Two-Loop Beta Functions in the LEFT: Dimension-Six Four-Fermion Operators in NDR

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The LEFT: Four-Fermion Operators

Up to dimension-six terms, the Lagrangian of the LEFT(n_q) reads

$$\mathcal{L}_{\rm LEFT}^{(n_q)} = \mathcal{L}_{\rm QCD+QED}^{(n_q)} + \sum_k L_k^{(3)} \mathcal{O}_k^{(3)} + \sum_k L_k^{(5)} \mathcal{O}_k^{(5)} + \sum_k L_k^{(6)} \mathcal{O}_k^{(6)} \ ,$$

with fermion content:

- LEFT(5): $u, c, d, s, b, e, \mu, \tau, \nu_{e,\mu,\tau}$ (heaviest: $m_b \approx 4.2$ GeV).
- LEFT(4): $u, c, d, s, e, \mu, \tau, \nu_{e,\mu,\tau}$ (heaviest: $m_\tau \approx 1.8$ GeV).
- LEFT(3): $u, d, s, e, \mu, \nu_{e,\mu,\tau}$ (heaviest: $m_\mu \approx 0.1$ GeV).

We focus on the large sector of dimension-six four-fermion operators:

$$\sum_k L_k^{(6)} \mathcal{O}_k^{(6)} = O_G + O_{\widetilde{G}} + \sum L_{\psi^4} \mathcal{O}_{\psi^4} \ .$$

The LEFT: Four-Fermion Operators

The general form of a four-fermion operator is:

$$\mathcal{O}_{\psi^4} \sim (\bar{\psi}_1 \Gamma \psi_2)\,(\bar{\psi}_3 \Gamma \psi_4)\ ,$$

Flavor conservation in QCD+QED $\implies$ disconnected flavor sectors:

Sector	**Flavors**	**Sector**	**Flavors**
$\Delta F = 2$	$\bar{\psi}_1\psi_2\,\bar{\psi}_1\psi_2$	$\Delta F = 1.5$	$\bar{\psi}_1\psi_2\,\bar{\psi}_1\psi_3$
$\Delta F = 1$	$\bar{\psi}_1\psi_2\,\bar{\psi}_3\psi_4$	$\Delta F = 1^{\bar{f}f}$	$\bar{\psi}_1\psi_2\,\bar{f}f$
$\lvert\Delta L\rvert = 2$	$\bar{\nu}_i^c\ell\,\bar{\psi}_3\psi_4$	$\Delta F = 0$	$\bar{f}f\,\bar{f}'f'$
$\lvert\Delta L\rvert = 4$	$\bar{\nu}_i^c\nu_j\,\bar{\nu}_k^c\nu_l$	$\lvert\Delta B\rvert = 1$	$\bar{q}_1^c q_2\,\bar{q}_3^c\ell$

Renormalization Group Evolution

The dependence of the Wilson coefficients (L_i) on the renormalization scale (μ) is described by the beta functions,

$$\frac{dL_i}{d\log\mu} = \gamma_{ji}\, L_j = \frac{1}{16\pi^2}\,\beta_i \equiv \frac{1}{16\pi^2}\,\dot{L}_i \ ,$$

where γ_{ij} are the components of the ADM, to be expanded as:

$$\gamma_{ij} = \tilde{\alpha}_s\,\gamma_{ij}^{(1,0)} + \tilde{\alpha}\,\gamma_{ij}^{(0,1)} + \tilde{\alpha}_s\tilde{\alpha}\,\gamma_{ij}^{(1,1)} + \tilde{\alpha}_s^2\,\gamma_{ij}^{(2,0)} + \tilde{\alpha}^2\,\gamma_{ij}^{(0,2)} + \cdots \ .$$

with $\tilde{\alpha}_s \equiv \alpha_s/4\pi$ and $\tilde{\alpha} \equiv \alpha/4\pi$.

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with $\tilde{\alpha}_s \equiv \alpha_s/4\pi$ and $\tilde{\alpha} \equiv \alpha/4\pi$.

