# Heat-kernel as a tool for one loop matching and running


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# An intro to heat kernel

- Heat-kernel was introduced nearly 70 years ago as a method to compute loops that respects gauge covariance throughout the calculation

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## On Gauge Invariance and Vacuum Polarization

Julian Schwinger
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(Received December 22, 1950)

This paper is based on the elementary remark that the extraction of gauge invariant results from a formally gauge invariant theory is ensured if one employs methods of solution that involve only gauge covariant quantities. We illustrate this statement in connection with the problem of vacuum polarization by a prescribed electromagnetic field. The vacuum current of a charged Dirac field, which can be expressed in terms of the Green's function of that field, implies an addition to the action integral of the electromagnetic field. Now these quantities can be related to the dynamical properties of a "particle" with space-time coordinates that depend upon a proper-time parameter. The proper-time equations of motion involve only electromagnetic field strengths, and provide a suitable gauge invariant basis for treating problems. Rigorous solutions of the equations of motion can be obtained for a constant field, and for a plane wave field. A renormalization of field strength and charge, applied to the modified lagrange function for constant fields, yields a finite, gauge invariant result which implies nonlinear properties for the electromagnetic field in the vacuum. The contribution of a zero spin charged field is also stated. After the same field strength renormalization, the modified physical quantities describing a plane wave in the vacuum reduce to just those of the maxwell field; there are no nonlinear phenomena for a single plane wave, of arbitrary strength and spectral composition. The results obtained for constant (that is, slowly varying fields), are then applied to treat the two-photon disintegration of a spin zero neutral meson arising from the polarization of the proton vacuum. We obtain approximate, gauge invariant expressions for the effective interaction between the meson and the electromagnetic field, in which the nuclear coupling may be scalar, pseudoscalar, or pseudovector in nature. The direct verification of equivalence between the pseudoscalar and pseudovector interactions only requires a proper statement of the limiting processes involved. For arbitrarily varying fields, perturbation methods can be applied to the equations of motion, as discussed in Appendix A, or one can employ an expansion in powers of the potential vector. The latter automatically yields gauge invariant results, provided only that the proper-time integration is reserved to the last. This indicates that the significant aspect of the proper-time method is its isolation of divergences in integrals with respect to the proper-time parameter, which is independent of the coordinate system and of the gauge. The connection between the proper-time method and the technique of "invariant regularization" is discussed. Incidentally, the probability of actual pair creation is obtained from the imaginary part of the electromagnetic field action integral. Finally, as an application of the Green's function for a constant field, we construct the mass operator of an electron in a weak, homogeneous external field, and derive the additional spin magnetic moment of $\alpha/2\pi$ magnetons by means of a perturbation calculation in which proper-mass plays the customary role of energy.

### I. INTRODUCTION

Quantum electrodynamics is characterized by several formal invariance properties, notably relativistic and gauge invariance. Yet specific calculations by conventional methods may yield results that violate these requirements, in consequence of the divergences inherent in present field theories. Such difficulties concerning relativistic invariance have been avoided by employing formulations of the theory that are explicitly invariant under coordinate transformations, and by maintaining this generality through the course of calculations. The preservation of gauge invariance has apparently been considered to be a more formidable task. It should be evident, however, that the two problems are quite analogous, and that gauge invariance difficulties naturally disappear when methods of solution are adopted that involve only gauge invariant quantities.

We shall illustrate this assertion by applying such a gauge invariant method to treat several aspects of the problem of vacuum polarization by a prescribed electromagnetic field. The calculation of the current associated with the vacuum of a charged particle field involves the construction of the Green's function for the particle field in the prescribed electromagnetic field. This vacuum current can be exhibited as the variation of an action integral with respect to the potential vector, which action effectively adds to that of the maxwell field in describing the behavior of electromagnetic fields in the vacuum. We shall relate these problems to the solution of particle equations of motion with a proper-time parameter. The equations of motion, which involve only electromagnetic field strengths, provide the desired gauge invariant basis for our discussion.

Explicit solutions can be obtained in the two situations of constant fields, and fields propagated with the speed of light in the form of a plane wave.[1] For constant (that is, slowly varying) fields, a renormalization of field strength and charge yields a modified lagrange function differing from that of the maxwell field by terms that imply a nonlinear behavior for the electromagnetic field. The result agrees precisely with one obtained some time ago by other methods and a somewhat different viewpoint.[2] The modified physical quantities characterizing the plane wave in the vacuum revert to those of the maxwell field after the same field strength renormalization. For weak arbitrarily varying fields, perturbation methods can be applied to the equations of motion. This will be discussed in Appendix A.

The consequences thus obtained are useful in connection with a class of problems in which gauge invari-

[1] That the Dirac equation can be solved exactly, in the field of a plane wave, was recognized by D. M. Volkow, Z. Physik **94**, 25 (1935).

