From Quantum Cohomology to Quantum K-Theory

Insights into Gromov-Witten invariants

Miroslava Mosso Rojas

An enumerative geometry problem

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A brief historical background

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- This contributed to the formalization of Gromov-Witten (GW) theory.
- The number of rational curves are the GW-invariants.

Gromov-Witten invariants

Gromov-Witten invariants

Given a projective variety X, we fix the homology class $\beta \in H_2(X; \mathbb{Z})$. Let $C \subset X$ of genus $g = 0$ and $p_i \in C$, where for each i there is an evaluation map:

$$ev_i : \overline{\mathscr{M}}_{0,n}(X, \beta) \longrightarrow X, \qquad ev_i(f : (C, p_1, \ldots, p_n) \longrightarrow X) = f(p_i),$$

this map evaluates f in each marked point p_i. The Gromov-Witten invariant for $X = \mathbb{P}^N$ definition is:

$$\langle \omega_1, \ldots, \omega_n \rangle_\beta = \int_{\overline{\mathscr{M}}_{0,n}(\mathbb{P}^N, \beta)} ev_1^*(\omega_1) \cdots ev_n^*(\omega_n)$$

where ev_i^* are the pullback maps and $\omega_i \in H^{ev}(X)$.

Localization Machinery

Localization machinery

Idea of the method

Represent various global values as a combination of local contributions.

When a smooth variety has a group action, cohomology classes localize to classes on the fixed point locus.

Equivariant / Atiyah-Bott localization method.

- Exact method that simplifies the computations to find the GW-invariants.

The Atiyah and Bott formula

Suppose we have a torus action $T = (\mathbb{C}^*)^{N+1}$ on a smooth manifold X. The fixed point locus V is a union of smooth connected components.

Let $\iota : V \hookrightarrow X$ be the inclusion map. Let N denote the normal bundle of V in X.

Since N is an equivariant vector bundle, it has an equivariant Euler class

$$e(N_{V/X}) \in H_T^*(V)$$

The Atiyah and Bott formula

The equivariant inclusion ι induces the map

$$\iota^* : H_T^*(X) \longrightarrow H_T^*(V)$$

The theorem of Atiyah and Bott says that the inverse of the Euler class of the normal bundle always exists along the fixed locus of a group action.

Then, letting V run over the fixed locus, for any equivariant class ϕ,

$$\int_X \phi = \sum_V \int_V \frac{\iota^* \phi}{e(N_{V/X})}$$

Kontsevich's approach

- The torus $T = (\mathbb{C}^*)^{N+1}$ defines an action on $\mathbb{P}^N$, and this induces an action on $\overline{\mathscr{M}}_{0,n}(\mathbb{P}^N, d)$.

