Modularity of Calabi-Yau Fourfolds and Applications to M-Theory Fluxes

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The Arithmetic of Calabi-Yau Manifolds

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Outline

1. Motivation: Flux Compactifications in String Theory
2. Modularity of Calabi-Yau Manifolds
3. The Deformation Method for Calabi-Yau Fourfolds
4. Example of a modular Calabi-Yau Fourfold
5. Conclusions

Motivation: Supersymmetric Flux Vacua

Type IIB string theory compactified on a family of CY 3-folds X_z^3

- Orientifold projection gives 4d, $\mathcal{N} = 1$ SUGRA as low energy EFT

Internal three-form fluxes $F, H \in H^3(X_z^3, \mathbb{Z})$ provide a superpotential

[Gukov, Vafa, Witten, 2000]

$$W = \int_{X_z^3} \Omega(z) \wedge (F - \tau H)$$

W furnishes a scalar potential

$$V(z, \tau) = e^K (|\nabla_z W|^2 + |\nabla_\tau W|^2)$$

Supersymmetric vacuum constraints: $V(z, \tau) = W(z, \tau) = 0$, i.e.

$$\partial_z W = 0 \quad , \quad \partial_\tau W = 0 \quad , \quad W = 0$$

implying in particular

$$F, H \in \left(H^{2,1}(X_z^3, \mathbb{C}) \oplus H^{1,2}(X_z^3, \mathbb{C}) \right) \cap H^3(X_z^3, \mathbb{Z})$$

[Taylor, Vafa, 2000]

Motivation: M-Theory Fluxes

M-theory compactified on a family of CY 4-folds X_z^4

- Low energy description: 3d, $\mathcal{N} = 1$ SUGRA

Superpotential from internal four-form flux $G \in H^4(X_z^4, \mathbb{Z})$:

$$W = \int_{X_z^4} \Omega(z) \wedge G$$

- Similarly: Supersymmetric flux vacua only if $\partial_z W = W = 0$ implying

$$G \in H^4(X_z^4, \mathbb{Z}) \cap \left(H^{4,0}(X_z^4, \mathbb{C}) \oplus H^{2,2}(X_z^4, \mathbb{C}) \oplus H^{0,4}(X_z^4, \mathbb{C})\right)$$

Question: Under which conditions does X_z^4 admit such a sublattice?

$\Rightarrow$ Modularity

[Kachru, Nally, Yang, 2020], [Candelas, de la Ossa, van Straten, 2021], [Candelas, De la Ossa, Kuusela, McGovern, 2023]

Modularity: Elliptic Curve $\mathcal{E}$

Consider an elliptic curve $\mathcal{E}/\mathbb{F}_p$ defined over finite fields:

$\mathcal{E}/\mathbb{F}_p = \Gamma^{\mathrm{fr}_p}$ set of fixed points of the frobenius action

$$\mathrm{fr}_p : (x_1, \ldots, x_n) \mapsto (x_1^p, \ldots, x_n^p)$$

Lefschetz fixed-point theorem:

$$N_p(\mathcal{E}) = |\Gamma^{\mathrm{fr}_p}| = \sum_{k=0}^{2} (-1)^k \mathrm{tr}(\mathrm{Fr}_p^k) = 1 - a_p + p$$

Frobenius endomorphisms on p-adic cohomology groups

$$\mathrm{Fr}_p^k : H^k(\mathcal{E}, \mathbb{Q}_p) \to H^k(\mathcal{E}, \mathbb{Q}_p)$$

Characteristic polynomial of Fr_p^1:

$$R_1^p(\mathcal{E}, T) = \det(\mathbb{1} - T(\mathrm{Fr}_p^1)^{-1}) = 1 - a_p T + pT^2$$

Modularity of Elliptic Curves:

$$f(\tau) = \sum_{p \text{ prime}} a_p q^p \qquad q = e^{2\pi i \tau}$$

defines a weight-two modular form.