[2] W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936). V. Weisskopf, Kgl. Danske Videnskab. Selskabs. Mat.-fys. Medd. **14**, No. 6 (1936).





## Three-Loop Yang-Mills $\beta$-Function *via* the Covariant Background Field Method

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**Abstract**

We demonstrate the effectivity of the covariant background field method by means of an explicit calculation of the 3-loop $\beta$-function for a pure Yang-Mills theory. To maintain manifest background invariance throughout our calculation, we stay in coordinate space and treat the background field non-perturbatively. In this way the presence of a background field does not increase the number of vertices and leads to a relatively small number of vacuum graphs in the effective action. Restricting to a covariantly constant background field in Fock-Schwinger gauge permits explicit expansion of all quantum field propagators in powers of the field strength only. Hence, Feynman graphs are at most logarithmically divergent. At 2-loop order only a single Feynman graph without subdivergences needs to be calculated. At 3-loop order 24 graphs remain. Insisting on manifest background gauge invariance at all stages of a calculation is thus shown to be a major labor saving device. All calculations were performed with *Mathematica* in view of its superior pattern matching capabilities. Finally, we describe briefly the extension of such covariant methods to the case of supergravity theories.



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## Chiral Perturbation Theory to One Loop*

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The Green's functions of QCD are expanded in powers of the external momenta and of the quark masses. The Ward identities of chiral symmetry determine the expansion up to and including terms of order $p^4$ (at fixed ratio $m_{\text{quark}}/p^2$) in terms of a few constants, which may be identified with the coupling constants of a unique effective low energy Lagrangian. The low energy representation of several Green's functions and form factors and of the $\pi\pi$ scattering amplitude are then calculated. The values of the low energy coupling constants are extracted from available experimental data. The corrections of order $M_\pi^2$ to the $\pi\pi$ scattering lengths and effective ranges turn out to be substantial and the improved low energy theorems agree very well with the measured phase shifts. The observed differences between the data and the uncorrected soft pion theorems may even be used to measure the scalar radius of the pion, which plays a central role in the low energy expansion. © 1984 Academic Press, Inc.

*Contents.* 

### 1. Introduction

If the quark masses are set equal to zero, the QCD Hamiltonian is symmetric under the chiral group $SU(N_f) \times SU(N_f)$. One assumes that the ground state of the

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# Heat kernel representation of the propagators

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A.A.Bel'kov et al., hep-ph/9506237

I.G.Avramidi, Nuc.Phys.B, 355(1991)

* Scalar propagator: $G_s(m,U) = \dfrac{-i}{D^2+m^2+U-i\epsilon}$

In Heat-kernel representation, $G_s(m,U) = \int_0^\infty dt\, K_s(t,m,U)$ where, $K_s(t,m,U) \equiv e^{-i(D^2+m^2+U)t}$

The kernel satisfies the Schrödinger equation, $i\partial_t K_s(t,m,U) = (D^2+m^2+U)\,K_s(t,m,U), \qquad K_s(0,m,U) = 1$

To get the form of the kernel, define: $K_s(t,m,U;x,y) = \langle y | K_s(t,m,U) | x \rangle$ and make the ansatz, $K_s(t,m,U;x,y) = K_0(t,m;x-y)\,H(t,U;x,y)$

$$\boxed{K_s(t,m,U;x,y) = \int \frac{d^dk}{(2\pi)^d} e^{-ik(x-y)} \left( e^{i(k^2-m^2)t} H(t,U;x,y) \right)}$$

$(4\pi t)^{-d/2} e^{-tM^2 - \frac{(x-y)^2}{4t}}$

The HK for the free operator $(D^2+m^2)^n$

$\sum \dfrac{(-t)^n}{n!} b_n(x,y)$

* Fermion propagator:

$$\boxed{K_f(t,m,U;x,y) = \int \frac{d^dk}{(2\pi)^d} e^{-ik(x-y)} \left( e^{i(k^2-m^2)t} (ik+m) H(t,U;x,y) \right)}$$

* Gauge propagator:

$$\boxed{K_v(t,m,U;x,y) = \int \frac{d^dk}{(2\pi)^d} e^{-ik(x-y)} \left( -\, e^{i(k^2-m^2)t} H(t,U;x,y) \right)}$$

Gersdorff et al., 2212:07451

# One loop effective action and heat-kernel coefficients

- Let's consider the part of UV Lagrangian that's bi-linear in $\Phi$: $\mathcal{L}^{\Phi} = \Phi^\dagger (D^2 + U + M^2)\Phi = \Phi^\dagger(\Delta)\Phi,$

