018

Modular forms of weight 2 and level 4

$$Y(a_1,\ldots,a_6|\tau) \equiv \frac{d}{d\tau}\left(\sum_{i=1}^{6} a_i \log \eta_i(\tau)\right)$$

$$\boxed{\sum_i a_i = 0}$$

Lowest weight forms arrange into:

$$Y_{\mathbf{2}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}$$ **doublet 2**

$$Y_{\mathbf{3'}}(\tau) = \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}$$ **triplet 3'**

$$Y_1(\tau) \equiv Y(1,1,\omega,\omega^2,\omega,\omega^2|\tau)$$
$$Y_2(\tau) \equiv Y(1,1,\omega^2,\omega,\omega^2,\omega|\tau)$$
$$Y_3(\tau) \equiv Y(1,-1,-1,-1,1,1|\tau)$$
$$Y_4(\tau) \equiv Y(1,-1,-\omega^2,-\omega,\omega^2,\omega|\tau)$$
$$Y_5(\tau) \equiv Y(1,-1,-\omega,-\omega^2,\omega,\omega^2|\tau)$$

Correct dimension (5)

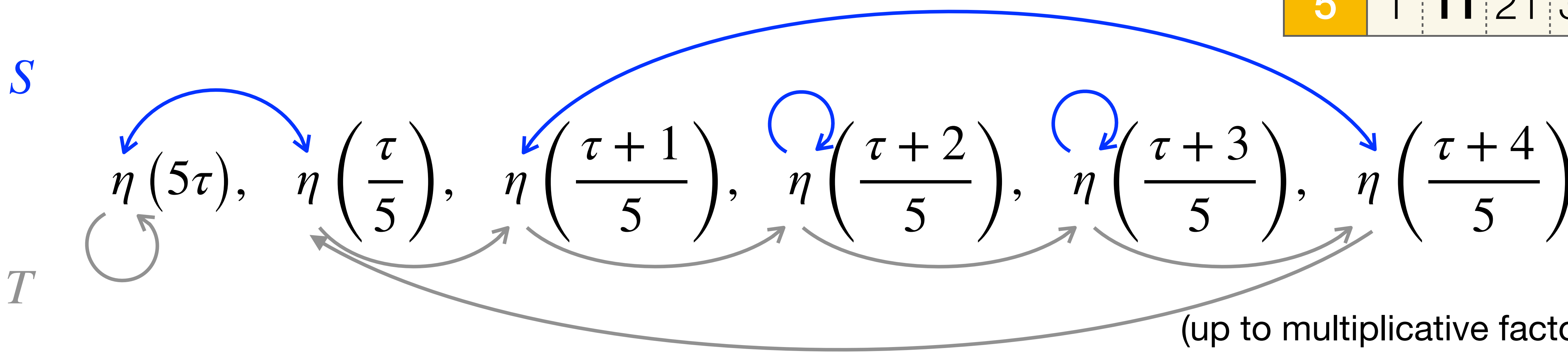
Products generate higher weight forms

From J.T. Penedo's talk at DISCRETE 2018

Modular forms of weight 2 and level 5

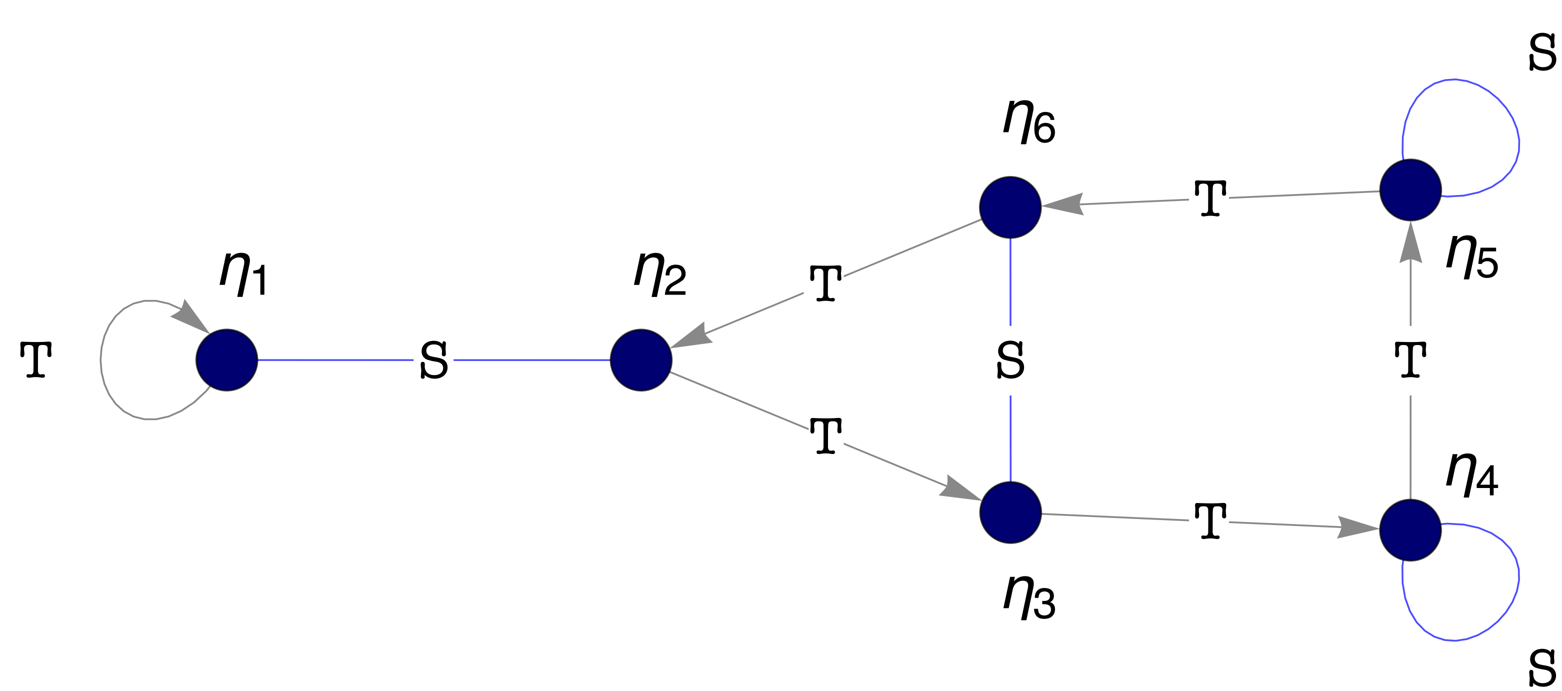
Level *N* = 5 $\left(\Gamma_5 \simeq A_5 : \quad S^2 = (ST)^3 = T^5 = I\right)$

N\k	0	2	4	6
5	1	**11**	21	31