The coefficients $\gamma_{ij}^{(n,m)}$ can be calculated using the renormalization matrices of the Wilson coefficients,

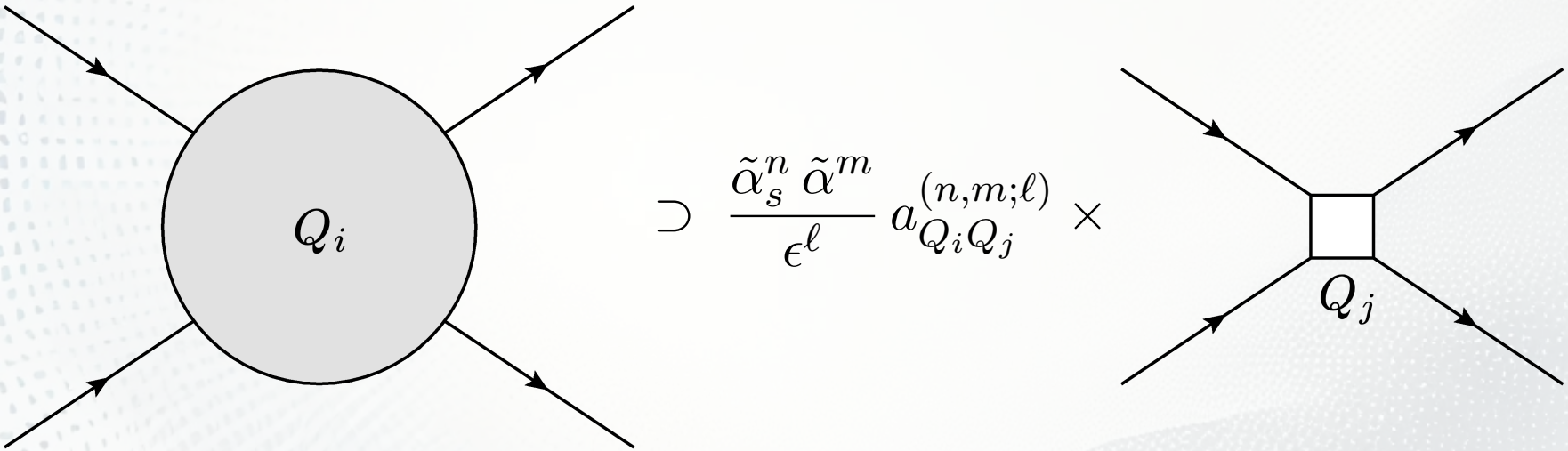
$$L_i^{(\text{bare})} = Z_{ij} L_j \quad \Longrightarrow \quad \gamma_{ij} = Z_{ik}\left(\frac{d\,Z^{-1}}{d\log\mu}\right)_{kj} \ .$$

Renormalization Group Evolution

The $\gamma_{ij}^{(n,m)}$ can then be rewritten in terms of matrix elements*,

$$\langle Q_i \rangle = \sum_{n,m=0}^{\infty} \tilde{\mu}^{2\epsilon(n+m)}\, \tilde{\alpha}_s^n\, \tilde{\alpha}^m\, \langle Q_i \rangle^{(n,m)} \quad ,$$

$$\langle Q_i \rangle^{(n,m)} = \sum_{k=0}^{n+m} \frac{1}{\epsilon^k} \left[a_{Q_i Q_j}^{(n,m;k)} \langle Q_j \rangle^{(0)} + a_{Q_i E_j}^{(n,m;k)} \langle E_j \rangle^{(0)} \right] \quad .$$



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For instance, at $O(\alpha_s)$ and $O(\alpha_s^2)$ one has

$$\gamma_{ij}^{(1,0)} = -2\hat{a}_{Q_i Q_j}^{(1,0;1)} - 2Z_{Q_i}^{(1,0;1)} \delta_{Q_i Q_j} \ ,$$

$$\gamma_{ij}^{(2,0)} = -4\hat{a}_{Q_i Q_j}^{(2,0;1)} + 4\hat{a}_{Q_i Q_k}^{(1,0;1)} \hat{a}_{Q_k Q_j}^{(1,0;0)} + 2\hat{a}_{Q_i E_k}^{(1,0;1)} \hat{a}_{E_k Q_j}^{(1,0;0)} - 4Z_{Q_i}^{(2,0;1)} \delta_{Q_i Q_j} \ ,$$

where $Z_{Q_i}^{(n,m;1)}$ contains the wave-function renormalization factors.

