Simplifying complicated tensor expressions arising in the study of EFTs

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Simple EFT expressions

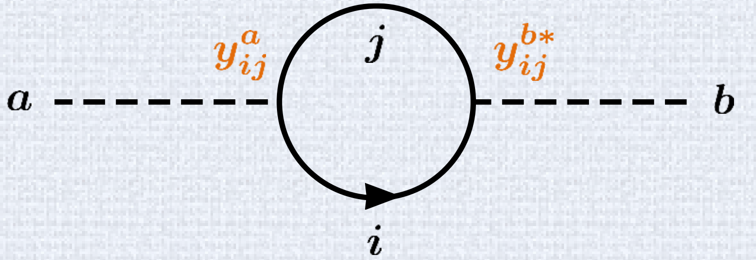
(Maybe a more fitting cover slide)

Operators, Wilson coefficients and fields as tensors

$$\mathscr{L} = \omega_{ij\cdots}^{(1)} \mathcal{O}_{ijk\cdots}^{(1)} + \omega_{ij\cdots}^{(2)} \mathcal{O}_{ijk\cdots}^{(2)} + \omega_{ij\cdots}^{(3)} \mathcal{O}_{ijk\cdots}^{(3)} + \cdots$$

Operators and Wilson coefficients (WC) are often tensors in flavor space

Amplitudes in perturbation theory are polynomials in these WCs



$$\propto y_{ij}^{a} y_{ij}^{b*} = \mathrm{Tr}\left[y^{a}\left(y^{b}\right)^{\dagger}\right]$$

The operators themselves are polynomials in the fields, which in turn are tensors with several indices: gauge, Dirac, flavor

Therefore, in computations with functional methods the fields are also extra tensors which appear in the expressions

Wilson coefficients have symmetries

... and they become more complicated for high-dim operators

None of the Standard Model couplings have symmetries in flavor space

The WC $\omega_{ij} = \omega_{ji}$ of the Weinberg operator $L_i L_j H H$ is the one with a symmetry

Nonetheless, this is still a 'simple' one.

The symmetries become more and more complicated as the number of repeated (flavored) fields increases

E.g.

$$\mathcal{O}_{ijkl} = \epsilon_{\alpha\beta\gamma}\epsilon_{nm}\epsilon_{pq}\left(Q^T_{i,\alpha n} C Q_{j,\beta p}\right)\left(Q^T_{k,\gamma q} C L_{l,m}\right)$$

$$\mathcal{O}_{ijkl} + \mathcal{O}_{jikl} - \mathcal{O}_{kijl} - \mathcal{O}_{kjil} = 0$$

the operator

$$\omega_{ijkl} + \omega_{jikl} - \omega_{jkil} - \omega_{kjil} = 0$$

the coupling

This is one possible way to describe the symmetry of this interaction. Not unique!

What is going one?

Symmetry of group contractions (Lorentz, gauge)

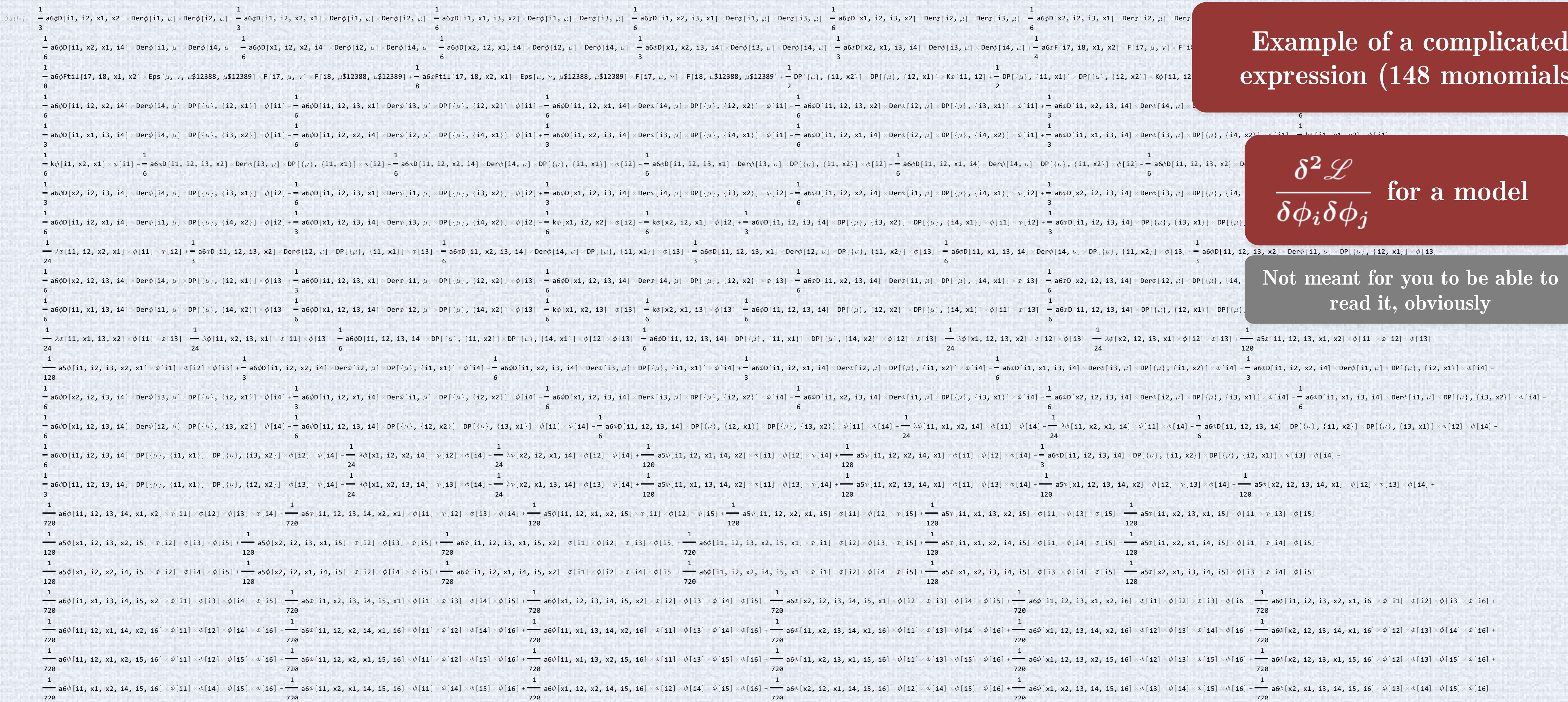
propagates to

Flavor symmetry of operator

propagates to

Flavor symmetry of Wilson coefficient

Example of a complicated expressions



Example of a complicated expression (148 monomials)

$$\frac{\delta^2 \mathscr{L}}{\delta\phi_i \delta\phi_j}$$ **for a model**

Not meant for you to be able to read it, obviously

Example of a complicated expressions

$$-4\, a6\phi D[i1, x1, i2, x2] \times Der\phi[i1, \mu] \times Der\phi[i2, \mu] + \frac{1}{2}\, a6\phi F[i7, i8, x1, x2] \times F[i7, \mu, \nu] \times F[i8, \mu, \nu] +$$
$$\frac{1}{4}\, a6\phi Ftil[i7, i8, x1, x2] \times Eps[\mu, \nu, \mu\$12388, \mu\$12389] \times F[i7, \mu, \nu] \times F[i8, \mu\$12388, \mu\$12389] +$$
$$DP[\{\mu\}, \{i1, x1\}] \times DP[\{\mu\}, \{i2, x2\}] \times K\phi[i1, i2] - m\phi 2[x1, x2] - k\phi[i1, x1, x2] \times \phi[i1] - \frac{1}{2}\, \lambda\phi[i1, i2, x1, x2] \times \phi[i1] \times \phi[i2] -$$
$$4\, a6\phi D[i1, i3, i2, x2] \times Der\phi[i1, \mu] \times DP[\{\mu\}, \{i2, x1\}] \times \phi[i3] - 4\, a6\phi D[i1, x2, i2, i3] \times Der\phi[i1, \mu] \times DP[\{\mu\}, \{i2, x1\}] \times \phi[i3] -$$
$$4\, a6\phi D[i1, i3, i2, x1] \times Der\phi[i1, \mu] \times DP[\{\mu\}, \{i2, x2\}] \times \phi[i3] - 4\, a6\phi D[i1, x1, i2, i3] \times Der\phi[i1, \mu] \times DP[\{\mu\}, \{i2, x2\}] \times \phi[i3] +$$
$$\frac{1}{6}\, a5\phi[i1, i2, i3, x1, x2] \times \phi[i1] \times \phi[i2] \times \phi[i3] - 4\, a6\phi D[i1, i3, i2, i4] \times DP[\{\mu\}, \{i1, x1\}] \times DP[\{\mu\}, \{i2, x2\}] \times \phi[i3] \times \phi[i4] +$$
$$\frac{1}{24}\, a6\phi[i1, i2, i3, i4, x1, x2] \times \phi[i1] \times \phi[i2] \times \phi[i3] \times \phi[i4]$$