- One-loop effective action after integrating out $\Phi$:

$$\mathscr{L}_{\text{eff, 1-loop}} = c_s\, \mathrm{tr} \log(-P^2 + U + M^2) = c_s\, \mathrm{tr} \int_0^\infty \frac{dt}{t} e^{-t\Delta} = c_s\, \mathrm{tr} \int_0^\infty \frac{dt}{t} K_s(t, x, x, \Delta)$$

$$= c_s \int_0^\infty \frac{dt}{t} (4\pi t)^{-d/2} e^{-tM^2} \sum_k \frac{(-t)^k}{k!} \mathrm{tr}[b_k] = \frac{c_s}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} M^{d-2k} \frac{(-1)^k}{k!} \Gamma[k - d/2] \underbrace{\mathrm{tr}[b_k]}_{\text{Heat-Kernel coefficients}}$$

- For $k \leq d/2$ the Gamma function has simple poles. Assuming $d = 4 - \epsilon$, we can write the divergent part of the effective-action:

$$\mathcal{L}_{div}^{(k)} = \frac{c_s}{(4\pi)^{2-\epsilon/2}} M^{d-2k} \frac{(-1)^k}{k!} \frac{(\epsilon/2 - 3 + k)!}{(\epsilon/2 - 1)!} \underbrace{\Gamma[\epsilon/2]}_{2/\epsilon - \gamma_E + \mathcal{O}(\epsilon)} \mathrm{tr}[b_k],$$

- Since the kernel satisfies the Schrödinger equation, we get a recursive relation to find the coefficients.

UB, Joydeep Chakrabortty, Shakeel Ur Rahaman, Kaanapuli Ramkumar; 2306.09103

# One loop effective action and heat-kernel coefficients

$$\mathscr{L}_{\text{eff, 1-loop}} = \frac{c_s}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} M^{d-2k} \frac{(-1)^k}{k!} \Gamma[k-d/2]\, \mathrm{tr}[b_k]$$

* Operators of the form $\mathcal{O}(D^r U^s)$ appear in the coefficient $b_n(x,x)$ where $n = r/2 + s$.

Start with the initial condition, $b_0(x,x) = I$

Use recursive relation, $\mathcal{O}(D^r U^s) \equiv [b_{r/2+s}][[U^s]] = \sum_{k=0}^{n=r/2+s} \frac{n!\,(n-1)!}{k!(2n-k)!} \{k\, D^{2(n-k)} \{U b_{k-1}[[U^{s-1}]]\} - T_{2(n-k)}\, b_k[[U^s]]\}$

$$\Rightarrow T_{\mu_1\mu_2\ldots\mu_m} = D_{\mu_1} T_{\mu_2\ldots\mu_m} + R_{\mu_2\ldots\mu_m,\mu_1},$$

in the above equation, $R_{\mu_2\ldots\mu_m,\mu_1} = [D_{\mu_2}\ldots D_{\mu_m}, D_{\mu_1}]$,

$$\Rightarrow R_{\mu_2\mu_3\ldots\mu_m,\mu_1} = G_{\mu_2\mu_1} D_{\mu_3}\ldots D_{\mu_m} + D_{\mu_2} D_{\mu_3\ldots\mu_m,\mu_1},$$

A few examples:

$$\begin{aligned}
\mathrm{tr}[b_0] &= \mathrm{tr}\, I, \\
\mathrm{tr}[b_1] &= \mathrm{tr}\, U, \\
\mathrm{tr}[b_2] &= \mathrm{tr}\left[U^2 + \frac{1}{6}(G_{\mu\nu})^2\right], \\
\mathrm{tr}[b_3] &= \mathrm{tr}\left[U^3 - \frac{1}{2}(U_{;\mu})^2 + \frac{1}{2} U\, G_{\mu\nu} G_{\mu\nu} - \frac{1}{10}(J_\nu)^2 + \frac{1}{15} G_{\mu\nu}\, G_{\nu\rho}\, G_{\rho\mu}\right], \\
\mathrm{tr}[b_4] &= \mathrm{tr}\Big[U^4 + U^2 U_{;\mu\mu} + \frac{4}{5} U^2 (G_{\mu\nu})^2 + \frac{1}{5}(U\, G_{\mu\nu})^2 + \frac{1}{5}(U_{;\mu\mu})^2 - \frac{2}{5} U\, U_{;\nu}\, J_\nu \\
&\quad - \frac{2}{5} U (J_\mu)^2 + \frac{2}{15} U_{;\mu\mu} (G_{\rho\sigma})^2 + \frac{4}{15} U\, G_{\mu\nu} G_{\nu\rho} G_{\rho\mu} + \frac{8}{15} U_{;\nu\mu}\, G_{\rho\mu} G_{\rho\nu} \\
&\quad + \frac{1}{35}(J_{\mu;\nu})^2 + \frac{16}{105} G_{\mu\nu} J_\mu J_\nu + \frac{1}{420}(G_{\mu\nu} G_{\rho\sigma})^2 + \frac{17}{210}(G_{\mu\nu})^2 (G_{\rho\sigma})^2 \\
&\quad + \frac{1}{105} G_{\mu\nu} G_{\nu\rho} G_{\rho\sigma} G_{\sigma\mu} + \frac{2}{35}(G_{\mu\nu} G_{\nu\rho})^2 + \frac{16}{105} J_{\nu;\mu} G_{\nu\sigma} G_{\sigma\mu}\Big],
\end{aligned}$$

