Invariants of Calabi-Yau Hybrid Models

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Motivation

- Smooth Calabi-Yau geometries only appear at some limiting points in the moduli space. What happens elsewhere?
- What replaces (topological) non-linear sigma models, curve counting, geometric D-branes, etc?
- What is the right language for formulating the equivalent problems in other regions of the moduli space?
- What are computational methods that work throughout the moduli space?
- This talk: Formulate and compute the analogues of genus-0 Gromov-Witten invariants for hybrid models.

Outline

Hybrid models

Genus-0 Correlators

Computation via GLSMs

Examples

- Hybrid phase of $\mathbb{P}^7[2,2,2,2]$
- A two-parameter model

Outlook

$N = (2,2)$ hybrid models I

- Consider a the total space of a rank n bundle over a compact space B:

$$Y = \mathrm{tot}(X \xrightarrow{\pi} B)$$

- And a holomorphic superpotential W such that

$$dW^{-1}(0) = B, \qquad \mathrm{dim} B = d.$$

- A hybrid model is an $N = (2,2)$ NLSM on Y with potential W, plus extra conditions so that the the IR CFT is well-defined ("good hybrids"). [Bertolini-Melnikov-Plesser 13]
  - Special cases:
    - B Calabi-Yau, $W = 0$: Calabi-Yau NLSM
    - $B = \mathrm{pt}$, $W \neq 0$: Landau-Ginzburg model
- The (scalar) fields can be decomposed in terms of base and fibre coordinates:

$$\phi^\alpha = (y^I, \varphi^i), \qquad I = 1, \dots, d, \quad i = 1, \dots, n$$

$N = (2,2)$ hybrid models II

- To have well-defined R-charges one needs a holomorphic Killing vector field V on Y such that

$$\mathcal{L}_V W = W, \qquad \mathcal{L}_V \pi^*(\omega) = 0, \quad \omega \in \Omega^\bullet(B)$$

  - $V = \sum_i q_i \varphi^i \frac{\partial}{\partial \varphi^i}$ ($q_i \ldots$ R-charges)
  - This defines a good hybrid.
  - It makes sure that the associated 2D CFT is well-defined.
- To realise Calabi-Yaus, we consider orbifolds.

[Vafa 89][Intriligator-Vafa 90][Kachru-Witten 93][Bertolini-Melnikov-Plesser 13]

  - There is a $\mathbb{Z}_N$-orbifold (compatible with R-symmetry).
  - The fibre scalars φ^i have charges $q_i = \frac{n_i}{d_i}$ with $N = \mathrm{lcm}(d_1, \ldots, d_n)$.
  - The base scalars y^I have charges $q_I = 0$.

State spaces I

- The state space is determined by the cohomology of operator(s) $\overline{Q}$ related to the SUSY generators.
  - Structure: $\overline{Q} = \overline{Q}_0 + \overline{Q}_w$ (base and fibre components)
  - Computation via spectral sequences. [Bertolini-Melnikov-Plesser 13]
  - For our purposes, we only need a subset of these states.
- The massless states of the (R,R) sector correspond to certain chiral ring elements of the CFT.
  - The CFT has (left) central charge $c = 3d + 3\sum_{i=1}^{n}(1 - 2q_i)$.
  - We combine the orbifold $\mathbb{Z}_N$ with the $\mathbb{Z}_2$ determining the GSO projection $\rightarrow \mathbb{Z}_{2N}$.
  - The states in the (R,R) sector arise for even $k \in \mathbb{Z}_{2N}$.
  - A subset of those corresponds to narrow states.
  - We characterise states by their left/right R-charges $(q, \overline{q}) \leftrightarrow$ grading on state space

State spaces II

- The R-charges of the vacuum $|k\rangle$ in the k-th twisted sector are [Bertolini-Melnikov-Plesser 13]

$$q_{|k\rangle} = \sum_\alpha \left[(q_\alpha - 1)(\tilde{\nu}_\alpha - \frac{1}{2}) - q_\alpha(\nu_\alpha - \frac{1}{2}) \right],$$

$$\overline{q}_{|k\rangle} = \sum_\alpha \left[q_\alpha(\tilde{\nu}_\alpha + \frac{1}{2}) + (q_\alpha - 1)(-\nu_\alpha + \frac{1}{2}) \right].$$

  - $\nu_\alpha = \frac{kq_\alpha}{2} \mod 1$, $\tilde{\nu}_\alpha = \frac{k(q_\alpha - 1)}{2} \mod -1$
- For $K_Y = \mathcal{O}_Y$, $X = \oplus_i L_i$ and k even, $|k\rangle$ is a section of

$$L_{|k\rangle} = \otimes_i L_i^{-\tau_i}, \qquad \tau_\alpha = \nu_\alpha - \tilde{\nu}_\alpha.$$

- Other states in the k-th twisted sector are obtained by acting with operators on $|k\rangle$.

