Integration on higher-genus Riemann surfaces

based on: 2306.08644 E. D'Hoker, M. Hidding, OS

2407.11476 & 2502.14769 E. D'Hoker, OS

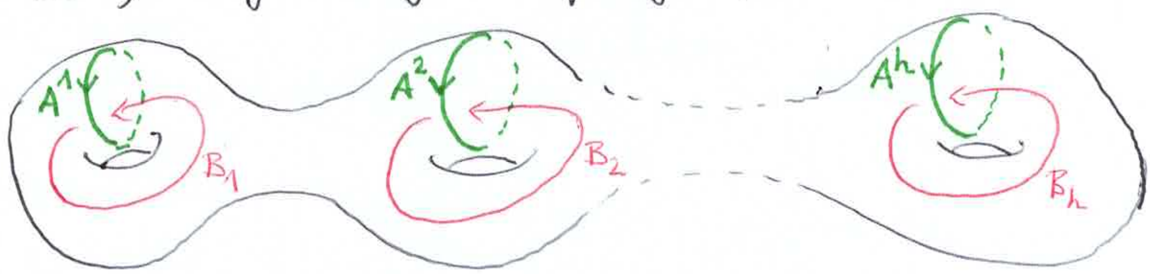
2501.07640 E. D'Hoker, B. Enriquez, OS, F. Zerbini

overview: white papers 2203.07088 & 2203.09099

outline: I) Context/motivation

II) Elliptic polylogarithms

III) Higher-genus polylogarithms

$\Sigma_h =$ 

I) Context / motivation

Old problem: $\int dx$ harder than $\frac{d}{dx}$, e.g.

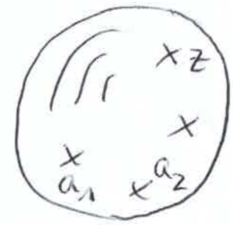
- for rational $R(x)$, also $\frac{dR(x)}{dx}$ is rational, but $\int dx\, R(x)$ may be not, e.g. $\int_1^z \frac{dx}{x} = \log(z)$
- next step $\int dx\, R(x) \log(x)$, successively build fct. space of "multiple polylogarithms" (MPLs) with

$$\underbrace{G(a_1,a_2,\dots,a_w}_{\in\,\mathbb{C}};z) = \int_0^z \frac{dx}{x-a_1}\, G(a_2,\dots,a_w;x)\,,\qquad G(\emptyset;x)=1$$

"weight" w "integration kernel"

e.g. $G(a;z) = \log\left(1-\frac{z}{a}\right)$ & $G(\vec{0}^{\,n-1},1;z) = -\mathrm{Li}_n(z)$

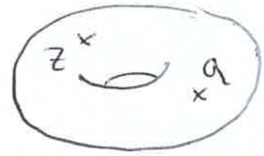
- $\{R(x)\cdot \text{MPLs}\}$ = completion of $\{R(x)\}$ to close under $\int dx$



Σ_0

$\Rightarrow$ takes care of integration on genus $h=0$ surface

- on $\Sigma_{h\geq 1}$, more tricky analogues of $R(x)$

e.g. @ $h=1$: 

$\Longleftrightarrow$ $y^2 = x^3 + bx + c$

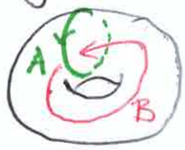
need iterated integrals of kernels $\frac{dx}{y}$, $\frac{x\,dx}{y}$, $\frac{dx}{y(x-a)}$

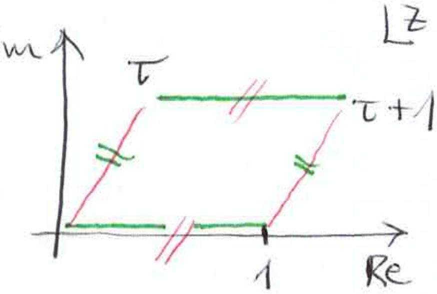
$\hookrightarrow$ elliptic MPLs (eMPLs)

- beyond torus / eMPLs
  * [Riemann surfaces] Σ_h @ genus $h\geq 2$ $\longleftarrow$ this talk
  * higher-dim varieties $\longleftarrow$ hopefully later