Kontsevich's approach

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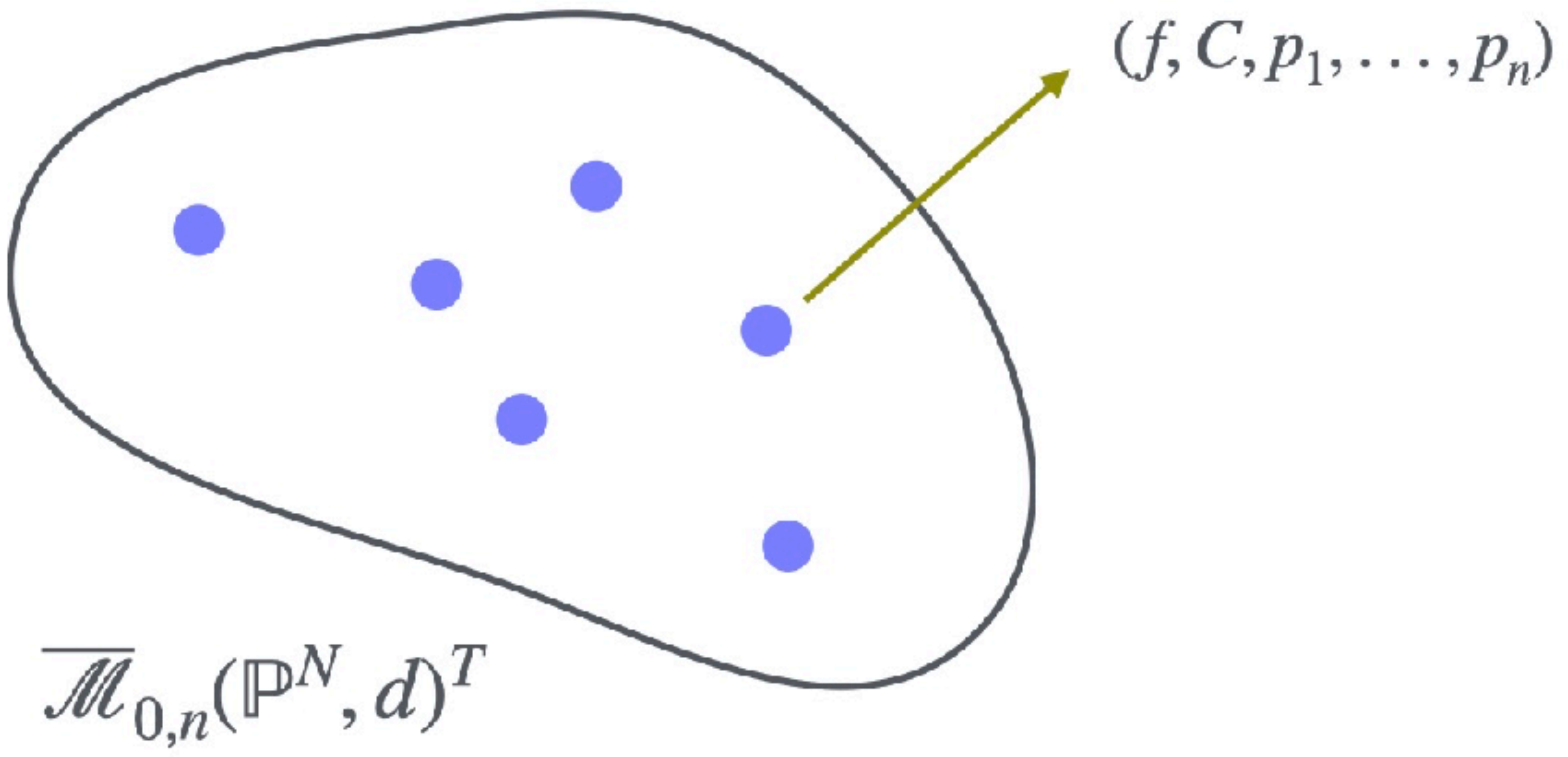
- A T-fixed point of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$ consists of a stable map $(f, C, p_1, \ldots, p_n)$