State spaces III

- To define a state in the (R,R) sector:
  - Split $\phi^\alpha = (y^{\alpha'}, \varphi^A)$ with $\tau_{\alpha'} < 1$ ("light") and $\tau_A \geq 1$ ("heavy").
  - By construction, $\tau_I < 1$, so that $y^{\alpha'} = (y^I, \varphi^{i'})$.
  - Decompose $X = X_k \oplus \oplus_A L_A$.
  - Define $Y_k = \mathrm{tot}(X_k \stackrel{\pi_k}{\to} B)$.
- An (R,R) state $|\Psi_u^s\rangle$ has the form

$$|\Psi_u^s\rangle = \Psi(y', \overline{x})^{\alpha'_1 \ldots \alpha'_s}_{\overline{I}_1 \ldots \overline{I}_u} \overline{\chi}_{\alpha'_1} \cdot \ldots \cdot \overline{\chi}_{\alpha'_s} \overline{\eta}^{\overline{I}_1} \cdot \ldots \cdot \overline{\eta}^{\overline{I}_u} |k\rangle.$$

with

$$q = q_{|k\rangle} + q_\Psi + s, \qquad \overline{q} = \overline{q}_{|k\rangle} + q_\Psi + u$$

State spaces IV

- Geometry of the (R,R) state:

$$|\Psi^s_u\rangle = \Psi(y', \overline{x})^{\alpha'_1\ldots\alpha'_s}_{\overline{I}_1\ldots\overline{I}_u}\overline{\chi}_{\alpha'_1}\cdot\ldots\cdot\overline{\chi}_{\alpha'_s}\overline{\eta}^{\overline{I}_1}\cdot\ldots\cdot\overline{\eta}^{\overline{I}_u}|k\rangle.$$

  - $\overline{\chi}_{\alpha'}$ transforms as a section of $T^*_{Y_k}$.
  - $\overline{\eta}^{\overline{I}}$ transforms as a section of $\pi^*_k(\overline{T}_B)$.
  - $|k\rangle$ is a section of $L_{|k\rangle} = \pi^*_k(\otimes_A L^*_A)$
  - Ψ^s_u is a $(0,u)$-form valued in $\mathcal{E}^s = \wedge^s T_{Y_k} \otimes L^*_{|k\rangle}$
- Next, we would have to compute the cohomology of $\overline{Q}$.
  - Spectral sequence with differentials

$$\overline{Q}_0 = -\overline{\eta}^{\overline{I}}\partial_{\overline{y}^{\overline{I}}}, \quad \overline{Q}_W = W_{\alpha'}(y')\chi^{\alpha'}$$

- In our case, this problem simplifies.

Narrow states

- Claim: Narrow states are states with even k where all the fibre fields φ^i have $\tau_i \geq 1$. [Erkinger-JK 22]
- Consequences:
  - $\{\varphi^i\} = \{\varphi^A\} \rightarrow L_{|k\rangle} = \pi_k^*(\otimes_i L_i^*) \rightarrow |k\rangle$ is $\mathbb{Z}_N$-invariant.
  - X_k is trivial $\rightarrow \{\chi_{\alpha'}\} = \{\chi_I\}$ are sections of T_B^* (and $\{\overline{\eta}^{\bar{I}}\}$ sections of $\overline{T}_B$)
  - $|\Psi_u^s\rangle$ are $(0,u)$-forms taking values in $\wedge^s T_B \otimes L_{|k\rangle}^*$.
  - Since $|k\rangle$ transforms as a section of a bundle over B, invariant states require a certain number of $\chi_I, \overline{\eta}^{\bar{I}}$-insertions.
  - In all our examples, $W_{\alpha'}(y') = 0 \rightarrow \overline{Q}_W = 0 \rightarrow \overline{Q}$ is equivalent to $\overline{Q}_0$.