$$\eta\left(5\tau\right), \quad \eta\left(\frac{\tau}{5}\right), \quad \eta\left(\frac{\tau+1}{5}\right), \quad \eta\left(\frac{\tau+2}{5}\right), \quad \eta\left(\frac{\tau+3}{5}\right), \quad \eta\left(\frac{\tau+4}{5}\right)$$

(up to multiplicative factors)



P.P. Novichkov, J.T. Penedo, S.T. Petcov, AT, JHEP **04** (2019) 174

Modular forms of weight 2 and level 5

$$X(a_1, \ldots, a_6 \,|\, \tau) \equiv \sum_{i=1}^{6} a_i \frac{\mathrm{d}}{\mathrm{d}\tau} \log \eta_i(\tau)\,, \qquad \sum_{i=1}^{6} a_i = 0$$

$$X_{\mathbf{5}}(\tau) = \begin{pmatrix} X_1(\tau) \\ X_2(\tau) \\ X_3(\tau) \\ X_4(\tau) \\ X_5(\tau) \end{pmatrix} \equiv c \begin{pmatrix} -\frac{1}{\sqrt{6}} X\left(-5,1,1,1,1,1 \,|\, \tau\right) \\ X(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta \,|\, \tau) \\ X(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2 \,|\, \tau) \\ X(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3 \,|\, \tau) \\ X(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4 \,|\, \tau) \end{pmatrix}, \qquad \zeta = e^{2\pi i/5}$$

A_5 quintet of weight 2 modular forms

$$11 = \mathbf{5} + \mathbf{3} + \mathbf{3}'$$

C. Franc, G. Mason, Ramanujan J. **41** (2016) 233

How to construct the triplets?

