Non-perturbative topological strings and arithmetic

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Two arithmetic problems

I will consider two different "arithmetic" problems in topological string theory.

The first one is a traditional one: the calculation of periods at the conifold point.

The second one concerns the nature of the numbers appearing in perturbative expansions. This can be also addressed in QFT, e.g. why multi-zeta values appear?

These are "perturbative" problems, but I'd like to show how **non-perturbative methods** give new perspectives on their arithmetic properties.

The TS/ST correspondence

The TS/ST correspondence is a conjectural strong-weak coupling duality between topological string theory on toric CYs and quantum mechanics

$$g_s = \frac{4\pi^2}{\hbar}$$

In this setting, to solve the **classical** problem of computing periods at the conifold point, we should solve a quantum-mechanical problem at strong coupling.

This looks hopeless, but in some cases we will be able to do it and obtain **closed formulae** for the periods!

Operators from mirror curves

In local mirror symmetry, mirror manifolds are encoded in algebraic curves, or equivalently on polytopes. The correspondence associates *g* operators to a polytope with *g* inner points. For simplicity I will mostly focus on *g=1*.

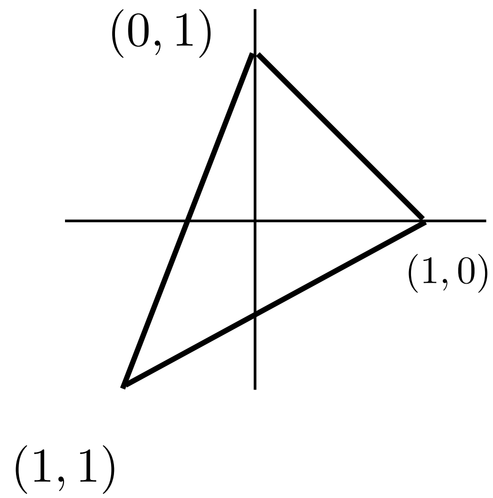
Set the inner point of the polytope at the origin

$$(n, m) \quad \textbf{vertex} \longrightarrow \quad \mathrm{e}^{m\mathsf{x}+n\mathsf{p}}$$

where x, p are canonical Heisenberg operators $[\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar$

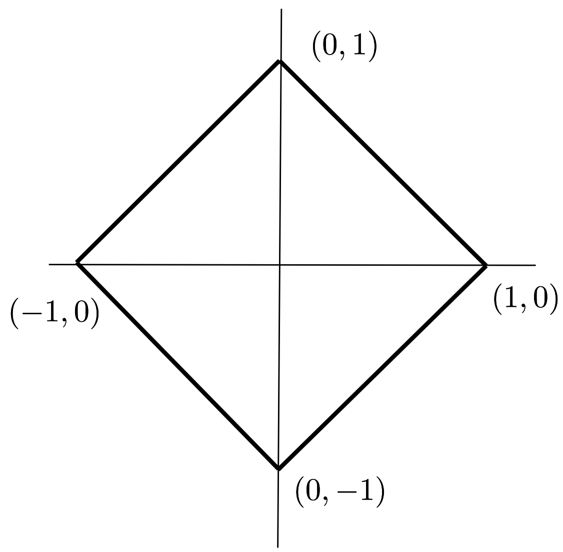
$$X \to \mathsf{O}_X = \sum_{\text{vertices}} \mathrm{e}^{n_i\mathsf{x}+m_i\mathsf{p}}$$

Local $\mathbb{P}^2$



$$\mathsf{O}_{\mathbb{P}^2} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{x}-\mathsf{p}}$$

Local $\mathbb{F}_0$



$$\mathsf{O}_{\mathbb{F}_0} = \mathrm{e}^{\mathsf{x}} + \xi_{\mathbb{F}_0} \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}}$$

Spectral theory

Theorem

[Grassi-Hatsuda-M.M., Kashaev-M.M., Laptev-Schimmer-Takhtakjan]

The operator $\rho_X = \mathsf{O}_X^{-1}$ on $L^2(\mathbb{R})$ is positive definite and of trace class

(assuming $\hbar > 0$ and some conditions on "mass parameters")