(a, c)-states, cohomology, and pairing

- Connection between (R,R) states and (a, c) states via spectral flow. [Lerche-Vafa-Warner 89]
  - The spectral flow operator $\mathcal{U}_{-\frac{1}{2},\frac{1}{2}}$ maps the (R,R) states to (a, c) states.
  - $(q, \overline{q})_{(a,c)} = (q - \frac{3}{2}, \overline{q} + \frac{3}{2})_{(R,R)}$.
- The narrow cohomology is

$$\mathcal{H}^{(a,c)}_{\mathrm{nar}} = \bigoplus_{\delta} H^*_{\delta}(B) = \bigoplus_{\delta,a} \mathcal{H}^{(a,c)}_{\delta,a}.$$

  - Coincides with mathematics results. [Clader 13][Chiodo-Nagel 15]
- Propose a topological pairing:

$$\langle \phi_{i,a}, \phi_{j,b} \rangle = \frac{1}{N} \delta_{i,N-j} \int_B a \wedge b.$$

  - Consistent with geometry and LG orbifolds.

Correlation functions

- Consider an A-twisted 2D topological SQFT/SCFT (eg. an A-twisted $(2,2)$ hybrid) coupled to 2D topological gravity (ie. topological string theory)
  - The state space of the SCFT is given by the elements of the (a,c) chiral ring. (not necessarily narrow!)
    - Here: $\phi_i \in \mathcal{H}^{(a,c)}_{\mathrm{nar}}$.
  - Associated to each state ϕ there is an infinite tower of gravitational operators $\tau_m(\phi_j)$.
- One can define genus g correlators with m insertions:

[Dijkgraaf, Verlinde^2, Witten, FJRW,...]

$$\langle \tau_{n_1}(\phi_{j_1}) \dots \tau_{n_m}(\phi_{j_m}) \rangle_{g,m,*} = \int_{\mathcal{M}_{g,m}} c_1(\mathcal{L}_1) \wedge \dots \wedge c_1(\mathcal{L}_m)$$

  - $\mathcal{L}_i$ are some line bundles over the punctures on the worldsheet
  - "$*$" and how to define the moduli space integral are model-dependent

Setup for this talk

- Focus on genus $g = 0$.
- Focus on (2, 2) SCFTs with central charge $(c, \overline{c}) = (9, 9)$.
  - This is the Calabi-Yau threefold case.
  - The (a, c)-ring elements have L/R R-charges $(q, \overline{q}) = (-i, i)$ $i \in \{0, 1, 2, 3\}$ with a unique element $\mathbf{1}$ with $(q, \overline{q}) = (0, 0)$.
- Consider good hybrid models.
  - Hybrid FJRW invariants $\langle \tau_{n_1}(\phi_{j_1}) \dots \tau_{n_m}(\phi_{j_m}) \rangle_{g,m,\beta}$ [Clader 13]
- The correlators satisfy topological recursion relations and selection rules.
  - Only give those relevant for genus 0.
- Goal: Compute the generating function of the genus-0 correlators

Recursion relations and selection rules I

- $U(1)_A$ R-symmetry selection rule:

$$\sum_{a=1}^{m}(n_a+\overline{q}_{j_a}) = (\hat{c}-3)(1-g)+m.$$

- String equation (puncture equation)

$$\langle\tau_{n_1}(\phi_{j_1})\ldots\tau_{n_m}(\phi_{j_m})\mathbf{1}\rangle_{g,m+1} = \sum_{i=1}^{m}\langle\tau_{n_i-1}(\phi_{j_i})\tau_{n_1}(\phi_{j_1})\ldots\widehat{\tau_{n_i}(\phi_{j_i})}\ldots\tau_{n_m}(\phi_{j_m})\rangle_{g,m}$$

- Dilaton equation

$$\langle\tau_{n_1}(\phi_{j_1})\ldots\tau_{n_m}(\phi_{j_m})\tau_1(\mathbf{1})\rangle_{g,m+1} = (2g-2+m)\langle\tau_{n_1}(\phi_{j_1})\ldots\tau_{n_m}(\phi_{j_m})\rangle_{g,m}$$

- These rules do not depend on the specifics of the model.

