Glauber singularities and factorization

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We are interested in PDF factorization in infrared safe observables in hadron-hadron collisions.

Our example is dimuon production, as in

J.C.Collins, D.E.Soper and G.F.Sterman,
"Soft Gluons and Factorization,"
Nucl. Phys. B 308, 833 (1988)

This paper works at all orders, but it may be helpful to look at just a first order example, emphasizing the Glauber singularities.

See also

J.C.Collins, D.E.Soper and G.F.Sterman,
"Factorization for Short Distance Hadron - Hadron Scattering,"
Nucl. Phys. B 261, 104 (1985)

Some notation

$$v = (v^+, v^-, v^1, v^2) = (v^+, v^-, \boldsymbol{v})$$

$$v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3)$$

$$p \cdot x = p^+ x^- + p^- x^+ - \boldsymbol{p} \cdot \boldsymbol{x}$$

- Translations in x^- generated by p^+.
- Translations in x^+ generated by p^-.

$$p^2 = 2p^+ p^- - \boldsymbol{p}^2$$

- $p^2 = 0$ implies

$$p^+ = \frac{\boldsymbol{p}^2}{2p^-} \qquad p^- = \frac{\boldsymbol{p}^2}{2p^+}$$

Definition of PDFs

- For quarks,

$$f_{i/h}(\xi, \mu_F) = \frac{1}{2} \int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-} \langle p | \bar{\psi}_i(0, y^-, \mathbf{0}) \gamma^+ F \psi_i(0) | p \rangle$$

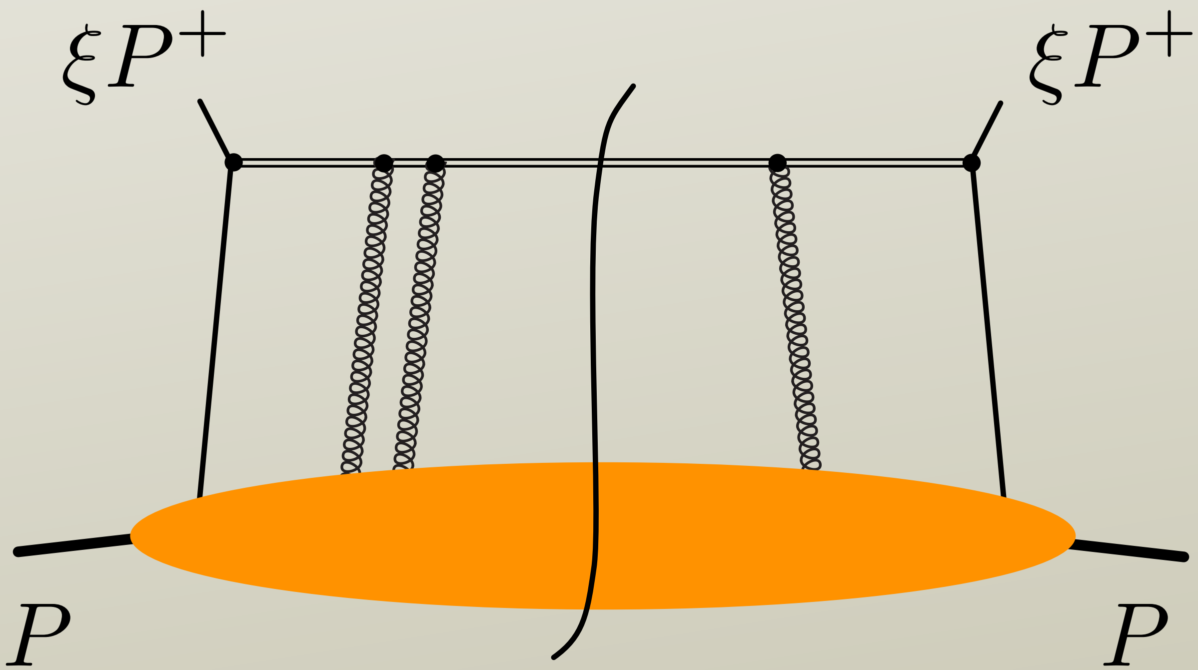
$$F = \mathcal{P} \exp\left(ig \int_0^{y^-} dz^- A_a^+(0, z^-, \mathbf{0})\, t_a \right)$$

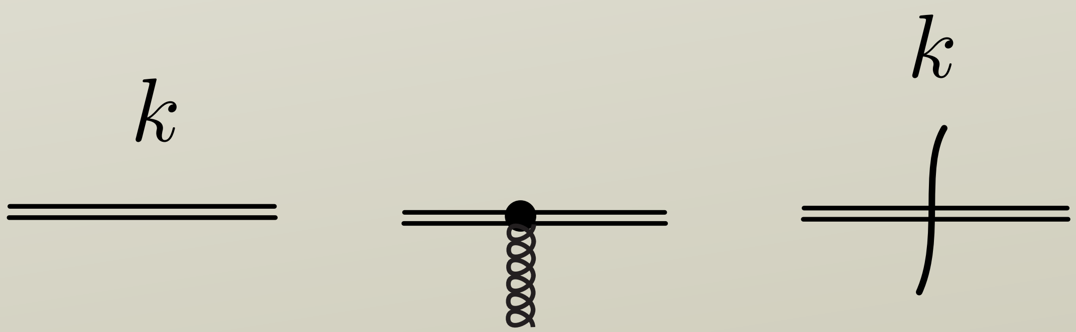
- For gluons, a similar definition.
- Renormalize with the $\overline{\mathrm{MS}}$ prescription with scale μ_F.
- Note that y^- is small, $y^- \sim 1/p^+$.

- Another way to write this:

$$f_{i/h}(\xi,\mu_F) = \frac{1}{2}\int \frac{dy^-}{2\pi}\, e^{i\xi p^+ y^-}\, \langle p|\bar{\psi}_i(0,y^-,\mathbf{0})\gamma^+ F \psi_i(0)|p\rangle$$

$$F = \bar{\mathcal{P}}\exp\left(-ig\int_{y^-}^{\infty} dz^-\, A_a^+(0,z^-,\mathbf{0})\, t_a\right)$$
$$\times\mathcal{P}\exp\left(ig\int_{0}^{\infty} dz^-\, A_a^+(0,z^-,\mathbf{0})\, t_a\right)$$





$$\frac{\mathrm{i}}{k\cdot u + \mathrm{i}\epsilon} \qquad \mathrm{i}gu^\alpha t^a_{ij} \qquad 2\pi\delta(k\cdot u)$$

Factorization

• Consider the cross section to produce a muon pair with virtuality Q^2 and rapidity y plus anything. Then factorization claims

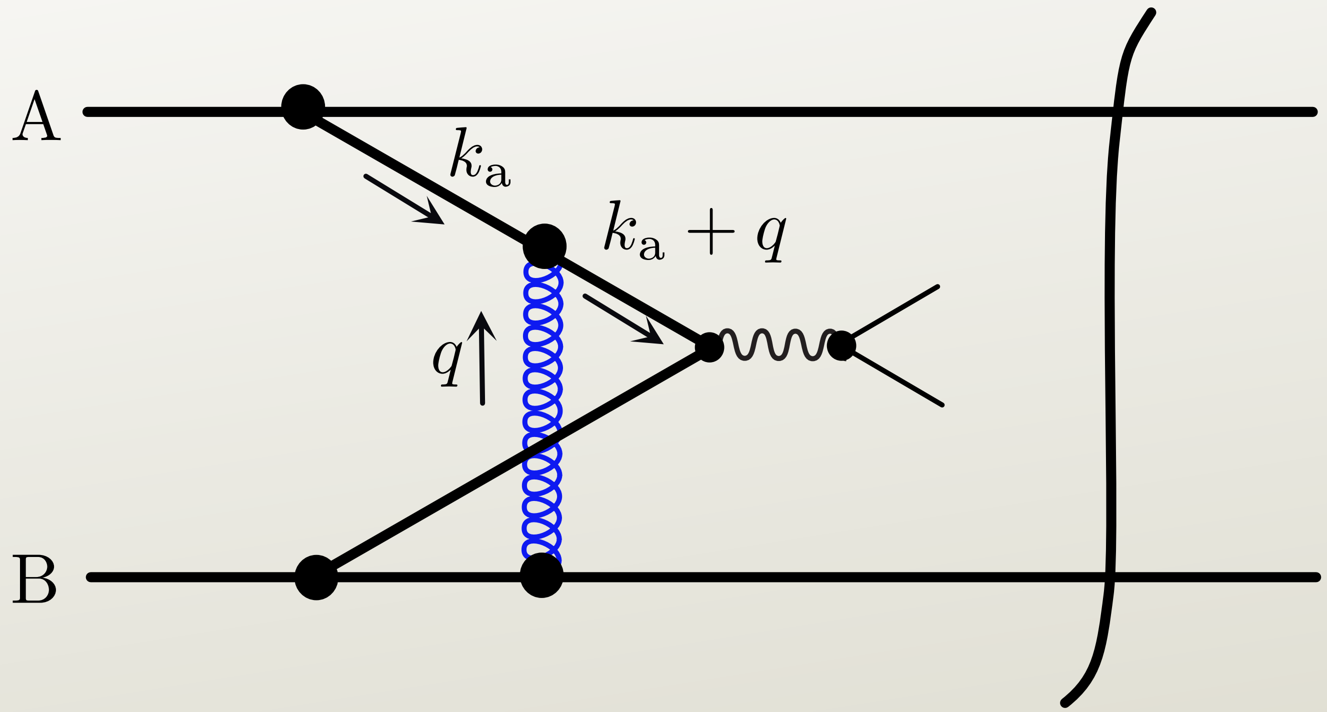
$$\frac{d\sigma}{dQ^2dy} = \sum_{a,b}\int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B\; f_{a/A}(\xi_A,\mu)\; f_{b/B}(\xi_B,\mu)\; \frac{d\hat{\sigma}_{ab}(\mu)}{dQ^2dy} + \mathcal{O}(m/\sqrt{Q^2})$$

$$y = \frac{1}{2}\log\left(\frac{q^+}{q^-}\right) \quad x_A = e^y\sqrt{Q^2/s} \quad x_B = e^{-y}\sqrt{Q^2/s}$$