$\boxed{U_{ij} = \frac{\delta^2 \mathcal{L}_{UV}}{\delta\Phi_i \delta\Phi_j}}$ $\boxed{G_{\mu\nu} = [P_\mu, P_\nu]}$

$\boxed{J_\mu = P_\nu G_{\nu\mu} = [P_\nu, [P_\nu, P_\mu]]}$

# One loop effective action up to dimension eight

$$
\begin{aligned}
\mathcal{L}_{\text{eff}}^{d\leq 8} = &\frac{c_s}{(4\pi)^2} M^4 \left[-\frac{1}{2}\left(\ln\left[\frac{M^2}{\mu^2}\right]-\frac{3}{2}\right)\right] + \frac{c_s}{(4\pi)^2}\,\mathrm{tr}\left\{ M^2\left[-\left(\ln\left[\frac{M^2}{\mu^2}\right]-1\right)U\right]\right.\\
&+ M^0\,\frac{1}{2}\left[-\ln\left[\frac{M^2}{\mu^2}\right]U^2 - \frac{1}{6}\ln\left[\frac{M^2}{\mu^2}\right](G_{\mu\nu})^2\right]\\
&+ \frac{1}{M^2}\frac{1}{6}\left[-U^3 - \frac{1}{2}(P_\mu U)^2 - \frac{1}{2}U\,(G_{\mu\nu})^2 - \frac{1}{10}(J_\nu)^2 + \frac{1}{15}G_{\mu\nu}\,G_{\nu\rho}\,G_{\rho\mu}\right]\\
&+ \frac{1}{M^4}\frac{1}{24}\left[U^4 - U^2(P^2U) + \frac{4}{5}U^2(G_{\mu\nu})^2 + \frac{1}{5}(U\,G_{\mu\nu})^2 + \frac{1}{5}(P^2U)^2\right.\\
&\qquad - \frac{2}{5}U\,(P_\mu U)\,J_\mu + \frac{2}{5}U(J_\mu)^2 - \frac{2}{15}(P^2U)(G_{\rho\sigma})^2 + \frac{1}{35}(P_\nu J_\mu)^2\\
&\qquad - \frac{4}{15}U\,G_{\mu\nu}G_{\nu\rho}G_{\rho\mu} - \frac{8}{15}(P_\mu P_\nu U)\,G_{\rho\mu}G_{\rho\nu} + \frac{16}{105}G_{\mu\nu}J_\mu J_\nu\\
&\qquad + \frac{1}{420}(G_{\mu\nu}G_{\rho\sigma})^2 + \frac{17}{210}(G_{\mu\nu})^2(G_{\rho\sigma})^2 + \frac{2}{35}(G_{\mu\nu}G_{\nu\rho})^2\\
&\qquad \left. + \frac{1}{105}G_{\mu\nu}G_{\nu\rho}G_{\rho\sigma}G_{\sigma\mu} + \frac{16}{105}(P_\mu J_\nu)G_{\nu\sigma}G_{\sigma\mu}\right]\\
&+ \frac{1}{M^6}\frac{1}{60}\left[-U^5 + 2\,U^3(P^2U) + U^2(P_\mu U)^2 - \frac{2}{3}U^2G_{\mu\nu}U\,G_{\mu\nu} - U^3(G_{\mu\nu})^2\right.\\
&\qquad + \frac{1}{3}U^2(P_\mu U)J_\mu - \frac{1}{3}U\,(P_\mu U)(P_\nu U)\,G_{\mu\nu} - \frac{1}{3}U^2J_\mu(P_\mu U)\\
&\qquad - \frac{1}{3}U\,G_{\mu\nu}(P_\mu U)(P_\nu U) - U\,(P^2U)^2 - \frac{2}{3}(P^2U)(P_\nu U)^2 - \frac{1}{7}((P_\mu U)G_{\mu\alpha})^2\\
&\qquad + \frac{2}{7}U^2G_{\mu\nu}G_{\nu\alpha}G_{\alpha\mu} + \frac{8}{21}U\,G_{\mu\nu}U\,G_{\nu\alpha}G_{\alpha\mu} - \frac{4}{7}U^2(J_\mu)^2 - \frac{3}{7}(U\,J_\mu)^2\\
&\qquad + \frac{4}{7}U\,(P^2U)(G_{\mu\nu})^2 + \frac{4}{7}(P^2U)U(G_{\mu\nu})^2 - \frac{2}{7}U\,(P_\mu U)J_\nu G_{\mu\nu}\\