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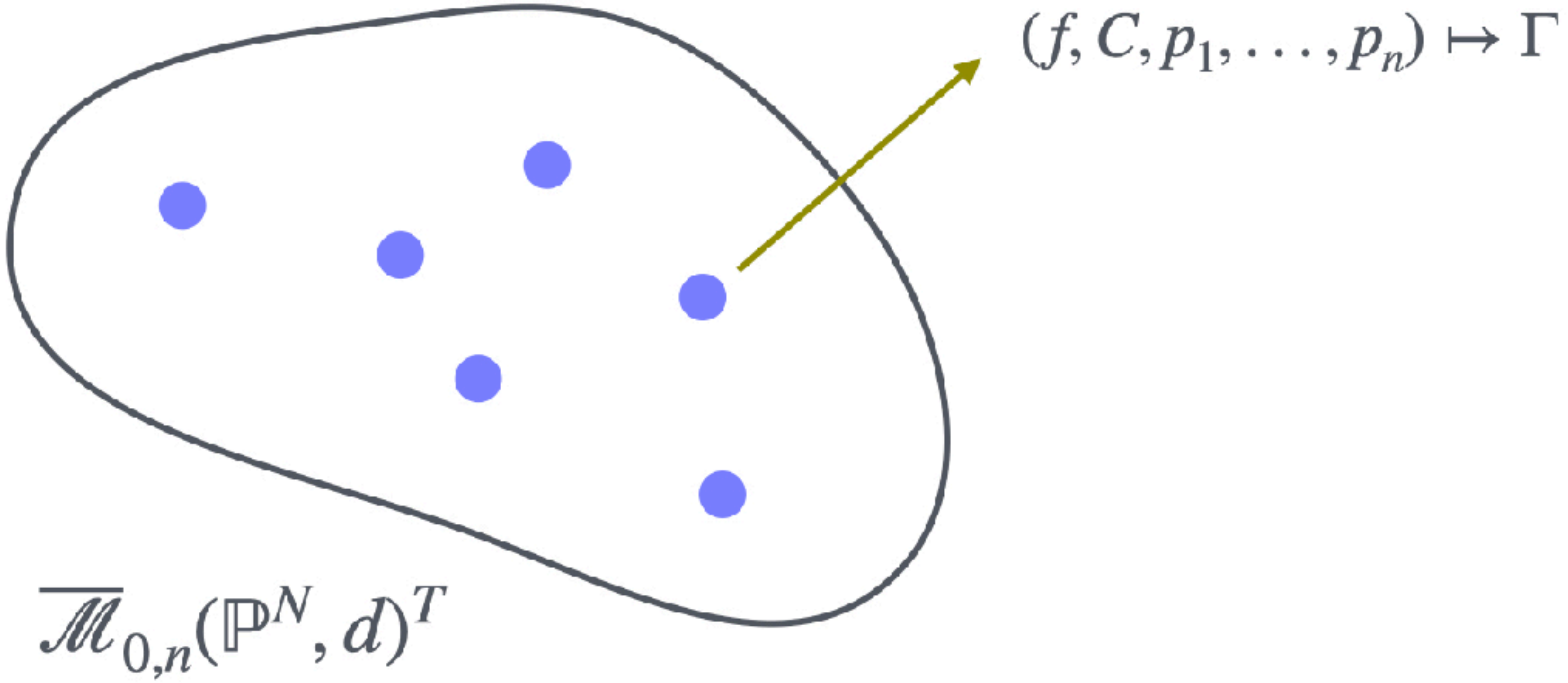
- We associate a tree Γ to a stable map.