Recursion relations and selection rules II

- Divisor axiom [Kontsevich-Manin 94][Hori 94]
  For "geometric operators" $\tilde{\phi} \in H^2(B)$ and $\beta \in H_2(B)$:

$$\langle \tau_{n_1}(\phi_{j_1}) \dots \tau_{n_m}(\phi_{j_m})\tilde{\phi}\rangle_{g,m+1,\beta} = \left(\int_\beta \tilde{\phi}\right) \langle \tau_{n_1}(\phi_{j_1}) \dots \tau_{n_m}(\phi_{j_m})\rangle_{g,m,\beta} + \dots$$

  - Our case: only $\mathbf{1}, \tau_1(\mathbf{1}), \tau_0(\phi_j) = \phi_j \leftrightarrow$ no '...'
  - Note! Not all ϕ_i are necessarily in $H^2(B)$.
- Quantum symmetry for $\mathbb{Z}_N$-orbifolds [Vafa 89]
  - There is a $\mathbb{Z}_N$ quantum symmetry acting on the state space.
  - Selection rule: [FJRW, Clader 13, Erkinger-JK 22]

$$2g - 2 + m - q_B(\beta) - \sum_{i=1}^{m} q_{\delta_{j_i}} = 0 \mod N$$

  - $q_B(\beta)$ is non-zero if the orbifold acts trivially on B (gerbes!)
  - $q_{\delta_{j_i}}$ accounts for the charge of the state in the fibre direction

J-function

- The J-function is the generating function of genus 0 correlators.

[Givental 97][Fan-Jarvis-Ruan 07][Clader 13][Romo-Scheidegger-JK 20][Erkinger-JK 22]

$$
\begin{aligned}
J =& e_0 + \sum_{i=1}^{h} t_i e_i \\
&+ \sum_{\beta} \sum_{\overline{q}_c < \hat{c}-1} \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_h \geq 0} \prod_{i=1}^{h} \frac{(t_i)^{k_i}}{k_i!} \langle \tau_n(\phi_c) \prod_{j=1}^{h} \phi_j^{k_j} \rangle_{0,|k|+1,\beta} \eta^{cb} e_b .
\end{aligned}
$$

  - Expansion in terms of basis elements e_k of $H^{(a,c)} = \bigoplus_{(q,\overline{q})} H^{(a,c)}_{(q,\overline{q})}$ of dimension $2h+2$. ($e_0 \leftrightarrow \mathbf{1}$)
  - t_i ... flat coordinates $\leftrightarrow$ marginal deformations of the CFT $\leftrightarrow$ ($\phi_i \leftrightarrow e_i$) with $(q,\overline{q}) = (-1,1)$ [Cecotti-Vafa 91]
  - $\eta_{ab} = \langle \phi_a \phi_b \rangle_{0,2,0}$ ... topological metric

J-function and I-function

- The I-function can also be expanded in terms of the e_k:

$$I(u) = I_0(u)e_0 + \sum_{i=1}^{h} I_i(u)e_i + I_c(u)e^c$$

- The J-function is related to the I-function via

$$J(t) = \frac{I(u(t))}{I_0(u(t))}$$

- The flat coordinates are given by

$$t_i(u) = \frac{I_i(u)}{I_0(u)}$$

  - This coincides with the mirror map.
  - We need to identify the states with $(q, \overline{q}) = (-1, 1)$.

J-function via the GLSM

- Hybrids arise as phases of GLSMs.
- The I-function can be extracted from GLSM partition functions. [Romo-Scheidegger-JK 20][Erkinger-JK 20][Erkinger-JK 22]
- Roadmap to extract invariants
    - Find a GLSM with a hybrid phase and compute the I-function.
    - Analyse the state space of the hybrid phase.
        - Use the information to associate each component of the I-function with a corresponding state in the CFT.
        - Identify the (a, c) states with with $(q, \overline{q}) = (-1, 1)$ that define the flat coordinates.
    - Compute the J-function via the I-function.
    - Use the selection rules and the string, dilaton and divisor equation to reduce the correlators to a minimal set.
    - Read off the invariants using the general structure of the J-function.