Modularity: The local Zeta-Function

Define the local zeta function

$$\zeta_p(X,T) := \exp\left(\sum_{r=1}^{\infty} N_{p^r}(X)\frac{T^r}{r}\right)$$

as generating function for the point counts $N_{p^r}(X) = |X/\mathbb{F}_{p^r}|$

Weil Conjectures: [Weil, 1949]

$$\zeta_p(X,T) = \frac{R_1^p(X,T)\cdots R_{2n-1}^p(X,T)}{R_0^p(X,T)\cdots R_{2n}^p(X,T)}$$

$R_k^p(X,T) = \det(\mathbb{1} - T(\mathrm{Fr}_p^k)^{-1})$ characteristic polynomials of the Frobenius endomorphisms $\mathrm{Fr}_p^k : H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)$

For elliptic curves:

$$\zeta_p(\mathcal{E},T) = \frac{1 - a_pT + pT^2}{(1-T)(1-pT)}$$

for $a_p = p + 1 - N_p(\mathcal{E})$

Modularity: Serre's Modularity Conjecture

Serre's Modularity Conjecture: [Serre, 1975]

One-to-one correspondence between two-dimensional representations $\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_{p^r})$ and modular forms that are Hecke eigenforms.

Note: $\langle \mathrm{fr}_p \rangle \subseteq \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is dense subgroup.

- $\mathrm{Fr}_p^k\colon H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)$: b^k-dimensional reps. of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$
- Reps. of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ can be lifted to reps. of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

For elliptic curves: Modularity of $\mathcal{E}$ is a special case of Serre as $b^1(\mathcal{E}) = 2$.

If $H^k(X,\mathbb{Q}_p) = \Lambda_p \oplus \Sigma_p$ such that $\mathrm{Fr}_p(\Lambda_p) \subseteq \Lambda_p$:

- $\mathrm{Fr}_p|_{\Lambda_p}\colon \Lambda_p \to \Lambda_p$ defines a $\dim(\Lambda_p)$-dimensional (sub-)rep. of fr_p
- If $H^k(X,\mathbb{Z}) = \Lambda \oplus \Sigma$, then $H^k(X,\mathbb{Q}_p) = \Lambda_p \oplus \Sigma_p$ for almost all p.

Modularity: Modular Calabi-Yau manifolds

We call a Calabi-Yau n-fold X *modular* if

$$H^n(X, \mathbb{Q}_p) = \Lambda_p \oplus \Sigma_p \quad , \quad \mathrm{Fr}_p(\Lambda_p) \subseteq \Lambda_p \quad , \quad \dim(\Lambda_p) = 2$$

for almost all primes p.

Properties of modular Calabi-Yau manifolds:

- $\mathrm{Fr}_p|_{\Lambda_p}$ defines a two-dimensional representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.
- The characteristic polynomial $R_n^p(X, T)$ factorizes

  $$R_n^p(X, T) = R_\Lambda^p(X, T) \cdot R_\Sigma^p(X, T)$$

  for almost all primes p.
- If X is modular, $R_\Lambda^p(X, T) = 1 - a_p p^\alpha T + p^\beta T^2$ for some $\alpha, \beta \in \mathbb{N}_0$.

  The a_p determine the corresponding modular form via

  $$f(\tau) = \sum_p a_p q^p \qquad q = e^{2\pi i \tau}$$

Modularity: Relation to Flux Vacua

Recall for type IIB string fluxes on a CY threefold X_3:

$$F, H \in H^3(X_3, \mathbb{Z}) \cap \left(H^{2,1}(X_3, \mathbb{C}) \oplus H^{1,2}(X_3, \mathbb{C})\right)$$

for a supersymmetric flux vacuum. Hence

$$\Lambda_{\text{flux}} := \langle F, H \rangle_{\mathbb{Q}} \subseteq H^3(X_3, \mathbb{Q})$$

defines a two-dimensional sub-representation

Necessary condition for supersymmetric flux vacua:

X_3 has to be a modular Calabi-Yau threefold!