*The matrix elements $\hat{a}^{(m,n;\ell)}$ include dimension-four counterterms but not higher-dimension.

Overview

Our main result is **the complete set of two-loop beta functions** describing the mixing among **four-fermion operators** in the **NDR scheme**, for the LEFT(n_q) with $n_q = 5, 4, 3$.

We express all results in the **JMS basis** of the LEFT.

Overview

Here is an example for LEFT(5):

$$\dot{L}^{S1,RR}_{ud\,ijkl} = g_s^2 \left[\frac{64}{9} L^{S1,RR}_{uddu\,ilkj} + \frac{112}{27} L^{S8,RR}_{uddu\,ilkj} - 16 L^{S1,RR}_{ud\,ijkl} + \frac{16}{9} L^{S8,RR}_{ud\,ijkl} \right]$$
$$+ e^2 \left[\frac{4}{27} L^{S1,RR}_{uddu\,ilkj} + \frac{16}{81} L^{S8,RR}_{uddu\,ilkj} - \frac{46}{9} L^{S1,RR}_{ud\,ijkl} \right]$$
$$+ \alpha_s^2 \left[\frac{9472}{81} L^{S1,RR}_{uddu\,ilkj} + \frac{16576}{243} L^{S8,RR}_{uddu\,ilkj} - \frac{2120}{9} L^{S1,RR}_{ud\,ijkl} + \frac{928}{81} L^{S8,RR}_{ud\,ijkl} \right]$$
$$+ \alpha_s \alpha \left[\frac{1888}{81} L^{S1,RR}_{uddu\,ilkj} + \frac{3808}{243} L^{S8,RR}_{uddu\,ilkj} + \frac{136}{27} L^{S1,RR}_{ud\,ijkl} + \frac{224}{81} L^{S8,RR}_{ud\,ijkl} \right]$$
$$+ \alpha^2 \left[\frac{98}{729} L^{S1,RR}_{uddu\,ilkj} + \frac{392}{2187} L^{S8,RR}_{uddu\,ilkj} + \frac{6391}{243} L^{S1,RR}_{ud\,ijkl} \right] .$$

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We express all results in the **JMS basis** of the LEFT.

We have obtained them using three main methods:

1. Change of Basis from the Literature
2. Symmetrization of Flavor Indices
3. Direct Reconstruction from the Tables of Diagram Poles

Method 1

Change of Basis from the Literature

Change of Basis: General Formalism

When changing operator basis,

$$\vec{Q}' = \hat{R}\left(\vec{Q} + \hat{W}\vec{E}\right) \ ,$$

the two-loop ADM transforms in a complicated way in order to preserve the renormalization scheme,

$$\begin{aligned}
\hat{\gamma}'^{(2,0)} =& \hat{R}\hat{\gamma}^{(2,0)}\hat{R}^{-1} + 4Z_{g_s}^{(1,0;1)}\Delta\hat{r}^{(1,0)} - \left[\Delta\hat{r}^{(1,0)}, \hat{\gamma}^{(1,0)}\right] \ , \\
\hat{\gamma}'^{(0,2)} =& \hat{R}\hat{\gamma}^{(0,2)}\hat{R}^{-1} + 4Z_{e}^{(0,1;1)}\Delta\hat{r}^{(0,1)} - \left[\Delta\hat{r}^{(0,1)}, \hat{\gamma}^{(0,1)}\right] \ , \\
\hat{\gamma}'^{(1,1)} =& \hat{R}\hat{\gamma}^{(1,1)}\hat{R}^{-1} - \left[\Delta\hat{r}^{(1,0)}, \hat{\gamma}^{(0,1)}\right] - \left[\Delta\hat{r}^{(0,1)}, \hat{\gamma}^{(1,0)}\right] \ ,
\end{aligned}$$

with $\Delta\hat{r}^{(1,0)}$ and $\Delta\hat{r}^{(0,1)}$ being one-loop matrices characterizing the shift.

Change of Basis: Results

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- In [1-3] they provide $\hat{\gamma}^{(2,0)}$ in the BMU basis for:
  - The $\Delta F = 2$ four-quark sectors ($\bar{d}_i d_j \bar{d}_i d_j$ and $\bar{u}_i u_j \bar{u}_i u_j$).
  - The $\Delta F = 1$ four-quark sectors ($\bar{u}_i u_j \bar{d}_k d_l$).
  - The d-type $\Delta F = 1^{\bar{f}f}$ sectors ($\bar{d}_i d_j \bar{f}_k f_k$).