GLSM data – I

- G ... a compact Lie group (will "break" to orbifold group)
- V ... space of chiral fields $\phi_i \in V$ (will become the y^α)
- $\rho_V : G \to GL(V)$ ... faithful complex representation
  - CY condition: $G \to SL(V)$
- $R : U(1)_V \to GL(V)$ ... vector R-symmetry
  - R_i ... R-charges
- $T \subset G$ ... maximal torus
  - Lie algebras: $\mathfrak{g} = Lie(G)$, $\mathfrak{t} = Lie(T)$
  - $Q_i^a \in \mathfrak{t}_{\mathbb{C}}^*$ ... gauge charges of chiral fields

GLSM Data – II

- $\mathtt{t}^a \in \mathfrak{g}_{\mathbb{C}}^*$ ... FI-theta parameters
  - $\mathtt{t}^a = \zeta^a - i\theta^a$ ζ^a : real, θ^a : 2π-periodic
  - $\mathtt{t}^a \leftrightarrow$ Kähler moduli of the CY $\leftrightarrow$ marginal deformations of the CFT
- $\sigma_a \in \mathfrak{g}_{\mathbb{C}}$ ... scalar components of the vector multiplet
- $W \in \mathrm{Sym}\, V^*$ ... superpotential (will become W of the hybrid)
  - G-invariant
  - (vector) R-charge 2
  - non-zero for compact CYs
- Classical potential

$$U = \frac{1}{8e^2}|[\sigma, \bar{\sigma}]|^2 + \frac{1}{2}\left(|Q(\sigma)\phi|^2 + |Q(\bar{\sigma})\phi|^2\right) + \frac{e^2}{2}D^2 + |F|^2$$

Phases of the GLSM

- Higgs branch: $\sigma = 0$
- D-terms

$$\mu(\phi) = \zeta$$

  - $\mu : V \to \mathfrak{g}^*$ ... moment map
- F-terms

$$dW = 0$$

- Phases: parameter space gets divided into chambers
- Classical vacua

$$X_\zeta = \{dW^{-1}(0)\} \cap \mu^{-1}(\zeta)/G$$

Sphere partition function

- Sphere partition function

[Benini-Cremonesi 12][Doroud-Gomis-LeFloch-Lee 12]

$$
\begin{aligned}
Z_{S^2} \; = \; & C \sum_{m \in \mathbb{Z}^{\mathrm{rk}G}} \int_\gamma d^{\mathrm{rk}G}\sigma \prod_{\alpha>0} \left(\frac{\alpha(m)^2}{4} + \alpha(\sigma)^2 \right) \\
& \cdot \prod_{j=1}^{\dim V} \frac{\Gamma\left(iQ_j(\sigma) - \frac{Q_j(m)}{2} + \frac{R_j}{2}\right)}{\Gamma\left(1 - iQ_j(\sigma) - \frac{Q_j(m)}{2} - \frac{R_j}{2}\right)} e^{-4\pi i \zeta(\sigma) - i\theta(m)}
\end{aligned}
$$

  - $\alpha > 0$ positive roots
  - γ ...integration contour (s.t. integral is convergent)
- Z_{S^2} computes the exact Kähler potential on $\mathcal{M}_K$.

[Jockers-Kumar-Lapan-Morrison-Romo 12][Gomis-Lee 12]

[Gerchkovitz-Gomis-Komargodski 14][Gomis-Hsin-Komargodski-Schwimmer-Seiberg-Theisen 15]

Z_{S^2} for hybrid phases

- For a hybrid model, the sphere partition function can be written as [Erkinger-JK 20]

$$Z_{S^2}^{hyb}(\mathtt{t}) = \sum_{\delta \in G^{hyb}} \int_B \frac{\Gamma_\delta(H)}{\Gamma_\delta^*(H)} |I_\delta(\mathtt{t}, H)|^2$$

  - $\delta \in G^{hyb}$... narrow sectors
  - Basis $H \in H^2(B)$
  - $\mathtt{t}$... FI-theta parameter of the GLSM
  - $\Gamma_\delta(H)$... Gamma class
- From this, we can extract (conjectural) expressions of the I-function (and the Gamma class...) and follow the roadmap above.

Example: Hybrid phase of $\mathbb{P}^7[2,2,2,2]$

- $U(1)$ GLSM

	$p_1,\ldots,p_4$	$x_1,\ldots,x_8$	FI
$U(1)$	-2	1	ζ
$U(1)_V$	0	1	

- Geometric phase at $\zeta \gg 0$.
- Hybrid phase at $\zeta \ll 0$ [Caldararu-Distler-Hellerman-Pantev-Sharpe 07]
  - base: B is $\mathbb{P}^3_{2222}$ with coordinates $(p_1,\ldots,p_4)$ (gerbe!)
  - fibre: $G^{hyb} = \mathbb{Z}_2$ Landau-Ginzburg orbifold with potential $W = \sum_i \langle p_i \rangle G_2^i(x_1,\ldots,x_8)$
  - The physical model is defined on $Y = \mathrm{tot}(\mathcal{O}(-1)^{\oplus 8} \to \mathbb{P}^3)$ with $\mathbb{Z}_2$ acting non-trivially on the fibre and trivially on the base.