Simplifies to a more manageable 14 terms

Simplifying complicated tensors polynomials

The problem of simplifying complicated tensor expressions is two-fold:

Dummy indices

T[i,a]U[a,j]

is not recognized as the same as

T[i,b]U[b,j]

Using the tensor symmetries

$$\alpha T_{ia}U_{aj} + \beta T_{ib}U_{jb}$$

only simplifies to

$$(\alpha - \beta)\, T_{ia}U_{aj}$$

if, for example, U is antisymmetric

I will introduce a recent code, **SimTeEx**, to put tensor expressions in canonical form, for arbitrarily complicated symmetries
(It has other functions to analyze tensor symmetries)

Sim Te Ex
Simplify Tensor Expressions

RF 2412.14390

Origin of the code: Study of a general EFT

Decouple the task of calculating amplitudes (RGEs, evanescent shifts, matching, regularization schemes issues, ...) from the details of a model, by studying a general EFT

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Idea: remove the gauge structure

The problem of flavor is more acute if there are few distinctions (other than flavor) among the fields

Best model to study flavor (most stringent test): A model with no gauge symmetry

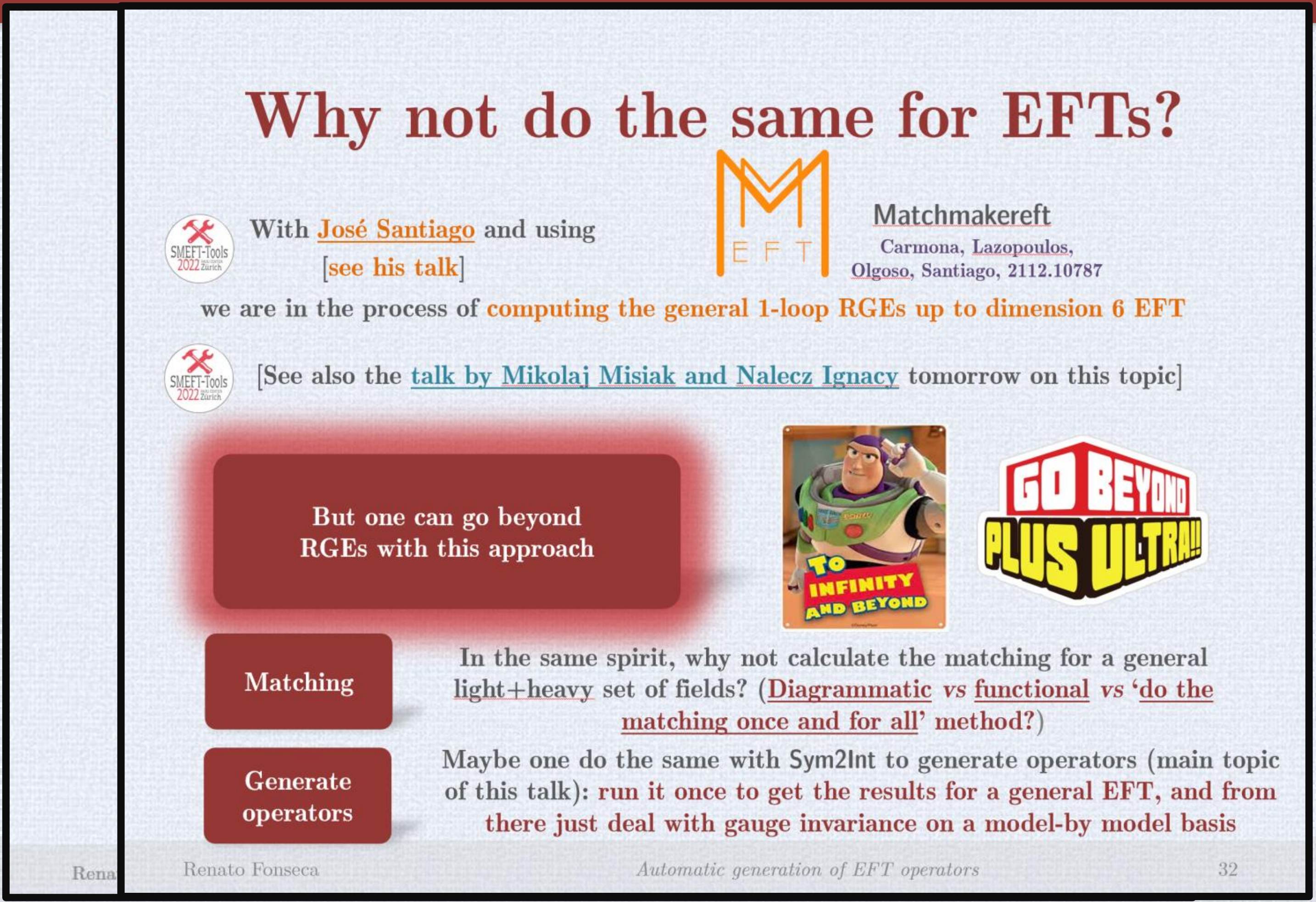
Model contains arbitrary number of copies/flavors of a left-handed Weyl spinors, real scalars, $F_{\mu\nu}$'s

This also describes the most general EFT one can have

SMEFT and other EFTs can be obtained from it by imposing gauge invariant on the various Wilson coefficients

Origin of the code: Study of a general EFT

Decouple the task of calculating amplitudes (RGEs, evanescent shifts, matching, regularization schemes issues, …) from the details of a model, by studying a general EFT



What is this general RGE idea?

Collect all scalars, fermions and vector fields into 3 multiplets. Be agnostic about the gauge group and how these fields transform under it

For a renormalizable model, this was done long ago together the the computation of its 2-loop RGEs

Jack, Osborn (1982,1983,1985)
Machacek, Vaughn (1983,1984,1985)
Luo, Wang, Xiao, hep-ph/0211440 (2003)

Martin, Vaughn, hep-ph/9311340 (1994)
Yamada, hep-ph/9401241 (1994)
(SUSY)

$$\mathscr{L}_{d\leq 4} = -\frac{1}{4}F^A_{\mu\nu}F^{B\,\mu\nu} + \frac{1}{2}D_\mu\phi_a D^\mu\phi_b + \bar{\psi}_i \mathrm{i}\not{D}\psi_j - \frac{1}{2}\left[(m_f)_{ij}\psi_i^T C\psi_j + \text{h.c.}\right]$$
$$-\frac{1}{2}(m_\phi^2)_{ab}\phi_a\phi_b - \frac{1}{2}\left[Y_{ija}\psi_i^T C\psi_j\phi_a + \text{h.c.}\right] - \frac{\kappa_{abc}}{3!}\phi_a\phi_b\phi_c - \frac{\lambda_{abcd}}{4!}\phi_a\phi_b\phi_c\phi_d$$

Coefficient × Operator

$$D_\mu\psi_i = \partial_\mu\psi_i - \mathrm{i}g t^A_{ij}V^A_\mu\psi_j$$
$$D_\mu\phi_a = \partial_\mu\phi_a - \mathrm{i}g\theta^A_{ab}V^A_\mu\phi_b$$

t^A and θ^A are Hermitian matrices (θ^A are also anti-symmetric)

In a particular model one must specify the shape of generic tensor coefficients shown here

The RGEs were given for these tensors

In practice, this usually involves simply enforcing gauge invariance on these tensor coefficients