- physics motivation: scattering amplitudes
  * string amplitudes [gravity / duality] $\leftarrow$ also serve as inspiration
  * Feynman integrals [precision GW / LHC]

II) Elliptic polylogs

parametrize $\Sigma_1 =$  as $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$:



- sv Green fct on $\Sigma_{h=1}$

$$G(z,\tau) = -\ln\left|\frac{\theta_1(z,\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi}{\mathrm{Im}\,\tau}(\mathrm{Im}\,z)^2 = G(z+1,\tau) = G(z+\tau,\tau)$$

- ∞ tower of sv integration kernels $f^{(n\in\mathbb{N})}(z,\tau) = f^{(n)}(z+1) = f^{(n)}(z+\tau)$

  * $f^{(1)}(z) = -\partial_z G(z) = \frac{1}{z} + O(z,\bar{z})$

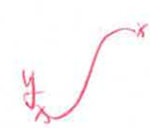
  * recursion @ higher modular weight

$$f^{(n+1)}(z) = -\int_{\Sigma_1} \frac{d^2x}{\mathrm{Im}\,\tau} f^{(1)}(z-x) f^{(n)}(x) \quad , \quad n \geq 1$$

  * mod. forms under $SL(2,\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$f^{(n)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{n} f^{(n)}(z,\tau)$$

- eMPLs = $\boxed{\text{homotopy-inv.}}$ iterated integrals of dz & $d\bar{z}$ & $dz\, f^{(n\geq 1)}(z-a)$ [Brown, Levin 1110.6917]

  * e.g. $\int_y^z dx \left(f^{(1)}(x) - \frac{\pi}{\mathrm{Im}\,\tau} \int_y^x (dx' - d\bar{x}') \right)$ only depends on endpts z, y & homotopy class of path 

  * generate homotopy-inv. combination via flat connection

• alternative: meromorphic & multi-valued kernels $g^{(n)}$

* $$\frac{\theta_1'(0)\,\theta_1(z+\alpha)}{\theta_1(z)\,\theta_1(\alpha)} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \alpha^{n-1} g^{(n)}(z)$$

e.g. $g^{(1)}(z) = \partial_z \log\theta_1(z) = g^{(1)}(z+\tau) + 2\pi i$

* related to $z \to z+\tau$ periodic $f^{(n)}$ via $(g^{(0)} = 1 = f^{(0)})$

$$\sum_{n=0}^{\infty} \alpha^{n-1} f^{(n)}(z) = \exp\left(2\pi i \frac{\operatorname{Im} z}{\operatorname{Im}\tau}\alpha\right) \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z)$$

* equiv. description of eMPLs: iterated int's of

$dz\, g^{(n\geq 0)}(z-a)$ —OR— earlier $\frac{dx}{y}, \frac{x\,dx}{y}, \frac{dx}{y(x-a)}$

automatically homotopy-inv.

• $\{\prod g^{(r)} \cdot \text{eMPLs}\}$ close under $\int dx$ ~~under~~ by Fay identities

$\in \mathbb{Q}$

$$g^{(a)}(x-y)\, g^{(b)}(x-z) = \sum_{\substack{c+d\\=a+b}} \left(p_{c,d}\, g^{(c)}(x-y)\, g^{(d)}(y-z) + q_{c,d}\, g^{(c)}(x-z)\, g^{(d)}(z-y) \right)$$

* cf. partial fraction $\frac{1}{(x-y)(x-z)} = \frac{1}{(x-y)(y-z)} + \frac{1}{(x-z)(z-y)} \to$ MPLs

* identical Fay id's for $g^{(n)} \to f^{(n)}$

III) Higher-genus polylogarithms

$H_1(\Sigma_h, \mathbb{Z})$-cycles with non-zero intersections $\#(A^I, B_J) = \delta^I_J$

• sv "Arakelov" Green fct. on $\Sigma_h \times \Sigma_h$

$\mathcal{G}(x,y) = -\log|E(x,y)|^2 +$ "sv completion"

- ∞ tower of sv kernels $f^{I_1\ldots I_n}{}_J(x,y)$, $n\in\mathbb{N}$ $\sum_{K=1}^{h}\overline{\omega_K(z)}\times((\mathrm{Im}\,\Omega)^{-1})^{IK}$