$\longrightarrow$ Discrete spectrum $\quad \mathrm{e}^{-E_n}, \quad n = 0, 1, \cdots$

$\longrightarrow$ All its traces are finite

$$\mathrm{Tr}\, \rho_X^\ell = \sum_{n \geq 0} \mathrm{e}^{-\ell E_n} < \infty, \qquad \ell = 1, 2, \cdots$$

Since the operator is of trace class, its **Fredholm determinant**

$$\Xi_X(\kappa) = \det\left(1 + \kappa \rho_X\right) = \prod_{n \geq 0} \left(1 + e^{\mu - E_n}\right)$$

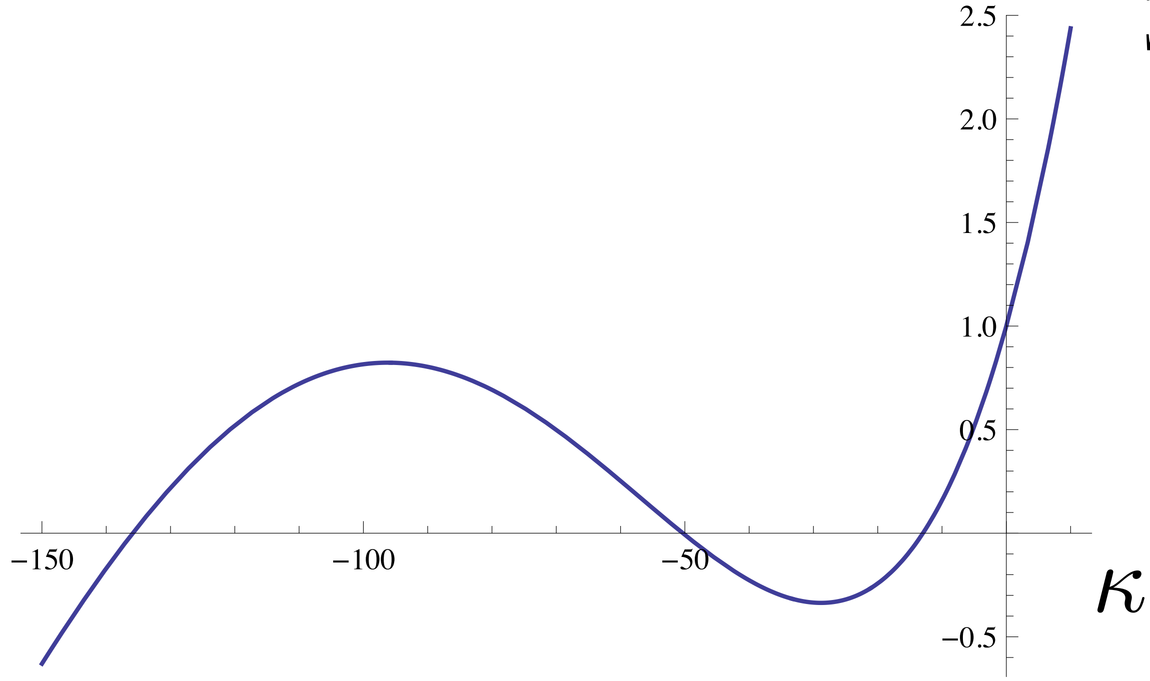
is an **entire** function of $\kappa = e^{\mu}$

It can be shown that κ is identified with the modulus of the CY, therefore this is an entire function on the CY moduli space

$$\Xi_X(\kappa) = 1 + \sum_{N=1}^{\infty} Z_X(N, \hbar) \kappa^N$$

"fermionic" spectral traces

$\Xi_{\mathbb{P}^2}(\kappa) \quad \hbar = 2\pi$



zeroes= -spectrum $-\mathrm{e}^{E_n}$

We want to recover conventional, perturbative topological strings from this quantum-mechanical problem