E.g.: in SM one has 45 Weyl fermions and 4 real scalars: the t^A are 45-dim; the θ^A are 4-dim. The Yukawa couplings are given by the most general Y tensor obeying

$$t^A_{ii'}Y_{i'ja} + t^A_{jj'}Y_{i'ja} + \theta^A_{aa'}Y_{ija'} = 0$$

In the SM, Y has 27 complex degrees of freedom

Dimension 5 Green basis

$$\mathscr{L}_5^{\text{phys}} = \left[\frac{1}{2}\left(a_{\psi F}^{(5)}\right)_{Aij}\psi_i^T C\sigma^{\mu\nu}\psi_j F_{\mu\nu}^A + \frac{1}{4}\left(a_{\psi\phi^2}^{(5)}\right)_{ijab}\psi_i^T C\psi_j\phi_a\phi_b + \text{h.c.}\right]$$
$$+\frac{1}{2}\left(a_{\phi F}^{(5)}\right)_{ABa}F^{A\,\mu\nu}F_{\mu\nu}^B\phi_a + \frac{1}{2}\left(a_{\phi\tilde{F}}^{(5)}\right)_{ABa}F^{A\,\mu\nu}\tilde{F}_{\mu\nu}^B\phi_a + \frac{1}{5!}\left(a_{\phi}^{(5)}\right)_{abcde}\phi_a\phi_b\phi_c\phi_d\phi_e$$
$$\mathscr{L}_5^{\text{red}} = \frac{1}{2}\left(r_{\phi\Box}^{(5)}\right)_{abc}(D_\mu D^\mu\phi_a)\phi_b\phi_c + \left[\frac{1}{2}\left(r_{\psi}^{(5)}\right)_{ij}(D_\mu\psi_i)^T C D^\mu\psi_j + \left(r_{\psi\phi}^{(5)}\right)_{ija}\bar{\psi}_i \mathrm{i}\not{D}\psi_j\phi_a + \text{h.c.}\right]$$

The Wilson coefficients have important symmetries (in some cases non-trivial)

$$\left(a_{\psi F}^{(5)}\right)_{ij} = -\left(a_{\psi F}^{(5)}\right)_{ji} \qquad \left(a_{\psi\phi^2}^{(5)}\right)_{ijab} = \left(a_{\psi\phi^2}^{(5)}\right)_{jiab} = \left(a_{\psi\phi^2}^{(5)}\right)_{ijba}$$
$$\left(a_{\phi F}^{(5)}\right)_{ABa} = \left(a_{\phi F}^{(5)}\right)_{BAa} \qquad \left(a_{\phi\tilde{F}}^{(5)}\right)_{ABa} = \left(a_{\phi\tilde{F}}^{(5)}\right)_{BAa} \qquad \left(a_{\phi}^{(5)}\right)_{abcde} = \text{fully symmetric}$$
$$\left(r_{\psi}^{(5)}\right)_{ij} = \left(r_{\psi}^{(5)}\right)_{ji} \qquad \left(r_{\phi\Box}^{(5)}\right)_{abc} = \left(r_{\phi\Box}^{(5)}\right)_{acb}$$

Integration-by-parts (IBPs) equations of motion (EOM) redundancies may affect only parts of these tensors (e.g. they can remove the symmetric part of some WC and leave untouched the anti-symmetric)

Dimension 6 Green basis

$$\mathscr{L}_6^{\text{phys}} = \frac{1}{3!}\left(a_{3F}^{(6)}\right)_{ABC}(F^A)_\mu^{\ \nu}(F^B)_\nu^{\ \rho}(F^C)_\rho^{\ \mu} + \frac{1}{3!}\left(a_{3\widetilde{F}}^{(6)}\right)_{ABC}(F^A)_\mu^{\ \nu}(F^B)_\nu^{\ \rho}(\widetilde{F}^C)_\rho^{\ \mu}$$
$$+\frac{1}{4}\left(a_{\phi F}^{(6)}\right)_{ABab}F^A_{\mu\nu}F^{B\,\mu\nu}\phi_a\phi_b + \frac{1}{4}\left(a_{\phi\widetilde{F}}^{(6)}\right)_{ABab}F^A_{\mu\nu}\widetilde{F}^{B\,\mu\nu}\phi_a\phi_b$$
$$+\frac{1}{4}\left(a_{\phi D}^{(6)}\right)_{abcd}(D_\mu\phi_a)(D^\mu\phi_b)\phi_c\phi_d + \frac{1}{6!}\left(a_{\phi}^{(6)}\right)_{abcdef}\phi_a\phi_b\phi_c\phi_d\phi_e\phi_f$$
$$+\frac{1}{2}\left(a_{\phi\psi}^{(6)}\right)_{ijab}\bar{\psi}_i\gamma^\mu\psi_j[\phi_a D_\mu\phi_b - \phi_b D_\mu\phi_a] + \frac{1}{4}\left(a_{\bar{\psi}\psi}^{(6)}\right)_{ijkl}(\bar{\psi}_i\gamma^\mu\psi_j)(\bar{\psi}_k\gamma_\mu\psi_l)$$
$$+\left[\frac{1}{2}\left(a_{\psi F}^{(6)}\right)_{Aija}F^A_{\mu\nu}\psi_i^T C\sigma^{\mu\nu}\psi_j\phi_a + \frac{1}{2!3!}\left(a_{\psi\phi}^{(6)}\right)_{ijabc}\psi_i^T C\psi_j\phi_a\phi_b\phi_c\right.$$
$$\left.+\frac{1}{4!}\left(a_{\psi\psi}^{(6)}\right)_{ijkl}(\psi_i^T C\psi_j)(\psi_k^T C\psi_l) + \text{h.c.}\right]$$

$$\mathscr{L}_6^{\text{red}} = \frac{1}{2!}\left(r_{2F}^{(6)}\right)_{AB}(D_\mu F^{A\,\mu\nu})(D^\rho F^B_{\rho\nu}) + \frac{1}{2!}\left(r_{FD\phi}^{(6)}\right)_{Aab}(D_\nu F^{A,\mu\nu})\left[(D_\mu\phi_a)\,\phi_b - (a\leftrightarrow b)\right]$$
$$+\frac{1}{2!}\left(r_{D\phi}^{(6)}\right)_{ab}(D_\mu D^\mu\phi_a)\,(D_\nu D^\nu\phi_b) + \frac{1}{3!}\left(r_{\phi D}^{(6)}\right)_{abcd}(D_\mu D^\mu\phi_a)\,\phi_b\phi_c\phi_d$$
$$+ \ldots$$

These tensors also have flavor symmetries

Note: not exactly the basis we used in the end

Complicated symmetries

Here are the ones of dimension 6 operators:

$$\left(a^{(6)}_{3F}\right)_{ABC} = \text{fully anti-symmetric}, \in \mathbb{R}$$

$$\left(a^{(6)}_{3\tilde{F}}\right)_{ABC} = \text{fully anti-symmetric}, \in \mathbb{R}$$

$$\left(a^{(6)}_{\phi\psi}\right)_{ijab} = -\left(a^{(6)}_{\phi\psi}\right)_{ijba} = \left[\left(a^{(6)}_{\phi\psi}\right)_{jiab}\right]^*$$

$$\left(a^{(6)}_{\bar{\psi}\psi}\right)_{ijkl} = \left(a^{(6)}_{\bar{\psi}\psi}\right)_{kjil} = \left(a^{(6)}_{\bar{\psi}\psi}\right)_{ilkj} = \left[\left(a^{(6)}_{\bar{\psi}\psi}\right)_{jilk}\right]^*$$

$$\left(a^{(6)}_{\phi D}\right)_{abcd} = \left(a^{(6)}_{\phi D}\right)_{bacd} = \left(a^{(6)}_{\phi D}\right)_{abdc} = \left(a^{(6)}_{\phi D}\right)_{cdab} \text{ and}$$
$$\left(a^{(6)}_{\phi D}\right)_{abcd} + \left(a^{(6)}_{\phi D}\right)_{adbc} + \left(a^{(6)}_{\phi D}\right)_{acdb} = 0, \left(a^{(6)}_{\phi D}\right)_{abcd} \in \mathbb{R}$$

$$\left(a^{(6)}_{\phi F}\right)_{ABab} = \left(a^{(6)}_{\phi F}\right)_{BAab} = \left(a^{(6)}_{\phi F}\right)_{ABba} \in \mathbb{R}$$

$$\left(a^{(6)}_{\phi \tilde{F}}\right)_{ABab} = \left(a^{(6)}_{\phi \tilde{F}}\right)_{BAab} = \left(a^{(6)}_{\phi \tilde{F}}\right)_{ABba} \in \mathbb{R}$$

$$\left(a^{(6)}_{\phi}\right)_{abcdef} = \text{fully symmetric} \in \mathbb{R}$$

$$\left(a^{(6)}_{\psi\psi}\right)_{ijkl} = \left(a^{(6)}_{\psi\psi}\right)_{jikl} = \left(a^{(6)}_{\psi\psi}\right)_{ijlk} = \left(a^{(6)}_{\psi\psi}\right)_{klij} \text{ and}$$
$$\left(a^{(6)}_{\psi\psi}\right)_{ijkl} + \left(a^{(6)}_{\psi\psi}\right)_{iljk} + \left(a^{(6)}_{\psi\psi}\right)_{iklj} = 0$$

$$\left(a^{(6)}_{\psi F}\right)_{Aija} = -\left(a^{(6)}_{\psi F}\right)_{Ajia}$$

$$\left(a^{(6)}_{\psi\phi}\right)_{ijabc} = \text{fully symmetric in } (i,j) \text{ and also } (a,b,c)$$