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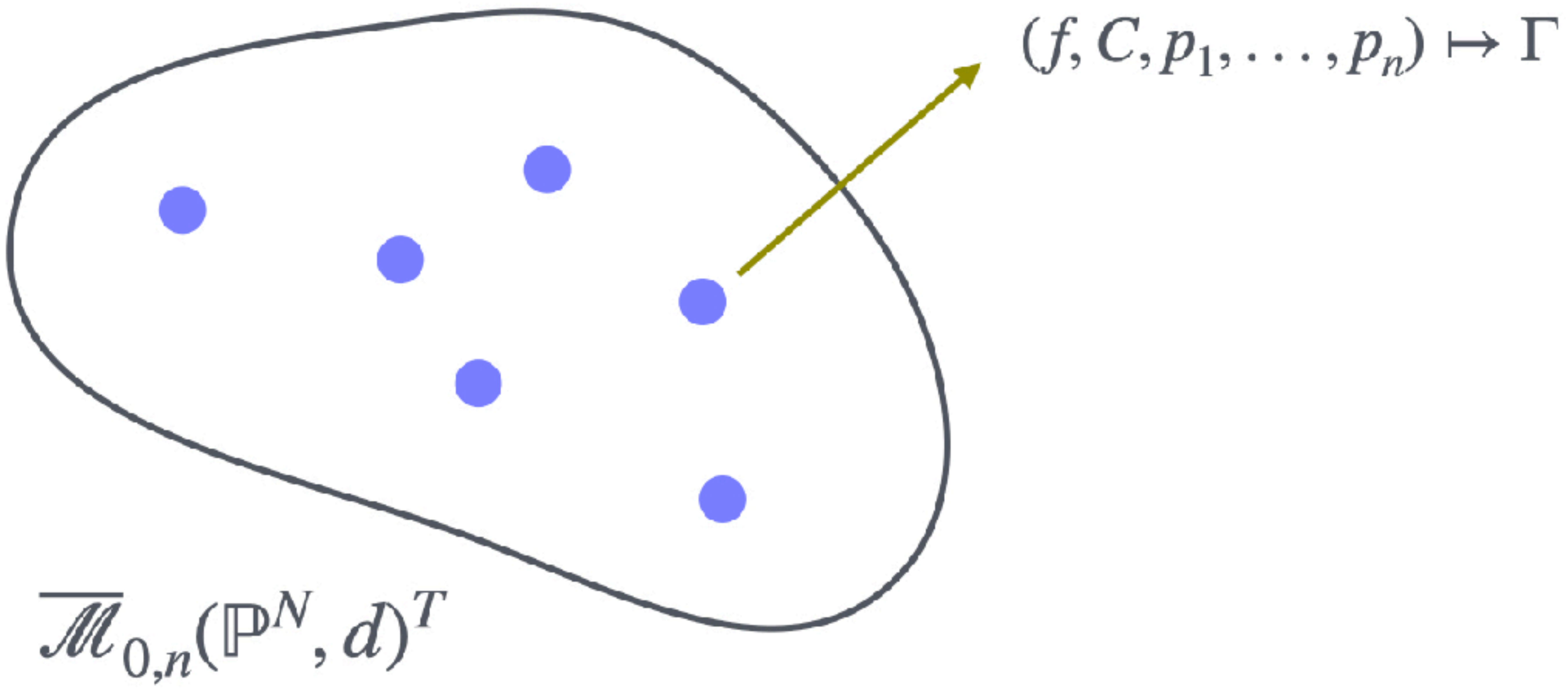
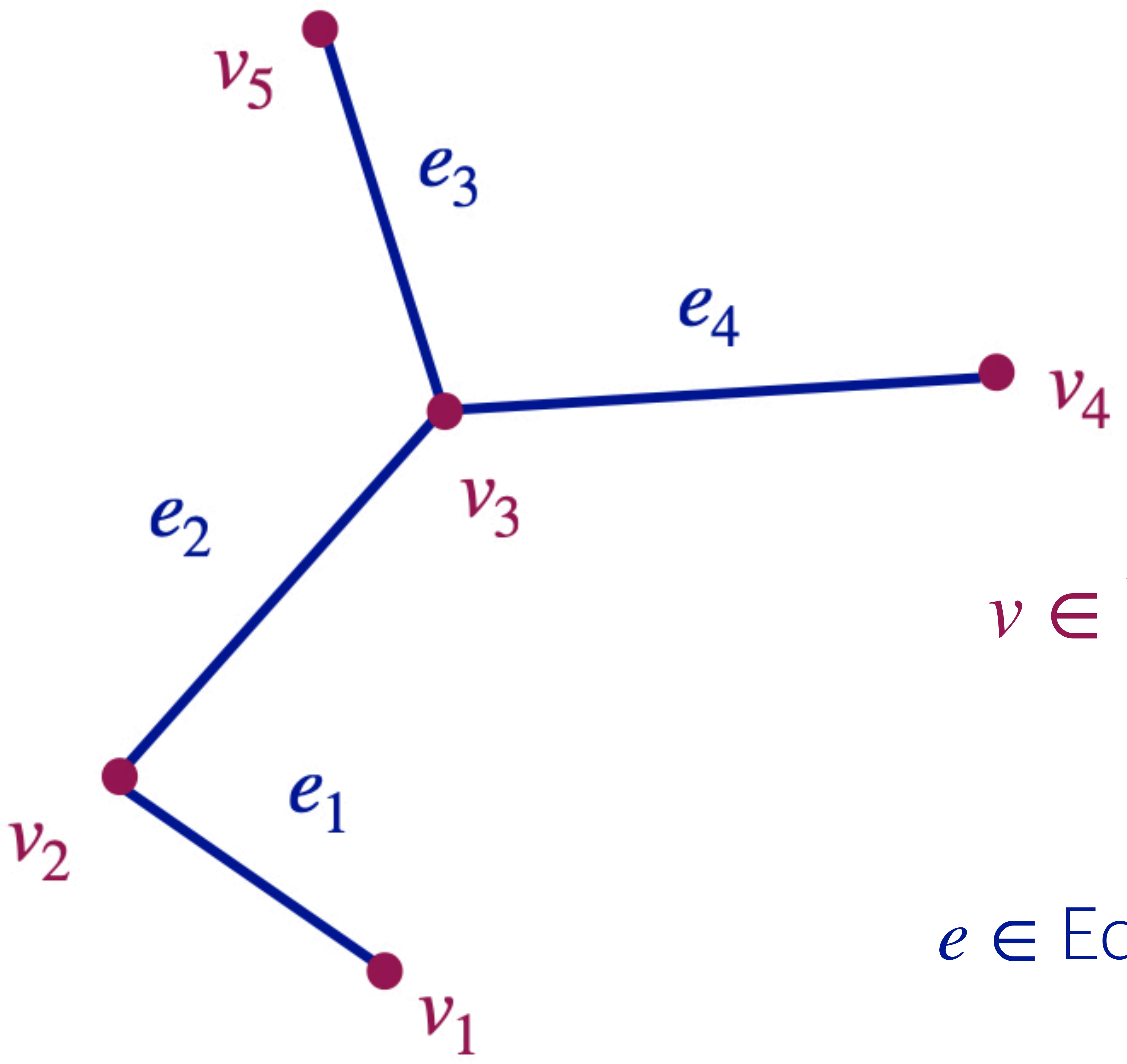
- We associate a tree Γ to a stable map.
- Each component C_i of C is mapped by f to a T-fixed point $(q_0, \ldots, q_N)$ of $\mathbb{P}^N$.



