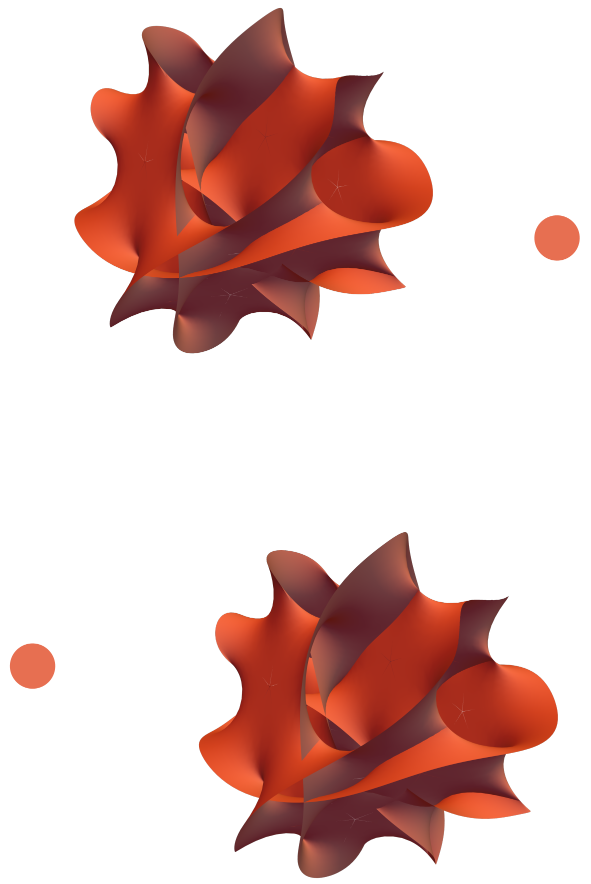
Ansatzing ε-factorized Differential Equations For Feynman Integrals

Apparent Singularities and an Application to 5PM Black Hole Scattering



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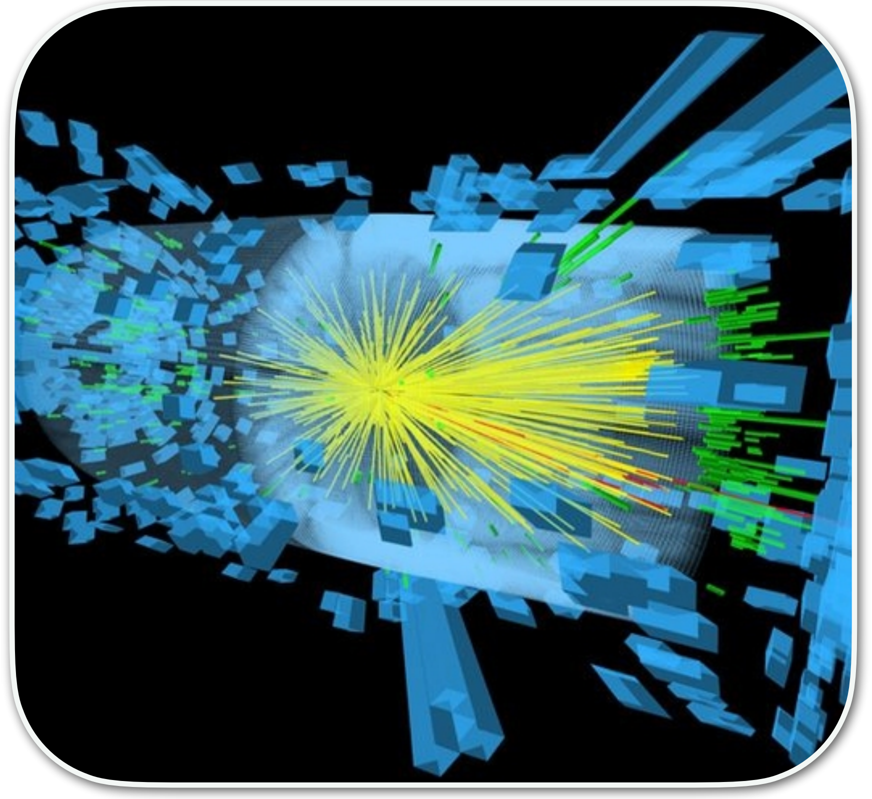
MITP Workshop Arithmetic of Calabi—Yau Manifolds
27/03/2025

Based on work in collaboration with
Hjalte Frellesvig, Roger Morales, Matthias Wilhelm, Stefan Weinzierl

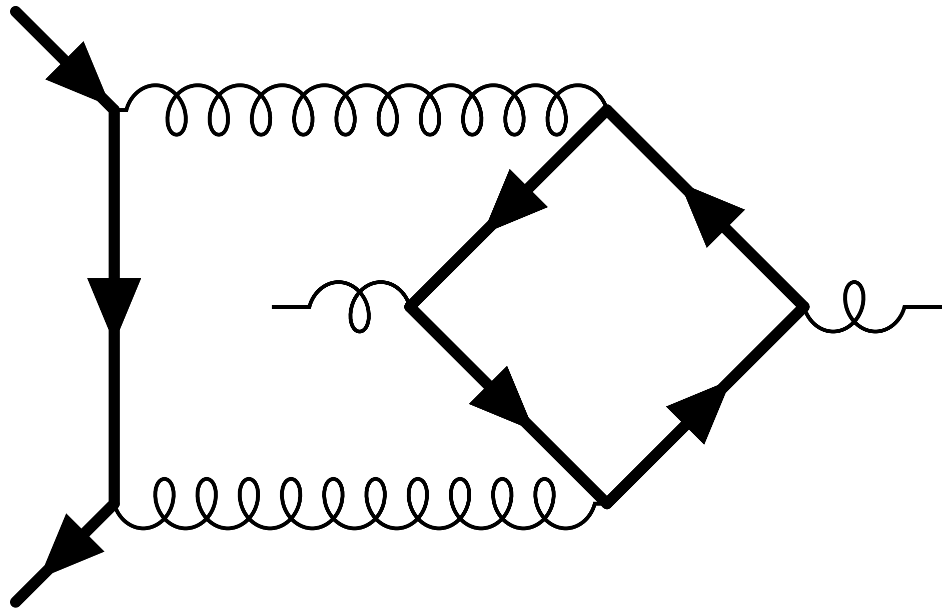
arXiv:2412.12057, JHEP 02(2025)209

Feynman Integrals

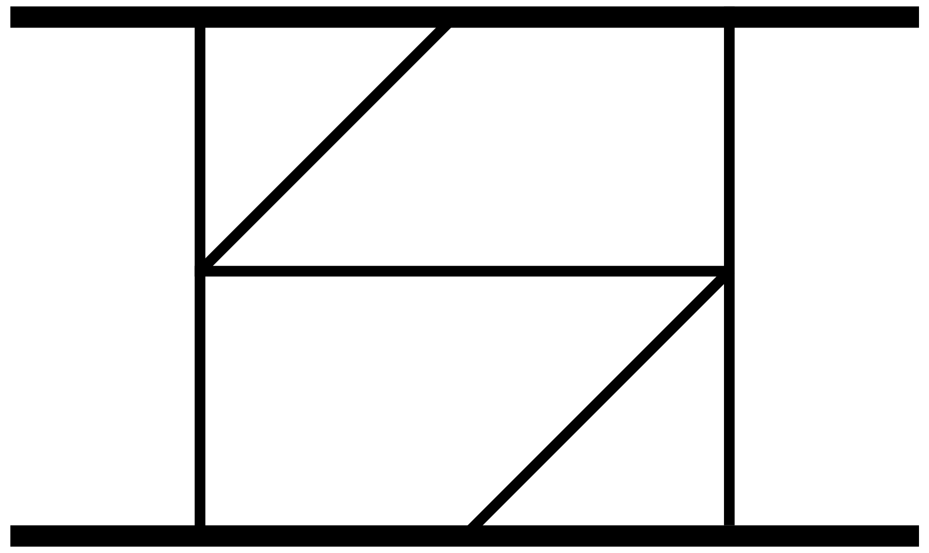
Building blocks of scattering amplitudes

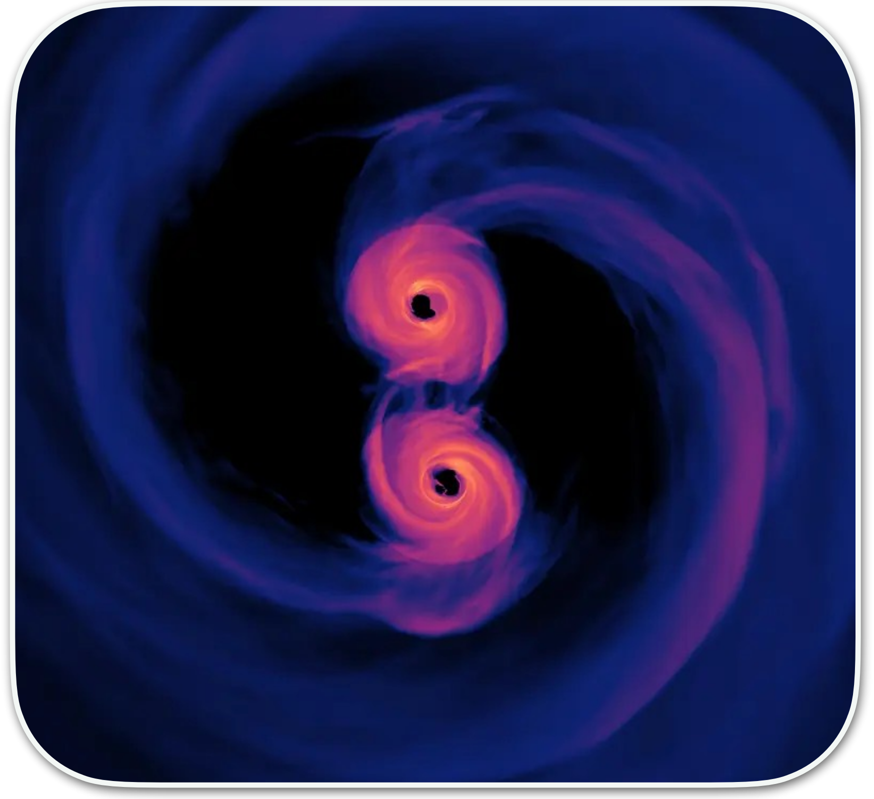


Particle Physics



Gravity





Scattering of…

Particles

Black holes

Theory independent building blocks capturing most loop-level complexity

Feynman Integrals

Basics

Dimensional regularization
$d = d_0 - 2\varepsilon$

$$I_{\nu_1 \ldots \nu_m} = \int \prod_i \frac{\mathrm{d}^d l_i}{i\pi^{d/2}} \frac{1}{\prod_j D_j^{\nu_j}}$$