Narrow state space I

- All eight x_i have $q_i = \frac{1}{2}$. In the twisted sectors the vacua $|k\rangle$ $k \in \mathbb{Z}_4$:

k	0	1	2	3
q	$-\frac{3}{2}$	0	$-\frac{3}{2}$	0
$\overline{q}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$
ν_i	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$\tilde{\nu}_i$	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$
τ_i	0	$\frac{1}{2}$	1	$\frac{3}{2}$

  - $k = 2\delta = 2$ is narrow.
- The (R,R) ground states are:

$$k = 2: \qquad |\Psi_3^0\rangle_{-\frac{3}{2},\frac{3}{2}} \quad |\Psi_2^1\rangle_{-\frac{1}{2},\frac{1}{2}} \quad |\Psi_1^2\rangle_{\frac{1}{2},-\frac{1}{2}} \quad |\Psi_0^3\rangle_{\frac{3}{2},-\frac{3}{2}}.$$

Narrow state space II

- Narrow states $\phi_{\delta,a}$

$\phi_{\delta,a}$	$\phi_{1,\mathbf{1}}$	$\phi_{1,H}$	ϕ_{1,H^2}	ϕ_{1,H^3}
$(q,\overline{q})$	$(0,0)$	$(-1,1)$	$(-2,2)$	$(-3,3)$

- Topological pairing

$$\langle \phi_{i,a}, \phi_{j,b} \rangle = \frac{1}{2}\delta_{i,2-j}\int_{\mathbb{P}^3} a \wedge b.$$

- *I*-function [Clader 13][Erkinger-JK 20]

$$I^{\zeta \ll 0}_{\delta=1}(\mathtt{t}, H) = \frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^4}{\Gamma\left(\frac{H}{2\cdot 2\pi i}+\frac{1}{2}\right)^8} \sum_{a=0}^{\infty} e^{\mathtt{t}\left(\frac{H}{2\cdot 2\pi i}+a\right)} \frac{\Gamma\left(a+\frac{H}{2\cdot 2\pi i}+\frac{1}{2}\right)^8}{\Gamma\left(1+2a+\frac{H}{2\pi i}\right)^4}.$$

$$I = I_1^{\mathbf{1}} e_{1,\mathbf{1}} + I_1^{H} e_{1,H} + I_2^{H^2} e_{1,H^2} + I_3^{H^3} e_{1,H^3}, \qquad u = e^{\frac{\mathtt{t}}{2}}$$

Correlators and selection rules

- Correlators allowed by $U(1)_A$-selection rule:

$$\langle(\phi_{1,H})^3\rangle_{0,3,0}, \quad \langle\tau_1(\phi_{1,1})(\phi_{1,H})^k\rangle_{0,k+1,\beta}, \quad \langle(\phi_{1,H})^{k+1}\rangle_{0,k+1,\beta}.$$

- Dilaton equation:

$$\langle\tau_1(\phi_{1,1})(\phi_{1,H})^k\rangle_{0,k+1,\beta} = (-2+k)\langle(\phi_{1,H})^k\rangle_{0,k,\beta}.$$

- Divisor equation:

$$\langle(\phi_{1,H})^k\rangle_{0,k,\beta} = \left(\int_\beta \phi_{1,H}\right)^k \langle\cdot\rangle_{0,0,\beta} = \beta^k\langle\cdot\rangle_{0,0,\beta}$$

- Quantum symmetry ($g=0$, m insertions):

$$-2-\beta = 0 \mod 2$$

J-function

- The J-function reduces to

$$\begin{aligned} J =& e_{1,\mathbf{1}} + t e_{1,H} \\ &+ 2e_{1,H^2}\frac{t^2}{2}\langle(\phi_{1,H})^3\rangle_{0,3,0} + 2e_{1,H^3}\frac{t^3}{6}\langle(\phi_{1,H})^3\rangle_{0,3,0} \\ &+ 2e_{1,H^2}\sum_{n\geq 2} e^{(2n-2)t}(2n-2)\langle\cdot\rangle_{0,0,2n-2} \\ &+ 2e_{1,H^3}\sum_{n\geq 2} e^{(2n-2)t}[(2n-2)t-2]\langle\cdot\rangle_{0,0,2n-2} \end{aligned}$$