Kontsevich's approach

$\Gamma =$



Connected components of $\overline{\mathscr{M}}_{0,n}(\mathbb{P}^N, d)^T$ are in one-to-one correspondence with connected trees Γ with labels i_v , S_v and d_e satisfying certain conditions.

$v \in \mathrm{Vert}(\Gamma) \longleftrightarrow C_v$ Point or connected components of C

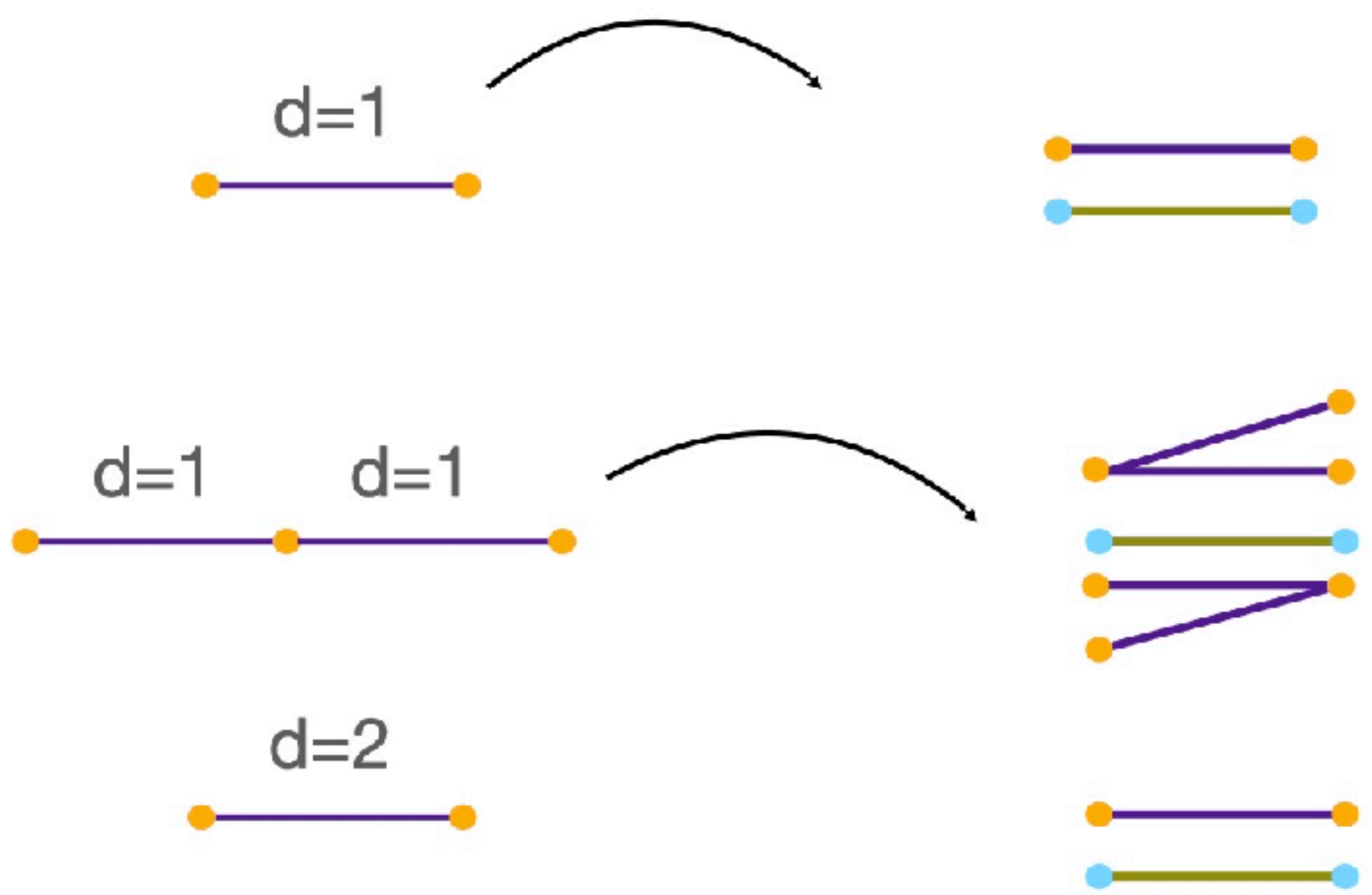
$e \in \mathrm{Edge}(\Gamma) \longleftrightarrow C_e$ Irreducible components of C mapped by f onto some coordinate line $l_e \subset \mathbb{P}^N$