Some of them are quite complicated

⊞ symmetry

$$\left(r^{(6)}_{\psi D}\right)_{ij} = \left(r^{(6)}_{\psi D}\right)_{ji}$$

$$\left(r^{(6)}_{D\phi}\right)_{ab} = \left(r^{(6)}_{D\phi}\right)_{ba} \in \mathbb{R}$$

$$\left(r^{(6)}_{2F}\right)_{AB} = \left(r^{(6)}_{2F}\right)_{BA} \in \mathbb{R}$$

$$\left(r^{(6)}_{DF\psi}\right)_{Aij} = \left[\left(r^{(6)}_{DF\psi}\right)_{Aji}\right]^*$$

$$\left(r^{(6)}_{F\psi}\right)_{Aij} = \left[\left(r^{(6)}_{F\psi}\right)_{Aji}\right]^*$$

$$\left(r^{(6)}_{\tilde{F}\psi}\right)_{Aij} = \left[\left(r^{(6)}_{\tilde{F}\psi}\right)_{Aji}\right]^*$$

$$\left(r^{(6)}_{FD\phi}\right)_{Aab} = -\left(r^{(6)}_{FD\phi}\right)_{Aba} \in \mathbb{R}$$

$$\left(r^{(6)}_{\phi\psi x}\right)_{ijab} = \left(r^{(6)}_{\phi\psi x}\right)_{ijba} = \left[\left(r^{(6)}_{\phi\psi x}\right)_{jiab}\right]^* \text{ for } x = 1,2$$

$$\left(r^{(6)}_{\phi D}\right)_{abcd} = \text{fully symmetric in } (b,c,d) \in \mathbb{R}$$

$$\left(r^{(6)}_{\psi\phi D1}\right)_{ija} = \left(r^{(6)}_{\psi\phi D1}\right)_{jia}$$

$$\left(r^{(6)}_{\psi\phi D2}\right)_{ija} = \left(r^{(6)}_{\psi\phi D2}\right)_{jia}$$

$$\left(r^{(6)}_{\psi\phi D3}\right)_{ija} = \text{no restrictions}$$

What do these tensors look like (in the SM)

fields

$$\psi = (u^c[\mathrm{R}], u^c[\mathrm{G}], u^c[\mathrm{B}], d^c[\mathrm{R}], d^c[\mathrm{G}], d^c[\mathrm{B}], Q[\mathrm{R},1], Q[\mathrm{R},2], Q[\mathrm{G},1], Q[\mathrm{G},2], Q[\mathrm{B},1], Q[\mathrm{B},2], e^c, L[1], L[2])^T$$

$$\phi = \left(H_R^+, H_R^0, H_I^+, H_I^0\right)^T \quad F_{\mu\nu} = \left(G_{\mu\nu}^1, G_{\mu\nu}^2, G_{\mu\nu}^3, G_{\mu\nu}^4, G_{\mu\nu}^5, G_{\mu\nu}^6, G_{\mu\nu}^7, G_{\mu\nu}^8, W_{\mu\nu}^1, W_{\mu\nu}^2, W_{\mu\nu}^3, B_{\mu\nu}\right)^T$$

θ^A

scalar representation matrices

SU(3)

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&\frac{i}{2}\\0&0&\frac{i}{2}&0\\0&-\frac{i}{2}&0&0\\-\frac{i}{2}&0&0&0 \end{pmatrix} \begin{pmatrix} 0&-\frac{i}{2}&0&0\\\frac{i}{2}&0&0&0\\0&0&0&-\frac{i}{2}\\0&0&\frac{i}{2}&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac{i}{2}&0\\0&0&0&-\frac{i}{2}\\-\frac{i}{2}&0&0&0\\0&\frac{i}{2}&0&0 \end{pmatrix} \begin{pmatrix} 0&0&\frac{i}{2}&0\\0&0&0&\frac{i}{2}\\-\frac{i}{2}&0&0&0\\0&-\frac{i}{2}&0&0 \end{pmatrix}$$

SU(3) SU(2) U(1)

What do these tensors look like (in the SM)

fields

$$\psi = (u^c[\mathrm{R}], u^c[\mathrm{G}], u^c[\mathrm{B}], d^c[\mathrm{R}], d^c[\mathrm{G}], d^c[\mathrm{B}], Q[\mathrm{R},1], Q[\mathrm{R},2], Q[\mathrm{G},1], Q[\mathrm{G},2], Q[\mathrm{B},1], Q[\mathrm{B},2], e^c, L[1], L[2])^T$$

$$\phi = \left(H_R^+, H_R^0, H_I^+, H_I^0\right)^T \quad F_{\mu\nu} = \left(G^1_{\mu\nu}, G^2_{\mu\nu}, G^3_{\mu\nu}, G^4_{\mu\nu}, G^5_{\mu\nu}, G^6_{\mu\nu}, G^7_{\mu\nu}, G^8_{\mu\nu}, W^1_{\mu\nu}, W^2_{\mu\nu}, W^3_{\mu\nu}, B_{\mu\nu}\right)^T$$

t^A

fermion representation Matrices (1 flavor)

<u>No surprise:</u>

These are **block diagonal matrices** because the fermions are in a reducible representation of the gauge group

$$\begin{pmatrix}
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
, \cdots ,
\begin{pmatrix}
-\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}
\end{pmatrix}$$

A=1 [first SU(3) generator]

A=12 [U(1) generator]

What do these tensors look like (in the SM)

fields

$$\psi = (u^c[\mathrm{R}], u^c[\mathrm{G}], u^c[\mathrm{B}], d^c[\mathrm{R}], d^c[\mathrm{G}], d^c[\mathrm{B}], Q[\mathrm{R},1], Q[\mathrm{R},2], Q[\mathrm{G},1], Q[\mathrm{G},2], Q[\mathrm{B},1], Q[\mathrm{B},2], e^c, L[1], L[2])^T$$

$$\phi = \left(H_R^+, H_R^0, H_I^+, H_I^0\right)^T \quad F_{\mu\nu} = \left(G^1_{\mu\nu}, G^2_{\mu\nu}, G^3_{\mu\nu}, G^4_{\mu\nu}, G^5_{\mu\nu}, G^6_{\mu\nu}, G^7_{\mu\nu}, G^8_{\mu\nu}, W^1_{\mu\nu}, W^2_{\mu\nu}, W^3_{\mu\nu}, B_{\mu\nu}\right)^T$$

Y_{ija}

$$-\frac{1}{2}\left[Y_{ija}\psi_i^T C\psi_j\phi_a + \text{h.c.}\right]$$

Yukawa couplings

Show here is $\frac{1}{\sqrt{2}}Y^*_{ij1}$ i.e. the interactions of H_R^+

u^cQ (rows 1–6, columns 7–12); u^cQ (rows 7–12, columns 1–3); d^cQ (rows 4–6, columns 7–12 and rows 7–12, columns 4–6); e^cL (rows 13–15, columns 13–15)

0	0	0	0	0	0	0	-Yu[f2, f1]	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-Yu[f2, f1]	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-Yu[f2, f1]	0	0	0
0	0	0	0	0	0	Yd[f2, f1]	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	Yd[f2, f1]	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	Yd[f2, f1]	0	0	0	0
0	0	0	Yd[f1, f2]	0	0	0	0	0	0	0	0	0	0	0
-Yu[f1, f2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	Yd[f1, f2]	0	0	0	0	0	0	0	0	0	0
0	-Yu[f1, f2]	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	Yd[f1, f2]	0	0	0	0	0	0	0	0	0
0	0	-Yu[f1, f2]	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	Ye[f2, f1]	0
0	0	0	0	0	0	0	0	0	0	0	0	Ye[f1, f2]	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Flavor is unexpanded (f1,f2 indices); otherwise, Y would be a 45x45x4 tensor

Our work

Write down a basis of operators for a general EFT up to dimension 6 (for now).

Derive the 1-loop (for now) RGEs for the physical Wilson coefficients

With these results, there will be no need to ever do physics again (calculate amplitudes of diagrams) to compute RGEs for a specific EFT

Only some algebra is needed in order to compute the the Wilson coefficient tensors (Y_{ija} , λ_{abcd}, etc)

RGEs of the bosonic operators up to dimension 6

2501.13185 [hep-ph]

Renormalization of general Effective Field Theories: Formalism and renormalization of bosonic operators

Renato M. Fonseca, Pablo Olgoso, José Santiago

We describe the most general local, Lorentz-invariant, effective field theory of scalars, fermions and gauge bosons up to mass dimension 6. We first obtain both a Green and a physical basis for such an effective theory, together with the on-shell reduction of the former to the latter. We then proceed to compute the renormalization group equations for the bosonic operators of this general effective theory at one-loop order.