  * $f^I{}_J(x,y) = \int_{\Sigma_h} \partial_x \mathcal{G}(x,z)\, \overline{\omega}^I(z)\, \omega_J(z) - \delta^I_J\, \partial_x \mathcal{G}(x,y)$

  * $f^{I_1\ldots I_n}{}_J(x,y) = \int_{\Sigma_h} \overline{\omega}^{I_1}(z)\, \partial_x \mathcal{G}(x,z)\, f^{I_2\ldots I_n}{}_J(z,y)$, $n\geq 2$

  * modular tensors under $Sp(2h,\mathbb{Z}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

    $$f^{I_1\ldots I_n}{}_J(x,y) \to (C\Omega+D)^{I_1}{}_{K_1}\ldots(C\Omega+D)^{I_n}{}_{K_n} f^{K_1\ldots K_n}{}_L(x,y)\left((C\Omega+D)^{-1}\right)^L{}_J$$

- higher-genus polylog's = homotopy-inv. iterated int's of $\omega_J(z)$ & $\overline{\omega}^J(z)$ & $f^{K_1\ldots K_n}{}_L(z,a)$ [DHS 2306.08644]

- alternative: meromorphic & multivalued $g^{I_1\ldots I_n}{}_J$

  * introduced by [Enriquez 1112.0864] via fct. properties

  * mostly adapt $f^{I_1\ldots I_r}{}_J$ expressions to [D'Hoker, OS, 2502.14769]

    $$\int_{\Sigma_h}\overline{\omega}^I \to \oint_{A^I} \quad\text{and}\quad \partial_x\mathcal{G}(x,y) \to -\partial_x \log E(x,y)$$

    $\Downarrow$

    $$g^I{}_J(x,y) = \oint_{A^I}^{(z)} \omega_J(z)\left(-\partial_x \log E(x,z)\right) + \delta^I_J\, \partial_x \log E(x,y) + i\pi\,\delta^I_J\,\omega_J(x)$$

    e.g.

    $$g^{I_1\ldots I_r}{}_J(x,y) = \oint_{A^{I_1}}^{(z)}\left(-\partial_x \log E(x,z)\right) g^{I_2\ldots I_r}{}_J(z,y) + (i\pi)^{\#}\begin{pmatrix}\text{lower}\\ \text{rank}\end{pmatrix}$$

  * direct rel's $f^{I_1\ldots I_n}{}_J(x,y) = g^{I_1\ldots I_n}{}_J(x,y) + \ldots$ [2501.07640]

- closure under $\int dx$ again by Fay-id's [2407.11476]

  $$g^{\vec{I}}{}_J(x,y)\, g^{\vec{K}}{}_L(z,x) = \sum\left(g(x,y)\,g(z,y) + \#\, g(x,z)\,g(z,y)\right)$$

  & identical eq's for $g^{\vec{P}}{}_a(a,b) \to f^{\vec{P}}{}_a(a,b)$

To be prepared on blackboards

- genus $h=1$ functions: Dedekind $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$

  & odd $\theta_1(z,\tau) = 2q^{1/8}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i z}q^n)(1-e^{-2\pi i z}q^n)$

- prime form $E(x,y) = (x-y) + \mathcal{O}((x-y)^3)$ @ arbitrary $h \geq 1$

  $$\underbrace{E(x,y)}_{\text{indep. on odd } \nu} = \frac{\theta_\nu\left(\int_y^x \omega\right)}{h_\nu(x)\, h_\nu(y)}$$

  odd characteristics ν

  Abel map $\int_y^x \omega_I \in \mathbb{C}^h$

  $$(h_\nu(y))^2 = \sum_{I=1}^{h} \omega_I(y) \frac{\partial}{\partial \zeta_I} \theta_\nu(\zeta)\Big|_{\zeta=0}$$

- abelian differentials $\omega_I(x)$ @ $I = 1, 2, \dots, h$

  normalized $\oint_{A^I} \omega_J = \delta^I_J$, period matrix $\oint_{B_I} \omega_J = \Omega_{IJ}$