- Change of coordinates:

$$t = \frac{I_1^H(u)}{I_1^{\mathbf{1}}(u)} = \log u + \frac{u^2}{2048} + \frac{157u^4}{268435456} + \mathcal{O}(u^6)$$

Invariants

- Read off invariants from e_{1,H^2}-coefficient which is $\frac{I_2^{H^2}}{I_1^1}$.
- The three-point correlator is:

$$\langle(\phi_{1,H})^3\rangle_{0,3,0} = \frac{1}{2}.$$

- The hybrid FJRW invariants are:

$2n-2$	2	4	6	8
$\langle\cdot\rangle_{2n-2}$	$\frac{1}{32768}$	$\frac{153}{17179869184}$	$\frac{343}{59373627899904}$	$\frac{3213145}{590295810358705651712}$

Relation to other invariants

- The hybrid phase can also be understood geometrically as a branched double cover over $\mathbb{P}^3$. [many papers]
- Instanton numbers [Sharpe 12]

i	1	2	3	4
N_i	64	1216	52032	3212992

- Relation to $\langle\cdot\rangle_{2n-2} = n_{2n-2}$:

$$2^{16k}2^5k^3n_{2k} = f_k(N)$$

  - where

$$\sum_{a=0}^{\infty}\frac{a^3N_aq^a}{1-q^a} = \sum_{k=0}^{\infty}f_k(N)q^k$$

Two-parameter example – I

- $U(1)^2$ GLSM

	p	x_6	x_4	x_5	x_1	x_2	x_3	FI
$U(1)_1$	-6	1	2	3	0	0	0	ζ_1
$U(1)_2$	0	-3	0	0	1	1	1	ζ_2

- Hybrid phase at $\zeta_1 \ll 0, \zeta_2 \gg 0$
  - base: B is $\mathbb{P}^2$ with coordinates (x_1, x_2, x_3)
  - fibre: $G^{hyb} = \mathbb{Z}_6$ Landau-Ginzburg orbifold with potential $W = \langle p \rangle G_{(6,0)}(\langle x_1, x_2 \rangle, \ldots, x_6)$
  - The physical model is defined on $Y = \mathrm{tot}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-3) \to \mathbb{P}^2)$ with $\mathbb{Z}_6$ acting on the fibre.
- Narrow states $\phi_{\delta,a}$

$\phi_{\delta,a}$	$\phi_{1,\mathbf{1}}$	$\phi_{1,H}$	ϕ_{1,H^2}	$\phi_{5,\mathbf{1}}$	$\phi_{5,H}$	ϕ_{5,H^2}
$(q,\overline{q})$	$(0,0)$	$(-1,1)$	$(-2,2)$	$(-1,1)$	$(-2,2)$	$(-3,3)$

Two-parameter example – II

- Correlators
  - $\beta = 0$:

$$\langle(\phi_{1,H})^2(\phi_{5,\mathbf{1}})\rangle_{0,3,0} = \frac{1}{6}$$

  - $\beta \neq 0$:

$$\langle(\phi_{1,H})^{k_1}(\phi_{5,\mathbf{1}})^{k_2}\rangle_{0,k_1+k_2,\beta} = \beta^{k_1}\langle(\phi_{5,\mathbf{1}})^{k_2}\rangle_{0,k_2,\beta}$$

  - Note: The divisor equation does not apply to $\phi_{2,\mathbf{1}}$.
    We get a hybrid of geometric (instanton) and FJRW invariants.
- Numbers

(k_2,β)	$(1,1)$	$(1,2)$	$(1,3)$	$(4,0)$	$(4,1)$
#	$\frac{113}{108}$	$\frac{196319}{62208}$	$\frac{360744024241}{19591041024}$	$\frac{1}{2}$	$\frac{379}{72}$

Open Questions

- Relation to torsion-refined invariants. [Katz-Klemm-Schimannek-Sharpe 22]
- Are there integer invariants?
- Invariants for hybrids that are not "good".
  - Mathematicians say that there is no J-function.
- Arithmetic of hybrids that arise in the moduli space of elliptic fibrations.
- D-branes and the associated categories. [Pryor-JK 24]