Luigi Carlo Bresciani will also talk about this topic on Wednesday

The package SimTeEx

Sim Simplify **Te** Tensor **Ex** Expressions

CanonicalForm

Main function of the program: Puts tensor polynomials in a canonical form

No symmetries

```
In[•]:= CanonicalForm[T[i, a] × U[a, j] + T[i, b] × U[b, j]]
Out[•]= 2 T[i, a] × U[a, j]
```

With symmetries

```
In[•]:= CanonicalForm[α T[i, a] × U[a, j] + β T[i, b] × U[j, b], {U[x, y] + U[y, x]}]
Out[•]= (α - β) T[i, a] × U[a, j]
```

$U_{xy} + U_{yx} = 0$

Format for the symmetries is a list of expressions which are =0 (but no need to write the "=0")

Fully general and simple input

1 Arbitrarily complicated symmetries can be fed into this function. I.e. fully general in this aspect.

2 Input needed is as intuitive as it gets

in my opinion

No need to figure out what are the Young projectors

(important aspect; more on this later)

No need to declare tensors

No need to declare indices used

On the generality

Riemann tensor symmetries

$R_{ijkl} = -R_{jikl}$, $R_{ijkl} = -R_{ijlk}$ and $R_{ijkl} + R_{iklj} + R_{iljk} = 0$

(Box symmetry ⊞)

Known quartic relation

$$R_{pqrs}R_{ptru}R_{tvqw}R_{uvsw} - R_{pqrs}R_{pqtu}R_{rvtw}R_{svuw}$$

$$-R_{mnab}R_{npbc}R_{mscd}R_{spda} + \frac{1}{4}R_{mnab}R_{psba}R_{mpcd}R_{nsdc} = 0$$

Peeters 2018

Possible input

```
In[•]:= expressionToSimplify = R[p, q, r, s] × R[p, t, r, u] × R[t, v, q, w] × R[u, v, s, w] -
          R[p, q, r, s] × R[p, q, t, u] × R[r, v, t, w] × R[s, v, u, w] -
          R[m, n, a, b] × R[n, p, b, c] × R[m, s, c, d] × R[s, p, d, a] +
          x  R[m, n, a, b] × R[p, s, b, a] × R[m, p, c, d] × R[n, s, d, c];
        symmetries = {R[f1, f2, f3, f4] + R[f2, f1, f3, f4], R[f1, f2, f3, f4] + R[f1, f2, f4, f3],
          R[f1, f2, f3, f4] + R[f1, f3, f4, f2] + R[f1, f4, f2, f3]};
        CanonicalForm[expressionToSimplify, symmetries]

Out[•]= 1/8 (-2 + 8 x) R[m, n, a, b] × R[m, p, c, d] × R[n, s, d, c] × R[p, s, b, a]
```

Even symmetries which make little sense can be given

```
In[•]:= CanonicalForm[x1 H[i, j] × T[j, k] + x2 H[i, a] × T[k, a], {T[a, b] - 2 T[b, a]}]

Out[•]= 0
```

$T_{ab} - 2T_{ba} = 0$ implies that T is identically 0

The algorithm used is sufficiently robust to deal with even these cases (no special code was needed)

On the simplicity

Other codes don't allow the user to give symmetries directly as equations

$$R_{ijkl} = -R_{jikl}\,, R_{ijkl} = -R_{ijlk} \text{ and } R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

Users must first figure out the associated Young tableaux (non-trivial and in fact not always possible)

If there are equivalent ways of expressing a tensor's symmetry which one is the appropriate one for SimTeEx?

?

It is up to the you, the user! SimTeEx doesn't care. Any equivalent set of equations is the same for the program

!

Same example with the Riemann tensor as in the last slide, but using an equivalent set of symmetry equations

```
symmetries2 = {R[f1, f2, f3, f4] + R[f3, f4, f2, f1],
   R[f1, f2, f3, f4] + R[f1, f3, f4, f2] + R[f1, f4, f2, f3]};
CanonicalForm[expressionToSimplify, symmetries2]
```

Out[ ]= $\frac{1}{8}(-2 + 8x)$ R[m, n, a, b] × R[m, p, c, d] × R[n, s, d, c] × R[p, s, b, a]

Anti-commuting tensors

Program also works for 'Grassmann tensors'

But why?

Just think for example of fermions, which can have many indices

$\psi_{ic\alpha f}$

SM(EFT) as an example:
i=Dirac/Weyl; c=color; α =isospin; f=flavor

Very simple example

Majorana mass for gauge singlets is symmetric

$m_{ij}\psi_i^T C\psi_j$

Necessary to declare the list of 'fermionic tensors'

Necessary to use non-commutative multiplication

```
In[ ]:= $CanonicalFormFermions = {ψ};

expression1 = ϵ[α, β] × ψ[α, i] ** ψ[β, j] × m[i, j];
expression2 = ϵ[α, β] × ψ[α, i] ** ψ[β, j] × m[j, i];
CanonicalForm[expression1 - expression2, {ϵ[f1, f2] + ϵ[f2, f1]}]

Out[ ]= 0
```

Lies

Simplify expression

Subjective; could involve for example minimizing the number of terms

$\neq$

Canonical Form

Should involve a function F which for two equivalent expressions expr_1 and expr_2

$$F(\text{expr}_1) = F(\text{expr}_2)$$

Second comment: the function I presented you does not have this property!

```
In[•]:= CanonicalForm[T[i] × T[i]]
        CanonicalForm[T[j] × T[j]]
        % === %%
Out[•]= T[i]^2
Out[•]= T[j]^2
Out[•]= False
```

For example, since it preserves the dummy index labels given by the user ...

(This is not the only reason)

However, it does obey the property

$$F(\text{expr}_1 - \text{expr}_2) = 0$$

A function with this property is called in the literature a normal form

Canonical Form $\neq$ **Normal Form**

Reason for this: convenience of the user

This behavior of the function CanonicalForm in SimTeEx was deliberate.

1 In order not to introduce new symbols, it reuses dummy index labels given by the user

2 Ensure that the result never has more monomials than the input

A canonical function F, as sometimes defined, does not need to have this property

Want a 'true' canonical function? Use the flag $TrueCanonicalForm

```
In[•]:= $TrueCanonicalForm = True;
sym = {T[a, b, c] + T[b, c, a] + T[c, a, b]};

CanonicalForm[T[c, a, b] × X[a, b], sym]
CanonicalForm[(-T[a, b, c] - T[b, c, a]) X[a, b], sym]
% === %%

Out[•]= -T[f1, f2, c] × X[f1, f2] - T[f1, c, f2] × X[f2, f1]

Out[•]= -T[f1, f2, c] × X[f1, f2] - T[f1, c, f2] × X[f2, f1]

Out[•]= True
```

Dummy indices (f1,f2,...) are re-labelled by the code

Output may have more monomials than the input

Some extra tools in

Sim Simplify | **Te** Tensor | **Ex** Expressions

(Functions beyond CanonicalForm)

Analyzing tensors with symmetries

SimTeEx comes with the following extra functions (GroupMath is needed)

YoungSymmetrizeTensor	Applies Young symmetrizer to a tensor
SnIrrepsInTensor	Returns the S_n irreps associated to a particular tensor symmetry
SingleProjector	Returns an Hermitian projector which contains the same symmetry as an input set of equations (condenses many conditions with one)
SameEquationsQ	Compares two sets of symmetry relations, symmetries 1 vs symmetries 2
SymmetriesOfNumericalTensor	Extracts a set of equations which describes the permutation symmetries of a numerical tensor

YoungSymmetrizeTensor and SymmetriesOfNumericalTensor

1	3
2	

```
In[ ]:= YoungSymmetrizeTensor[Y[p, q, r], {{1, 3}, {2}}]
```

Out[ ]= $\frac{1}{3}$ Y[p, q, r] − $\frac{1}{3}$ Y[q, p, r] − $\frac{1}{3}$ Y[q, r, p] + $\frac{1}{3}$ Y[r, q, p]

1	2

```
In[ ]:= YoungSymmetrizeTensor[S[x1, x2], {{1, 2}}]
```

Out[ ]= $\frac{1}{2}$ S[x1, x2] + $\frac{1}{2}$ S[x2, x1]