[Kachru, Nally, Yang, 2020], [Candelas, De la Ossa, Kuusela, McGovern, 2023]

M-Theory fluxes on a CY fourfold X_4:

$$G \in H^4(X_4, \mathbb{Z}) \cap \left(H^{4,0}(X_4, \mathbb{C}) \oplus H^{2,2}(X_4, \mathbb{C}) \oplus H^{0,4}(X_4, \mathbb{C})\right)$$

Modular Fourfolds: Possibilities for Modularity

Two different choices of two-dimensional sub-reps. $\Lambda \subseteq H^4(X_4, \mathbb{Q})$:

"Attractor points":

$$\Lambda_{\text{att}} \subseteq H^{4,0}(X_4, \mathbb{C}) \oplus H^{2,2}(X_4, \mathbb{C}) \oplus H^{0,4}(X_4, \mathbb{C})$$

Mimic the behaviour of rank-two attractor points of Calabi-Yau threefolds

"Attractive K3-points":

$$\Lambda_{\text{AK3}} \subseteq H^{3,1}(X_4, \mathbb{C}) \oplus H^{1,3}(X_4, \mathbb{C})$$

Geometric origin from (a tate-twist of) an attractive K3 surface

Sufficient condition for M-theory flux vacua:

X_4 attractor point $\Rightarrow$ any $G \in \Lambda_{\text{att}}$ defines a suitable M-theory flux

[Jockers, S.K., Kuusela, 2023]

Modular Fourfolds: Computing $R_n^p(X, T)$

Recall that $R_n^p(X, T) = \det(\mathbb{1} - T(\mathrm{Fr}_p^n)^{-1})$

For a family of CY-threefolds X_z:

$$\nabla \mathrm{Fr}_p^3 = p \mathrm{Fr}_p^3 \nabla$$

∇: Gauss-Manin connection on the vector bundle given by $H^3(X_z, \mathbb{Q}_p)$

Let $F_p(z)$ denote the matrix representing Fr_p^3 and $U_p(z) = F_p^{-1}(z)$, then

$$U_p(z) = E(\tilde{z}^p)^{-1} U_p(0) E(\tilde{z}) \qquad \tilde{z} := \mathrm{Teich}_p(z)$$

for $E_{ij}(z) = \mathcal{D}_i \varpi_j(z)$ the period matrix

- The $\mathcal{D}_i$ are generators of the Picard-Fuchs ideal
- For $h^{2,1} = 1$: $\mathcal{D}_i = \theta_z^i = (z\partial_z)^i$ for $i = 0, \ldots, 4$.

[Candelas, de la Ossa, Elmi, van Straten, 2019]
[Candelas, de la Ossa, Kuusela, 2024]

Modular Fourfolds: Computing $R_n^p(X, T)$

The Hodge diamond of a Calabi-Yau threefold

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

- horizontal and vertical Hodge structure are separated
- The horizontal Hodge structure is determined by $\mathcal{D}_i\Omega(z)$

$$\zeta_p(X, T) = \frac{R_3^p(X, T)}{(1-T)(1-pT)^{h^{1,1}}(1-p^2T)^{h^{1,1}}(1-p^3T)}$$

Modular Fourfolds: Computing $R_n^p(X,T)$

The Hodge diamond of a Calabi-Yau fourfold

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

- $H^{2,2}(X,\mathbb{C}) = H_h^{2,2}(X,\mathbb{C}) \oplus H_v^{2,2}(X,\mathbb{C}) \oplus H_\perp^{2,2}(X,\mathbb{C})$
- Assume Fr_p^4 to be compatible with this decomposition

$$\zeta_p(X,T) = \frac{R_3^p(X,T)R_5^p(X,T)}{(1-T)(1-pT)^{h^{1,1}}R_4^p(X,T)(1-p^3T)^{h^{1,1}}(1-p^4T)}$$