[1] A. J. Buras, P. H. Weisz, 1990.

[2] A. J. Buras, M. Jamin, M. E. Lautenbacher and P. H. Weisz, 1992 (hep-ph/9211304 [hep-ph]).

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- Many authors provide $\hat{\gamma}^{(2,0)}$ and $\hat{\gamma}^{(1,1)}$ for the "SM sector" of the d-type $\Delta F = 1^{\bar{f}f}$ sector in both the BMU and the CMM bases.

  They are helpful, but do not allow for a direct change of basis!

Method 2

Symmetrization of Flavor Indices

Flavor Symmetrization: General Formalism

This method is an extension of the one used in [3] to obtain $\hat{\gamma}^{(2,0)}$ for the four-quark $\Delta F = 2$ sector **starting from a different sector** $(\Delta F = 1)$.

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The key point is that we start with operators having no symmetries in their flavor indices, and then **symmetrize** them via a change of basis.

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The key point is that we start with operators having no symmetries in their flavor indices, and then **symmetrize** them via a change of basis.

Example:

Starting with the basis of a four-quark $\Delta F = 1$ sector, we perform a change of basis into a "$\pm$" basis:

$$\begin{matrix}(\bar{u}_i \Gamma u_j)(\bar{d}_k \Gamma d_l) \\ (\bar{u}_i^\alpha \Gamma u_j^\beta)(\bar{d}_k^\beta \Gamma d_l^\alpha)\end{matrix} \quad \Longrightarrow \quad \begin{matrix}\frac{1}{2}(\bar{u}_i \Gamma u_j)(\bar{d}_k \Gamma d_l) + \frac{1}{2}(\bar{u}_i \Gamma d_l)(\bar{d}_k \Gamma u_j) \\ \frac{1}{2}(\bar{u}_i \Gamma u_j)(\bar{d}_k \Gamma d_l) - \frac{1}{2}(\bar{u}_i \Gamma d_l)(\bar{d}_k \Gamma u_j)\end{matrix}$$

The operators on the RHS are (anti)symmetric under $u_j \leftrightarrow d_l$!

Flavor Symmetrization: General Formalism

The "+" operators in a "±" basis mimic the flavor symmetries that are inherent to the target flavor sectors. In this case, $\Delta F = 1.5$ $(\bar{d}_i d_j \bar{d}_k d_j)$.

The γ_{ij} describing the mixing among the "+" operators are then **numerically identical** to the full ADM in those flavor sectors!

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Starting with the four-quark $\Delta F = 1$ sector $(\bar{u}_i u_j \bar{d}_k d_l)$ we just obtained from [3]:

- Symmetrizing under both $\bar{u}_i \leftrightarrow \bar{d}_k$ and $u_j \leftrightarrow d_l$ yields the ADM for $\Delta F = 2$ (e.g. $\bar{d}_i d_j \bar{d}_i d_j$).

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- Symmetrizing under **either** $\bar{u}_i \leftrightarrow \bar{d}_k$ **or** $u_j \leftrightarrow d_l$ yields the ADM for $\Delta F = 1.5$ $(\bar{d}_i d_j \bar{d}_k d_j)$.

  We do this to extend the results of the literature onto the four-quark $\Delta F = 1.5$ sector!

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This works in general, starting with any of the $\Delta F = 1$ sectors.

Method 3

Direct Reconstruction from the Tables of Pole Coefficients

Tables of Diagram Poles: General Formalism

Some information is available in the literature regarding the separate contributions to the ADM from each diagram. These are the **tables of pole coefficients**.

We are interested in the coefficients associated to the $1/\epsilon$ poles.

They are built such that

$$\gamma_{ij}^{(m,n)} = -4Z_{Q_i}^{(m,n,1)}\delta_{Q_iQ_j} - 4\sum_{D\,\in\,\text{diags.}} C_{D,Q_iQ_j}^{(m,n)} T_{D,Q_iQ_j}^{(m,n;1)} \,.$$

For each class of $n+m$-loop diagram D:

- The overall color and charge factors are factorized in the $C_{D,Q_iQ_j}^{(m,n)}$.
- The rest is included in the $T_{D,Q_iQ_j}^{(m,n;1)}$, given in the tables.