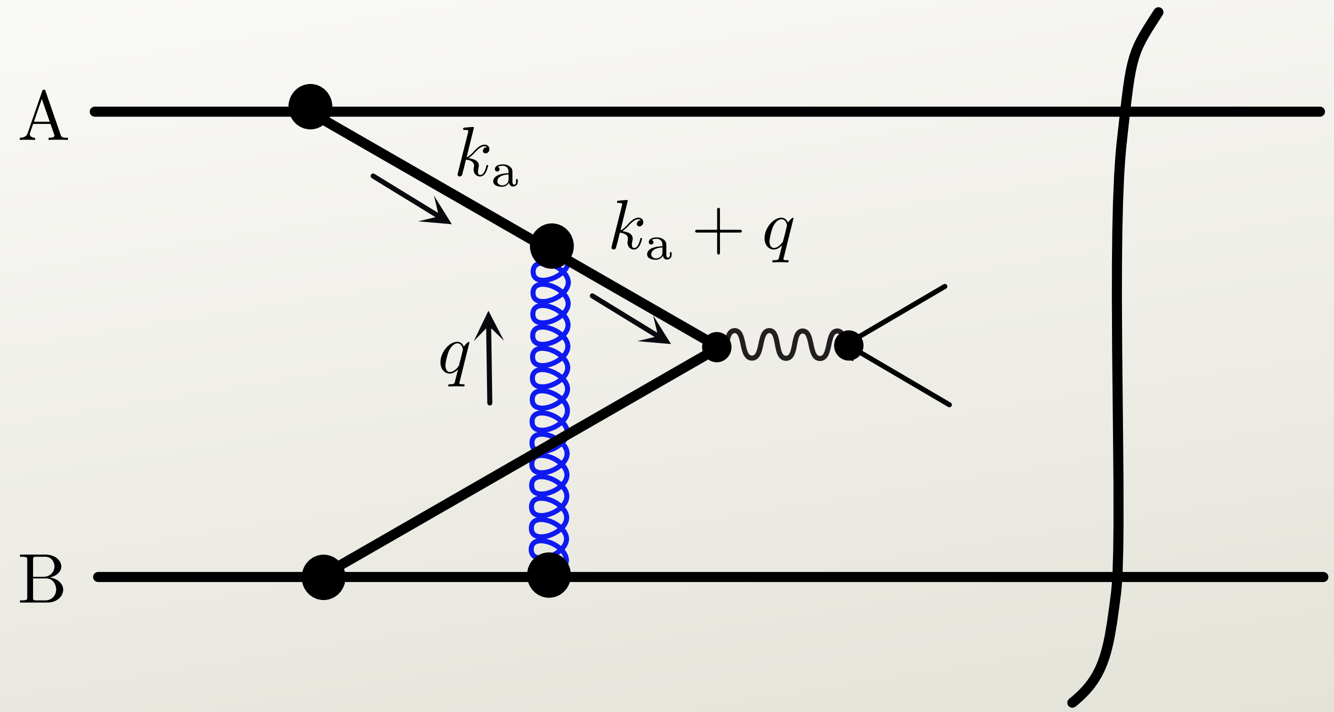
- This is the inclusive cross section. Unmeasured hadrons in the final state are allowed.

• There are power suppressed corrections.

A simple contribution



- Hadron A moves in the + direction.
- Hadron B moves in the − direction.
- I show some of the parton lines.
- They can be quarks, antiquarks, or gluons.
- Choose integration region: large q^-.



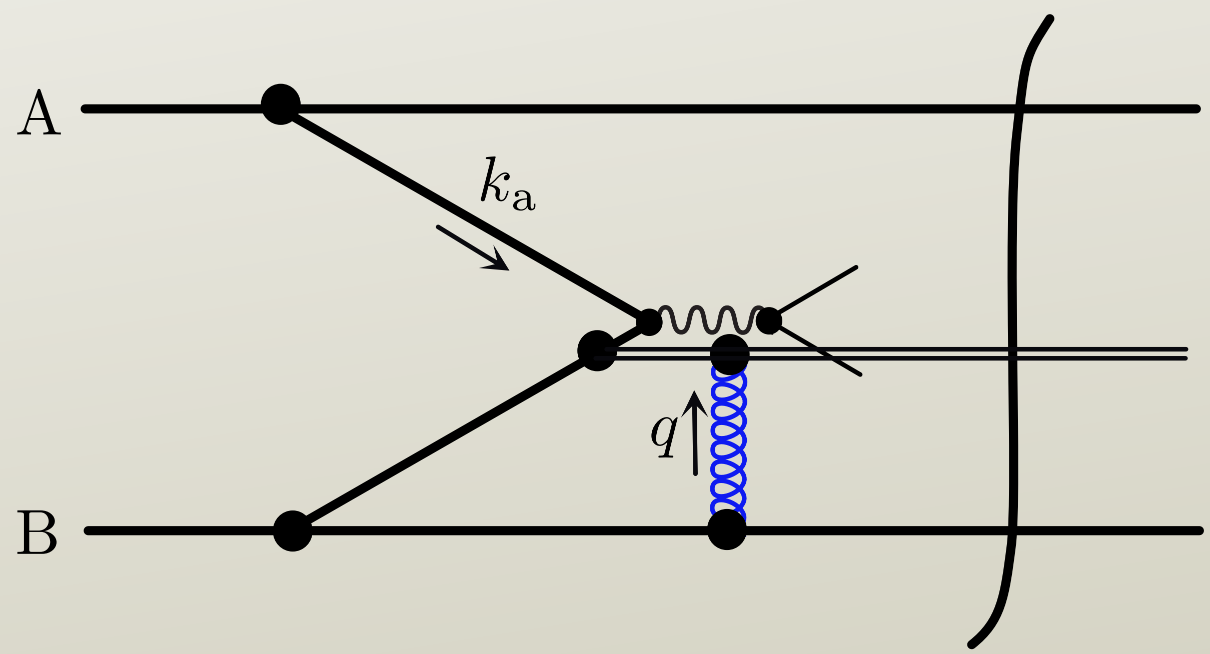
- A gluon with momentum q is exchanged from a spectator line in B to the active line in A.
- Let q have large q^-, small q^+ and $\boldsymbol{q}$.
- We get

$$\frac{\not{k}_{\mathrm{a}} + \not{q}}{(k_{\mathrm{a}} + q)^2 + \mathrm{i}\epsilon} \not{\varepsilon} \frac{\not{k}_{\mathrm{a}}}{k_{\mathrm{a}}^2 + \mathrm{i}\epsilon} \sim \frac{k_{\mathrm{a}}^+ \gamma^-}{2k_{\mathrm{a}}^+ q^-} \varepsilon^- \gamma^+ \frac{k_{\mathrm{a}}^+ \gamma^-}{k_{\mathrm{a}}^2 + \mathrm{i}\epsilon}$$

$$\sim \frac{\varepsilon^-}{q^-} \frac{\not{k}_{\mathrm{a}}}{k_{\mathrm{a}}^2 + \mathrm{i}\epsilon}$$

- This gives us the gluon attached to the eikonal line representing the PDF in hadron B.

$$\frac{\varepsilon^-}{q^-} \frac{\not{k}_{\mathrm{a}}}{k_{\mathrm{a}}^2 + \mathrm{i}\epsilon}$$

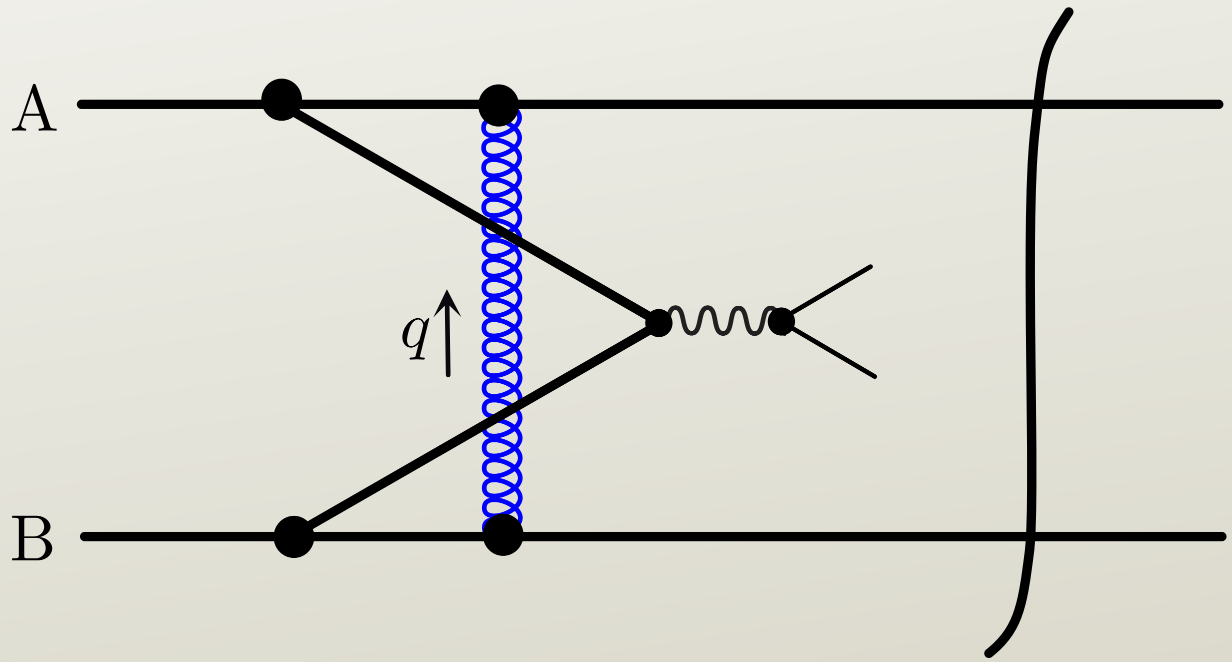


- What could go wrong?

A more difficult case

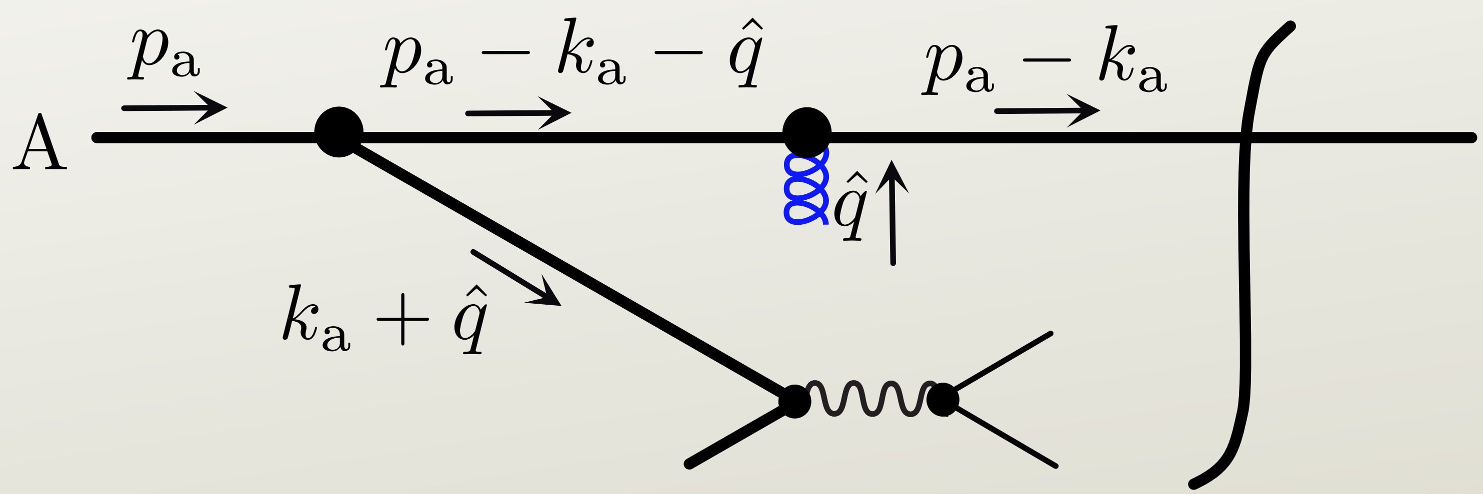
- Integration region with all components of q small.