Propagators $\nu_j \in \mathbb{Z}$
Numerators: $\nu_j < 0$

For simple integrals fine
For systematic study, parametric representation useful

Feynman Parametrization

$$I_{\nu_1 \ldots \nu_m} \sim \int \left(\prod \mathrm{d}\alpha_i \alpha_i^{\nu_i - 1} \right) \delta(1 - \sum \alpha_i) \frac{U^{\nu-(l+1)d/2}}{F^{\nu - ld/2}}$$

$U(\alpha_i)$, $F(\alpha_i)$ are Symanzik polynomials, predictable from graph, independent of d

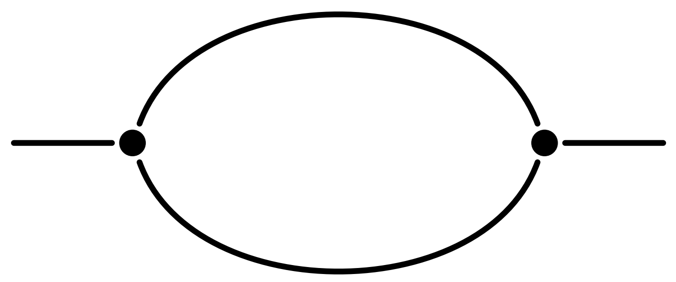
What we want:

Laurent series of $I_{\nu_1 \ldots \nu_m}$ in ε

Feynman Integrals

A simple example

Bubble integral



$$I_{\nu_1\nu_2} \sim \int \frac{\mathrm{d}l^d}{i\pi^{d/2}} \frac{1}{(l^2 - m_1^2)^{\nu_1}((l-p)^2 - m_2^2)^{\nu_2}}$$

$$\sim \int (\mathrm{d}\alpha_1 \alpha_1^{\nu_1 - 1})(\mathrm{d}\alpha_2 \alpha_2^{\nu_2 - 1})\delta(1 - \alpha_1 - \alpha_2)\frac{U^{\nu - d}}{F^{\nu - d/2}}$$

with $U = \alpha_1 + \alpha_2$
$F = -\alpha_1\alpha_2 p^2 + U(m_1^2\alpha_1 + m_2^2\alpha_2)$

$$I_{11} = \frac{e^{\gamma_e \varepsilon}\Gamma\left(2 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2} - 1\right)^2}{\Gamma(d-2)}(-p^2)^{\frac{d}{2}-2} = \frac{1}{\varepsilon} + (2 - \log(-p^2)) + O\left(\varepsilon^1\right)$$

for $d = 4 - 2\varepsilon$

Feynman Integrals

Towards differential equations

Two important features:

1. Integration-by-parts relations $\int \left(\prod_i \mathrm{d}^D l_i \right) q^\mu \frac{\partial}{\partial l_j^\mu} \left(\frac{1}{\prod_j D_j^{\nu_j}} \right) = 0$

   Total derivative

   →Generate linear relations between Feynman integrals

   →Can find a minimal basis of Feynman integrals: **Master Integrals**

2. Derivatives of Feynman Integrals are again Feynman Integrals

   w.r.t. external kinematics

Differential equations for Feynman Integrals

Separating Dimension and Kinematics

"Main" tool to evaluate Feynman Integrals

Basis of Master Integrals $I = \{I_1, \dots, I_n\}$
Kinematic variables $x = \{x_1, \dots, x_n\}$

Matrix of differential 1-forms

$$\mathrm{d}I = A(x, \varepsilon) I$$

Find "good" basis $J = UI$ such that ε factorizes

$$\mathrm{d}J = \varepsilon \tilde{A}(x) J$$

Solution given by path-ordered exponential

$$J = \mathbb{P}\exp\left(\varepsilon \int \tilde{A}\right) J_0$$

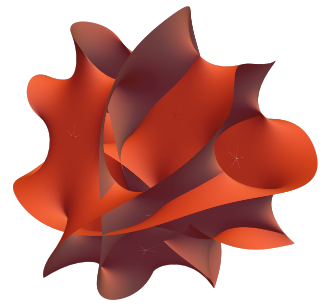
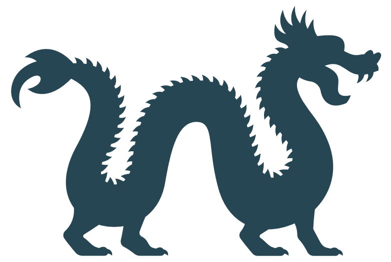
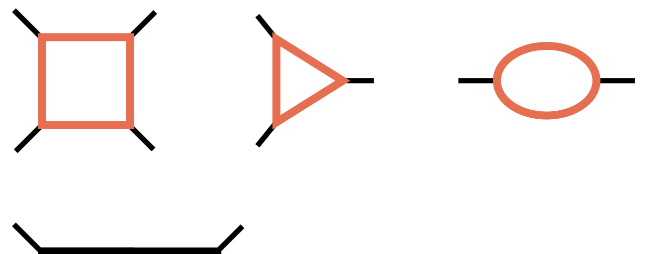
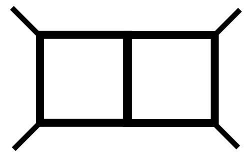
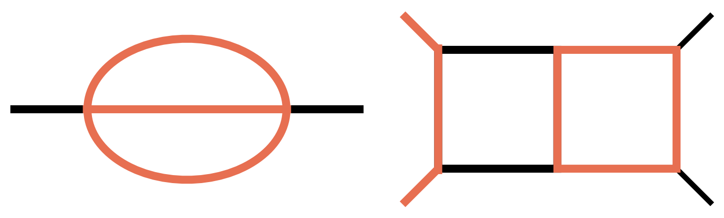
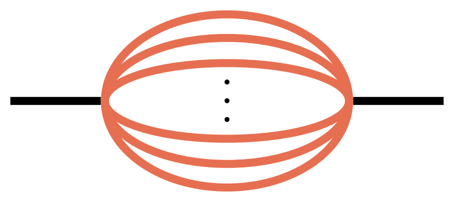
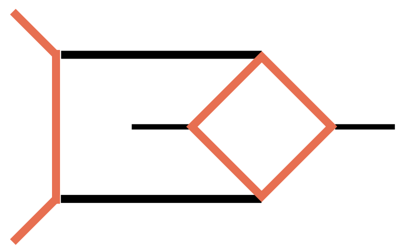
Let $\mathcal{C}(t)$ be an integration contour
with $t \in [0,1]$ $\mathcal{C}(0) = x_0$ $\mathcal{C}(1) = x$

$$J = \varepsilon^0 J_0 + \varepsilon^1 \int_0^1 \mathrm{d}t \tilde{A}(t) J_0 + \varepsilon^2 \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \tilde{A}(t_1)\tilde{A}(t_2) J_0 + \mathcal{O}(\varepsilon^3)$$

A zoo of geometries

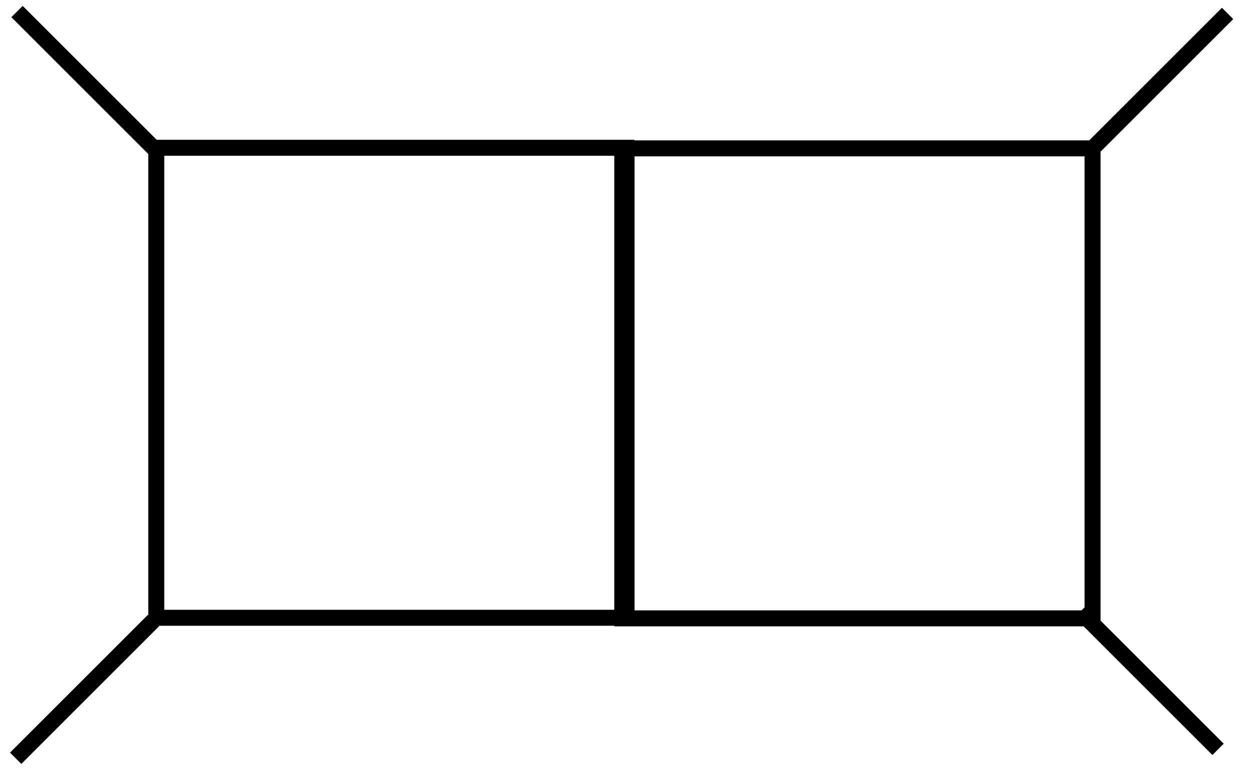
Integrals associated to geometries

Determines suitable function space

					
Geometry	$\mathbb{P}^1$	Elliptic curves	Calabi—Yaus	Higher-genus Curves	???
Functions	(poly)logarithms $\log$, Li_n, MPL	Elliptic functions K, E, Π, eMPL modular forms	? Expansions, in some cases modular forms [talks by Sara and Claude]	Higher-genus functions heMPL Siegel modular forms	
Examples	  Most planar massless integrals				Here be dragons

How do we identify geometry of integrals?