Tables of Diagram Poles: General Formalism

An example of one such table, from A. J. Buras and P. H. Weisz, 1990 [1]:

D	M	$1/\epsilon^2$	$(1/\epsilon)_{\mathrm{NDR}}$	$(1/\epsilon)_{\mathrm{HV}}$	$(1/\epsilon)_{\mathrm{DRED}}$	D′
4	2	-1	$\frac{5}{2}$	$\frac{5}{2}$	2	23
5	2	-16	16	16	16	10
6	2	-1	$\frac{5}{2}$	$\frac{5}{2}$	2	24
7	2	–	-4	-4	-4	16
8	2	–	-4	-4	-4	14
9	2	–	-4	-4	-4	15
10	4	-2	3	-1	2	6
11	4	8	14	14	8	9
12	4	-2	3	-1	2	5
13	4	2	-3	1	-2	12
14	4	-8	-2	-2	8	13
15	4	2	-3	1	-2	11
16	4	8	-14	2	–	3
17	4	8	14	-2	–	7
18	4	-2	3	3	4	2
19	4	-2	3	3	4	1
20	4	8	14	-2	–	8
21	4	8	-14	2	–	4
22	1	-1	–	–	$-\frac{1}{2}$	17
23	1	-16	–	–	–	19
24	1	-1	–	–	$-\frac{1}{2}$	18
25	4	6	-11	1	-4	21
26	4	-24	2	2	20	20
27	4	6	-11	1	-4	22
28	4	–	–	–	–	28
29	2	–	$\frac{5}{2}N - n_{\mathrm{F}}$	$-\frac{5}{6}N + \frac{1}{3}n_{\mathrm{F}}$	–	25
30	2	$5N - 2n_{\mathrm{F}}$	$-\frac{11}{3}N + \frac{2}{3}n_{\mathrm{F}}$	$-\frac{11}{3}N + \frac{2}{3}n_{\mathrm{F}}$	$-\frac{11}{3}N + \frac{2}{3}n_{\mathrm{F}}$	27
31	2	–	$\frac{5}{2}N - n_{\mathrm{F}}$	$-\frac{5}{6}N + \frac{1}{3}n_{\mathrm{F}}$	–	26

Tables of Diagram Poles: General Formalism

$$\gamma_{ij}^{(m,n)} = -4Z_{Q_i}^{(m,n,1)}\delta_{Q_iQ_j} - 4\sum_{D\,\in\,\text{diags.}} C_{D,Q_iQ_j}^{(m,n)} T_{D,Q_iQ_j}^{(m,n;1)} \,.$$

The $1/\epsilon$ pole coefficients $T_{D,Q_iQ_j}^{(m,n;1)}$ were originally calculated in [1-3] for $\hat{\gamma}^{(2,0)}$ in the four-quark $\Delta F = 1$ sector (current-current diagrams), and in the SM subsector of $\Delta F = 1^{\bar{f}f}$ (penguin diagrams).

However, the color/charge factors can be easily recomputed, and the $T_{D,Q_iQ_j}^{(m,n;1)}$ are essentially universal!

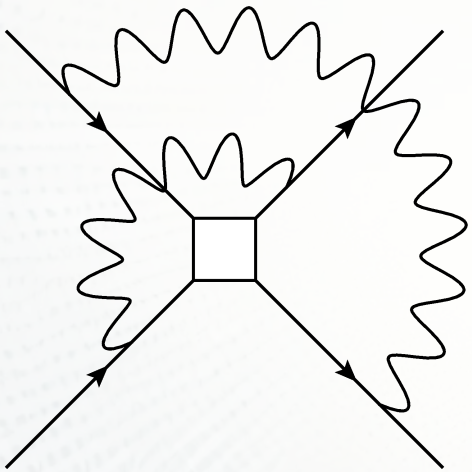
We find that this reduced number of coefficients is enough to reconstruct **all two-loop contributions to the ADM, and for all four-fermion sectors!**