P.P. Novichkov, J.T. Penedo, S.T. Petcov, AT, JHEP **04** (2019) 174

Jacobi theta function

Jacobi theta function:

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n z\right)$$

Useful properties:

S. Kharchev, A. Zabrodin, J. Geom. Phys. **94** (2015) 19

$$\theta_3(z+1,\tau) = \theta_3(z,\tau)\,, \quad \theta_3(z+\tau,\tau) = e^{-\pi i(2z+\tau)}\theta_3(z,\tau)$$

$$\theta_3(z+1/2,\tau) = \theta_4(z,\tau)\,, \quad \theta_3(z+\tau/2,\tau) = e^{-\pi i(z+\tau/4)}\theta_2(z,\tau)$$

$$\theta_3(z,\tau+1) = \theta_4(z,\tau)\,, \quad \theta_3(z/\tau,-1/\tau) = \sqrt{-i\tau}\,e^{\pi i z^2/\tau}\theta_3(z,\tau)$$

θ_1, θ_2 and θ_4 are auxiliary theta functions

Alternative construction invoking Klein forms has been worked out in

G.-J. Ding, S.F. King, X.-G. Liu, PRD **100** (2019) 115005

P.P. Novichkov, J.T. Penedo, S.T. Petcov, AT, JHEP **04** (2019) 174

12 seed functions

$$\alpha_{1,-1}(\tau) \equiv \theta_3\left(\frac{\tau+1}{2}, 5\tau\right) \qquad \alpha_{2,-1}(\tau) \equiv e^{2\pi i\tau/5}\,\theta_3\left(\frac{3\tau+1}{2}, 5\tau\right)$$

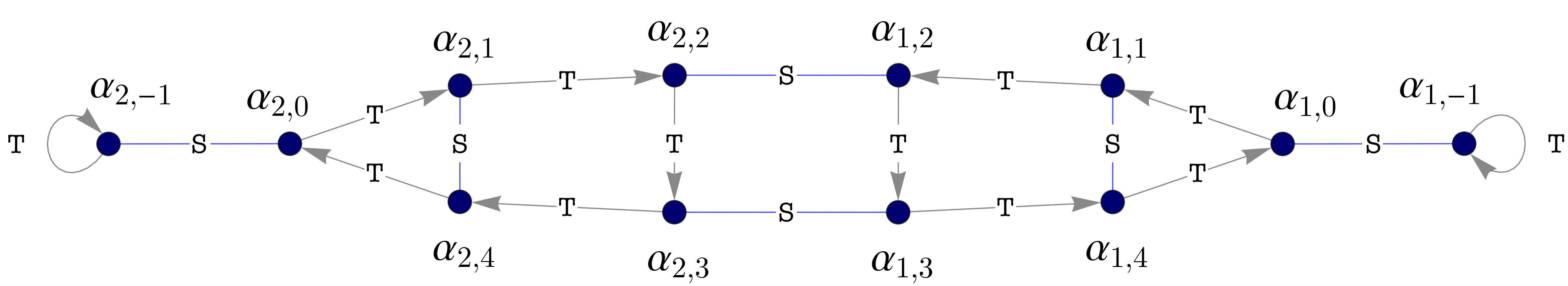
$$\alpha_{1,0}(\tau) \equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau}{5}\right) \qquad \alpha_{2,0}(\tau) \equiv \theta_3\left(\frac{\tau+7}{10}, \frac{\tau}{5}\right)$$

$$\alpha_{1,1}(\tau) \equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+1}{5}\right) \qquad \alpha_{2,1}(\tau) \equiv \theta_3\left(\frac{\tau+8}{10}, \frac{\tau+1}{5}\right)$$

$$\alpha_{1,2}(\tau) \equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+2}{5}\right) \qquad \alpha_{2,2}(\tau) \equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau+2}{5}\right)$$

$$\alpha_{1,3}(\tau) \equiv \theta_3\left(\frac{\tau+2}{10}, \frac{\tau+3}{5}\right) \qquad \alpha_{2,3}(\tau) \equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+3}{5}\right)$$

$$\alpha_{1,4}(\tau) \equiv \theta_3\left(\frac{\tau+3}{10}, \frac{\tau+4}{5}\right) \qquad \alpha_{2,4}(\tau) \equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+4}{5}\right)$$