YoungSymmetrizeTensor

```
In[ ]:= SymmetriesOfNumericalTensor[LeviCivitaTensor[3]]

Out[ ]= {tensor[id1, id2, id3] + tensor[id3, id2, id1],
         tensor[id1, id3, id2] - tensor[id3, id2, id1]}
```

SymmetriesOfNumericalTensor

```
In[ ]:= SymmetriesOfNumericalTensor[TensorProduct[LeviCivitaTensor[2], LeviCivitaTensor[2]]]

Out[ ]= {tensor[id1, id2, id3, id4] - tensor[id4, id3, id2, id1],
         tensor[id1, id2, id4, id3] + tensor[id4, id3, id2, id1],
         tensor[id1, id4, id2, id3] - tensor[id4, id2, id3, id1] + tensor[id4, id3, id2, id1]}
```

Tries to return a set of simple equations which together completely characterize the symmetries of a numerical tensor

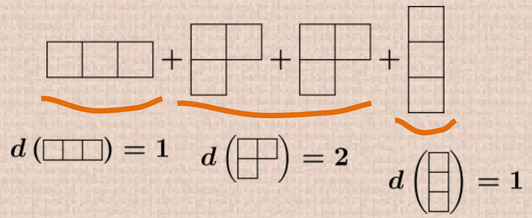
SnIrrepsInTensor

-quick summary

Irreps of S_n can be identified with partitions λ of n, or graphically with Young diagrams with n boxes

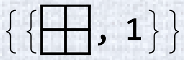
A general tensor with n equal indices, (with no symmetries at all) can be split into $d(\lambda)$ parts transforming as λ

E.g.: general 3-index tensor



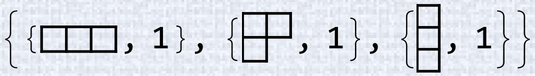
A tensor with symmetries is obviously not a general one; It will contain only some of these components

```
In[ ]:= SnIrrepsInTensor[{R[f2, f1, f3, f4] + R[f1, f2, f3, f4],
          R[f1, f2, f3, f4] + R[f1, f2, f4, f3], R[f1, f2, f3, f4] - R[f3, f4, f1, f2],
          R[f1, f2, f3, f4] + R[f1, f3, f4, f2] + R[f1, f4, f2, f3]}]
```

Out[ ]=


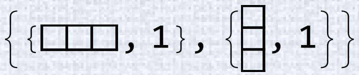
Riemann tensor: has a box symmetry

```
In[ ]:= SnIrrepsInTensor[{YoungSymmetrizeTensor[Y[p, q, r], {{1, 2}, {3}}]}]
```

Out[ ]=


Rank-3 tensor with a mixed symmetry component removed

```
In[ ]:= SnIrrepsInTensor[{YoungSymmetrizeTensor[X[p, q, r], {{1, 2}, {3}}], YoungSymmetrizeTensor[X[p, q, r], {{1, 3}, {2}}]}]
```

Out[ ]=


Rank-3 tensor with both mixed symmetry components removed

With this info it is trivial, for example, to compute the number of independent tensor components

SameEquationsQ

Imagine having a set of equations $E=\{eq_1, eq_2, ..., eq_I\}$ describing what you think are the symmetries of a tensor. Someone else comes along with a different set of equations, $E'=\{eq'_1, eq'_2, ..., eq'_J\}$.

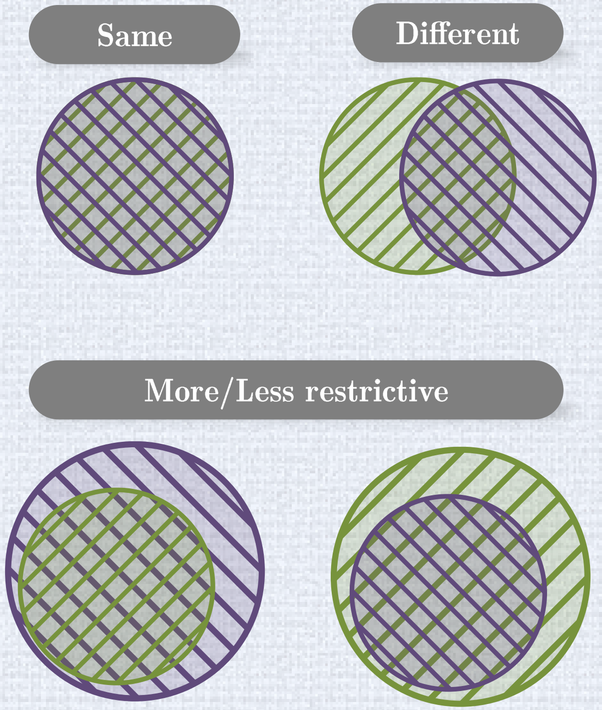
Are they the same? Maybe E contains all of the restrictions in E' and more, or vice-versa. Ou maybe they are just different.

```
eqs1 = {P[k1, k2] - Q[k1, k2]};
eqs2 = {P[k1, k2] + P[k2, k1] - (Q[k1, k2] + Q[k2, k1]),
   P[k1, k2] - P[k2, k1] - (Q[k1, k2] - Q[k2, k1])};
eqs3 = {P[k1, k2] + P[k2, k1] - (Q[k1, k2] + Q[k2, k1])};
eqs4 = {P[k1, k2] - Q[k2, k1]};

SameEquationsQ[eqs1, eqs2]
SameEquationsQ[eqs1, eqs3]
SameEquationsQ[eqs1, eqs4]
Out[ ]= Same system of equations

Out[ ]= Equations #1 are more restrictive

Out[ ]= Equations #1 and #2 are different
```



This function is a bit more general than others in SimTeEx in the sense that it allows symmetry conditions between different tensors (P and Q in this example)

SameEquationsQ

That was a very simple example (to illustrate what is at stake)
But note that in general it can be non-trivial to see that two sets of conditions are (non)equivalent

```
In[•]:= RiemannSyms1 = {R[f2, f1, f3, f4] + R[f1, f2, f3, f4],
    R[f1, f2, f3, f4] + R[f1, f2, f4, f3], R[f1, f2, f3, f4] - R[f3, f4, f1, f2],
    R[f1, f2, f3, f4] + R[f1, f3, f4, f2] + R[f1, f4, f2, f3]};
RiemannSyms2 = {R[f1, f2, f3, f4] + R[f1, f2, f4, f3],
    -R[f2, f1, f3, f4] + R[f1, f3, f4, f2] + R[f1, f4, f2, f3]};
SameEquationsQ[RiemannSyms1, RiemannSyms2]

Out[•]= Same system of equations
```

E.g.: Riemann tensor

In[•]:= RiemannSyms3 = $\Big\{\frac{11}{12}$ R[f1, f2, f3, f4] + $\frac{1}{12}$ R[f1, f2, f4, f3] − $\frac{1}{24}$ R[f1, f3, f2, f4] + $\frac{1}{24}$ R[f1, f3, f4, f2] + $\frac{1}{24}$ R[f1, f4, f2, f3] −
$\frac{1}{24}$ R[f1, f4, f3, f2] + $\frac{1}{12}$ R[f2, f1, f3, f4] − $\frac{1}{12}$ R[f2, f1, f4, f3] + $\frac{1}{24}$ R[f2, f3, f1, f4] − $\frac{1}{24}$ R[f2, f3, f4, f1] − $\frac{1}{24}$ R[f2, f4, f1, f3] +
$\frac{1}{24}$ R[f2, f4, f3, f1] + $\frac{1}{24}$ R[f3, f1, f2, f4] − $\frac{1}{24}$ R[f3, f1, f4, f2] − $\frac{1}{24}$ R[f3, f2, f1, f4] + $\frac{1}{24}$ R[f3, f2, f4, f1] − $\frac{1}{12}$ R[f3, f4, f1, f2] +
$\frac{1}{12}$ R[f3, f4, f2, f1] − $\frac{1}{24}$ R[f4, f1, f2, f3] + $\frac{1}{24}$ R[f4, f1, f3, f2] + $\frac{1}{24}$ R[f4, f2, f1, f3] − $\frac{1}{24}$ R[f4, f2, f3, f1] + $\frac{1}{12}$ R[f4, f3, f1, f2] −
$\frac{1}{12}$ R[f4, f3, f2, f1]$\Big\}$;
SameEquationsQ[RiemannSyms1, RiemannSyms3]

Out[•]= Same system of equations

One can always condense the information in a list of symmetry conditions into just 1. This leads me to a final function included in SimTeEx.

SingleProjector

```
SingleProjector[<null conditions with the tensor symmetries>]
```

Returns the unique hermitian projector P such that the condition P(tensor)=tensor is equivalent to the set of null equations given as input.