- Look at this graph.



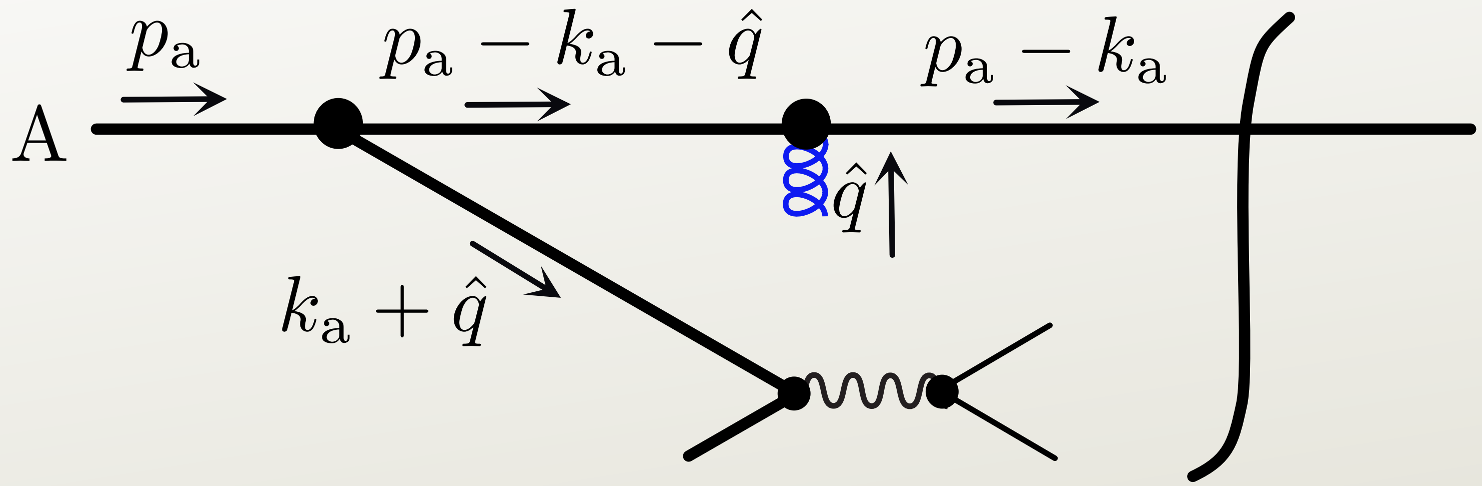
- Look at A partons first.

The A partons



- We consider all components of q to be small.
- p_a^+ and k_a^+ are large with $k_a^+ < p_a^+$.
- p_a^- and k_a^- are small.
- Neglect q^+ compared to p_a^+ and k_a^+.

$$\hat{q} = (0, q^-, \boldsymbol{q})$$



$$G_{\rm A} = \int \frac{dk_{\rm a}^-}{2\pi}\ \frac{1}{2k_{\rm a}^+(k_{\rm a}^- + q^-) - (\boldsymbol{k}_{\rm a} + \boldsymbol{q})^2 + {\rm i}\epsilon}$$

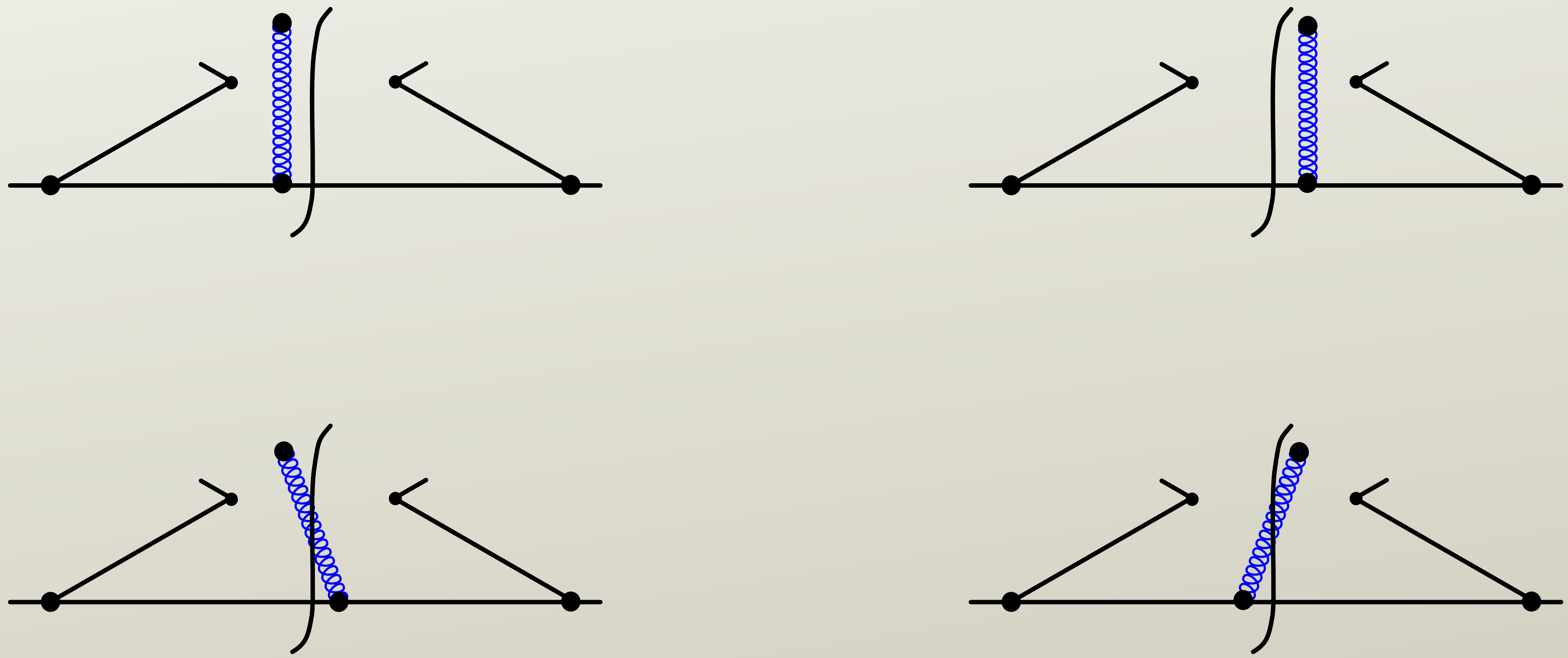
$$\times \frac{J\cdot\varepsilon}{2(p_{\rm a}^+ - k_{\rm a}^+)(p_{\rm a}^- - k_{\rm a}^- - q^-) - (\boldsymbol{p}_{\rm a} - \boldsymbol{k}_{\rm a} - \boldsymbol{q})^2 + {\rm i}\epsilon}$$

$$\times(2\pi)\,\delta(2(p_{\rm a}^+ - k_{\rm a}^+)(p_{\rm a}^- - k_{\rm a}^-) - (\boldsymbol{p}_{\rm a} - \boldsymbol{k})^2)$$

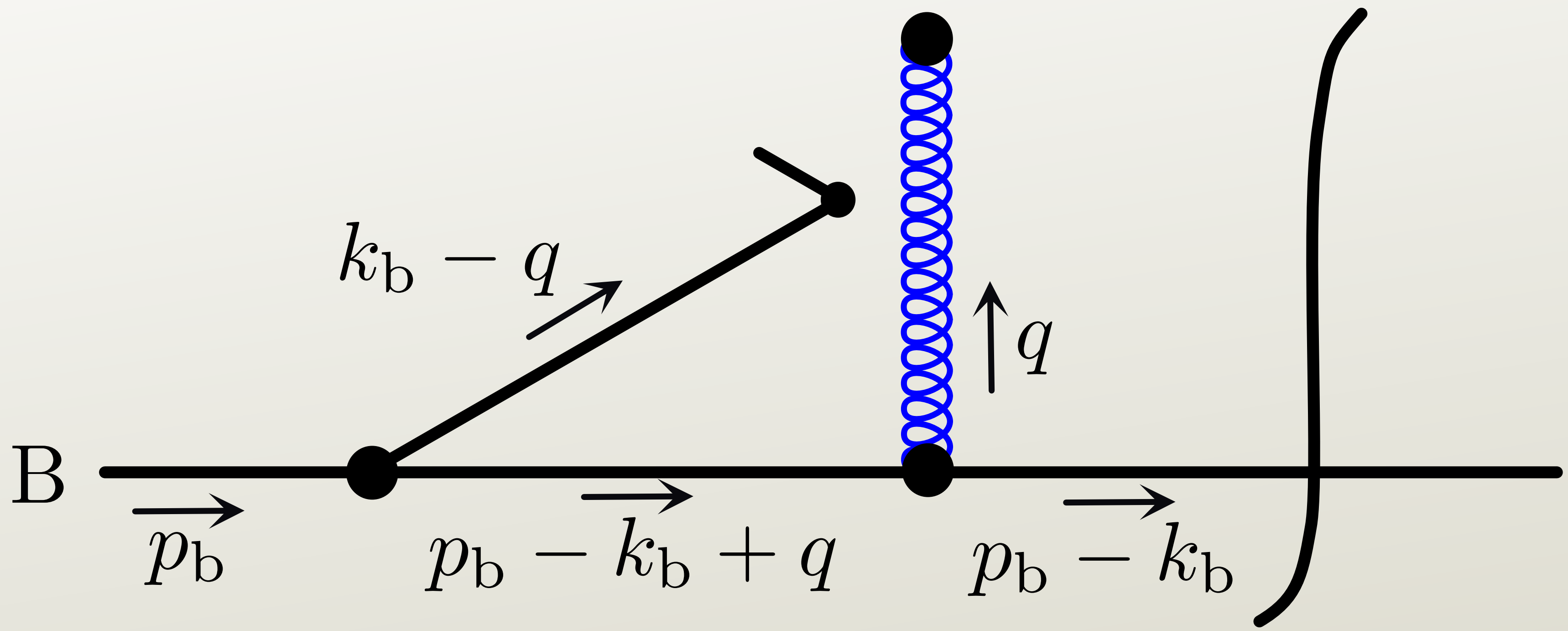
- There are poles in q^- very close to $q^- = 0$, with opposite ${\rm i}\epsilon$.
- It seems that something has gone wrong.