&\qquad - \frac{2}{7}(P_\mu U)U\,G_{\mu\nu}J_\nu - \frac{4}{7}U\,(P_\mu U)G_{\mu\nu}J_\nu - \frac{4}{7}(P_\mu U)U\,J_\nu G_{\mu\nu}\\
&\qquad + \frac{4}{21}U\,G_{\mu\nu}(P^2U)G_{\mu\nu} + \frac{11}{21}(P_\alpha U)^2(G_{\mu\nu})^2 - \frac{10}{21}(P_\mu U)J_\nu U\,G_{\mu\nu}\\
&\qquad - \frac{10}{21}(P_\mu U)G_{\mu\nu}U\,J_\nu - \frac{2}{21}(P_\mu U)(P_\nu U)G_{\mu\alpha}G_{\alpha\nu} + \frac{10}{21}(P_\nu U)(P_\mu U)G_{\mu\alpha}G_{\alpha\nu}\\
&\qquad - \frac{1}{7}(G_{\alpha\mu}(P_\mu U))^2 - \frac{1}{42}((P_\alpha U)G_{\mu\nu})^2 - \frac{1}{14}(P_\mu P^2U)^2 - \frac{4}{21}(P^2U)(P_\mu U)J_\mu\\
&\qquad \left. + \frac{4}{21}(P_\mu U)(P^2U)J_\mu + \frac{2}{21}(P_\mu U)(P_\nu U)(P_\mu J_\nu) - \frac{2}{21}(P_\nu U)(P_\mu U)(P_\mu J_\nu)\right]\\
&+ \frac{1}{M^8}\frac{1}{120}\left[U^6 - 3\,U^4(P^2U) - 2\,U^3(P_\nu U)^2 + \frac{12}{7}U^2(P_\mu P_\nu U)(P_\nu P_\mu U)\right.\\
&\qquad + \frac{26}{7}(P_\mu P_\nu U)U\,(P_\mu U)(P_\nu U) + \frac{26}{7}(P_\mu P_\nu U)(P_\mu U)(P_\nu U)U + \frac{9}{7}(P_\mu U)^2(P_\nu U)^2\\
&\qquad + \frac{9}{7}U\,(P_\mu P_\nu U)U\,(P_\nu P_\mu U) + \frac{17}{14}((P_\mu U)(P_\nu U))^2 + \frac{8}{7}U^3G_{\mu\nu}U\,G_{\mu\nu}\\
&\qquad + \frac{5}{7}U^4(G_{\mu\nu})^2 + \frac{18}{7}G_{\mu\nu}(P_\mu U)U^2(P_\nu U) + \frac{9}{14}(U^2G_{\mu\nu})^2\\
&\qquad + \frac{18}{7}G_{\mu\nu}U\,(P_\mu U)(P_\nu U)U + \frac{18}{7}(P_\mu P_\nu U)(P_\mu U)U\,(P_\nu U)\\
&\qquad + \left(\frac{8}{7}G_{\mu\nu}U\,(P_\mu U)U\,(P_\nu U) + \frac{26}{7}G_{\mu\nu}(P_\mu U)U\,(P_\nu U)U\right)\\
&\qquad \left. + \left(\frac{24}{7}G_{\mu\nu}(P_\mu U)(P_\nu U)U^2 - \frac{2}{7}G_{\mu\nu}U^2(P_\mu U)(P_\nu U)\right)\right]\\
&+ \frac{1}{M^{10}}\frac{1}{210}\left[-U^7 - 5\,U^4(P_\nu U)^2 - 8\,U^3(P_\mu U)U(P_\mu U) - \frac{9}{2}(U^2(P_\mu U))^2\right]\\
&\left. + \frac{1}{M^{12}}\frac{1}{336}\left[U^8\right]\right\}.
\end{aligned}
$$

Your favourite BSM Lagrangian

↓

Find: $U = \dfrac{\delta^2 \mathscr{L}^\Phi}{\delta \Phi^2}$

↓

Use this formula to get Green's basis like structures

↓

Remove redundancies to recast them into Warsaw-like structures

$$\boxed{U_{ij} = \frac{\delta^2 \mathcal{L}_{UV}}{\delta\Phi_i \delta\Phi_j}} \quad \boxed{G_{\mu\nu} = [P_\mu, P_\nu],}$$

$$\boxed{J_\mu = P_\nu G_{\nu\mu} = [P_\nu, [P_\nu, P_\mu]].}$$

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# One loop effective action up to dimension eight