P.P. Novichkov, J.T. Penedo, S.T. Petcov, AT, JHEP **04** (2019) 174

Modular forms of weight 2 and level 5

$$Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}\,|\,\tau) \equiv \sum_{i,j} c_{i,j}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(\tau)\,, \qquad \sum_{i,j} c_{i,j} = 0$$

$$Y_{\mathbf{5}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{\sqrt{6}}Y\left(-5,1,1,1,1,1;-5,1,1,1,1,1\,|\,\tau\right) \\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,|\,\tau) \\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,|\,\tau) \\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,|\,\tau) \\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,|\,\tau) \end{pmatrix}$$

$$Y_{\mathbf{3}}(\tau) = \begin{pmatrix} Y_6(\tau) \\ Y_7(\tau) \\ Y_8(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}Y\left(-\sqrt{5},-1,-1,-1,-1,-1;\sqrt{5},1,1,1,1,1\,|\,\tau\right) \\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,-1,-\zeta^4,-\zeta^3,-\zeta^2,-\zeta\,|\,\tau) \\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,-1,-\zeta,-\zeta^2,-\zeta^3,-\zeta^4\,|\,\tau) \end{pmatrix}$$

$$Y_{\mathbf{3'}}(\tau) = \begin{pmatrix} Y_9(\tau) \\ Y_{10}(\tau) \\ Y_{11}(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}Y\left(\sqrt{5},-1,-1,-1,-1,-1;-\sqrt{5},1,1,1,1,1\,|\,\tau\right) \\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,-1,-\zeta^3,-\zeta,-\zeta^4,-\zeta^2\,|\,\tau) \\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,-1,-\zeta^2,-\zeta^4,-\zeta,-\zeta^3\,|\,\tau) \end{pmatrix}$$

A_5 quintet and triplets of weight 2 modular forms

P.P. Novichkov, J.T. Penedo, S.T. Petcov, AT, JHEP **04** (2019) 174

Modular forms of higher weight at level 5

Weight 4: $Y_i Y_j$ 66 combinations - 45 constraints
= 21 independent combinations

N\k	0	2	4	6
5	1	11	**21**	31

$$Y^{(4)}_{\mathbf{1}} = Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5$$

$$Y^{(4)}_{\mathbf{3}} = \begin{pmatrix} -2Y_1Y_6 + \sqrt{3}\,Y_5Y_7 + \sqrt{3}\,Y_2Y_8 \\ \sqrt{3}\,Y_2Y_6 + Y_1Y_7 - \sqrt{6}\,Y_3Y_8 \\ \sqrt{3}\,Y_5Y_6 - \sqrt{6}\,Y_4Y_7 + Y_1Y_8 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{3'}} = \begin{pmatrix} \sqrt{3}\,Y_1Y_6 + Y_5Y_7 + Y_2Y_8 \\ Y_3Y_6 - \sqrt{2}\,Y_2Y_7 - \sqrt{2}\,Y_4Y_8 \\ Y_4Y_6 - \sqrt{2}\,Y_3Y_7 - \sqrt{2}\,Y_5Y_8 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{4}} = \begin{pmatrix} 2Y_4^2 + \sqrt{6}\,Y_1Y_2 - Y_3Y_5 \\ 2Y_2^2 + \sqrt{6}\,Y_1Y_3 - Y_4Y_5 \\ 2Y_5^2 - Y_2Y_3 + \sqrt{6}\,Y_1Y_4 \\ 2Y_3^2 - Y_2Y_4 + \sqrt{6}\,Y_1Y_5 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{5},1} = \begin{pmatrix} \sqrt{2}\,Y_1^2 + \sqrt{2}\,Y_3Y_4 - 2\sqrt{2}\,Y_2Y_5 \\ \sqrt{3}\,Y_4^2 - 2\sqrt{2}\,Y_1Y_2 \\ \sqrt{2}\,Y_1Y_3 + 2\sqrt{3}\,Y_4Y_5 \\ 2\sqrt{3}\,Y_2Y_3 + \sqrt{2}\,Y_1Y_4 \\ \sqrt{3}\,Y_3^2 - 2\sqrt{2}\,Y_1Y_5 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{5},2} = \begin{pmatrix} \sqrt{3}\,Y_5Y_7 - \sqrt{3}\,Y_2Y_8 \\ -Y_2Y_6 - \sqrt{3}\,Y_1Y_7 - \sqrt{2}\,Y_3Y_8 \\ -2Y_3Y_6 - \sqrt{2}\,Y_2Y_7 \\ 2Y_4Y_6 + \sqrt{2}\,Y_5Y_8 \\ Y_5Y_6 + \sqrt{2}\,Y_4Y_7 + \sqrt{3}\,Y_1Y_8 \end{pmatrix}$$

P.P. Novichkov, J.T. Penedo, S.T. Petcov, AT, JHEP **04** (2019) 174