1. **Output is a projector P, i.e. P(tensor)=tensor [note: not P(tensor)=0] with $P^2=P$**
2. $P^\dagger = P$
3. **There is always one, and only one, $P \in S_n$ algebra which fully describes a tensor's symmetry via de relation P(tensor)=tensor and satisfying $P^\dagger = P^2 = P$**

Convinced myself of this with a constructive argument (therefore not particularly elegant)

On this last point, in relation to the uniqueless one can see, for example, that $P^2=P$ is important. Just take a symmetry 2-index tensor T: its symmetry can be described by

$$P(T_{ij}) = T_{ij} \quad \text{for} \quad P = P_S + aP_A \quad \text{with} \qquad P_S \equiv \frac{1}{2}(e+(12)) \quad P_A \equiv \frac{1}{2}(e-(12))$$

any 'a' different from 1 will do

SingleProjector

2 $P^\dagger = P$ ← What does is an adjoint projector?

For two tensors A and B of equal rank define the inner product

$$\langle A, B\rangle = A^*_{i_1 i_2 \cdots i_n} B_{i_1 i_2 \cdots i_n}$$

... and take some P in the S_n algebra,

$$P = \sum_{\pi \in S_n} c_\pi \pi$$

$$\langle A, P(B)\rangle \equiv \langle P^\dagger(A), B\rangle = \sum_{\pi\in S_n} c_\pi A^*_{i_1 i_2 \cdots i_n} B_{\pi(i_1 i_2 \cdots i_n)}$$

$$= \sum_{\pi\in S_n} c_\pi A^*_{\pi^{-1}(i_1 i_2 \cdots i_n)} B_{i_1 i_2 \cdots i_n}$$

$$= \sum_{\pi\in S_n} \left[c^*_{\pi^{-1}} A_{\pi(i_1 i_2 \cdots i_n)}\right]^* B_{i_1 i_2 \cdots i_n}$$

→

$$P^\dagger = \sum_{\pi\in S_n} c^*_{\pi^{-1}} \pi$$

So note: if there is a term $\omega_{ijk\cdots}\mathcal{O}_{ijk\cdots}$ and the operator has a symmetry $P\left(\mathcal{O}_{ijk\cdots}\right) = \mathcal{O}_{ijk\cdots}$ then the Wilson coefficient <u>can</u> (not unique!) be taken to have a symmetry $P^{\dagger *}\left(\omega_{ijk\cdots}\right) = \omega_{ijk\cdots}$

Not $P\left(\omega_{ijk\cdots}\right) = \omega_{ijk\cdots}$ **!** But if P is Hermitian (and real, as it is extremely often the case) then this is true: the coefficient can be taken to have the same symmetry as the operator

SingleProjector

```
In[•]:= SingleProjector[{R[f2, f1, f3, f4] + R[f1, f2, f3, f4],
          R[f1, f2, f3, f4] + R[f1, f2, f4, f3], R[f1, f2, f3, f4] - R[f3, f4, f1, f2],
          R[f1, f2, f3, f4] + R[f1, f3, f4, f2] + R[f1, f4, f2, f3]}]
```

Out[•]= $\frac{1}{12}$ R[f1, f2, f3, f4] − $\frac{1}{12}$ R[f1, f2, f4, f3] + $\frac{1}{24}$ R[f1, f3, f2, f4] − $\frac{1}{24}$ R[f1, f3, f4, f2] − $\frac{1}{24}$ R[f1, f4, f2, f3] + $\frac{1}{24}$ R[f1, f4, f3, f2] − $\frac{1}{12}$ R[f2, f1, f3, f4] + $\frac{1}{12}$ R[f2, f1, f4, f3] − $\frac{1}{24}$ R[f2, f3, f1, f4] + $\frac{1}{24}$ R[f2, f3, f4, f1] + $\frac{1}{24}$ R[f2, f4, f1, f3] − $\frac{1}{24}$ R[f2, f4, f3, f1] − $\frac{1}{24}$ R[f3, f1, f2, f4] + $\frac{1}{24}$ R[f3, f1, f4, f2] + $\frac{1}{24}$ R[f3, f2, f1, f4] − $\frac{1}{24}$ R[f3, f2, f4, f1] + $\frac{1}{12}$ R[f3, f4, f1, f2] − $\frac{1}{12}$ R[f3, f4, f2, f1] + $\frac{1}{24}$ R[f4, f1, f2, f3] − $\frac{1}{24}$ R[f4, f1, f3, f2] − $\frac{1}{24}$ R[f4, f2, f1, f3] + $\frac{1}{24}$ R[f4, f2, f3, f1] − $\frac{1}{12}$ R[f4, f3, f1, f2] + $\frac{1}{12}$ R[f4, f3, f2, f1]

```
In[•]:= YoungSymmetrizeTensor[Y[p, q, r], {{1, 3}, {2}}]
        SingleProjector[{Y[p, q, r] - %}]
```

1	3
2	

Out[•]= $\frac{1}{3}$ Y[p, q, r] − $\frac{1}{3}$ Y[q, p, r] − $\frac{1}{3}$ Y[q, r, p] + $\frac{1}{3}$ Y[r, q, p]

Out[•]= $\frac{1}{3}$ Y[p, q, r] − $\frac{1}{6}$ Y[p, r, q] − $\frac{1}{6}$ Y[q, p, r] − $\frac{1}{6}$ Y[q, r, p] − $\frac{1}{6}$ Y[r, p, q] + $\frac{1}{3}$ Y[r, q, p]

Note: Normal Young projectors are usually not Hermitian

Implementation details

Other codes

ATENSOR
Ilyin, Kryukov (1996)

Redberry
Bolotin, Poslavsky (2013)

Cadabra
Peeters (2018)

xPerm (in xAct)
Marťn-Garća (2008)

All handle dummy indices and mono-term symmetries (more on this in a bit). I think that from this list only Cadabra can handle multi-term symmetries (the complicated symmetries)

Cadabra

Declare the symmetry of the tensors via Young projectors

```
A_{m n p}::TableauSymmetry(shape={2,1}, indices={0,2,1}).
ex:= A_{m n p};
```

$$A_{mnp}$$

```
young_project_tensor(_);
```

$$\frac{1}{3}A_{mnp} + \frac{1}{3}A_{pnm} - \frac{1}{3}A_{nmp} - \frac{1}{3}A_{pmn}$$

cadabra.science/manual/young_project_tensor.html

Simplification of expressions with multi-term symmetries: Replace everywhere A with this expression

More info on this:
cadabra.science/notebooks/tensor_monomials.html

But note: some tensor symmetries cannot be expression with Young projectors (if I have time I'll talk about it later)

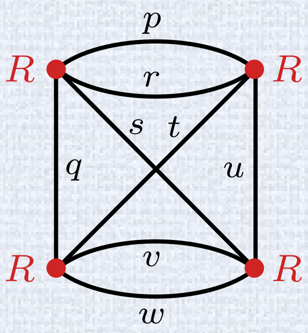
SimTeEx: dummy indices and graphs

Graphs are a natural way to deal with dummy indices in tensor monomials

Take the first monomial of

$$R_{pqrs}R_{ptru}R_{tvqw}R_{uvsw} - R_{pqrs}R_{pqtu}R_{rvtw}R_{svuw}$$
$$-R_{mnab}R_{npbc}R_{mscd}R_{spda} + \frac{1}{4}R_{mnab}R_{psba}R_{mpcd}R_{nsdc} = 0$$

If we assume for a moment that R is a fully symmetric tensor (so order of the indices is irrelevant), we could represent the first monomial as follows



$$R_{pqrs}R_{ptru}R_{tvqw}R_{uvsw}$$

One must label the vertices of the diagram(the monomial might involve more than one rank-4 tensor)

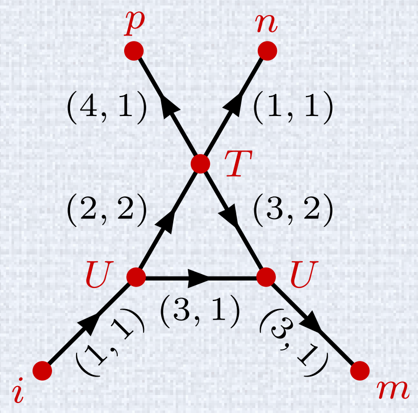
But edges are unlabeled. That is the whole point of using a graph: the dummy labels are irrelevant

... but not all tensor are fully symmetric

In general one must distinguish the indices of each vertex (index #1, #2, ...).
In a graph representation, this can be achieved by labelling the edges (and giving them a direction).