* To integrate out heavy fermions instead of scalars, one has to bosonize the Dirac operator to recast the one effective action in terms of a second order elliptic operator:

$$
\begin{aligned}
S_{\text{eff}}^{(1)}\big|_{\text{fermion}} &= -i\,\ln\mathrm{Det}[\not{P} - M_f] = -\frac{i}{2}\big\{\ln\mathrm{Det}[\not{P} - M_f] + \ln\mathrm{Det}[-\not{P} - M_f]\big\} \\
&= -\frac{i}{2}\big\{\ln\mathrm{Det}\big[-\not{P}^2 + M_f^2\big] - \mathcal{A}\big\},
\end{aligned}
$$

where $\mathscr{A}$ is defined as:

$$
\mathcal{A} = \ln\mathrm{Det}\big[-\not{P}^2 + M_f^2\big] - \ln\mathrm{Det}[\not{P} - M_f] - \ln\mathrm{Det}[-\not{P} - M_f]
$$

( **Cognola et al., hep-th/9910038**) shows that this function for massive operator in even dimension can also be expressed in terms of heat-kernel coefficients

$$
\mathcal{A} = 2\sum_{j=1}^{d/2} \frac{(-1)^j\, M_f^{2j}\, Q_j}{j!}\,[b_{d/2-j}], \qquad \text{with} \qquad Q_j = \sum_{l=1}^{j} \frac{1}{2l-1}.
$$

* This shows that the effective action for integrating out heavy scalars can be easily modified to get the fermionic one loop effective action by substituting $U_f = \dfrac{i}{4}[\gamma^\mu, \gamma^\nu]\, G_{\mu\nu}$.

Chakrabortty et al., 2308.03849

# Deriving one loop counter terms

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- At one loop, the divergent contribution comes from:

$$\mathcal{L}^{(k)}_{div} = \frac{c_s}{(4\pi)^{2-\epsilon/2}} M^{d-2k} \frac{(-1)^k}{k!} \frac{(\epsilon/2-3+k)!}{(\epsilon/2-1)!} \underbrace{\Gamma[\epsilon/2]}_{2/\epsilon - \gamma_E + \mathcal{O}(\epsilon)} \mathrm{tr}[b_k],$$

$$\mathcal{L}_{(1)} = \alpha c_s M^{-\epsilon} \left( \Gamma[\epsilon/2-2] M^4 \tilde{b}_0 - \Gamma[\epsilon/2-1] M^2 \tilde{b}_1 + \frac{1}{2}\Gamma[\epsilon/2]\tilde{b}_2 \right),$$

$$\alpha = \frac{1}{16\pi^2}$$

- Let's consider a real SQFT described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\phi D^2 \phi + \frac{1}{2} M^2 \phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{c_6}{6!}\phi^6 + \frac{c_8}{8!}\phi^8.$$

$$\boxed{U = \frac{\delta^2 \mathscr{L}}{\delta \phi^2}}$$

The one loop divergent contribution is:

$$\begin{aligned}
\mathcal{L}_{(1)} &= \frac{\alpha}{2}\left( \Gamma[\epsilon/2-2] M^4 - \Gamma[\epsilon/2-1] M^2 U + \frac{1}{2}\Gamma[\epsilon/2]\, U^2 \right), \\
&= \frac{\alpha}{2}\Bigg( \Gamma[\epsilon/2-2] M^4 - \Gamma[\epsilon/2-1] M^2 \left[ \frac{\lambda}{2}\phi^2 + \frac{c_6}{4!}\phi^4 + \frac{c_8}{6!}\phi^6 \right] \\
&\quad + \frac{1}{2}\Gamma[\epsilon/2] \left[ \frac{\lambda^2}{4}\phi^4 + \frac{\lambda c_6}{4!}\phi^6 + \left( \frac{c_6^2}{576} + \frac{\lambda c_8}{6!} \right)\phi^8 \right] \Bigg).
\end{aligned}$$

UB, Joydeep Chakrabortty, Kaanapuli Ramkumar; 2404.02734

# Computing loops with heat kernel

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- Let's start with a different form for the Green's function:

$$G(x,y) = \sum_{n=0}^{\infty} g_n(x,y)\tilde{b}_n(x,y)\,,$$

With

$$g_n(x,y) = \int_0^\infty dt\, \frac{1}{(4\pi t)^{d/2}} e^{\frac{z^2}{4t}} e^{-M^2 t} \frac{(-t)^n}{n!}$$
$$= \frac{(-1)^n 2^{\frac{d}{2}-n}}{(4\pi)^{\frac{d}{2}} n!} \left(\frac{M}{z}\right)^{\frac{d}{2}-n-1} \mathcal{K}_{\frac{d}{2}-n-1}(Mz),$$

$$z_\mu = x_\mu - y_\mu$$

- The general structure of the loop integral is given by:

$$\mathcal{L}_{(L-loop)} = S_f \int d^d x_1 \ldots \int d^d x_n\, V_{(m_1)}(x_1)\ldots V_{(m_n)}(x_n) G(x_1,x_1)^{p_1} \ldots G(x_n,x_n)^{p_n} G(x_m,x_n)^{q_{mn}}.$$