The rest of the graph

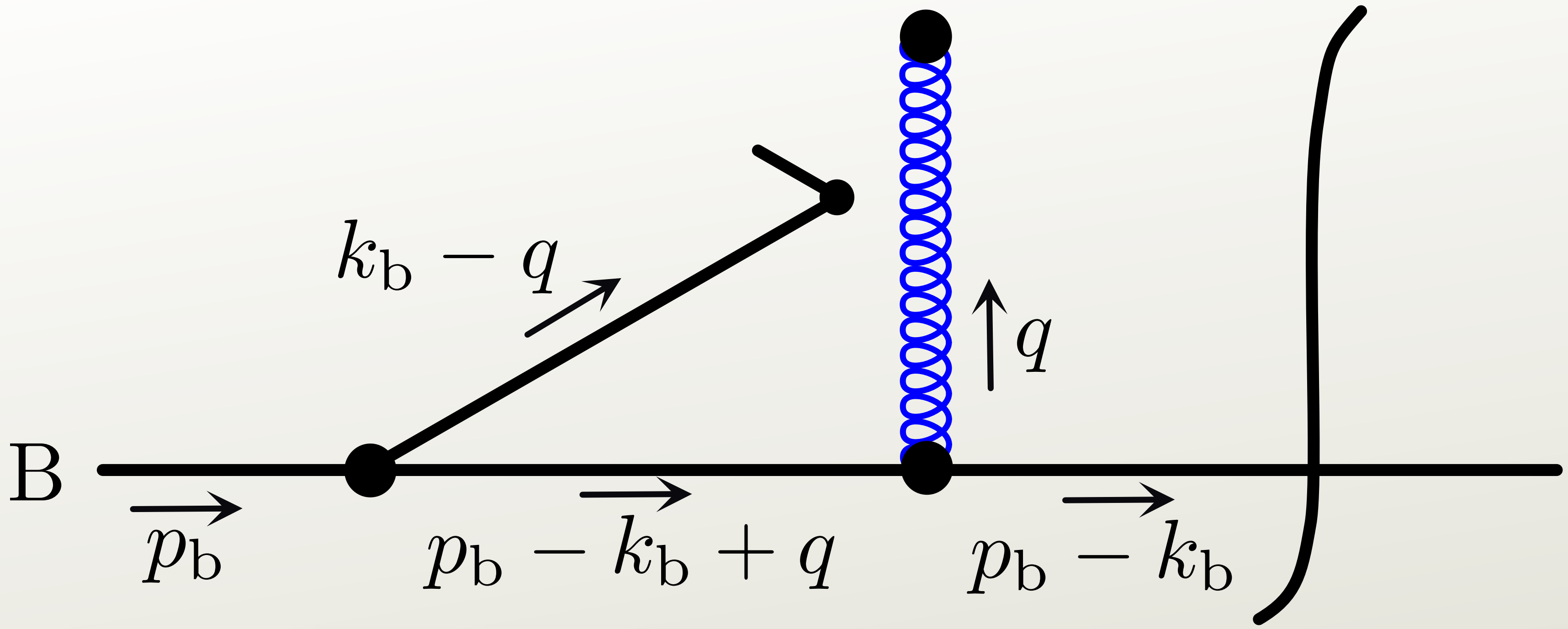
• There are four graphs with the soft gluon attaching to the spectator line.



• We will need relations among these.

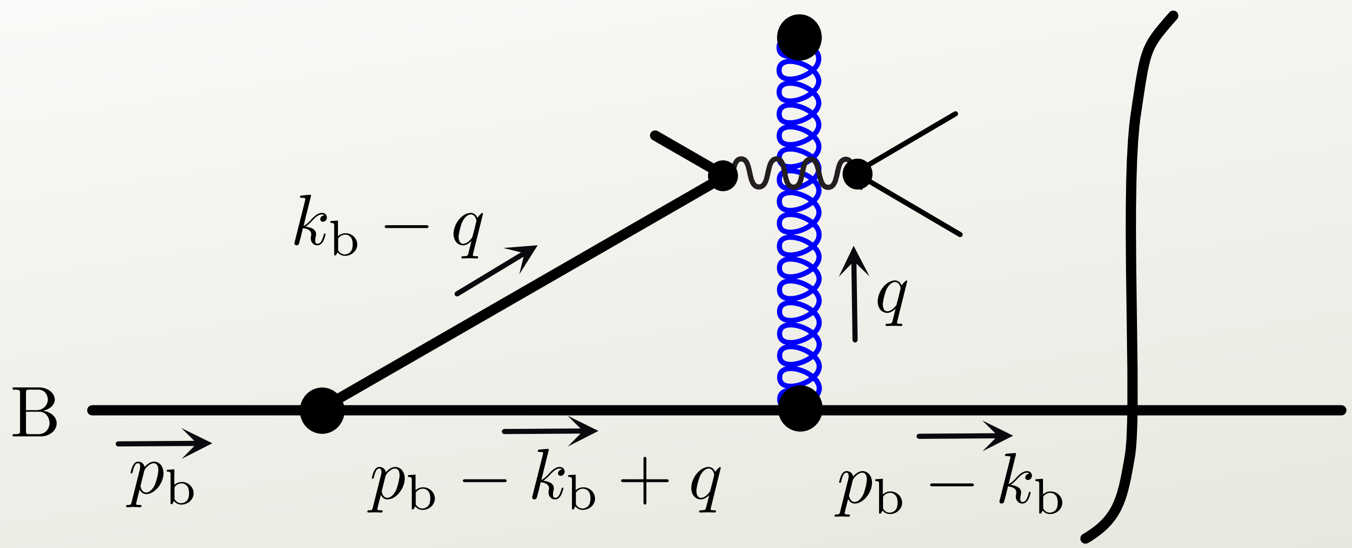


- p_b^- and k_b^- are large with $k_b^- < p_b^-$.
- p_b^+ and k_b^+ are small.
- Neglect q^- compared to p_b^- and k_b^-.
- Here the gluon propagator is included.
- We have an integral over q^+.



$$
\begin{aligned}
G_{\rm R} = \int \frac{dk_{\rm b}^{+}}{2\pi} \int \frac{dq^{+}}{2\pi}\, &\frac{\rm i}{2q^{+}q^{-} - \boldsymbol{q}^{2} + {\rm i}\epsilon} \\
&\times \frac{1}{2k_{\rm b}^{-}(k_{\rm b}^{+} - q^{+}) - (\boldsymbol{k}_{\rm b} - \boldsymbol{q})^{2} + {\rm i}\epsilon} \\
&\times \frac{J(q)}{2(p_{\rm b}^{-} - k_{\rm b}^{-})(p_{\rm b}^{+} - k_{\rm b}^{+} + q^{+}) - (\boldsymbol{p}_{\rm b} - \boldsymbol{k}_{\rm b} + \boldsymbol{q})^{2} + {\rm i}\epsilon} \\
&\times (2\pi)\,\delta(2(p_{\rm b}^{-} - k_{\rm b}^{-})(p_{\rm b}^{+} - k_{\rm b}^{+}) - (\boldsymbol{p}_{\rm b} - \boldsymbol{k}_{\rm b})^{2})
\end{aligned}
$$

- There are poles in q^{+} very close to $q^{+} = 0$, with opposite ${\rm i}\epsilon$.
- It seems that something has gone wrong again.



$$G_{\rm R} = \frac{1}{4k_{\rm b}^-(p_{\rm b}^- - k_{\rm b}^-)}\int\frac{dk_{\rm b}^+}{2\pi}\int\frac{dq^+}{2\pi}\,\frac{\rm i}{2q^+q^- - \boldsymbol{q}^2 + {\rm i}\epsilon}$$
$$\times\left[k_{\rm b}^+ - q^+ - \frac{(\boldsymbol{k}_{\rm b} - \boldsymbol{q})^2}{2k_{\rm b}^-} + {\rm i}\epsilon\right]^{-1}$$
$$\times\left[p_{\rm b}^+ - k_{\rm b}^+ + q^+ - \frac{(\boldsymbol{p}_{\rm b} - \boldsymbol{k}_{\rm b} + \boldsymbol{q})^2}{2(p_{\rm b}^- - k_{\rm b}^-)} + {\rm i}\epsilon\right]^{-1} J\cdot\varepsilon$$
$$\times(2\pi)\,\delta(2(p_{\rm b}^- - k_{\rm b}^-)(p_{\rm b}^+ - k_{\rm b}^+) - (\boldsymbol{p}_{\rm b} - \boldsymbol{k}_{\rm b})^2)$$

- Write this as

$$G_{\rm R} = G_{\rm IS} + G_{\rm FS}$$

$$G_{\mathrm{IS}} = \frac{J\cdot\varepsilon}{4k_{\mathrm{b}}^{-}(p_{\mathrm{b}}^{-}-k_{\mathrm{b}}^{-})}\int\frac{dk_{\mathrm{b}}^{+}}{2\pi}\int\frac{dq^{+}}{2\pi}\,\frac{\mathrm{i}}{2q^{+}q^{-}-\boldsymbol{q}^{2}+\mathrm{i}\epsilon}$$

$$\times\left[p_{\mathrm{b}}^{+}-\frac{(\boldsymbol{k}_{\mathrm{b}}-\boldsymbol{q})^{2}}{2k_{\mathrm{b}}^{-}}-\frac{(\boldsymbol{p}_{\mathrm{b}}-\boldsymbol{k}_{\mathrm{b}}+\boldsymbol{q})^{2}}{2(p_{\mathrm{b}}^{-}-k_{\mathrm{b}}^{-})}+\mathrm{i}\epsilon\right]^{-1}$$

$$\times\left[k_{\mathrm{b}}^{+}-q^{+}-\frac{(\boldsymbol{k}_{\mathrm{b}}-\boldsymbol{q})^{2}}{2k_{\mathrm{b}}^{-}}+\mathrm{i}\epsilon\right]^{-1}$$

$$\times(2\pi)\,\delta(2(p_{\mathrm{b}}^{-}-k_{\mathrm{b}}^{-})(p_{\mathrm{b}}^{+}-k_{\mathrm{b}}^{+})-(\boldsymbol{p}_{\mathrm{b}}-\boldsymbol{k}_{\mathrm{b}})^{2})$$

$$G_{\mathrm{FS}} = \frac{J\cdot\varepsilon}{4k_{\mathrm{b}}^{-}(p_{\mathrm{b}}^{-}-k_{\mathrm{b}}^{-})}\int\frac{dk_{\mathrm{b}}^{+}}{2\pi}\int\frac{dq^{+}}{2\pi}\,\frac{\mathrm{i}}{2q^{+}q^{-}-\boldsymbol{q}^{2}+\mathrm{i}\epsilon}$$

$$\times\left[p_{\mathrm{b}}^{+}-\frac{(\boldsymbol{k}_{\mathrm{b}}-\boldsymbol{q})^{2}}{2k_{\mathrm{b}}^{-}}-\frac{(\boldsymbol{p}_{\mathrm{b}}-\boldsymbol{k}_{\mathrm{b}}+\boldsymbol{q})^{2}}{2(p_{\mathrm{b}}^{-}-k_{\mathrm{b}}^{-})}+\mathrm{i}\epsilon\right]^{-1}$$

$$\times\left[p_{\mathrm{b}}^{+}-k_{\mathrm{b}}^{+}+q^{+}-\frac{(\boldsymbol{p}_{\mathrm{b}}-\boldsymbol{k}_{\mathrm{b}}+\boldsymbol{q})^{2}}{2(p_{\mathrm{b}}^{-}-k_{\mathrm{b}}^{-})}+\mathrm{i}\epsilon\right]^{-1}$$

$$\times(2\pi)\,\delta(2(p_{\mathrm{b}}^{-}-k_{\mathrm{b}}^{-})(p_{\mathrm{b}}^{+}-k_{\mathrm{b}}^{+})-(\boldsymbol{p}_{\mathrm{b}}-\boldsymbol{k}_{\mathrm{b}})^{2})$$