Symanzik Polynomials



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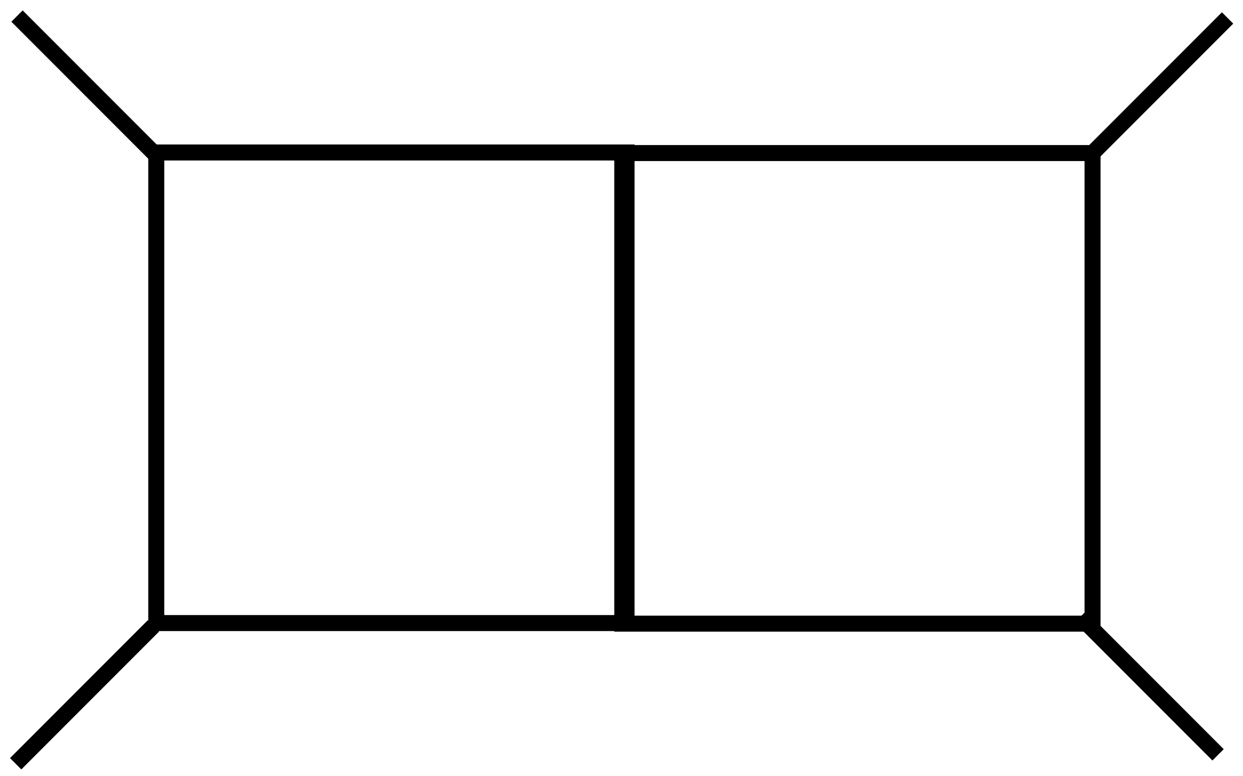
Study varieties defined by Symanzik polynomials U **and** F

In general complicated…

How do we identify geometry of integrals?

Maximal Cuts



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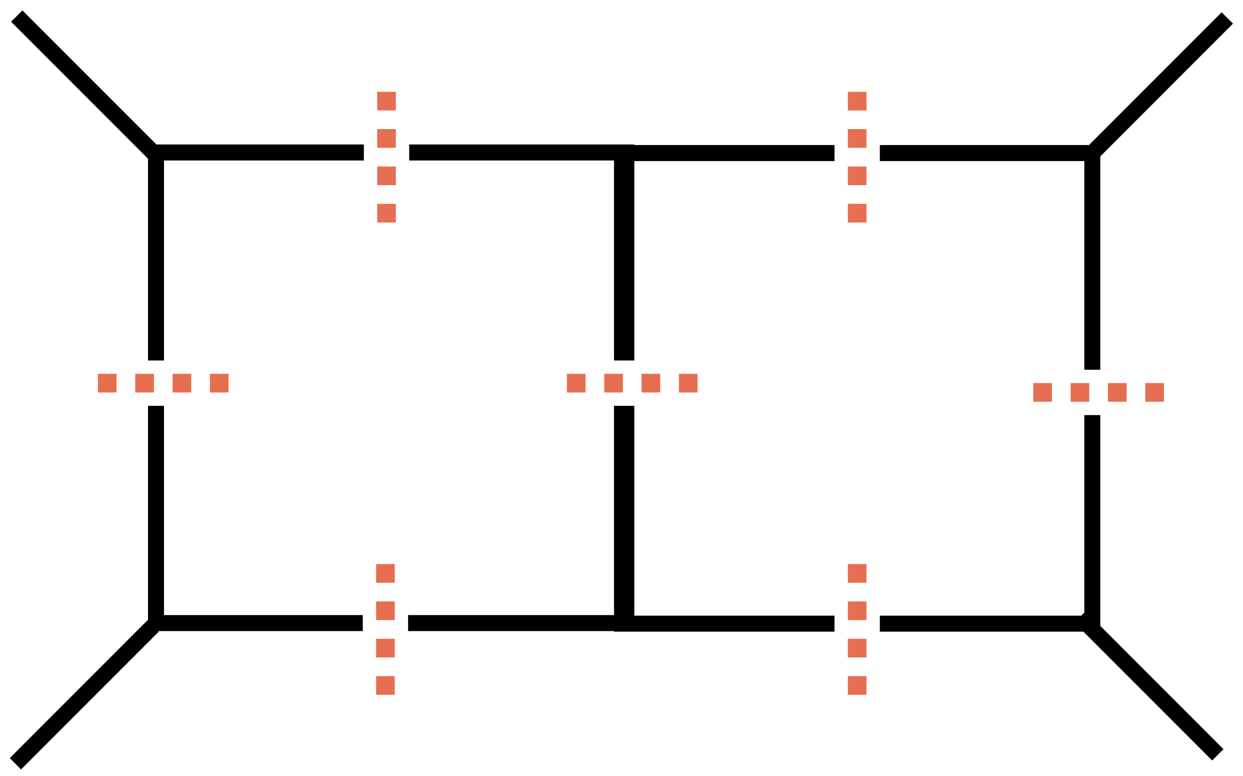


$I/.\{\frac{1}{D_j} \to \delta(D_j)\}$

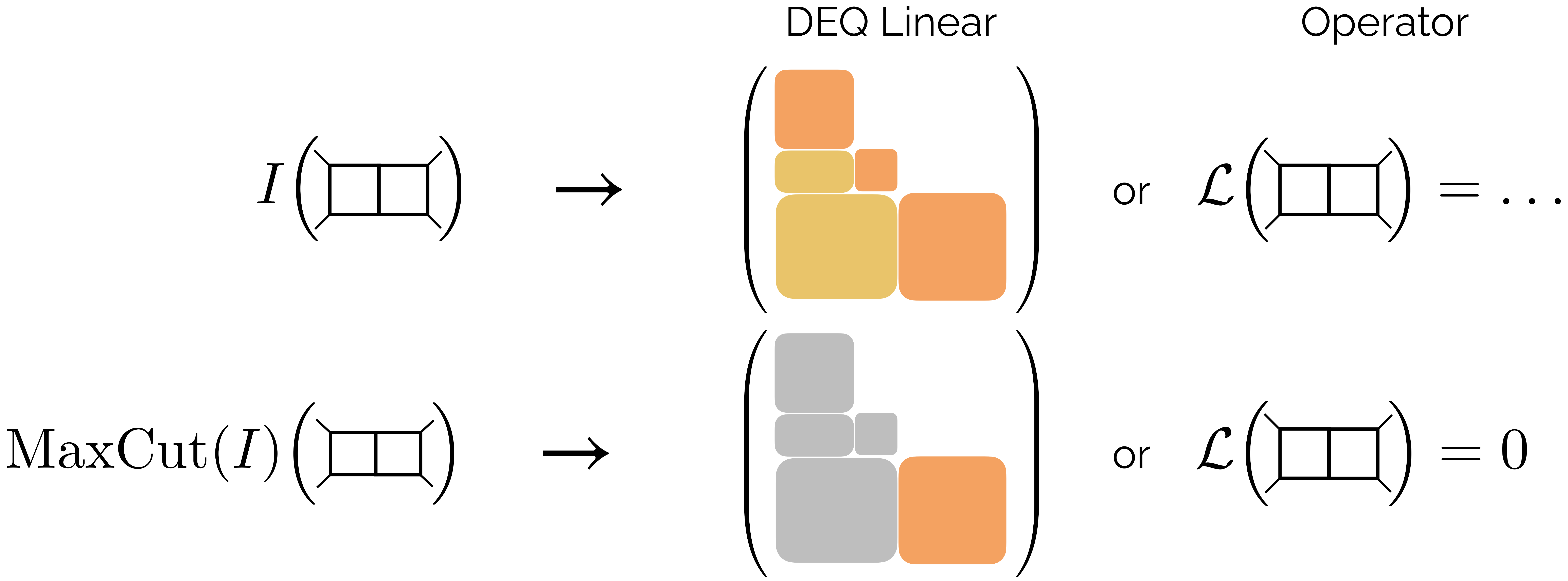
Skeletonized version of integral

Much simpler to extract geometry

Max-Cuts

Differential Equations

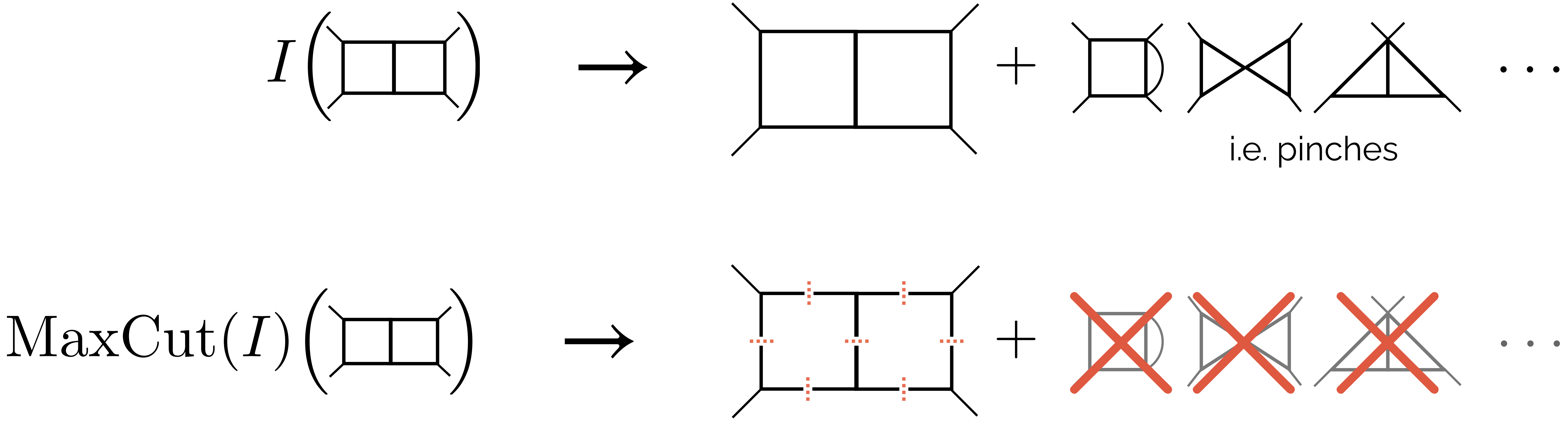
In differential equations for Feynman integrals, maximal cuts are homogeneous part