I recall that the conventional genus *g* topological string free energies compute Gromov-Witten invariants

$$F_g^{\mathrm{LR}}(t) = \sum_d N_{g,d} \mathrm{e}^{-dt}$$

However, I will be interested in the topological string free energies in the **conifold frame**, which are obtained by a symplectic transformation of the above, and contain the same information [Aganagic-Bouchard-Klemm]. They are naturally expressed in terms of the flat coordinate vanishing at the conifold locus

$$F_g(\lambda)$$

A conjecture

Let us consider the following 't Hooft-like limit

$$N, \hbar \to \infty \qquad \lambda = N g_s \text{ fixed}$$

Then, the large *N* asymptotics of the fermionic spectral trace is given by the topological string free energy in the conifold frame

$$\log Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g(\lambda) g_s^{2g-2}$$

Many checks, no proof (expected to be very hard)

Back to the periods

$$F_0(\lambda) = \frac{\lambda^2}{2}\left(\log(\lambda) - \frac{3}{2}\right) + C\lambda + \cdots$$

↑

$\propto$ value of the A-period at the conifold

Can we obtain this from the quantum-mechanical problem? Fermionic spectral traces can be written as **matrix integrals**

[Fredholm, 1903]

$$Z_X(N,\hbar) = \frac{1}{N!}\int \mathrm{d}x_1 \cdots \mathrm{d}x_N \, \det_{i,j} \rho_X(x_i, x_j)$$

↑

integral kernel

Explicit formulae for integral kernels are hard to obtain!

But, surprisingly, the operators from mirror curves can be often diagonalized exactly in terms of **Faddeev's quantum dilogarithm**! [Kashaev-M.M.]

$$\Phi_{\mathsf{b}}(x) = \exp\left(\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{e}^{-2\mathrm{i}xz}}{4\sinh(z\mathsf{b})\sinh(z\mathsf{b}^{-1})}\frac{\mathrm{d}z}{z}\right)$$

and $\hbar \propto \mathsf{b}^2$

Since this function is manifestly self-dual, we will be able to obtain the strong-coupling behaviour $\hbar \to \infty$ in terms of the behavior as $\hbar \to 0$, which is governed by the **classical** dilogarithm

$$\log \Phi_{\mathsf{b}}\left(\frac{x}{2\pi\mathsf{b}}\right) \sim \frac{1}{2\pi\mathrm{i}\mathsf{b}^2}\mathrm{Li}_2(-\mathrm{e}^x)$$

e.g. for local $\mathbb{P}^2$ one has

$$\rho_{\mathbb{P}^2}(x,y) = \frac{\Phi_b\left(x + \mathrm{i}b/3\right)}{\Phi_b\left(y - \mathrm{i}b/3\right)} \frac{\mathrm{e}^{\pi b(x+y)/3}}{2b\cosh\left[\pi\left(\frac{x-y}{b} + \frac{\mathrm{i}}{6}\right)\right]} \qquad b^2 = \frac{3\hbar}{2\pi}$$

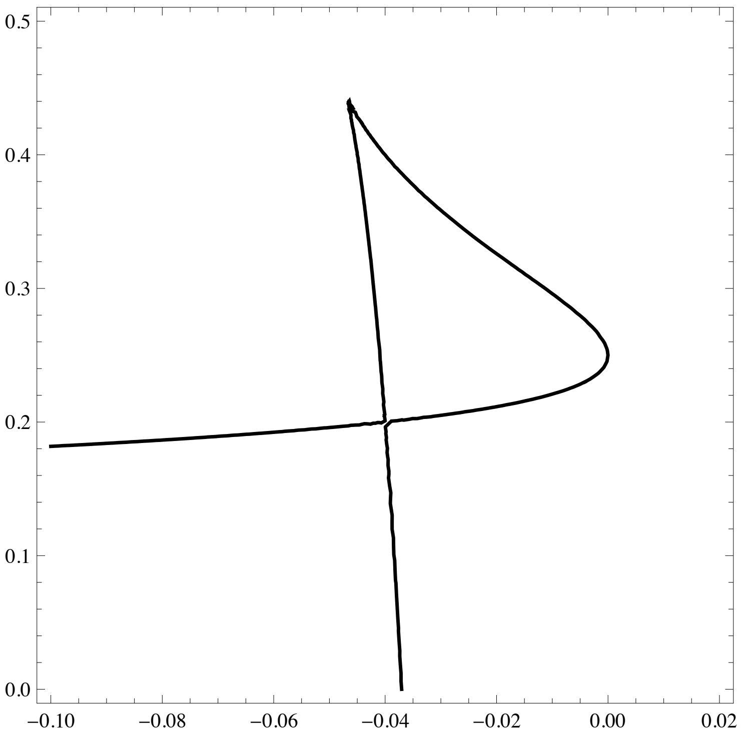
In this way one finds, by explicit computation,