Initial state contribution

- We can perform the integration over q^+ for $G_{\rm IS}$:

$$G_{\rm IS} = \frac{J\cdot\varepsilon}{4k_{\rm b}^-(p_{\rm b}^- - k_{\rm b}^-)}\int\frac{dk_{\rm b}^+}{2\pi}\int\frac{dq^+}{2\pi}\,\frac{\rm i}{2q^+q^- - \boldsymbol{q}^2 + {\rm i}\epsilon}$$
$$\times\left[p_{\rm b}^+ - \frac{(\boldsymbol{k}_{\rm b}-\boldsymbol{q})^2}{2k_{\rm b}^-} - \frac{(\boldsymbol{p}_{\rm b}-\boldsymbol{k}_{\rm b}+\boldsymbol{q})^2}{2(p_{\rm b}^- - k_{\rm b}^-)} + {\rm i}\epsilon\right]^{-1}$$
$$\times\left[k_{\rm b}^+ - q^+ - \frac{(\boldsymbol{k}_{\rm b}-\boldsymbol{q})^2}{2k_{\rm b}^-} + {\rm i}\epsilon\right]^{-1}$$
$$\times(2\pi)\,\delta(2(p_{\rm b}^- - k_{\rm b}^-)(p_{\rm b}^+ - k_{\rm b}^+) - (\boldsymbol{p}_{\rm b}-\boldsymbol{k}_{\rm b})^2)$$

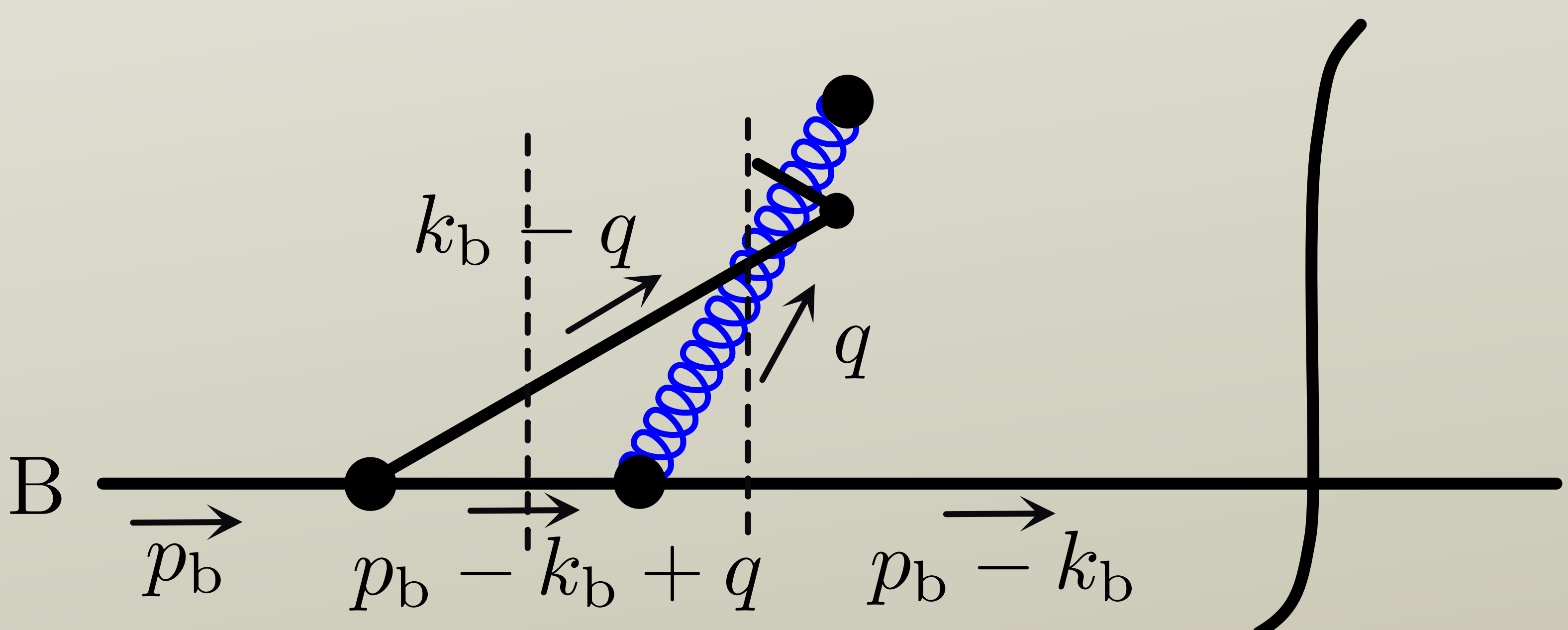
- $G_{\rm IS} = 0$ if $q^- < 0$.
- For $q^- > 0$, close contour in lower half q^+ plane.

$$G_{\mathrm{IS}} = \frac{\theta(q^- > 0) J\cdot\varepsilon}{4k_{\mathrm{b}}^-(p_{\mathrm{b}}^- - k_{\mathrm{b}}^-)} \int \frac{dk_{\mathrm{b}}^+}{2\pi}$$

$$\times \left[p_{\mathrm{b}}^+ - \frac{(\boldsymbol{k}_{\mathrm{b}} - \boldsymbol{q})^2}{2k_{\mathrm{b}}^-} - \frac{(\boldsymbol{p}_{\mathrm{b}} - \boldsymbol{k}_{\mathrm{b}} + \boldsymbol{q})^2}{2(p_{\mathrm{b}}^- - k_{\mathrm{b}}^-)} + \mathrm{i}\epsilon \right]^{-1}$$

$$\times \left[k_{\mathrm{b}}^+ - \frac{\boldsymbol{q}^2}{2q^-} - \frac{(\boldsymbol{k}_{\mathrm{b}} - \boldsymbol{q})^2}{2k_{\mathrm{b}}^-} + \mathrm{i}\epsilon \right]^{-1}$$

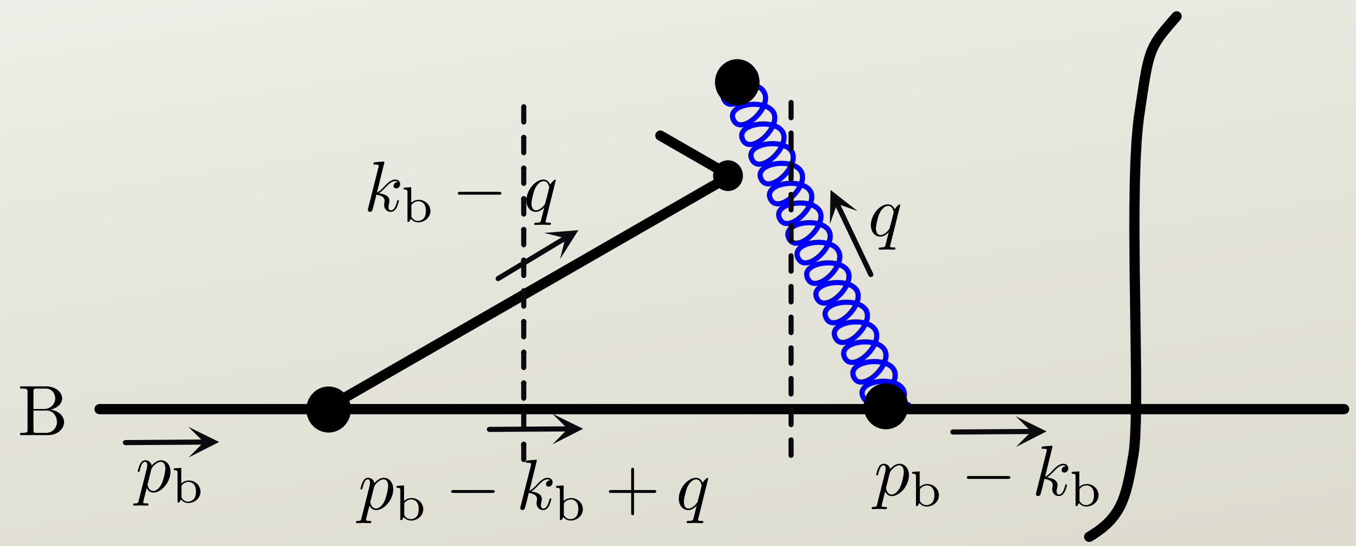
$$\times (2\pi)\, \delta(2(p_{\mathrm{b}}^- - k_{\mathrm{b}}^-)(p_{\mathrm{b}}^+ - k_{\mathrm{b}}^+) - (\boldsymbol{p}_{\mathrm{b}} - \boldsymbol{k}_{\mathrm{b}})^2)$$



$$D = p_{\mathrm{b}}^+ - \sum \frac{\boldsymbol{k}_i^2}{2k_i^-}$$

Final state contribution

- After integrating over q^+, G_{FS} is



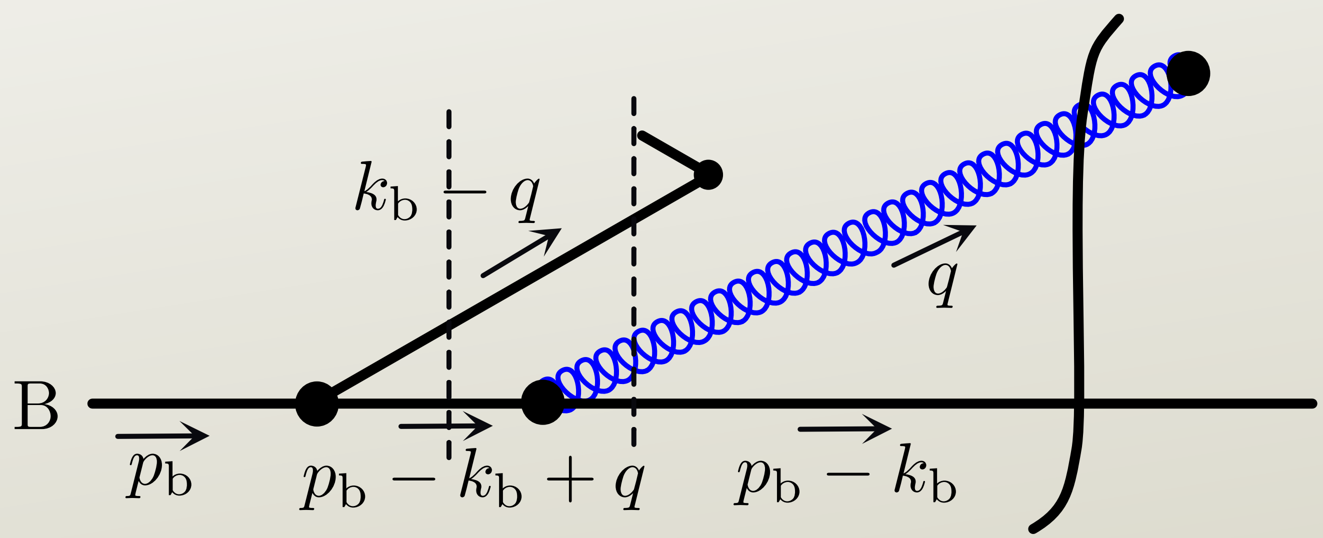
$$D = p_{\mathrm{b}}^+ - \sum \frac{\boldsymbol{k}_i^2}{2k_i^-}$$

- G_{FS} needs $q^- < 0$.