Max-Cuts

Graphs

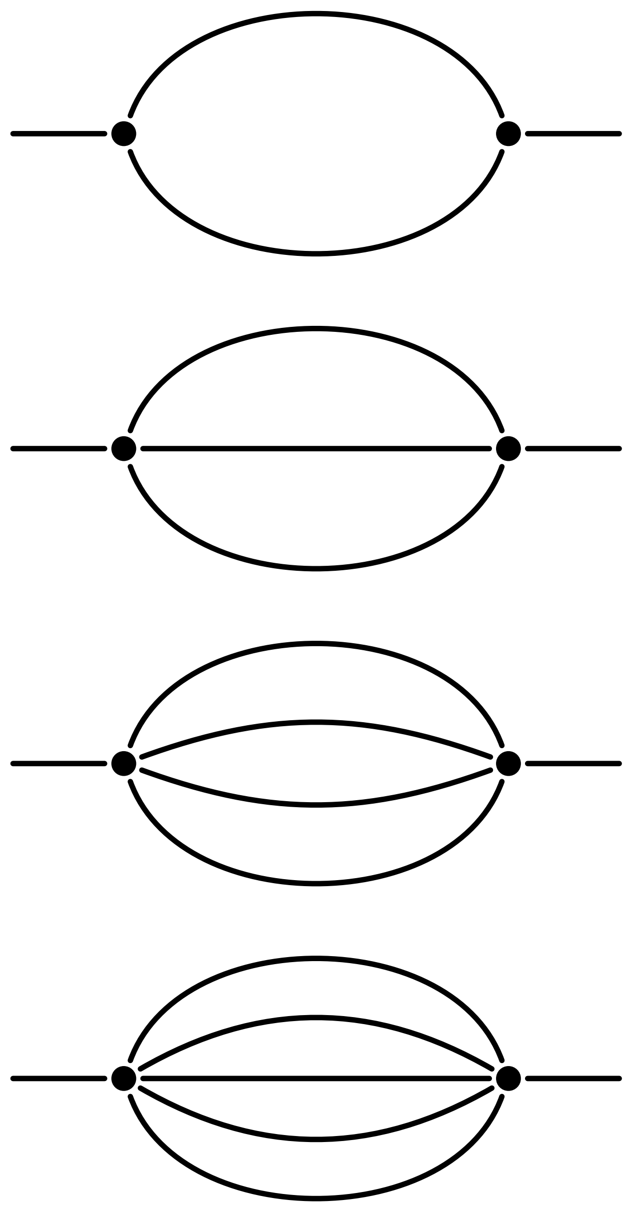
Inhomogeneities translate into (sub)graphs



$\mathrm{MaxCut}(I)|_{\varepsilon=0}$ ~ period of geometry

Bananas: A Calabi—Yau Prototype

Calabi–Yau…



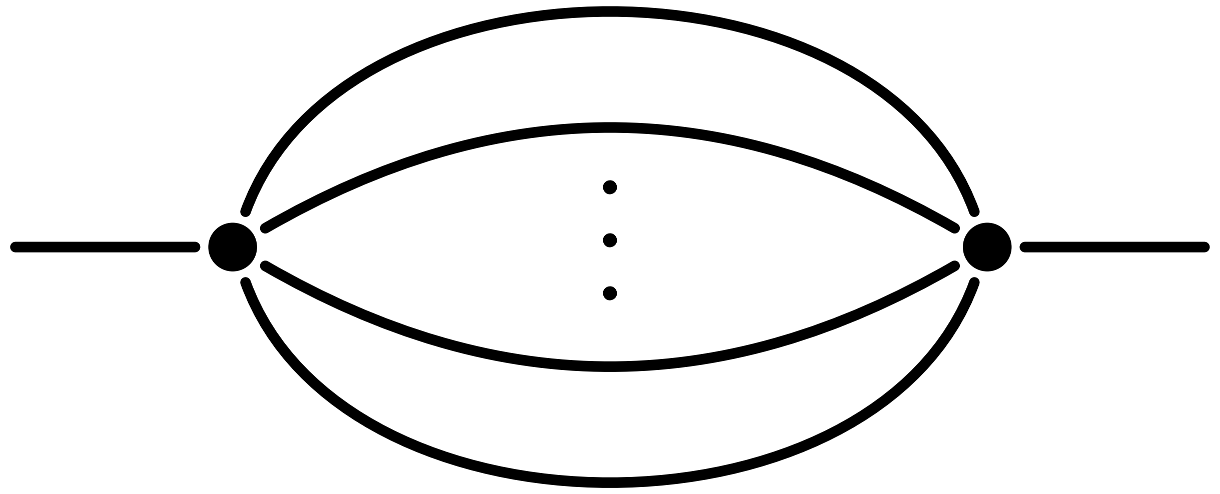
0-fold

1-fold = elliptic curve
"Sunrise"

2-fold = K3 surface

3-fold

⋮



ℓ-loop Banana integral

$\hat{=}$

$(\ell - 1)$ -fold Calabi–Yau manifold

Simplification: Equal-mass → single scale

Kinematic variable $x = \frac{p^2}{m^2}$

ε-factorized Differential Equations for Calabi—Yau Integrals

$$\mathrm{d}I = \varepsilon A I$$

Just two ingredients

One Seed Integral I
via Picard—Fuchs
operator

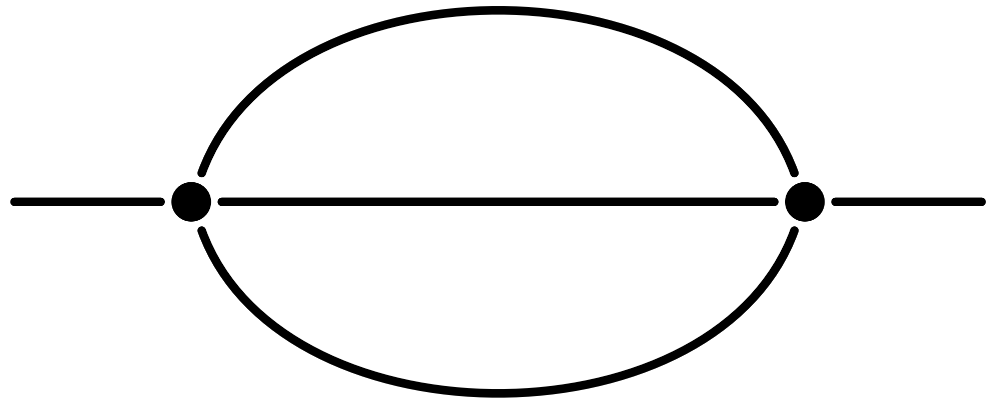
+

Ansatz for
differential equation A

Idea: Fix Ansatz by eliminating non-ε-factorizing terms

An elliptic example

Sunrise integral in $d = 2 - 2\varepsilon$



The seed integral: $I = I_{111}$

Picard—Fuchs operator (homogenous ~ maximal cut)

$$I''(x) + \left(\frac{2-2\epsilon}{2x} + \frac{2\epsilon+1}{x-9} + \frac{2\epsilon+1}{x-1}\right) I'(x) + \frac{(-2\epsilon-1)(x(-2\epsilon-2)-2\epsilon+6)}{2(x-9)(x-1)x} I = 0$$

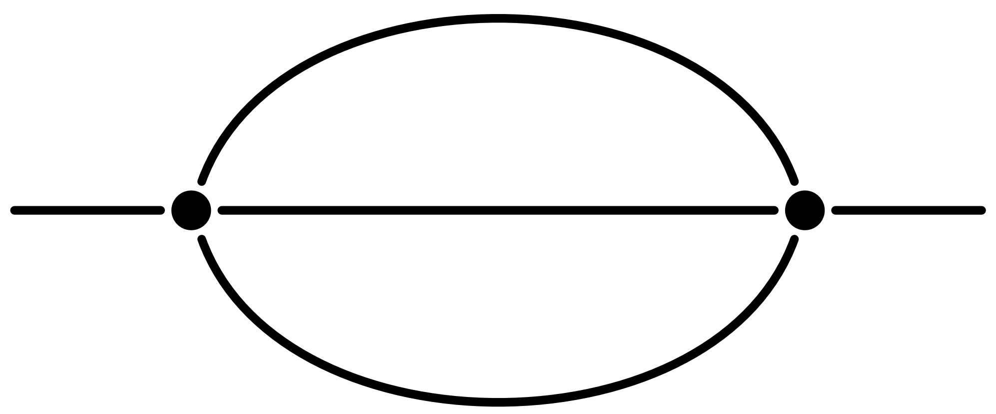
Ansatz:

$$M_1 = \frac{I}{\varpi}$$

$$M_2 = \frac{1}{\varepsilon} J(x) \frac{\mathrm{d}M_1}{\mathrm{d}x} + F(x) M_1$$