Local $\mathbb{P}^2$ $\quad 2\pi t = 9\,\mathrm{Im}\,\mathrm{Li}_2\left(\mathrm{e}^{\mathrm{i}\pi/3}\right)$ [Rodríguez-Villegas]

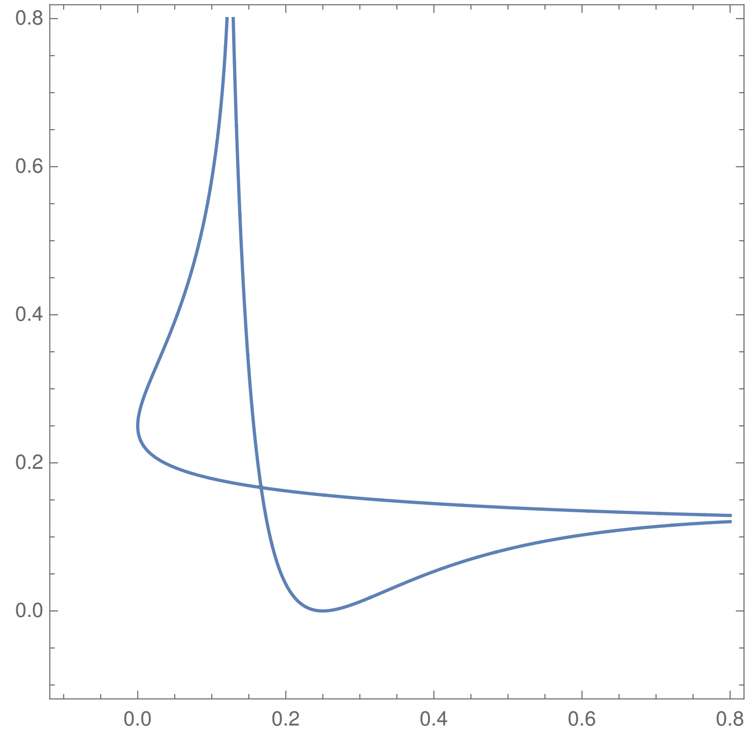
Local $\mathbb{F}_0$ $\quad 2\pi t = 16\,\mathrm{Im}\,\mathrm{Li}_2\left(\mathrm{i}\xi_{\mathbb{F}_0}^{1/4}\right)$ [Kasteleyn]

Higher genus curves

The most powerful results are obtained at higher genus. It turns out that the TS/ST correspondence predicts the existence of a special point in the conifold locus, the **maximal conifold point**



resolved $\mathbb{C}^3/\mathbb{Z}_5$



$Y^{3,0}$ geometry

Using explicit results about the relevant operators, one can predict the value of the periods at the maximal conifold point in many cases, in terms of special values of the dilogarithm.

Many of these predictions have been verified rigorously by [Doran, Kerr, Sinha Babu et al.]

Open issues

Currently, we do not know explicitly the integral kernels of the operators in all cases, even for elliptic curves. This limits the applicability of the TS/ST correspondence to compute periods.

The simplest case where the integral kernel is not known is local $\mathbb{F}_1$. Explicit evaluations of the period in this case [Kerr-M.M.] indicate that one might need additional ingredients (special functions?) to obtain it .

Another issue is to understand better the geometry (and existence) of the maximal conifold point, as required by the TS/ST correspondence.

Deconstructing the perturbative series

The coefficients appearing in perturbative series often have special number-theoretic properties. For example, in QFT, values of (multiple) zeta functions often appear.