- We leave G_{FS} for later.

A second initial state contribution

- What about $G_{\rm IS}$ when the gluon enters the final state?



- To make $G_{\rm IS,cut}$, in $G_{\rm IS}$ replace

$$\int \frac{dq^+}{2\pi}\, \frac{\rm i}{2q^+q^- - \boldsymbol{q}^2 + {\rm i}\epsilon} \to \int \frac{dq^+}{2\pi}\, 2\pi\delta_+(2q^+q^- - \boldsymbol{q}^2)$$

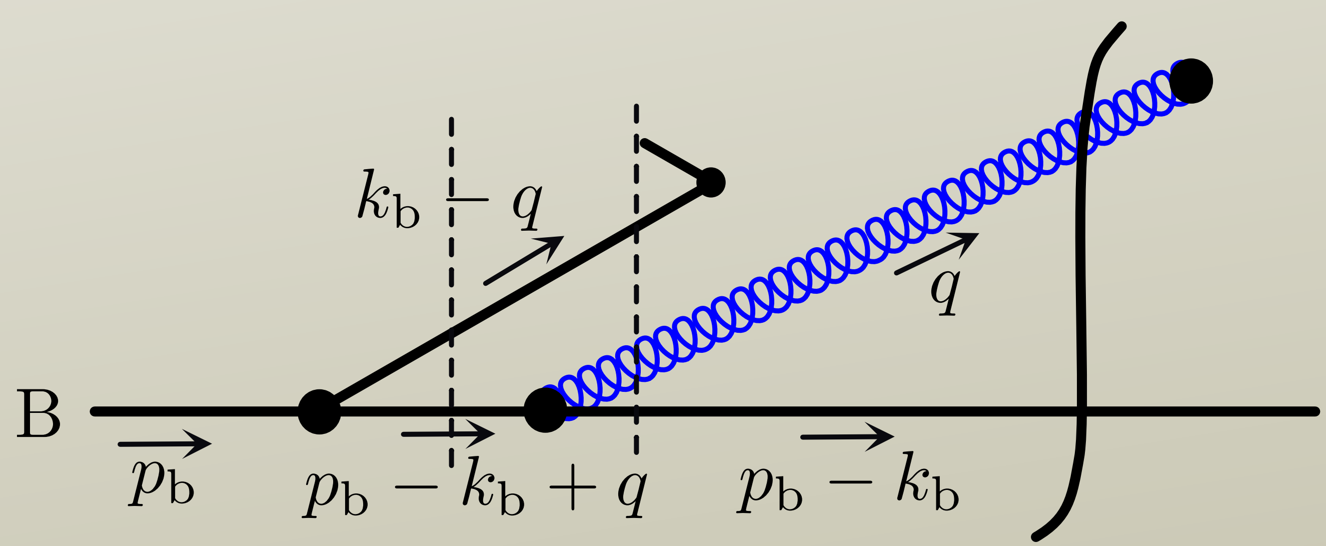
- This gives

$$G_{\mathrm{IS,cut}} = \frac{\theta(q^- > 0) J\cdot\varepsilon}{4k_{\mathrm{b}}^-(p_{\mathrm{b}}^- - k_{\mathrm{b}}^-)} \int \frac{dk_{\mathrm{b}}^+}{2\pi}$$

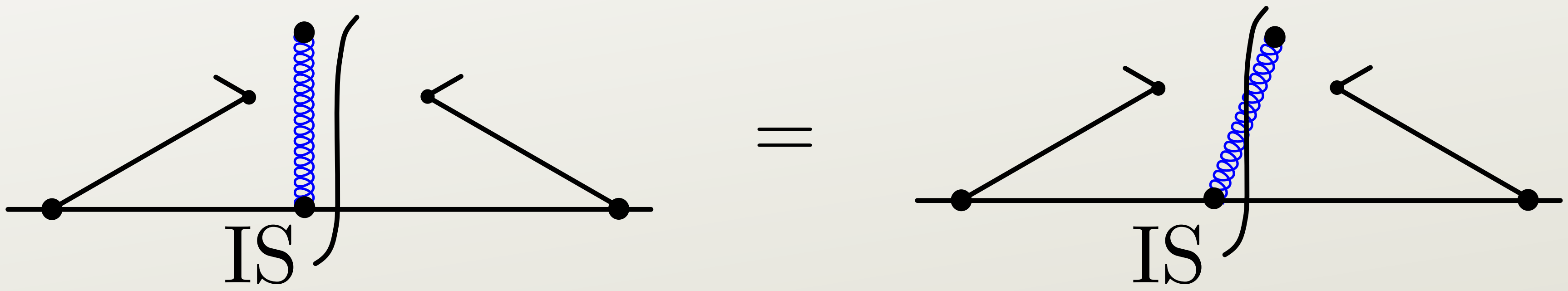
$$\times \left[p_{\mathrm{b}}^+ - \frac{(\boldsymbol{k}_{\mathrm{b}} - \boldsymbol{q})^2}{2k_{\mathrm{b}}^-} - \frac{(\boldsymbol{p}_{\mathrm{b}} - \boldsymbol{k}_{\mathrm{b}} + \boldsymbol{q})^2}{2(p_{\mathrm{b}}^- - k_{\mathrm{b}}^-)} + \mathrm{i}\epsilon \right]^{-1}$$

$$\times \left[k_{\mathrm{b}}^+ - \frac{\boldsymbol{q}^2}{2q^-} - \frac{(\boldsymbol{k}_{\mathrm{b}} - \boldsymbol{q})^2}{2k_{\mathrm{b}}^-} + \mathrm{i}\epsilon \right]^{-1}$$

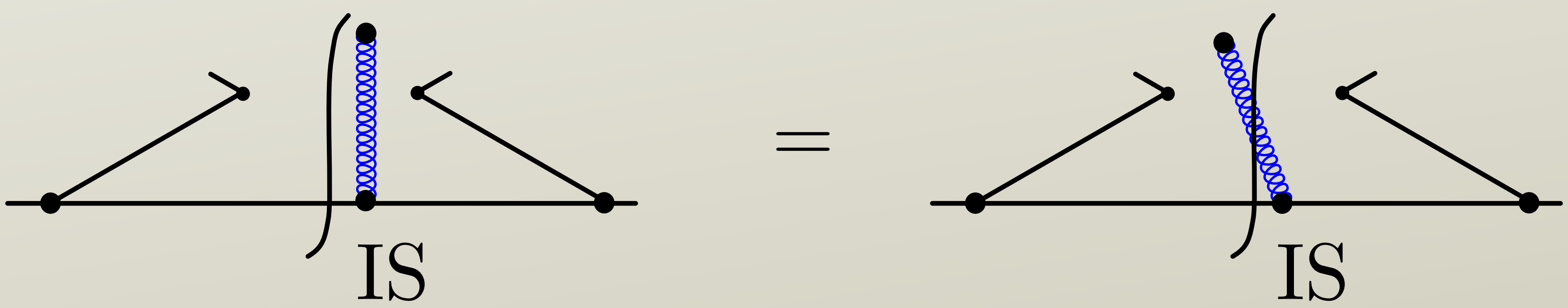
$$\times (2\pi)\, \delta(2(p_{\mathrm{b}}^- - k_{\mathrm{b}}^-)(p_{\mathrm{b}}^+ - k_{\mathrm{b}}^+) - (\boldsymbol{p}_{\mathrm{b}} - \boldsymbol{k}_{\mathrm{b}})^2)$$



- We see that $G_{\mathrm{IS}} = G_{\mathrm{IS,cut}}$.

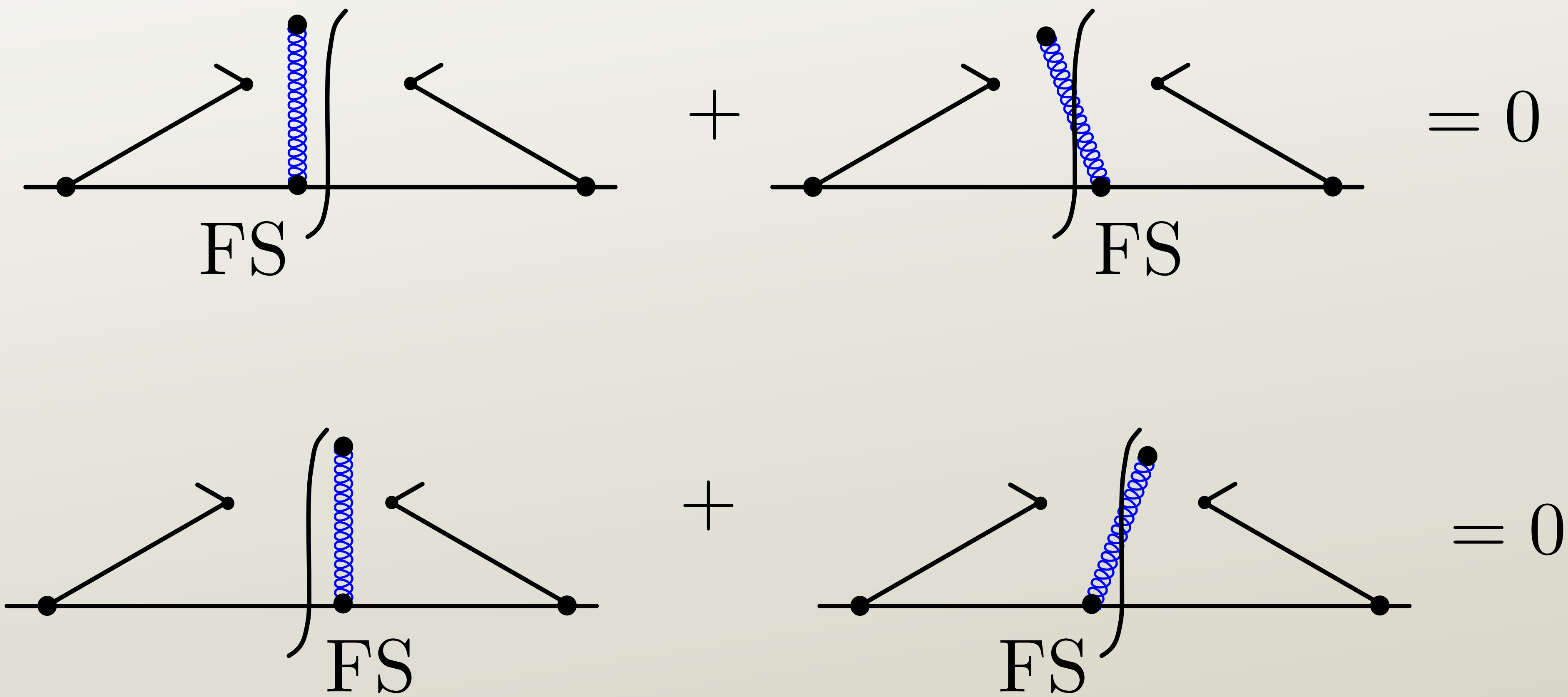


- The same applies to the conjugate amplitudes.