Modular Fourfolds: Computing $R_n^p(X,T)$

The Hodge diamond of a Calabi-Yau fourfold

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

- $H^{2,2}(X,\mathbb{C}) = H_h^{2,2}(X,\mathbb{C}) \oplus H_v^{2,2}(X,\mathbb{C}) \oplus H_\perp^{2,2}(X,\mathbb{C})$
- Assume Fr_p^4 to be compatible with this decomposition

$$\zeta_p(X,T) = \frac{R_3^p(X,T)R_5^p(X,T)}{(1-T)(1-pT)^{h^{1,1}}R_h^p(X,T)R_\perp^p(X,T)(1-p^3T)^{h^{1,1}}(1-p^4T)}$$

Modular Fourfolds: Computing $R_p^n(X, T)$

Under this assumption, the deformation method can be applied to compute $R_h^p(X, T)$:

$$R_h^p(X_z, T) = \det(\mathbb{1} - TU_p(z))$$

for

$$U_p(z) = E^{-1}(\tilde{z}^p)\tilde{U}_p(0)E(\tilde{z})$$

As in the threefold case

- $E_{ij}(z) = \mathcal{D}_i\varpi_j(z)$ is the period matrix
- $\tilde{z} := \mathrm{Teich}_p(z)$

Note: R_h^p probes only the $2(h^{3,1}+1)+h_h^{2,2}$ dimensional subspace

$$\langle\ \Omega(z), \mathcal{D}_i\Omega(z)\ \rangle \subseteq H^4(X, \mathbb{Q}_p)$$

A Modular Example

Consider the family of Hulek-Verrill fourfolds

$$X_{z_i} := \left\{ 1 + \frac{z_1}{x_1} + \cdots + \frac{z_6}{x_6} = 0 \ , \ 1 + x_1 + \cdots + x_6 = 0 \right\} \subset (\mathbb{C}^\star)^6 \subset \mathbb{P}^7$$

[Hulek, Verrill, 2005]

- To obtain an effective one-parameter model, restrict to the diagonal $z_1 = \cdots = z_6 = z$ which we denote by X_z
- Fundamental period:

$$\varpi_0(z) = \sum_{n_1 + \cdots + n_6 = k} \binom{k}{\mathbf{n}}^2 z^k$$

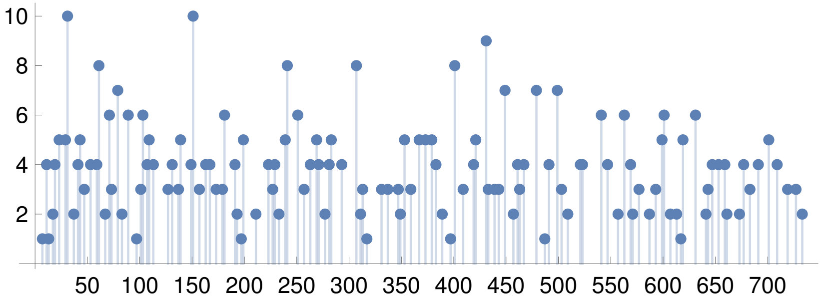
  is annihilated by a degree five Picard-Fuchs operator
- This subfamily realizes an effective model of horizontal Hodge type $(1, 1, 1, 1, 1)$

A Modular Example

For $z \in \bar{\mathbb{Q}}$, X_z is modular if $R_h^p(X_{\bar{z}}, T)$ factorizes quadratically for any $\bar{z} \in \mathbb{F}_p$ representing z, i.e.

$$\bar{z} \equiv z \mod p$$

Histogram of points $\bar{z} \in \mathbb{F}_p$ s.t. $R_h^p(X_{\bar{z}}, T)$ factorizes quadratically



Here: $7 \leq p \leq 733$

A Modular Example

Reconstruction of the modular point $z \in \bar{\mathbb{Q}}$:

Collection of points $\bar{z} \in \mathbb{F}_p$ with quadratic factorization

prime p	$\bar{z} \in \mathbb{F}_p$			
$p = 11$	1	6	8	10
$p = 13$	1			
$p = 17$	1	15		

prime p	$\bar{z} \in \mathbb{F}_p$			
$p = 19$	1	2	7	17
$p = 23$	1	4	5	12
...				