An elliptic example

Sunrise integral in $d = 2 - 2\varepsilon$



Ansatz + Integral: Two together lead to differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F(x) & 1 \\ * & * \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

determined by ansatz and Picard—Fuchs operator contains terms $\varepsilon^{<0}$

Matching: Requiring $\varepsilon^{<0}$ to vanish

→ Constraints on $F(x), J(x), \varpi(x)$

Solving constraints: Constraints consistently solvable

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \varepsilon \begin{pmatrix} \eta_2 & 1 \\ \eta_4 & \eta_2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

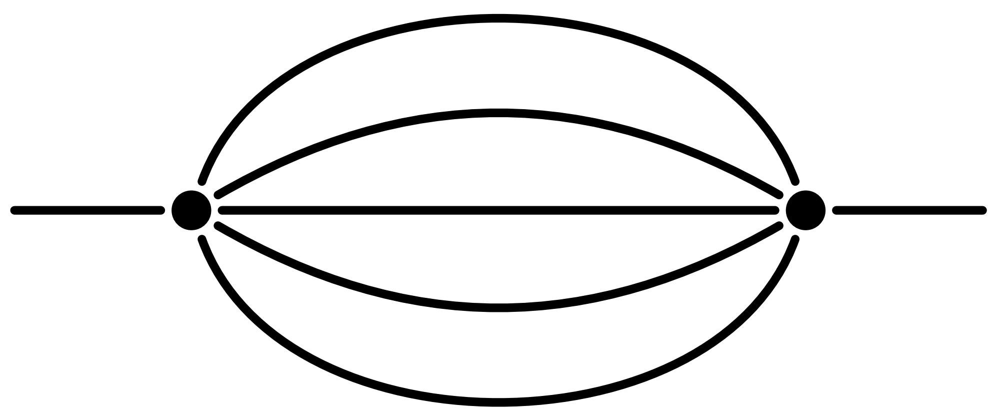
$\varpi = \psi_1 \quad J(x) = \frac{\mathrm{d}\tau}{\mathrm{d}x} \quad \tau = \frac{\psi_2}{\psi_1}$

ψ_i: period of elliptic curve

η_i: modular form, weight i

A CY 3-fold example

Four-loop banana integral in $d = 2 - 2\varepsilon$



Simplest non-trivial Calabi—Yau integral
Associated to Hulek—Verrill Calabi—Yau 3-fold

The seed integral: $I = I_{11111}$

Picard—Fuchs operator
(homogenous ~ maximal cut)

$$I^{(4)}(x)+\frac{2\left(5x^3\epsilon+5x^3-105x^2\epsilon-140x^2+259x\epsilon+777x+225\epsilon-450\right)I'''(x)}{(x-25)(x-9)(x-1)x}+\frac{\left(35x^3\epsilon^2+60x^3\epsilon+25x^3-343x^2\epsilon^2-945x^2\epsilon-518x^2-363x\epsilon^2+1554x\epsilon+1839x-225\epsilon^2+675\epsilon-450\right)I''(x)}{(x-25)(x-9)(x-1)x^2}+\frac{(2\epsilon+1)\left(25x^2\epsilon^2+40x^2\epsilon+15x^2-42x\epsilon^2-322x\epsilon-196x-207\epsilon^2-78\epsilon+285\right)I'(x)}{(x-25)(x-9)(x-1)x^2}+\frac{(2\epsilon+1)(3\epsilon+1)(4\epsilon+1)(x\epsilon+x+3\epsilon-5)}{(x-25)(x-9)(x-1)x^2}$$

Ansatz:

$$M_1 = \frac{1}{\varpi} I_{11111}$$

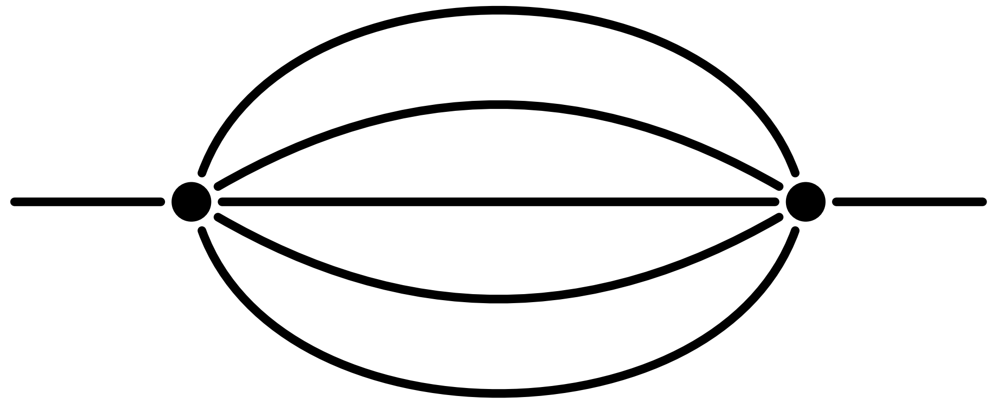
$$M_2 = \frac{1}{\varepsilon} J \frac{\mathrm{d}M_1}{\mathrm{d}x} + F_{11} M_1$$

$$M_3 = \frac{1}{\varepsilon} \frac{J}{K_1} \frac{\mathrm{d}M_2}{\mathrm{d}x} + F_{21} M_1 + F_{22} M_2$$

$$M_4 = \frac{1}{\varepsilon} J \frac{\mathrm{d}M_3}{\mathrm{d}x} + F_{31} M_1 + F_{32} M_2 + F_{33} M_3$$

A CY 3-fold example

Four-loop banana integral in $d = 2 - 2\varepsilon$



Ansatz + Integral:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & 1 & 0 & 0 \\ F_{21} & F_{22} & K_1 & 0 \\ F_{31} & F_{32} & F_{33} & 1 \\ * & * & * & * \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$$

Matching: Requiring $\varepsilon^{<0}$ to vanish
→ Differential Constraints on $F_{ij}(x), K_1(x), J(x), \varpi(x)$

Solving constraints: Constraints consistently solvable

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \varepsilon \begin{pmatrix} f_2 & 1 & 0 & 0 \\ f_4 & f_2' & K_1 & 0 \\ f_6 & f_4' & f_2' & 1 \\ f_8 & f_6 & f_4 & f_2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$$

$\varpi = \omega_1 \quad J(x) = \frac{\mathrm{d}\tau}{\mathrm{d}x} \quad \tau = \frac{\omega_2}{\omega_1}$

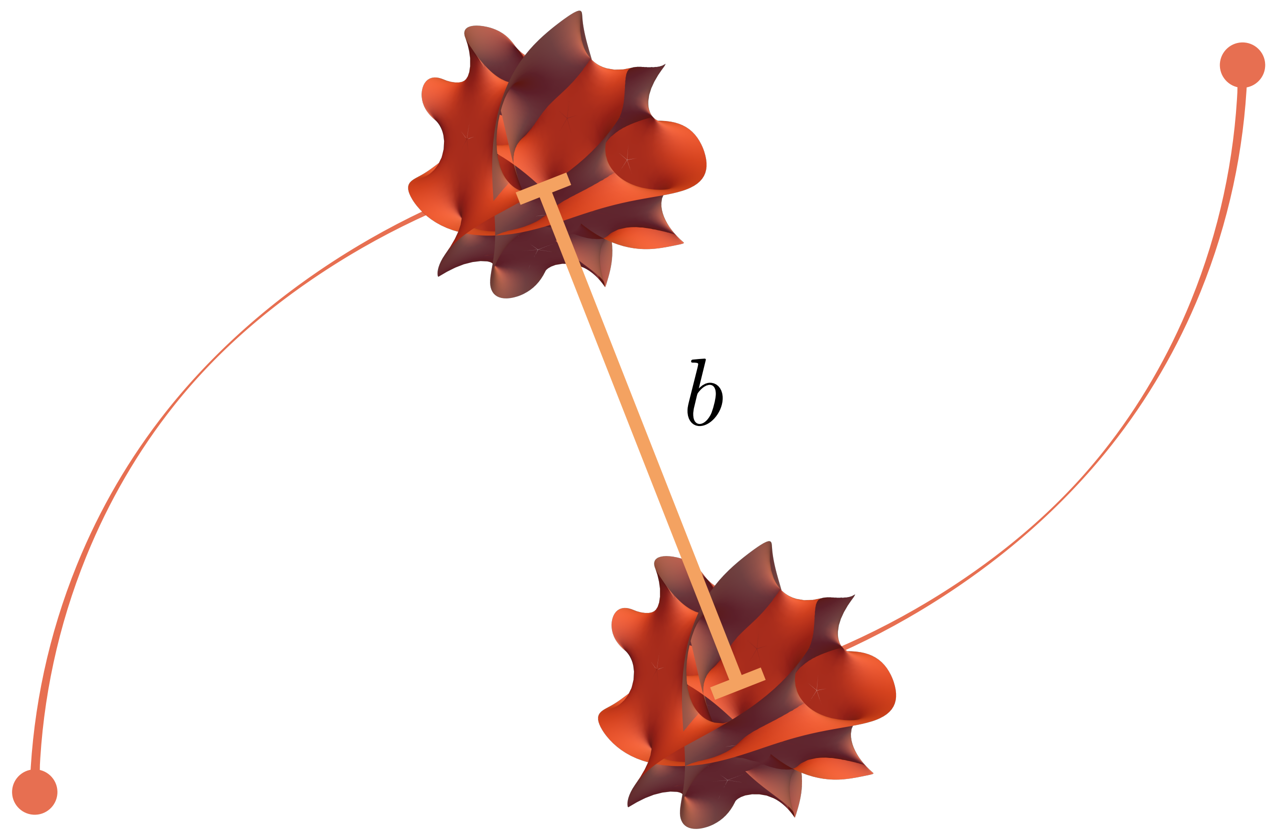
ω_i: period of Calabi—Yau
K_1: Yukawa coupling
f_i: automorphic form, weight i ?

Banana Integrals are very idealized
Can we find a “real-world” application?