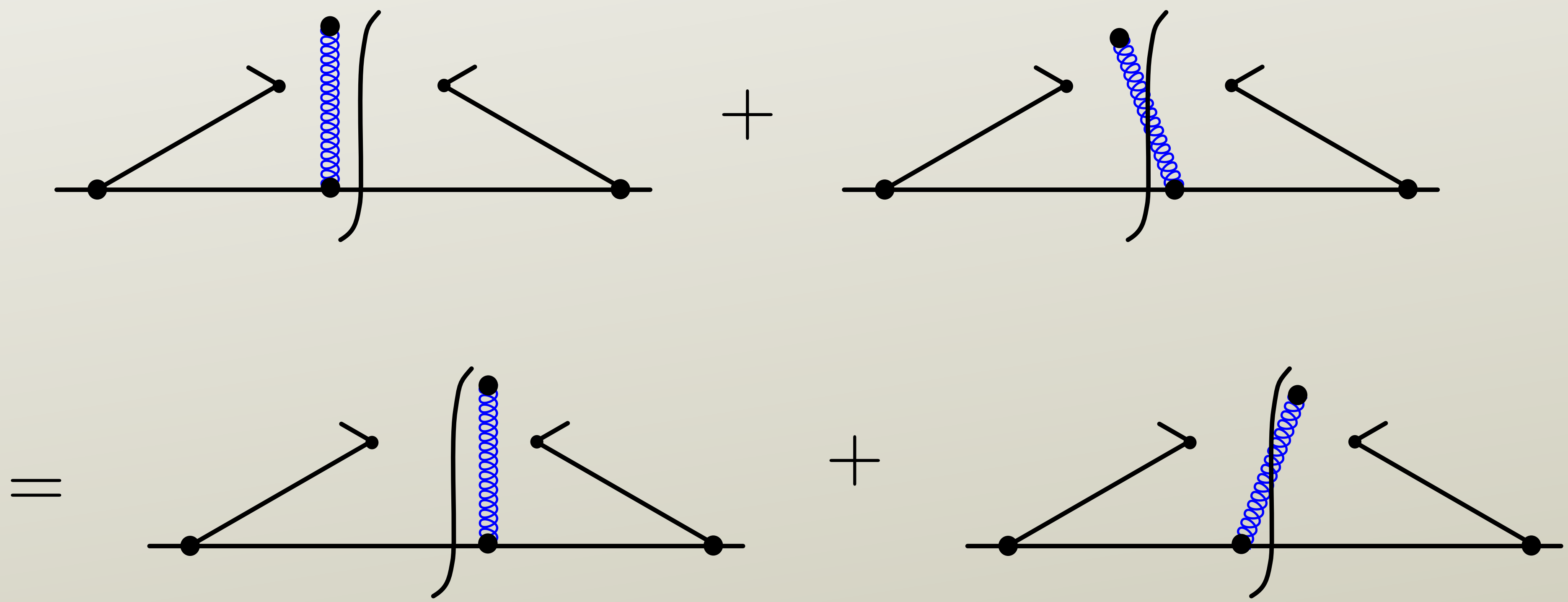
Cancelations

- The final state contributions cancel.



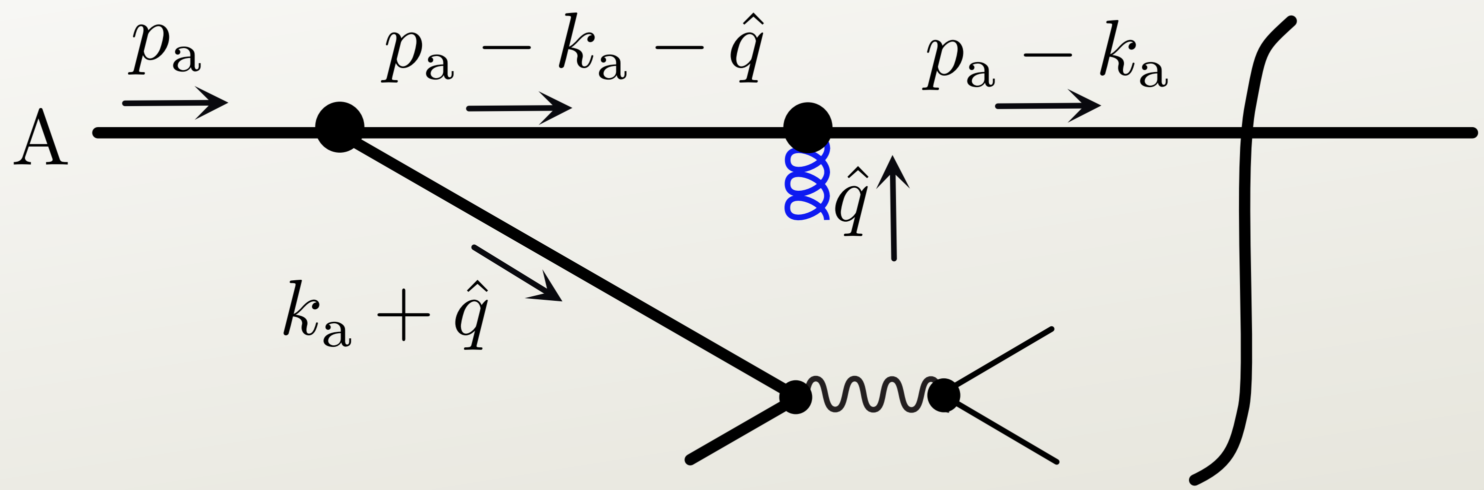
- I omit the details of this.

• Thus when we analyze the hadron A factor, the rest of the graph is independent of whether the gluon attaches to the left of the cut or the right of the cut.



• We can now use this property to analyze the factor for hadron A.

Hadron A



$$G_{\mathrm{A}} = \frac{J \cdot \varepsilon}{8k_{\mathrm{a}}^{+}(p_{\mathrm{a}}^{+} - k_{\mathrm{a}}^{+})^2} \int \frac{dk_{\mathrm{a}}^{-}}{2\pi}$$

$$\times \left[k_{\mathrm{a}}^{-} + q^{-} - \frac{(\boldsymbol{k}_{\mathrm{a}} + \boldsymbol{q})^2}{2k_{\mathrm{a}}^{+}} + \mathrm{i}\epsilon \right]^{-1}$$

$$\times \left[p_{\mathrm{a}}^{-} - k_{\mathrm{a}}^{-} - q^{-} - \frac{(\boldsymbol{p}_{\mathrm{a}} - \boldsymbol{k}_{\mathrm{a}} - \boldsymbol{q})^2}{2(p_{\mathrm{a}}^{+} - k_{\mathrm{a}}^{+})} + \mathrm{i}\epsilon \right]^{-1}$$

$$\times (2\pi)\, \delta\!\left(p_{\mathrm{a}}^{-} - k_{\mathrm{a}}^{-} - \frac{(\boldsymbol{p}_{\mathrm{a}} - \boldsymbol{k}_{\mathrm{a}})^2}{2(p_{\mathrm{a}}^{+} - k_{\mathrm{a}}^{+})} \right)$$

- Write this as $G_{\mathrm{A}} = G_{\mathrm{IS}} + G_{\mathrm{FS}}$.

$$
\begin{aligned}
G_{\mathrm{IS}} = \frac{J\cdot\varepsilon}{8k_{\mathrm{a}}^{+}(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})^2}\int\frac{dk_{\mathrm{a}}^{-}}{2\pi}
&\times\left[p_{\mathrm{a}}^{-}-\frac{(\boldsymbol{k}_{\mathrm{a}}+\boldsymbol{q})^2}{2k_{\mathrm{a}}^{+}}-\frac{(\boldsymbol{p}_{\mathrm{a}}-\boldsymbol{k}_{\mathrm{a}}-\boldsymbol{q})^2}{2(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})}+\mathrm{i}\epsilon\right]^{-1} \\
&\times\left[k_{\mathrm{a}}^{-}+q^{-}-\frac{(\boldsymbol{k}_{\mathrm{a}}+\boldsymbol{q})^2}{2k_{\mathrm{a}}^{+}}+\mathrm{i}\epsilon\right]^{-1} \\
&\times(2\pi)\,\delta\left(p_{\mathrm{a}}^{-}-k_{\mathrm{a}}^{-}-\frac{(\boldsymbol{p}_{\mathrm{a}}-\boldsymbol{k}_{\mathrm{a}})^2}{2(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})}\right)
\end{aligned}
$$

$$
\begin{aligned}
G_{\mathrm{FS}} = \frac{J\cdot\varepsilon}{8k_{\mathrm{a}}^{+}(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})^2}\int\frac{dk_{\mathrm{a}}^{-}}{2\pi}
&\times\left[p_{\mathrm{a}}^{-}-\frac{(\boldsymbol{k}_{\mathrm{a}}+\boldsymbol{q})^2}{2k_{\mathrm{a}}^{+}}-\frac{(\boldsymbol{p}_{\mathrm{a}}-\boldsymbol{k}_{\mathrm{a}}-\boldsymbol{q})^2}{2(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})}+\mathrm{i}\epsilon\right]^{-1} \\
&\times\left[p_{\mathrm{a}}^{-}-k_{\mathrm{a}}^{-}-q^{-}-\frac{(\boldsymbol{p}_{\mathrm{a}}-\boldsymbol{k}_{\mathrm{a}}-\boldsymbol{q})^2}{2(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})}+\mathrm{i}\epsilon\right]^{-1} \\
&\times(2\pi)\,\delta\left(p_{\mathrm{a}}^{-}-k_{\mathrm{a}}^{-}-\frac{(\boldsymbol{p}_{\mathrm{a}}-\boldsymbol{k}_{\mathrm{a}})^2}{2(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})}\right)
\end{aligned}
$$

Another final state contribution

• Manipulate $G_{\rm FS}$.