- Two different kinds of singularities can appear from Green's function: at coincidence limit $z \to 0$ and at non-coincidence limit $z \neq 0$

- In both cases, in $d = 4 - \epsilon$, the first three terms in the series contribute to divergence:

$$G(x,y) = g_0(x,y)\,\tilde{b}_0 + g_1(x,y)\,\tilde{b}_1 + g_2(x,y)\,\tilde{b}_2 + \alpha\mathcal{F}(x,y)\,,$$

$$\alpha = \frac{1}{16\pi^2}$$

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# Divergences at two loop

* At the coincidence limit the singularities can be extracted from:

$$\alpha = \frac{1}{16\pi^2}$$

$$g_0(x,y) = \alpha\,\pi^{2-d/2}\Big[2^{4-d}M^{d-2}\Gamma[1-d/2] - 2^{2-d}z^2M^d\Gamma[-d/2] + \frac{1}{8}M^4z^{6-d}\Gamma[d/2-3] - M^2z^{4-d}\Gamma[d/2-2] + 4z^{6-d}\Gamma[d/2-1]\Big] + \mathcal{O}(z^4)\,,$$

$$g_1(x,y) = \alpha\,\pi^{2-d/2}\Big[2^{2-d}z^2M^{d-2}\Gamma[1-d/2] - 2^{4-d}M^{d-4}\Gamma[2-d/2] + \frac{1}{4}M^2z^{6-d}\Gamma[d/2-3] - z^{4-d}\Gamma[d/2-2]\Big] + \mathcal{O}(z^4)\,,$$

$$g_2(x,y) = \alpha\,\pi^{2-d/2}\Big[-2^{1-d}z^2M^{d-4}\Gamma[2-d/2] + 2^{3-d}M^{d-6}\Gamma[3-d/2] + \frac{1}{8}z^{6-d}\Gamma[d/2-3]\Big] + \mathcal{O}(z^4)\,.$$

* In non-coincidence limit different combination of component Green's functions can be divergent

For example,

$$g_0(x,y)^2 = \alpha\frac{2^\epsilon\pi^{\epsilon/2}\Gamma\left[1-\frac{\epsilon}{2}\right]^2\Gamma\left[\frac{\epsilon}{2}\right]}{\Gamma[2-\epsilon]}\delta(z),$$

$$g_0(x,y)^2g_1(x,y) = -\alpha^2(4\pi)^\epsilon\Gamma\left[1-\frac{\epsilon}{2}\right]^2\left(\frac{\Gamma\left[-\frac{\epsilon}{2}\right]\Gamma[\epsilon]}{\Gamma\left[2-\frac{3\epsilon}{2}\right]} + M^{-\epsilon}\frac{\Gamma\left[\frac{\epsilon}{2}\right]^2}{\Gamma[2-\epsilon]}\right)\delta(z),$$

$$g_0(x,y)^3 = \alpha^2(4\pi)^\epsilon\Gamma\left[1-\frac{\epsilon}{2}\right]^2\left(\frac{\Gamma\left[1-\frac{\epsilon}{2}\right]\Gamma[\epsilon-1]}{\Gamma\left[3-\frac{3\epsilon}{2}\right]}D^2 - 3M^2\frac{\Gamma\left[-\frac{\epsilon}{2}\right]\Gamma[\epsilon]}{\Gamma\left[2-\frac{3\epsilon}{2}\right]} + 3M^{2-\epsilon}\frac{\Gamma\left[\frac{\epsilon}{2}\right]\Gamma\left[\frac{\epsilon}{2}-1\right]}{\Gamma[2-\epsilon]}\right)\delta(z),$$