Calabi—Yaus in Gravity

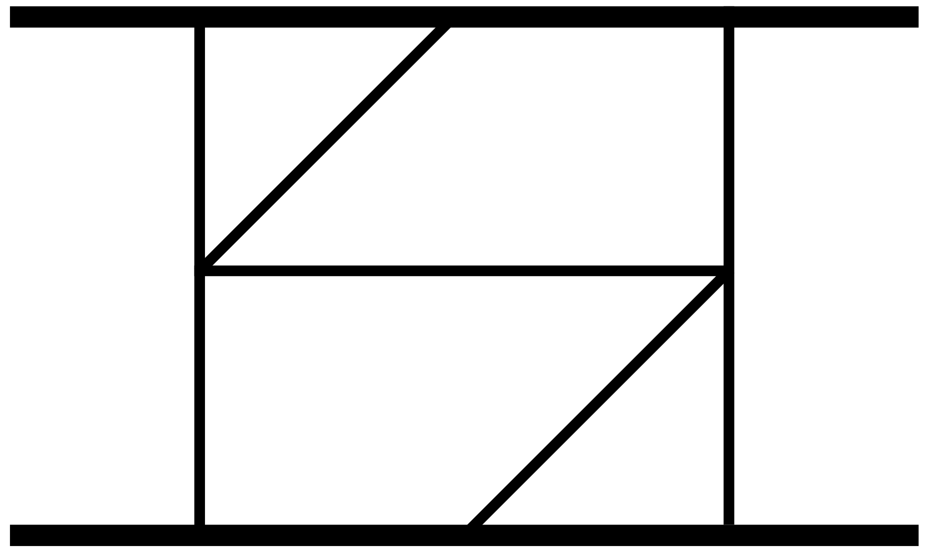
Scattering of black holes

Black holes modeled as massive scalars



Impact parameter $|b| \sim 1/|q|$

$\sim$

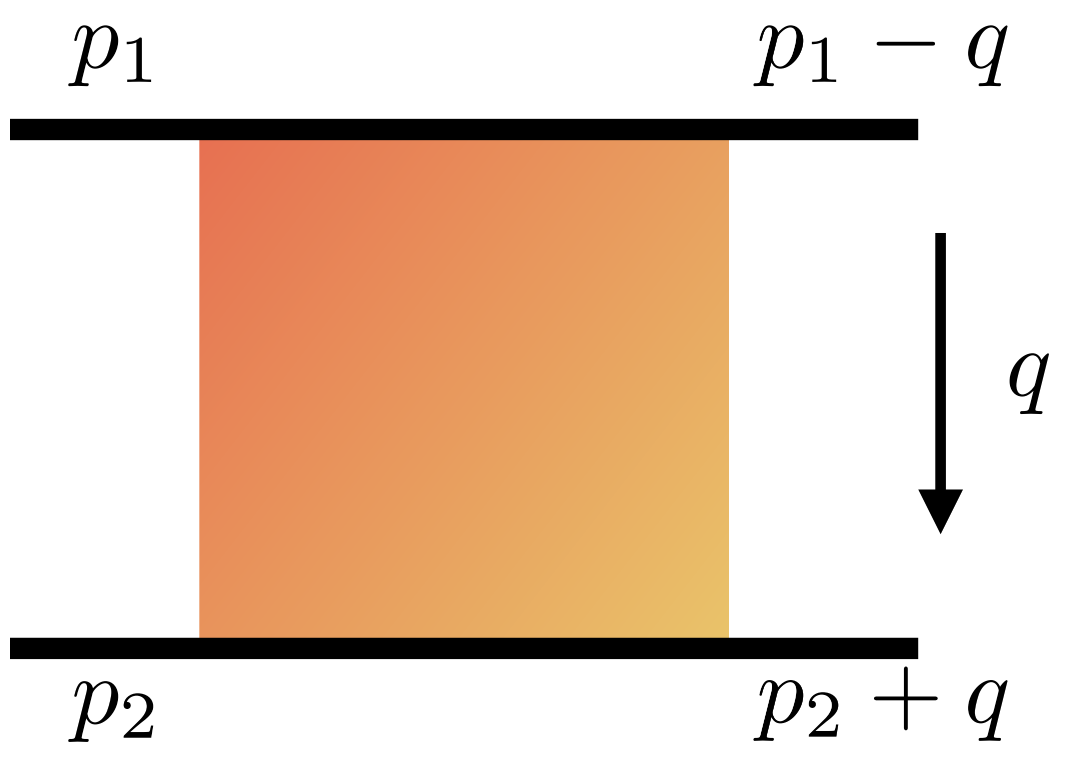


Assume long range interaction $r_s/|b| \ll 1$, thus $Gm|q| \ll 1$ $\longrightarrow$ Compute corrections in Post-Minkowskian expansion in G^n

Extract classical effects from seemingly quantum description

Integrals for Black Holes

Classical limit described by soft $|q|$ limit



$$p_1 = \overline{p}_1 - q/2$$
$$p_2 = \overline{p}_2 + q/2$$
$$\overline{p}_i \cdot q = 0$$
$$|q| \ll 1$$

Scalar propagator:

$$\frac{1}{(k+p_i)^2 - m_i^2} \sim \frac{1}{m_i}\frac{1}{2u_i \cdot k} + \mathcal{O}(q^2)$$

$u_i = \frac{\overline{p}_i}{\overline{m}_i} \quad u_i \cdot q = 0 \quad u_i^2 = 1$

$\overline{m_i}^2 = \overline{p}_i^2 = m_i^2 - q^2/4$

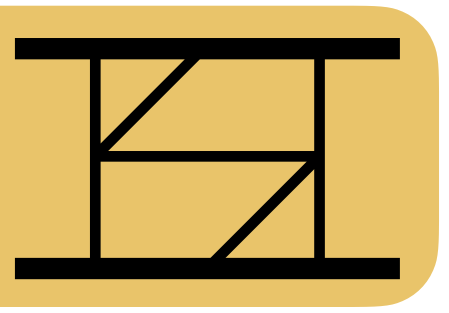
At L loops: order $|q|^{L-2} G^{L+1}$

Single kinematic scale: $y = u_1 \cdot u_2$

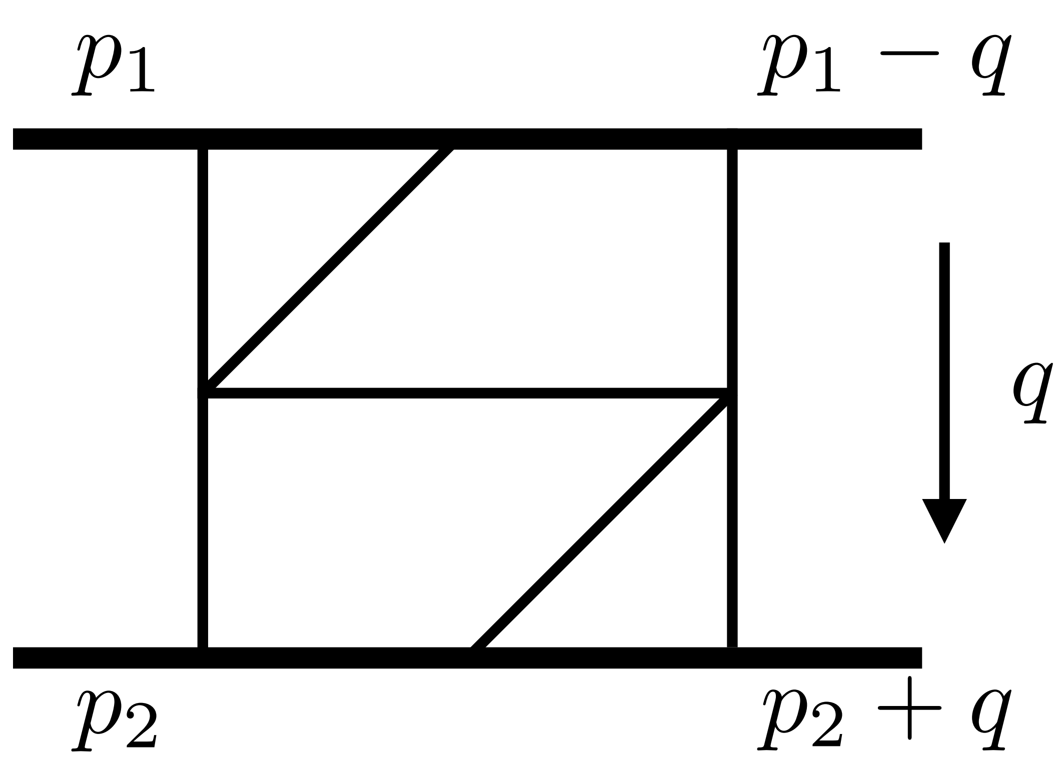
Great test cases:
- Relevance for gravitational wave physics
- Single kinematic scale
- At 3 and 4 loop: Calabi—Yau 2-folds (K3) and 3-folds appear
  [Frellesvig, Morales, Wilhelm, '23; Klemm, Nega, Sauer, Plefka, '24]
- Integrals with subtopologies (think DEQ with inhomogeneity)

For most, methods from Banana integrals are sufficient

except for one!



An Integral for 2-Self-force Correction



$\sim$

$$I_{\nu_1,\nu_2,\ldots,\nu_{11}} = \int \frac{\mathrm{d}^d k_1\, \mathrm{d}^d k_2\, \mathrm{d}^d k_3\, \mathrm{d}^d k_4}{\rho_1^{\nu_1} \rho_2^{\nu_2} \cdots \rho_{11}^{\nu_{11}}}$$

In $d = 4$, $I = I_{11111111111}$ is annihilated by

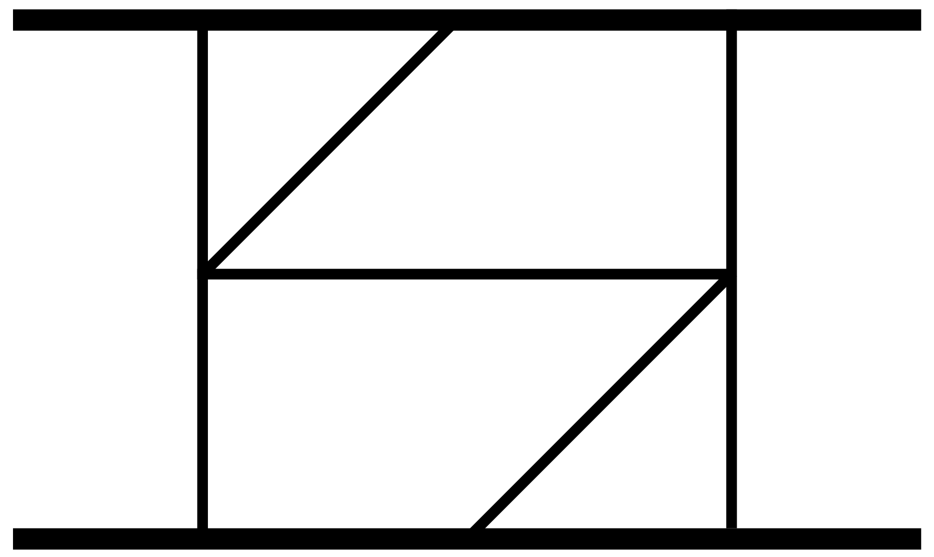
$$I^{(4)}(x) - \frac{(10y^2+2)}{y-y^3} I^{(3)}(x) + \frac{(25y^4+y^2+2)}{y^2\left(y^2-1\right)^2} I^{''}(x) + \frac{(15y^4-6y^2-1)}{y\left(y^2-1\right)^3} I^{'}(x) + \frac{4y^4-y^2+4}{4y^2\left(y^2-1\right)^3} I(x) = 0$$

Calabi—Yau operator (up to normalization): Associated to Calabi—Yau 3-fold

So where is the problem?