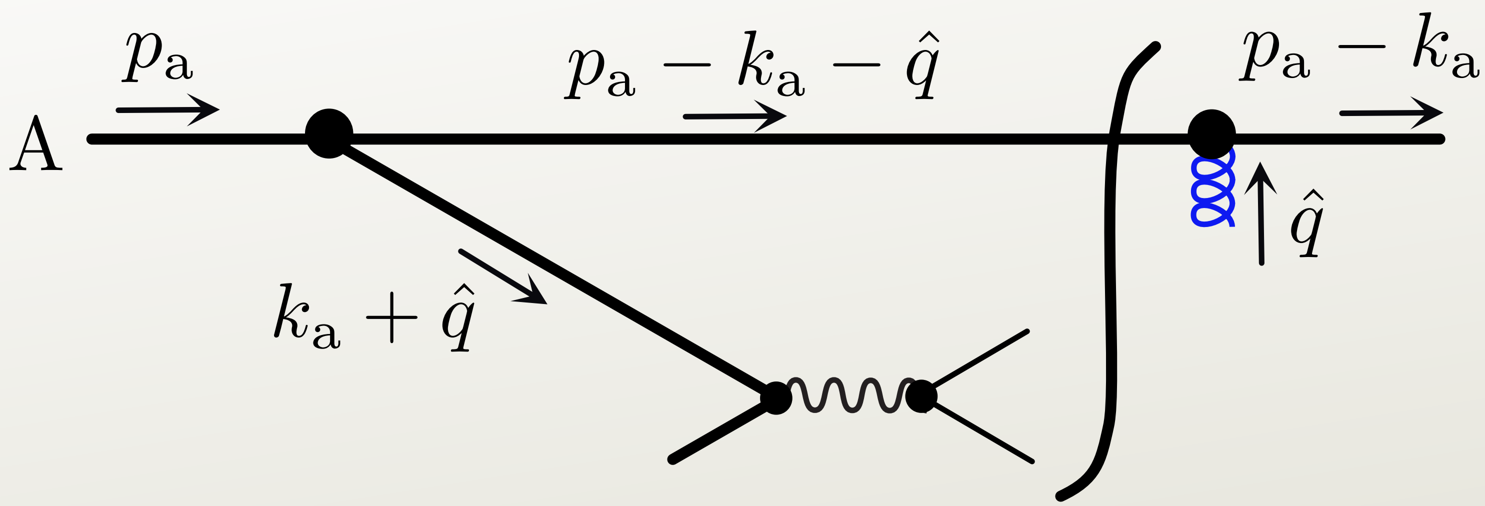
• Write the delta function as a difference of two denominator factors with opposite $\mathrm{i}\epsilon$ terms. Only one of these contributes.

$$
\begin{aligned}
G_{\rm FS} = {} & \frac{J\cdot\varepsilon}{8k_{\rm a}^+(p_{\rm a}^+ - k_{\rm a}^+)^2}\int\frac{dk_{\rm a}^-}{2\pi} \\
& \times\left[p_{\rm a}^- - \frac{(\boldsymbol{k}_{\rm a}+\boldsymbol{q})^2}{2k_{\rm a}^+} - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)} + \mathrm{i}\epsilon\right]^{-1} \\
& \times\left[p_{\rm a}^- - k_{\rm a}^- - q^- - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)} + \mathrm{i}\epsilon\right]^{-1} \\
& \times(-\mathrm{i})\left[p_{\rm a}^- - k_{\rm a}^- - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)} - \mathrm{i}\epsilon\right]^{-1}
\end{aligned}
$$

• Write the middle denominator factor as a delta function.

$$
\begin{aligned}
G_{\rm FS} = -\frac{J\cdot\varepsilon}{8k_{\rm a}^{+}(p_{\rm a}^{+}-k_{\rm a}^{+})^2}\int\frac{dk_{\rm a}^{-}}{2\pi}
\\
\times\left[p_{\rm a}^{-}-\frac{(\boldsymbol{k}_{\rm a}+\boldsymbol{q})^2}{2k_{\rm a}^{+}}-\frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^{+}-k_{\rm a}^{+})}+{\rm i}\epsilon\right]^{-1}
\\
\times(2\pi)\delta\left(p_{\rm a}^{-}-k_{\rm a}^{-}-q^{-}-\frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^{+}-k_{\rm a}^{+})}\right)
\\
\times\left[p_{\rm a}^{-}-k_{\rm a}^{-}-\frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a})^2}{2(p_{\rm a}^{+}-k_{\rm a}^{+})}-{\rm i}\epsilon\right]^{-1}
\end{aligned}
$$

- This is the negative of a graph $G_{\rm FS,cut}$ with a different cut.



$$G_{\rm FS,cut} = \frac{J\cdot\varepsilon}{8k_{\rm a}^+(p_{\rm a}^+ - k_{\rm a}^+)^2}\int\frac{dk_{\rm a}^-}{2\pi}$$

$$\times\left[p_{\rm a}^- - \frac{(\boldsymbol{k}_{\rm a}+\boldsymbol{q})^2}{2k_{\rm a}^+} - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)} + {\rm i}\epsilon\right]^{-1}$$

$$\times(2\pi)\delta\left(p_{\rm a}^- - k_{\rm a}^- - q^- - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)}\right)$$

$$\times\left[p_{\rm a}^- - k_{\rm a}^- - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)} - {\rm i}\epsilon\right]^{-1}$$

• If we include the factors for the Born level conjugate amplitude for hadron A, $G_{\mathrm{FS,cut}}$ is one of our graphs.

• We need to multiply both G_{FS} and $G_{\mathrm{FS,cut}}$ by the rest of the graph, describing the gluon propagator and hadron B.

• But we have seen that the rest of the graph does not depend on where the final state cut is.

• Thus, including the rest of the graph,

$$G_{\mathrm{FS}} + G_{\mathrm{FS,cut}} = 0$$

Integration contour for initial state contribution

- We conclude that we need only $G_{\rm IS}$.

$$
\begin{aligned}
G_{\rm IS} = {} & \frac{J\cdot\varepsilon}{8k_{\rm a}^+(p_{\rm a}^+ - k_{\rm a}^+)^2} \int \frac{dk_{\rm a}^-}{2\pi} \\
& \times \left[p_{\rm a}^- - \frac{(\boldsymbol{k}_{\rm a}+\boldsymbol{q})^2}{2k_{\rm a}^+} - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)} + {\rm i}\epsilon\right]^{-1} \\
& \times \left[k_{\rm a}^- + q^- - \frac{(\boldsymbol{k}_{\rm a}+\boldsymbol{q})^2}{2k_{\rm a}^+} + {\rm i}\epsilon\right]^{-1} \\
& \times (2\pi)\,\delta\!\left(p_{\rm a}^- - k_{\rm a}^- - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)}\right)
\end{aligned}
$$

- This has no singularities in the upper half q^- plane.

• We can thus deform the q^- contour for $G_{\rm A}$ into the upper half plane.

$$
\begin{aligned}
G_{\rm A} = \frac{J\cdot\varepsilon}{8k_{\rm a}^+(p_{\rm a}^+ - k_{\rm a}^+)^2}\int\frac{dk_{\rm a}^-}{2\pi}
&\times\left[k_{\rm a}^- + q^- - \frac{(\boldsymbol{k}_{\rm a}+\boldsymbol{q})^2}{2k_{\rm a}^+} + {\rm i}\epsilon\right]^{-1}\\
&\times\left[p_{\rm a}^- - k_{\rm a}^- - q^- - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a}-\boldsymbol{q})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)} + {\rm i}\epsilon\right]^{-1}\\
&\times(2\pi)\,\delta\!\left(p_{\rm a}^- - k_{\rm a}^- - \frac{(\boldsymbol{p}_{\rm a}-\boldsymbol{k}_{\rm a})^2}{2(p_{\rm a}^+ - k_{\rm a}^+)}\right)
\end{aligned}
$$

• $G_{\rm A}$ has a final state pole in the upper half plane.

• We have seen that the contributions from deforming past this pole will cancel when we include all of the graphs.

Consequence of the contour deformation

$$
\begin{aligned}
G_{\mathrm{A}} = \frac{J\cdot\varepsilon}{8k_{\mathrm{a}}^{+}(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})^{2}} \int \frac{dk_{\mathrm{a}}^{-}}{2\pi}
& \\
\times \left[k_{\mathrm{a}}^{-} + q^{-} - \frac{(\boldsymbol{k}_{\mathrm{a}}+\boldsymbol{q})^{2}}{2k_{\mathrm{a}}^{+}} + \mathrm{i}\epsilon\right]^{-1} & \\
\times \left[p_{\mathrm{a}}^{-} - k_{\mathrm{a}}^{-} - q^{-} - \frac{(\boldsymbol{p}_{\mathrm{a}}-\boldsymbol{k}_{\mathrm{a}}-\boldsymbol{q})^{2}}{2(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})} + \mathrm{i}\epsilon\right]^{-1} & \\
\times (2\pi)\,\delta\left(p_{\mathrm{a}}^{-} - k_{\mathrm{a}}^{-} - \frac{(\boldsymbol{p}_{\mathrm{a}}-\boldsymbol{k}_{\mathrm{a}})^{2}}{2(p_{\mathrm{a}}^{+}-k_{\mathrm{a}}^{+})}\right) &
\end{aligned}
$$

- With large q^{-}, we can set $\boldsymbol{q} \to 0$.

- With $\boldsymbol{q} = 0$ in G_{A}, we can now return the q^- contour to the real axis.

- $G_{\mathrm{A}}(q)$, is $G_{\mathrm{A}}(\tilde{q})$, where

$$\tilde{q} = (0, q^-, \mathbf{0})$$

- With $\boldsymbol{q} = 0$ the final state poles (which cancel) are

$$\frac{1}{q^- - \mathrm{i}\epsilon}$$

Using Ward identities

• Now we can apply Ward identities.

$$J \cdot \varepsilon = J^+ \varepsilon^- = \frac{J^+ q^-}{q^- - \mathrm{i}\epsilon}\, \varepsilon^-$$

• This essentially attaches a future oriented eikonal line to each vertex.

• Eikonal lines at internal vertices cancel.

• This leaves only the eikonal line at the annihilation vertex.

• Now summing over whether the soft gluon attaches to the left of the final state cut or to the right, the entire effect of exchanging the soft gluon cancels.

Intuitive picture

- The soft gluons see a parton jet with internal structure in the initial state.
- Since the soft gluons have zero transverse momentum, they cannot resolve the structure.
- They see only a color singlet incoming object.
- After the annihilation they see an object with unresolved structure, but with color.
- The color is the anticolor of the annihilating quark.