Local Calabi-Yau manifolds

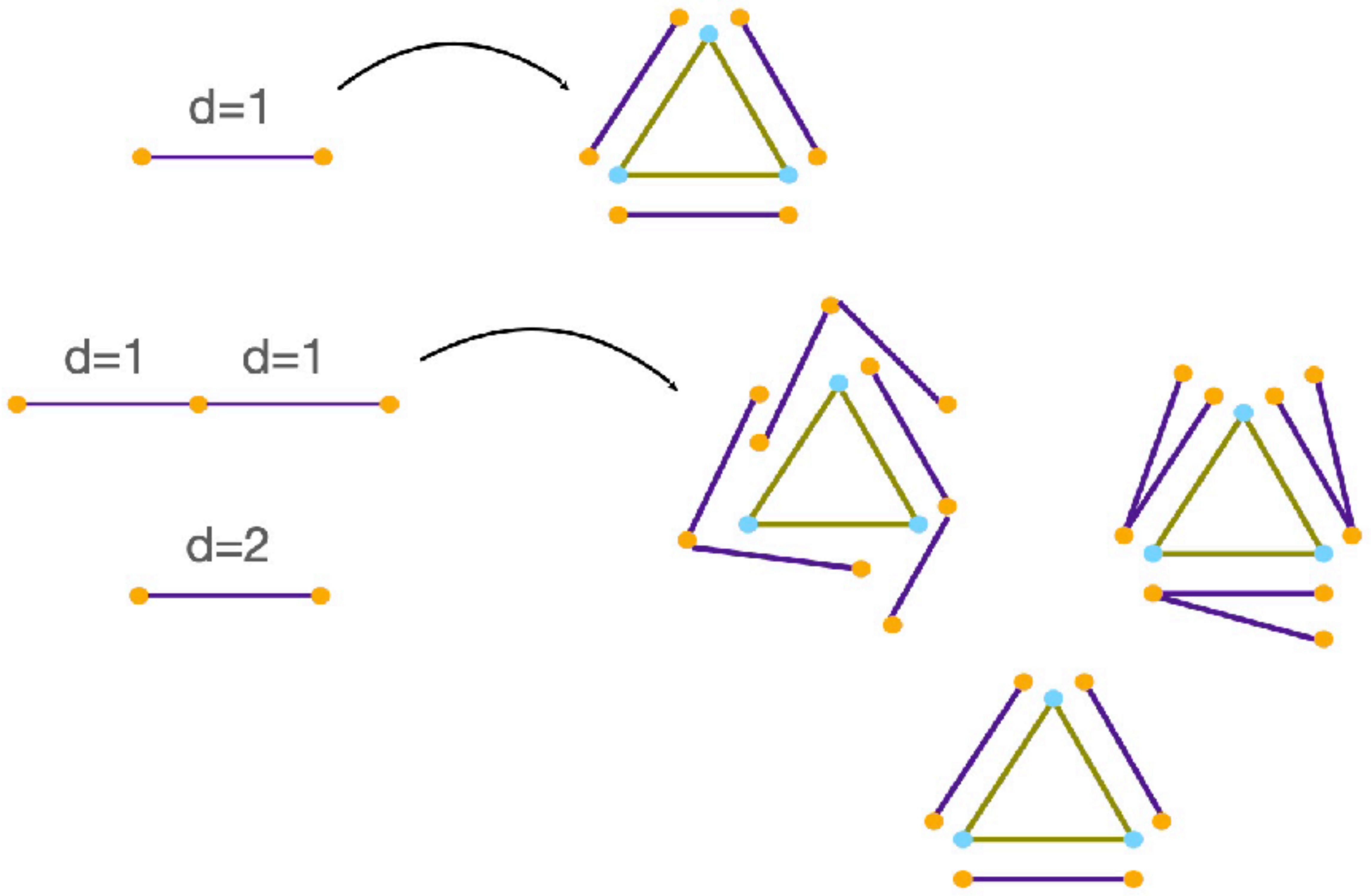
Examples

C-Y manifolds as the target space

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$$



$$\mathcal{O}(-3) \longrightarrow \mathbb{P}^2$$



K-theoretic G-W invariants

K-theoretic invariants

The $g = 0$ K-theoretic invariants of a Kähler manifold X are holomorphic Euler characteristics over the moduli space of stable maps $\overline{\mathscr{M}}_{0,m}(X, \beta)$ is given by:

$$\langle t_1(q), \ldots, t_m(q) \rangle_{0,m,\beta} = \chi_{\overline{\mathscr{M}}_{0,m}(X,\beta)} \left(ev_1^* t_1(L_1) \otimes \ldots \otimes ev_m^* t_m(L_m) \otimes \mathcal{O}^{vir} \right)$$

where $t_i(q)$ are Laurent polynomials in q with coefficients in $K(X)$.

$\mathcal{O}^{vir}$ is the virtual structure sheaf of $\overline{\mathscr{M}}_{0,m}(X, \beta)$

The K-theoretic invariants are encoded in the K-theoretic Givental J-function.

K-theoretic invariants

Let X be a smooth projective variety. Let $K(X) = K^0(X, \mathbb{Q})$ be the Grothendieck group of holomorphic vector bundles with rational coefficients.

The $\mathcal{J}$-function is a generating function that includes the previous correlators, it has the following form:

$$\mathcal{J}_X(t(q,Q)) := (1-q)\phi_0 + t(q,Q) + \sum_{d,n}\sum_{\alpha} \phi^{\alpha} Q^d \left\langle t_1(q,Q), \ldots, t_n(q,Q), \frac{\phi_\alpha}{1-qL} \right\rangle_{0,n,d}$$

Where $\{\phi_\alpha\}$ is the basis of the ring $K(X)$ for $\alpha = 0,\ldots,N$ with $\phi_0 = 1 = [\mathcal{O}_X]$ the identity element. Let $\{\phi^\alpha\}$ the dual basis. Let Q^d be the Novikov variables and q is a formal parameter.