An Integral for 2-Self-force Correction

In $d = 4 - 2\varepsilon$, $I = I_{1111111111}$ is annihilated by



$$
\begin{aligned}
&\mathcal{L}^{(5)} I(y) = I^{(5)}(y) \\
&- I^{(4)}(y) \frac{y\left(y^2\left(16\epsilon^3+60\epsilon^2-532\epsilon-51\right)-800\epsilon^3+3552\epsilon^2+1060\epsilon+70\right)}{\left(y^2-1\right)\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)} \\
&+ I'''(y) \frac{\left(y^4\left(16\epsilon^4-64\epsilon^3-1184\epsilon^2+2576\epsilon+255\right)-3y^2\left(352\epsilon^4-2936\epsilon^3+6228\epsilon^2+2062\epsilon+141\right)+16\left(800\epsilon^4+220\epsilon^3+262\epsilon^2+75\epsilon+5\right)\right)}{\left(y^2-1\right)^2\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)} \\
&+ I''(y) \frac{y\left(4y^4\left(32\epsilon^4+108\epsilon^3-1020\epsilon^2+1009\epsilon+105\right)-y^2\left(128\epsilon^5+9776\epsilon^4-30320\epsilon^3+30120\epsilon^2+12588\epsilon+919\right)+6400\epsilon^5+81104\epsilon^4+8368\epsilon^3+2\cdots\right.}{\left(y^2-1\right)^3\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)} \\
&+ I'(y) \frac{\left(4y^6\left(208\epsilon^4+1200\epsilon^3-3312\epsilon^2+1604\epsilon+183\right)-y^4\left(64\epsilon^6+2176\epsilon^5+73232\epsilon^4-125184\epsilon^3+35788\epsilon^2+26808\epsilon+2155\right)+y^2\left(8384\epsilon^6+14662\cdots\right.\right.}{4\left(y^2-1\right)^4\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\cdots\right.} \\
&+ I(y) \frac{12y^5(1-2\epsilon)^2\left(4\epsilon^2+32\epsilon+3\right)-y^3\left(64\epsilon^6+1152\epsilon^5+20112\epsilon^4-19136\epsilon^3-980\epsilon^2+1192\epsilon+107\right)+4y\left(800\epsilon^6+400\epsilon^5+12432\epsilon^4-920\epsilon^3+23\cdots\right.}{4\left(y^2-1\right)^4\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)}
\end{aligned}
$$

Two new features compared to Bananas:

factorization $\mathcal{L}^{(5)} \overset{\varepsilon \to 0}{=} \mathcal{L}^{(1)} \mathcal{L}^{(4)}$
evanescent Master integral in $d = 4$

1. Operator with different dimensions for $\varepsilon \to 0$
2. Operator has unphysical singularity, quadratic in ε

$$y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)=0$$

Apparent Singularity Singularity of operator at which all solutions are non-singular

Apparent Singularities in Feynman Integrals

Check empirically: **While undesirable, almost all integrals have them**

When encountering them, you have some options

Option 1: Go back and make a “better" choice for integral

After extensive scan over candidate integrals: no luck

Option 2: Add additional integrals not belonging to geometry

~ generalization of integrals of the third kind

Requires additional analysis at integrand level

Option 3: Go back and make a better Ansatz

Ansatzing ε-factorized DEQs

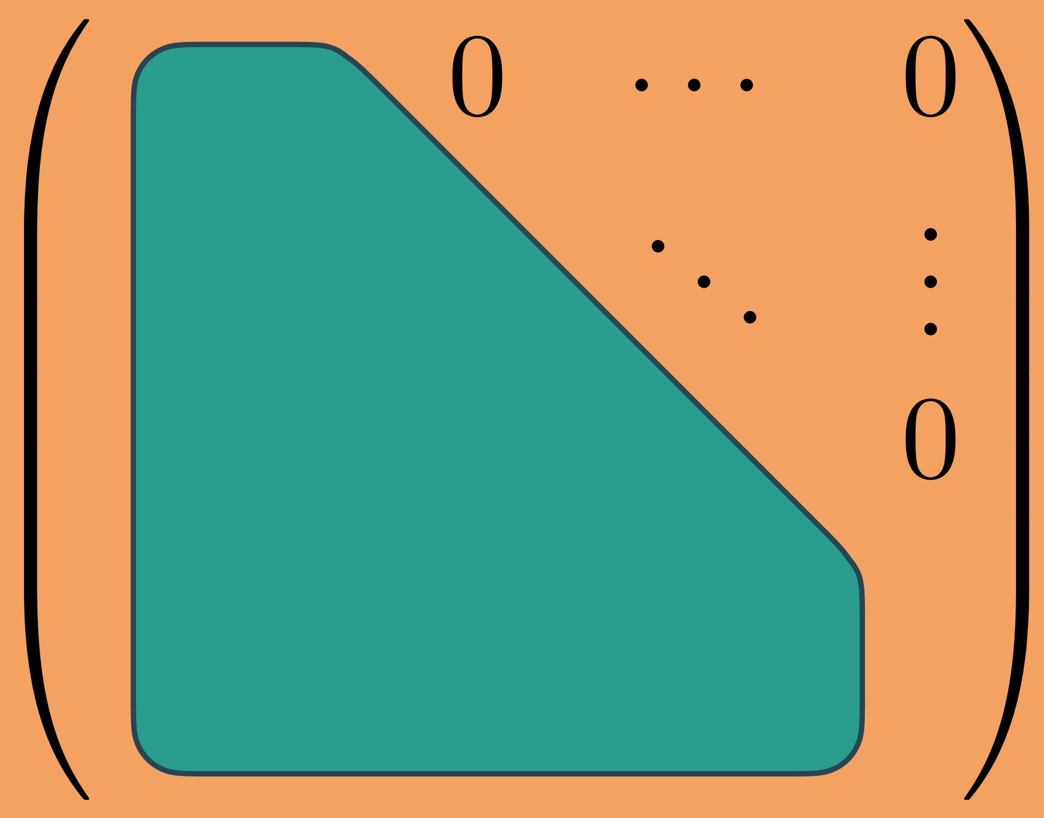
(Revisited)

For Banana Integrals we made the Ansatz

$$
\begin{aligned}
M_1 &= \frac{1}{\varpi} I_{11111} \\
M_2 &= \frac{1}{\varepsilon} J \frac{\mathrm{d}M_1}{\mathrm{d}x} + F_{11} M_1 \\
M_3 &= \frac{1}{\varepsilon} \frac{J}{K_1} \frac{\mathrm{d}M_2}{\mathrm{d}x} + F_{21} M_1 + F_{22} M_2 \\
M_4 &= \frac{1}{\varepsilon} \frac{J}{K_2} \frac{\mathrm{d}M_2}{\mathrm{d}x} M_3 + F_{31} M_1 + F_{32} M_2 + F_{33} M_3 \\
&\vdots
\end{aligned}
$$

all $M_{i>1}$ have operators with apparent singularities

$A \sim \varepsilon$



Let us reverse arrow: don't specify Masters but rather shape of differential equation

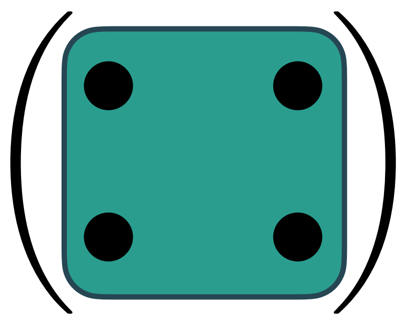
Tune shape for operator of M_1 to have properties we want

Some Examples

Assume $\frac{\mathrm{d}}{\mathrm{d}x} M = \varepsilon J(x) A\, M$

Sectors with 2 Master Integrals

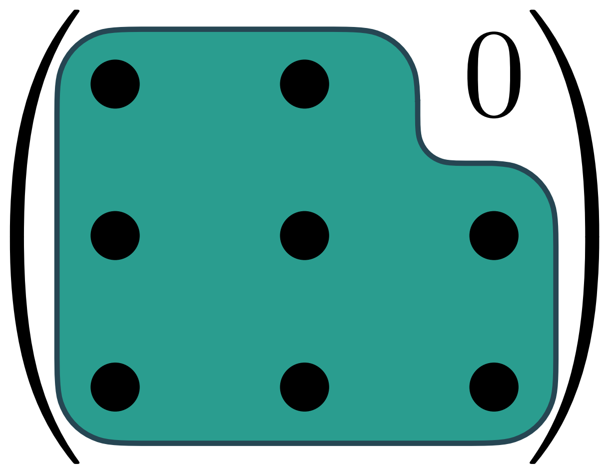
Only one choice

$A =$ 

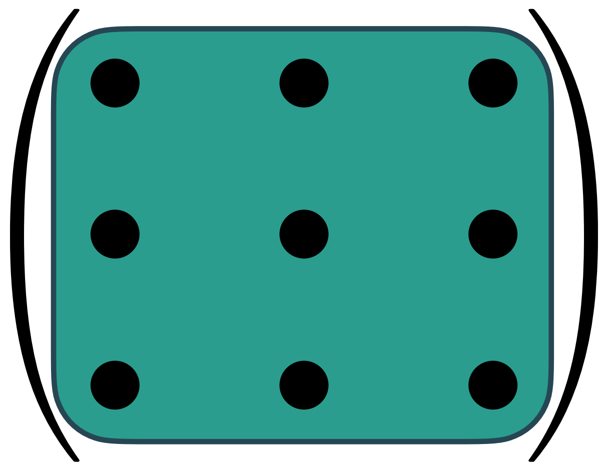
Operator of M_1 has no ε-dependent apparent singularity