(Rational) solution $z \in \mathbb{Q}$ s.t. X_z is modular:

$$z = 1$$

- Tested for all primes $11 \leq p \leq 733$
- No additional modular point found

A Modular Example

Coefficients a_p of quadratic factor

$$R^p_\Lambda(X_1, T) = 1 - a_p p T + p^2 T^2$$

give q-expansion of a unique Hecke eigenform in $S_3(\Gamma_0(15), \chi_{-15})$

[Bönisch, Fischbach, Klemm, Nega, Safari, 2020]

There is a two-dimensional sublattice

$$\Lambda_{\mathsf{AK3}} = \left[H^{3,1}(X_1,\mathbb{C}) \oplus H^{1,3}(X_1,\mathbb{C})\right] \cap H^4(X_1,\mathbb{Z})$$

In particular

$$\nabla_z \Pi(z)|_{z=1} = -\frac{3}{28}\left(2\frac{L_3(1)}{\pi^2}\begin{pmatrix} 12 \\ 5 \\ 20 \\ -50 \\ 10 \end{pmatrix} + i\frac{L_3(2)}{\pi^3}\begin{pmatrix} 0 \\ 5 \\ 24 \\ -80 \\ 20 \end{pmatrix}\right)$$

as expected from *Deligne's Conjecture*

How about fluxes?

- Attractive K3-point! $\Rightarrow \Lambda_{\mathsf{AK3}}$ does **not** provide suitable fluxes
- However: $G := C \cdot \mathrm{Re}(\Omega(z=1)) \in \Sigma\,, \quad C \in \mathbb{R}$

$$\Sigma = \left[H^{4,0}(X_1,\mathbb{C}) \oplus H^{2,2}_h(X_1,\mathbb{C}) \oplus H^{0,4}(X_1,\mathbb{C})\right] \cap H^4(X_1,\mathbb{Z})$$

A modular Example

Geometric Interpretation: There is a birational description

$$X_z \overset{\text{bir.}}{\sim} \mathcal{E}_{\varphi,z} \times_{\mathbb{P}^1} \mathrm{K3}_{\varphi,z}$$

where

$$\mathcal{E}_{\varphi,z} = \left\{ 1 + y_0 + y_1 + y_2 = 0 \ , \ 1 + \tfrac{\varphi}{y_0} + \tfrac{z}{y_1} + \tfrac{z}{y_2} = 0 \right\} \subset \mathbb{P}^4$$
$$\mathrm{K3}_{\varphi,z} = \left\{ 1 + y_3 + y_4 + y_5 + y_6 = 0 \ , \ 1 + \tfrac{\varphi/z}{y_3} + \tfrac{1}{y_4} + \tfrac{1}{y_5} + \tfrac{1}{y_6} = 0 \right\} \subset \mathbb{P}^5$$

are the lower dimensional analogs of X_z

For $z = 1$,

- $\mathcal{E}_{1,z}$ is a singular fibre of Kodaira-type I_3
- $\mathrm{K3}_{1,z}$ is attractive (and hence modular)

This observation singles out a pair of cycles which is responsible for modularity on X_z

Conclusions

Modularity is a useful tool to search for M-theory flux vacua

- "Attractor-point" $\Rightarrow$ Sublattice Λ_{att} realizes suitable fluxes
- Modularity is not a necessary criterion.

Restrictions and Assumptions

- Analysis restricted to horizontal part $H_h^4(X, \mathbb{C})$ of cohomology
- Need to assume that Fr_p^4 is compatible with this decomposition
- Search restricted to algebraic moduli space $z \in \bar{\mathbb{Q}}$

Based on the numerics:

- Construction of Frobenius action is self-consistent
- Modular structure in accordance with Deligne's conjecture and geometric interpretation

A posteori justification for the assumptions