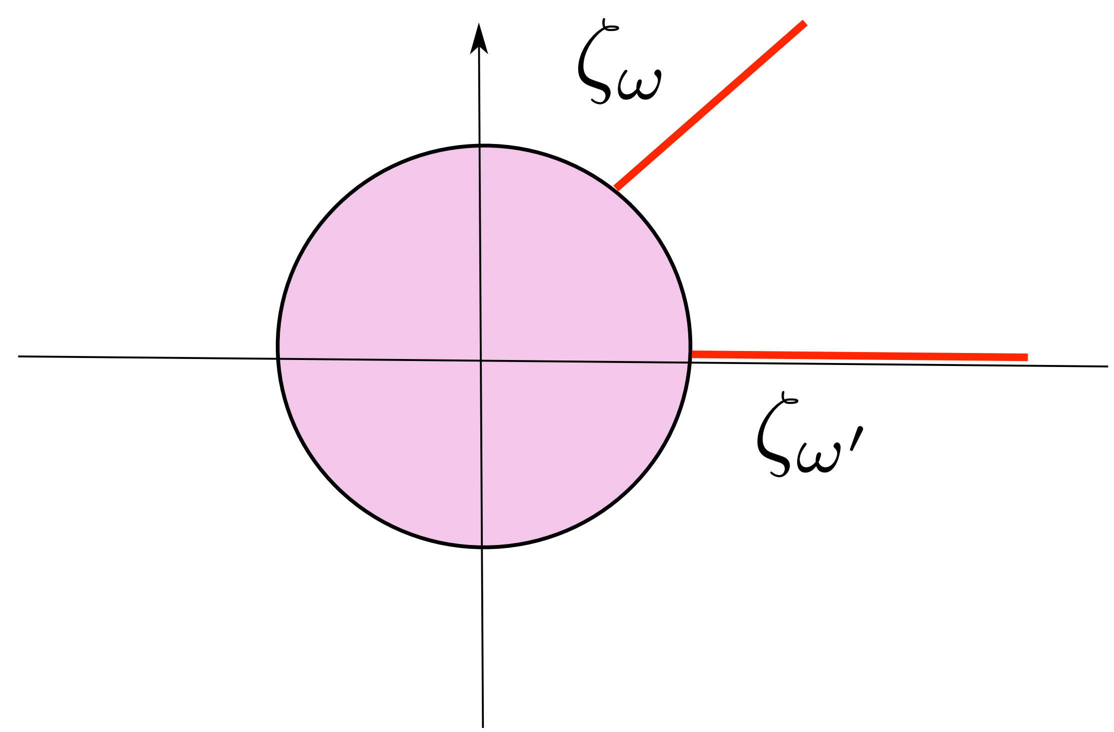
An interesting perspective on this fact of life is given by the **theory of resurgence**. The starting point of this theory is a **factorially divergent**, or Gevrey-1 series:

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \qquad\qquad a_n \sim n!$$

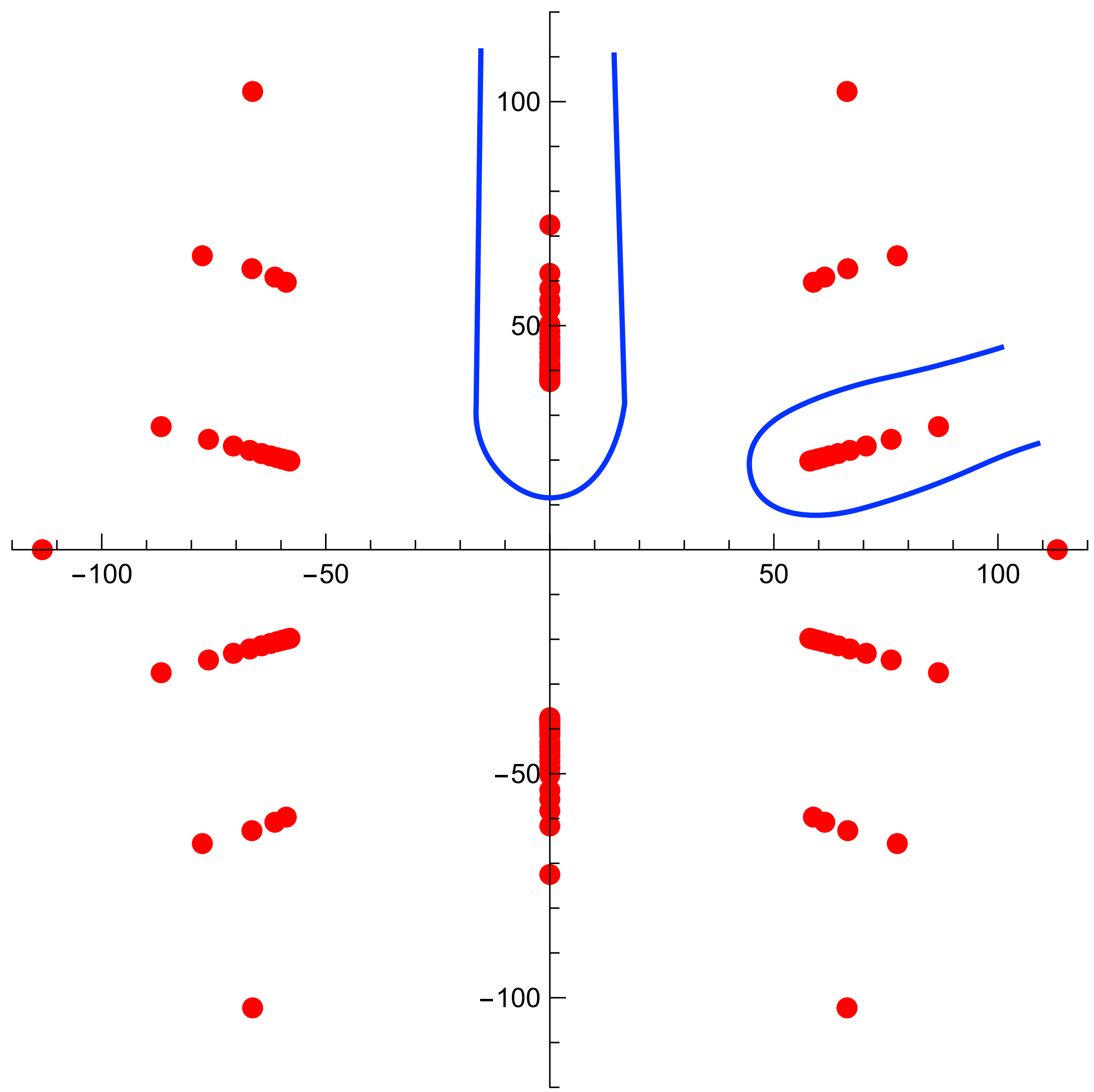
This series is formal, but one can obtain a holomorphic germ at the origin by Borel transform:

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \xrightarrow{\text{Borel transform}} \widehat{\varphi}(\zeta) = \sum_{n \geq 0} \frac{a_n}{n!} \zeta^n$$

The Borel transform is analytic at the origin, but has **singularities** in the complex plane



We then have a "resurgent" version of Darboux' theorem:



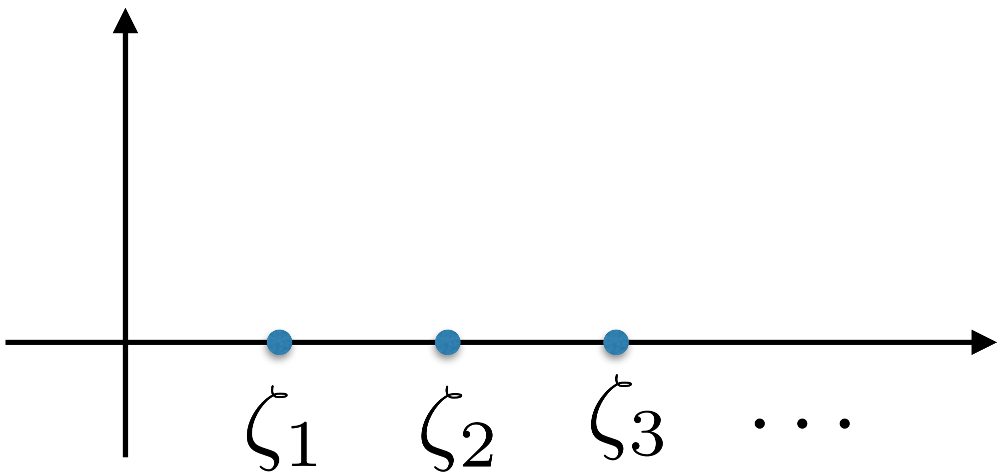
$$\frac{a_k}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_0 \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta$$