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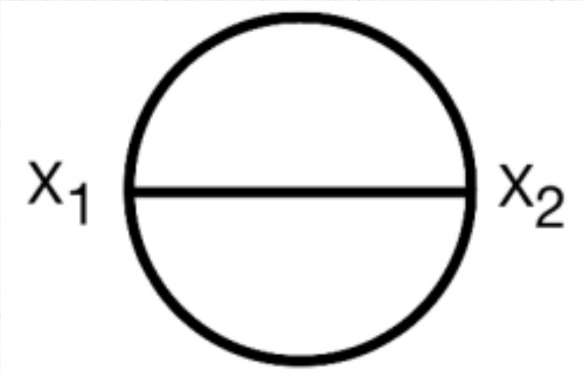
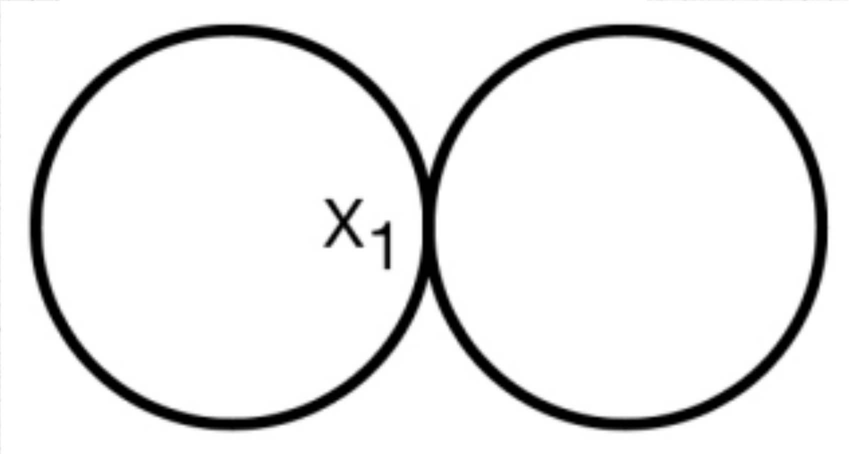
# Divergences at two loop

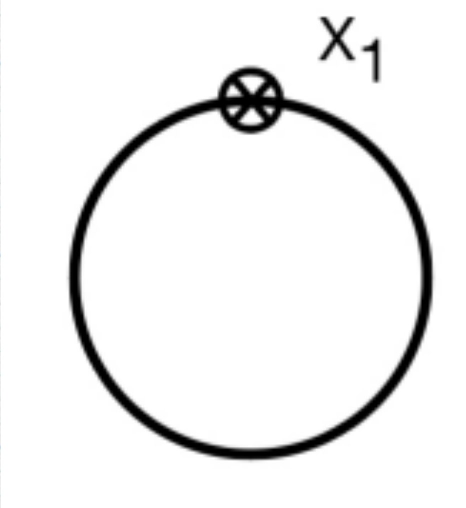
- There are three diagrams that one needs to compute at two loop:




$$\mathcal{L}^{a}_{(2)} \subset -\frac{1}{12}\int d^d x_1\, d^d x_2\, V_{(3)}(x_1)\, G(x_1,x_2)^3\, V_{(3)}(x_2)$$




$$\mathcal{L}^{b}_{(2)} \subset \frac{1}{8}\int d^d x_1\, V_{(4)}(x_1)\, G(x_1,x_1)^2$$




$$\mathcal{L}^{ct}_{(2)} \subset \frac{1}{2}\int d^d x_1\, V^{(ct-1)}_{(2)}(x_1)\, G(x_1,x_1),$$

$$\phi^2\Big|^{div} = \alpha^2\left[\frac{M^2\lambda^2}{4\epsilon} - \frac{1}{\epsilon^2}\left(M^2\lambda^2 + \frac{M^4 c_6}{4}\right)\right]\phi^2,$$

$$\phi^4\Big|^{div} = \alpha^2\left[\frac{1}{\epsilon}\left(\frac{5c_6\lambda M^2}{72} + \frac{\lambda^3}{8}\right) - \frac{1}{\epsilon^2}\left(\frac{c_8 M^4}{48} + \frac{11c_6\lambda M^2}{24} + \frac{3\lambda^3}{8}\right)\right]\phi^4,$$

$$\phi^6\Big|^{div} = \alpha^2\left[\frac{1}{\epsilon}\left(\frac{43c_6\lambda^2}{864} - \frac{c_8\lambda^3 M^2}{1440} + \frac{c_8\lambda M^2}{240} + \frac{c_6^2 M^2}{288}\right) - \frac{1}{\epsilon^2}\left(\frac{3c_6\lambda^2}{16} + \frac{11c_8\lambda M^2}{360} + \frac{5c_6^2 M^2}{144}\right)\right]\phi^6,$$

$$\phi^8\Big|^{div} = \alpha^2\left[\frac{1}{\epsilon}\left(\frac{c_8\lambda^2}{432} + \frac{c_6^2\lambda}{160}\right) - \frac{1}{\epsilon^2}\left(\frac{31c_8\lambda^2}{2880} + \frac{29c_6^2\lambda}{1152}\right)\right]\phi^8,$$

# Summary

* Heat kernel provides a way to compute loops in a gauge covariant way.
* It can be used to get a compact and universal form for the one loop effective action.
* The fermionic one loop effective action can be derived from the same formula by modifying the interaction terms
* The single poles can be extracted from the one loop effective action from the poles of the gamma function at $k = 0,1,2$.
* The poles at the two loop can rise from both coincidence limit and non-coincidence limit of the Green's function. There is no need perform any kind of spatial or momentum integral here.

**Thanks!**