K-theoretic invariants

The $\mathcal{J}$-function for $X = \mathbb{P}^N$. Considering the action of the torus T^{N+1} of diagonal matrices $diag(\Lambda_0, \ldots, \Lambda_N)$ such that $\Lambda_i \in \mathbb{C}^*$, acting on $\mathbb{P}^N$.

The case in which the input $t(q, Q)$ is 0, the $\mathcal{J}$-function has the following form:

$$\mathcal{J}_{\mathbb{P}^N}(0) := 1 - q + \sum_{i,d} \phi_i Q^d \left\langle \frac{\phi^i}{1 - qL} \right\rangle_{0,1,d} = (1-q) \sum_{d \geq 0} \frac{Q^d}{\Pi_{i=0}^N (1 - qP\Lambda_i^{-1})(1 - q^2 P\Lambda_i^{-1}) \cdots (1 - q^d P\Lambda_i^{-1})}$$

where P is the hyperplane class.

Connecting quantum cohomology and quantum K-theory

Connecting Quantum Cohomology and Quantum K-theory

[H. Jockers and P. Mayr, 1808.02040] [S. Garoufalidis and E. Scheidegger, 2101.07490] Studied and explored the following lifting:

BPS partition functions of $\mathcal{N}=2$ susy 3d gauge theories

(UV) model

3d world-volumes $\Sigma \times_q S^1$

BPS index on the 3d world-volume

IR limit

Givental's permutation equivariant quantum K-theory on Kähler manifolds

Holomorphic Euler characteristics

1-dim lifting

Small radius limit

$q = e^{-\beta\hbar}$

$q \to 1$

2d $\mathcal{N} = (2,2)$ GLSM

(UV) model

2d world-sheet Σ

IR limit

Quantum cohomology on Kähler manifolds

GW-invariants

Connecting Quantum Cohomology and Quantum K-theory

QUANTUM COHOMOLOGY (2d)	*counterpart*	QUANTUM K-THEORY (3d)
GW-invariants	↔	K-theoretic GW-invariants
Intersection numbers on moduli spaces of stable maps from Σ to X.	↔	Holomorphic Euler characteristics of vector bundles over those moduli spaces.
Rational numbers.	↔	Integer numbers.
Counting instanton's world-sheets.	↔	BPS index.
Recovered from the small radius limit of the 3d theory.	↔	q-deformation of quantum cohomology.

Connecting Quantum Cohomology and Quantum K-theory

Quantum K-theory on local Calabi-Yau 3-folds **(3d)**

K-theoretic GW-invariants

→

Givental's J-function for:

$g = 0$ → [H. Jockers and P. Mayr] [S. Garoufalidis and E. Scheidegger]

G-V invariants

Degeneracies of BPS states

5d target space (obtained by an M-theory compactification of X)

$g > 0$ → [You-Cheng Chou, L. Herr, Y.-P. Lee] [Givental]

↕

Quantum cohomology on local Calabi-Yau 3-folds **(2d)**

GW-invariants

→

Calculate the localization results of

[A. Klemm, E. Zaslow,....]

for $g \geq 0$

Work in progress

We focus on the topological A-model in string theory.

Quantum K-theory on local Calabi-Yau 3-folds

(3d)

K-theoretic GW-invariants via the Atiyah-Bott localization method ⟷ *Reconstruct from Givental's J-function the combinatorics of graphs for:*

$g = 0$ *(we have some results)*

$g > 0$ *(2nd work)*

↕

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CONJECTURE

Gopakumar-Vafa invariants

Quantum cohomology on local Calabi-Yau 3-folds (2d)

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THANK YOU

=)

[some images were taken from: https://art.djnavarro.net/]