Let us work with the simpler example $U_{ijk}U_{klm}T_{njlp}$

Edge labels (n_1, n_2) indicate that the index in slot #n_1 of the departing tensor contracts with the index in slot # n_2 of the incident tensor. For example the (3,2) implies that the third index of T contract with the second index of one of the U tensors (the one with the external m index).

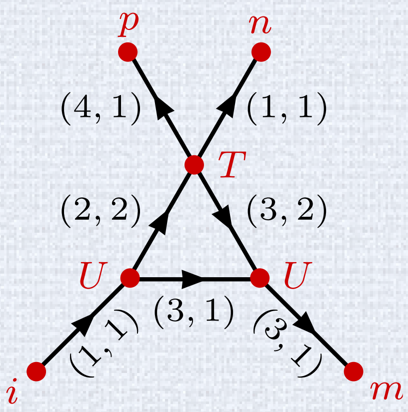


It is important to give a direction to the edges so that one can interpret the two numbers which label each edge.

(This is just one possible possibility of doing things)

External indices: vertices with 1 line

How to represent these graphs?



Using an adjacency matrix for example:

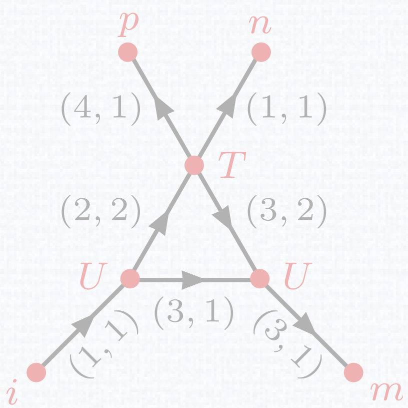
$$\mathscr{M}=\begin{pmatrix} 0&1&1&1&1&0&0\\ 1&0&1&0&0&1&0\\ 1&1&0&0&0&0&1\\ 1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0 \end{pmatrix}$$

However, because the vertices and edges need to be labelled, this must be generalized. I used the following representation for each graph

$$\left\{\{T,U,U,p,n,i,m\},\mathscr{M}^{(\mathrm{gen})}\right\}$$

Representation of $U_{ijk}U_{klm}T_{njlp}$

$$\mathscr{M}^{(\mathrm{gen})}=\begin{pmatrix}
\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix}\\
\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix}\\
\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\end{pmatrix}\\
\begin{pmatrix}0&0&0&1\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & (0) & (0) & (0) & (0)\\
\begin{pmatrix}1&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & (0) & (0) & (0) & (0)\\
\begin{pmatrix}0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & (0) & (0) & (0) & (0)\\
\begin{pmatrix}0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & \begin{pmatrix}0&0&1\end{pmatrix} & (0) & (0) & (0) & (0)
\end{pmatrix}$$



How to repre

Let's not loose sight of the goal: the purpose of all this work is to have a representation which allows us to put each monomial in a canonical form.

Using an adjacency matrix

If there are no tensor symmetries, we can do that by permuting rows/columns associated to equal tensors and systematically picking a 'minimal' adjacency matrix.

However, because the vertices and edges need to be labelled, this must be generalized. I used the following representation for each graph

$$\left\{\{T, U, U, p, n, i, m\}, \mathscr{M}^{(\text{gen})}\right\}$$

Representation of $U_{ijk}U_{klm}T_{njlp}$

$$\mathscr{M}^{(\text{gen})} = \begin{pmatrix}
\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\end{pmatrix} \\
\begin{pmatrix}0&0&0&1\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & (0) & (0) & (0) & (0) \\
\begin{pmatrix}1&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & (0) & (0) & (0) & (0) \\
\begin{pmatrix}0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & (0) & (0) & (0) & (0) \\
\begin{pmatrix}0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\end{pmatrix} & \begin{pmatrix}0&0&1\end{pmatrix} & (0) & (0) & (0) & (0)
\end{pmatrix}$$

Polynomials: vector space spanned by graphs

Tensor polynomials are linear combinations of these graphs

$$P = \sum_i c_i g_i$$

If we find that some graphs are equivalent, say $g_1 = g_2$, then we can simplify the polynomial

$$c_1 g_1 + c_2 g_2 + \cdots \to (c_1 + c_2)\, g_1 + \cdots .$$

Very simple <u>example</u>: `2 m[a,q,q,b] T[a,b] + A m[c1,m,m,c2] T[c1,c2]`

$$2g_1 + Ag_2 \qquad \text{with} \qquad g_1 = g_2 = \left\{ \{m, T\}, \left(\begin{matrix} \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 1&0\\0&0\\0&0\\0&1 \end{pmatrix} \\ \begin{pmatrix} 1&0&0&0\\0&0&0&1 \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{matrix} \right) \right\}$$

In this example this is trivial to see by eye

So we could simplify the original expression to

`(2+A) m[a,q,q,b] T[a,b]` (reusing index names) ... or maybe `(2+A) m[c1,m,m,c2] T[c1,c2]`

Symmetries

This was to deal with dummy indices, assuming tensors have symmetries

What if they do?

Mono-term symmetries

The easy ones

$$T_{\pi(i_1 i_2 \cdots i_n)} = \sigma T_{i_1 i_2 \cdots i_n}$$

Examples

$$T_{ab} = -T_{ba}$$

$$T_{abcd} = T_{badc}$$

$$T_{abcd} = T_{bacd} \text{ and } T_{abcd} = -T_{abdc}$$

$$T_{abc} = \omega T_{bca} \text{ with } \omega \equiv \exp\left(\frac{2\pi i}{3}\right)$$

They can be accounted for by allowing
the exchange of some graph edges
(and maybe tracking a sign/phase while doing so)

Multi-term symmetries

$$T_{\pi_1(i_1 i_2 \cdots i_n)} + T_{\pi_2(i_1 i_2 \cdots i_n)} + \cdots + T_{\pi_p(i_1 i_2 \cdots i_n)} = 0 \text{ with } p > 2$$

These are the complicated ones

The hard ones: multiterm symmetries

Let us take a simple example

$$T_{abc} + T_{bca} + T_{cab} = 0$$

The symmetry

$$xT_{abc} + yT_{bca} + zT_{cab}$$

The expression to simplify
(note: no dummy indices)

Plan: assign an order of preference among the monomials, e.g. $T_{abc} < T_{bca} < T_{cab}$

T_{abc} is the least preferred, so it should be replaced by the other monomials whenever it appears

$$xT_{abc} + yT_{bca} + zT_{cab} \rightarrow (y-x)\, T_{bca} + (z-x)\, T_{cab}$$

Simple to do, right?

1. Result depends on order of preference among the different permutations of T
2. Strategy can lead to more monomial in the output than in the input (e.g. $x = 1, y = z = 0$)
3. Now take $U_{abc}\left(xT_{abc} + yT_{bca} + zT_{cab}\right)$ where the new U tensor has no symmetries. Since these are now dummy indices, one cannot just use the simple strategy of setting some order like $T_{abc} < T_{bca} < T_{cab}$. Instead one should order/set a preference for the graphs we have been discussing (associated to each monomial)

The rest is then linear algebra

Multi-term symmetries are null relations among graphs g_i

$$0=\sum_{i=1}^{k} n_i^{(a)} g_i$$

$$P=\sum_{i=1}^{k} c_i g_i \rightarrow P'=P+\sum_a \omega_a \sum_{i=1}^{k} n_i^{(a)} g_i \equiv \sum_{i=1}^{k} c_i' g_i$$

Original expression

New expression obtained by adding the null relations

Aim could be, for example, to set as many of the c_i' coefficients to 0

$$\overbrace{\begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_k \\ \hline 0 & n_1^{(1)} & n_2^{(1)} & \cdots & n_k^{(1)} \\ 0 & n_1^{(2)} & n_2^{(2)} & \cdots & n_k^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}}^{g_1 \quad g_2 \quad \cdots \quad g_k} \xrightarrow{\text{RREF}} \overbrace{\begin{pmatrix} 1 & c_1' & c_2' & \cdots & c_k' \\ \hline 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}}^{g_1 \quad g_2 \quad \cdots \quad g_k}$$

For a given ordering of the graphs g_i, the problem is solved by putting a matrix reduced row echelon form (RREF)

(Getting the absolute minimum number of terms in the final expression might require testing all graph ordering)

To conclude ...

Having tools to understand and handle tensor expressions can be important in the study of EFTs

I've described SimTeEx, which can be used to simplify tensor expressions

It was created to help study the general EFT (RGEs, evanescent operators, matching, ...), where operator symmetries are particularly complicated

This last part (study of the general EFT) is a work in progress.
SimTeEx is now public and can be used for other applications.

Thank you