contour deformation →

contribution of singularities

A simple (pole) example

Borel transform
with only simple poles



$$\widehat{\varphi}(\zeta_\omega + \xi) = -\frac{\mathsf{S}_\omega}{2\pi\xi} + \cdots$$

Stokes constants

$$a_k = \frac{k!}{2\pi} \sum_\omega \frac{\mathsf{S}_\omega}{\zeta_\omega^{k+1}}$$

In the context of topological strings, this approach was developed by Claudia Rella in her thesis

In physics, the singularities of the Borel transform correspond to non-perturbative sectors of the theory, so we can in principle reconstruct the coefficients of the perturbative series from non-perturbative information!

An example in topological strings

Topological strings are complicated, but they simplify very much in what we called the **replica limit** [M.M.-Schwick]

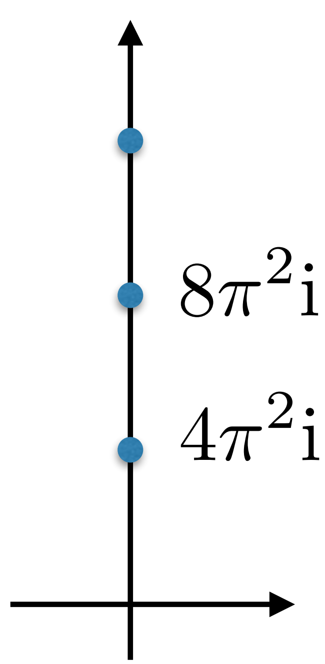
In the TS/ST correspondence, it corresponds to the *N=0* limit of the spectral traces. It just keeps a single coefficient in the conifold expansion, at all genera.

$$F_g(\lambda) = \frac{B_{2g}}{2g(2g-2)} \lambda^{2-2g} + \boxed{c_g} \lambda + \cdots \qquad g \geq 2$$

Let us consider the asymptotic series of these coefficients and ask about its Borel transform

$$R(g_s) = \sum_{g \geq 2} c_g g_s^{2g-2}$$

In the case of local $\mathbb{P}^2$, one can use [Gu-M.M., Rella] to find that the Borel transform has simple poles on the imaginary axis, so the formulae above apply and we get [M.M.-Schwick]



$$\begin{aligned} c_g &= 4\pi\Gamma(2g) \sum_{n \geq 1} \frac{n \mathsf{S}_n}{(4\pi^2 \mathrm{i} n)^{2g}} \\ &= \sqrt{3}(-1)^g \frac{B_{2g} B_{2g-1}(2/3)}{(2g)!} \end{aligned}$$

This is a simple calculation and it has not been extended to more complicated examples, but I find it conceptually important.

First, it shows that knowledge of the so-called resurgent structure (i.e. singularities and Stokes constants) allows to reconstruct perturbative data.

But there is more physics in it.

Singularities in the Borel transform are conjecturally related to the spectrum of BPS states: their location is the mass of the state, and their Stokes constants are BPS invariants. It is known that the singular part of $F_g(\lambda)$ is determined by the BPS state becoming **massless** at the conifold [Strominger, Vafa].

The above calculation shows that higher coefficients in $F_g(\lambda)$ could be obtained from appropriate sums over **massive** BPS states. This sum gives in addition the "arithmetic structure" of the coefficient.

Conclusions

Perhaps the most powerful non-perturbative approaches to topological strings are the TS/ST correspondence and the theory of resurgence.

These approaches not only have implications for perturbation theory, but also for its arithmetic properties. They are still not fully developed (lack of explicit integral kernels, more complicated singularities…), but further developments will have implications for arithmetic!