Sectors with 3 Master Integrals

Two choices

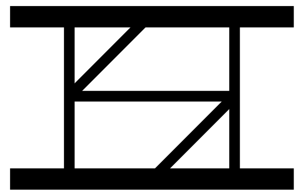
$A =$ 

Operator of M_1 has no ε-dependent apparent singularity

$A =$ 

Operator of M_1 has apparent singularity linear in ε

Our Four-loop Integral

For  with 5 Master Integrals

Assume:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & F_{12} & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & F_{24} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} & 0 \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix}$$

Can check: Picard—Fuchs of M_1 has apparent singularity, quadratic in ε

Fits singularity of operator of $I_{11111111111}$

Matching Ansatz

(Revisited)

In case of Bananas

$$M_1 = \frac{I}{\varpi}$$
$$M_2 = \frac{1}{\varepsilon} J(x) \frac{\mathrm{d}M_1}{\mathrm{d}x} + F(x) M_1$$

→

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F(x) & 1 \\ * & * \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

→ **Eliminate non-ε-factorizing pieces**

For more general Ansatz: Matching of operators

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

→

$$\mathcal{L}_{M_1} = P(x) \mathcal{L}_{I/\varpi}$$

→ **Match coefficients at each order in ε**

Operator of normalized seed integral

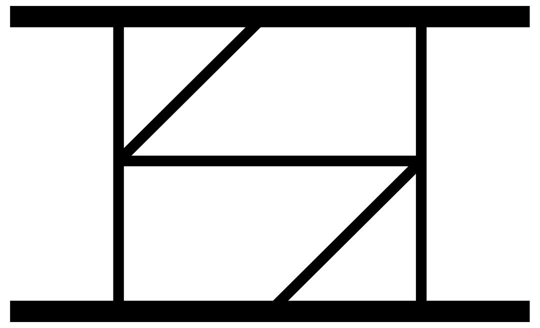
Some Benefits

- Only minimal information required:
  → Shape of DEQ and seed integral
- No apriori knowledge of possible kernels
- In principle not limited to Calabi—Yau: all information fixed by Ansatz

But, knowing geometry helps!

Predict good seed and some entries ahead of computation

ε-factorized DEQ

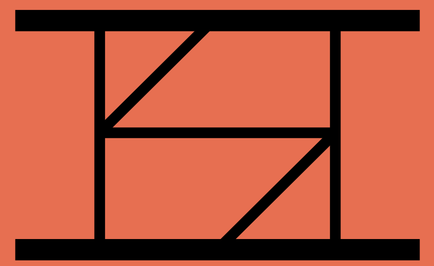
For  performed matching

$$I = I_{1111111111}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & F_{12} & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & F_{24} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} & 0 \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix}$$

$$\mathcal{L}_{M_1} = P(x)\mathcal{L}_{I/\varpi}$$

→ Consistently solvable differential constraints for $F_{ij}(x), J(x), \varpi(x)$

→ ε-factorized DEQ for 

Kernels

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & 1 & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & K & 0 \\ F_{31} & F_{32} & F_{33} & F_{23} & 0 \\ F_{41} & F_{42} & F_{32} & F_{22} & 1 \\ F_{51} & F_{41} & F_{31} & F_{21} & F_{11} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix}$$

Can work with self-dual (persymmetric) Ansatz

→ Reduces number of coefficients to 11

Can find closed form expressions in terms of

- Holomorphic period ϖ_0 (and derivative)
- Yukawa coupling K (and derivative)
- Change of variable to mirror map $J = \mathrm{d}x/\mathrm{d}\tau$

Four integral kernels…

$$F_{23} = K \int_{\tau_0}^{\tau} \mathrm{d}\tau_2 \int_{x_0}^{x_2(\tau_2)} \mathrm{d}x_1 \frac{\left(3{x_1}^4 - 10{x_1}^2 + 3\right)\varpi_0(x_1)}{\left({x_1}^2+1\right)^3}$$

$$F_{22} = -\int_{\tau_0}^{\tau} \mathrm{d}\tau_2 \; K(\tau_2) \int_{x_0}^{x_2(\tau_2)} \mathrm{d}x_1 (\ldots)$$

$$F_{42} = -\int_{\tau_0}^{\tau} \mathrm{d}\tau_2 \int_{x_0}^{x_2(\tau_2)} \mathrm{d}x_1 (\ldots)$$

$$F_{32} = \int_{x_0}^{x} \frac{16\left(2x_1^4 - 11x_1^2 + 2\right)\varpi(x_1)}{\left(x_1^2+1\right)^3} + \ldots \mathrm{d}x_1$$

…and remaining algebraically dependent

Integrals

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & 1 & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & K & 0 \\ F_{31} & F_{32} & F_{33} & F_{23} & 0 \\ F_{41} & F_{42} & F_{32} & F_{22} & 1 \\ F_{51} & F_{41} & F_{31} & F_{21} & F_{11} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix}$$

Shape of differential equation: Similar to Banana with insertion of M_3

Deriving change of basis we find:

Standard derivative basis ("banana-like")

$$M_1 = \frac{\varepsilon^3}{\varpi_0} I_{\text{seed}} \,,$$

$$M_2 = \frac{1}{\varepsilon}\frac{\mathrm{d}M_1}{\mathrm{d}\tau} - F_{11}M_1 \,,$$

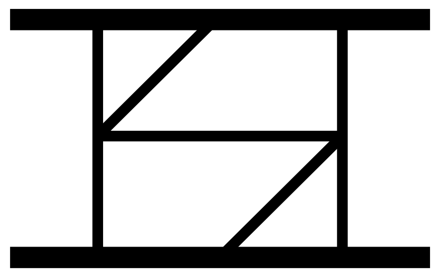
$$M_3 = C_1\, I_{\text{extra}} + C_2\, M_1 + C_3\, M_2 \,,$$

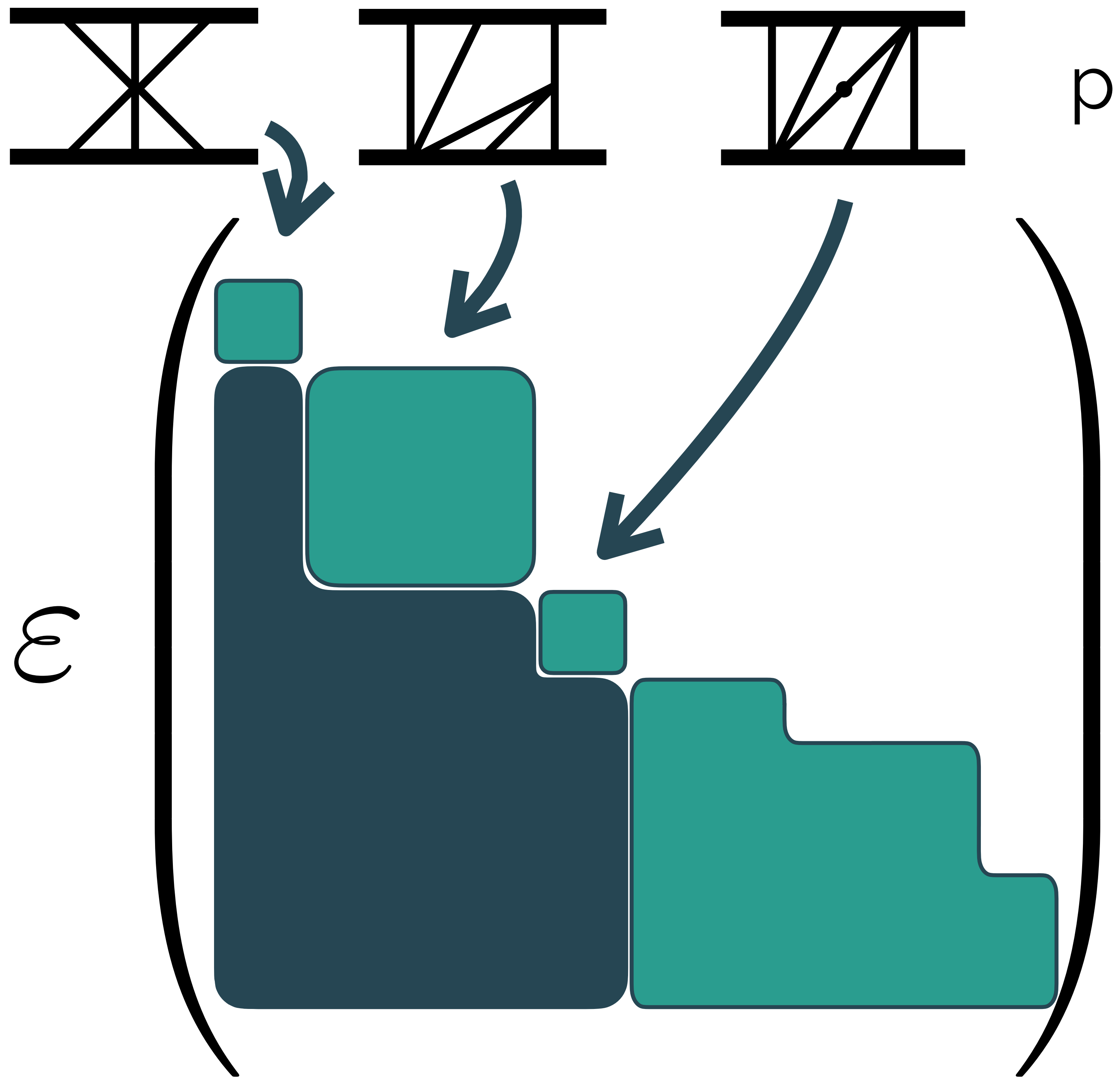
$$M_4 = \frac{1}{K}\left(\frac{1}{\varepsilon}\frac{\mathrm{d}M_2}{\mathrm{d}\tau} - F_{21}M_1 - F_{22}M_2 - F_{23}M_3\right),$$

$$M_5 = \frac{1}{\varepsilon}\frac{\mathrm{d}M_4}{\mathrm{d}\tau} - F_{41}M_1 - F_{42}M_2 - F_{43}M_3 - F_{44}M_4 \,,$$

Contains integral I_{extra} with additional pole

Beyond Maxcut

Full integral  depends on subsectors



purely polylogarithmic

Fully ε-factorized

Conclusions

- Extended existing methods to tackle new features in multi-loop integrals
    - **Operators with apparent singularities**
    - Allows to **work with sub-optimal seed integral or sectors with extra integrals**
- In principle independent of geometry (though it helps)
    - **Kernels fixed by differential constraints → easy series expansion**
- Applied method to derive ε**-factorized differential equation for "real-world" four-loop integral for 5PM correction**
- Can we understand resulting DEQ entries better?

Thank you!