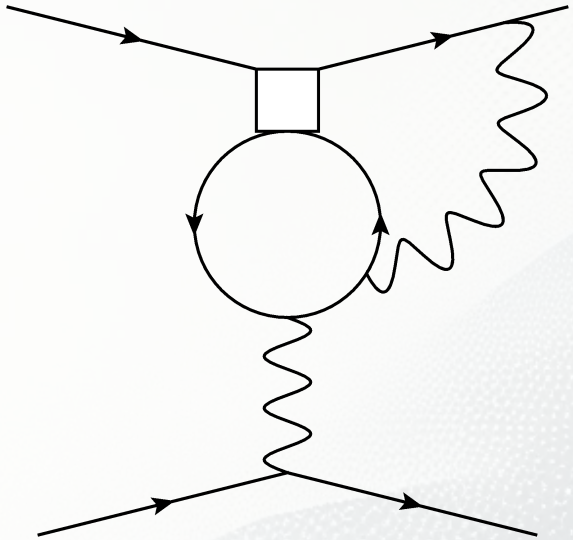
Tables of Diagram Poles: General Formalism

For that, we separate the contributions to the two-loop ADM in **current-current** and **penguin**, depending on the diagrams involved:

$$\hat{\gamma}^{(m,n)} = \hat{\gamma}_{cc}^{(m,n)} + \hat{\gamma}_{p}^{(m,n)} \ ,$$

where $\hat{\gamma}_{cc}^{(m,n)}$ includes contributions from current-current diagrams alone*, $\hat{a}^{(m,n;\ell)}$, while $\hat{\gamma}_{p}^{(m,n)}$ contains the rest.





* Including also the wave-function renormalization factors.

Tables of Diagram Poles: Current-Current Contributions

The tables of current-current operators, combined with the appropriate color/charge factors, directly provide the current-current ADM for a set of operators like:

$$Q_1 = (\bar{\psi}_1 \gamma^\mu P_L \psi_2)(\bar{\psi}_3 \gamma_\mu P_L \psi_4) \,, \qquad Q_2 = (\bar{\psi}_1 \gamma^\mu P_L T^A \psi_2)(\bar{\psi}_3 \gamma_\mu P_L T^A \psi_4) \,,$$
$$Q_3 = (\bar{\psi}_1 \gamma^\mu P_L \psi_2)(\bar{\psi}_3 \gamma_\mu P_R \psi_4) \,, \qquad Q_4 = (\bar{\psi}_1 \gamma^\mu P_L T^A \psi_2)(\bar{\psi}_3 \gamma_\mu P_R T^A \psi_4) \,,$$
$$Q_5 = (\bar{\psi}_1 P_L \psi_2)(\bar{\psi}_3 P_R \psi_4) \,, \qquad Q_6 = (\bar{\psi}_1 P_L T^A \psi_2)(\bar{\psi}_3 P_R T^A \psi_4) \,,$$
$$Q_7 = (\bar{\psi}_1 P_R \psi_2)(\bar{\psi}_3 P_R \psi_4) \,, \qquad Q_8 = (\bar{\psi}_1 P_R T^A \psi_2)(\bar{\psi}_3 P_R T^A \psi_4) \,,$$
$$Q_9 = (\bar{\psi}_1 \sigma^{\mu\nu} P_R \psi_2)(\bar{\psi}_3 \sigma_{\mu\nu} P_R \psi_4) \,, \qquad Q_{10} = (\bar{\psi}_1 \sigma^{\mu\nu} P_R T^A \psi_2)(\bar{\psi}_3 \sigma_{\mu\nu} P_R T^A \psi_4) \,,$$

for any four fermions $\psi_1, \psi_2, \psi_3, \psi_4$. This includes all two-loop contributions to the current-current ADM: $O(\alpha_s^2)$, $O(\alpha_s \alpha)$ and $O(\alpha^2)$.

This provides the ADM for **all** $\Delta F = 1$ sectors, directly in the JMS basis (except in the four-fermion case, which requires an additional change of basis). This is also true in the case of all $|\Delta L| = 2$ sectors.

Tables of Diagram Poles: Current-Current Contributions

These results can then be extended to other sectors using changes of basis and **flavor symmetrization**!

With this, we complete the derivation **all NLO current-current contributions to the ADM for the complete set of four-fermion operators** in the LEFT.

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With this, we complete the derivation **all NLO current-current contributions to the ADM for the complete set of four-fermion operators** in the LEFT.

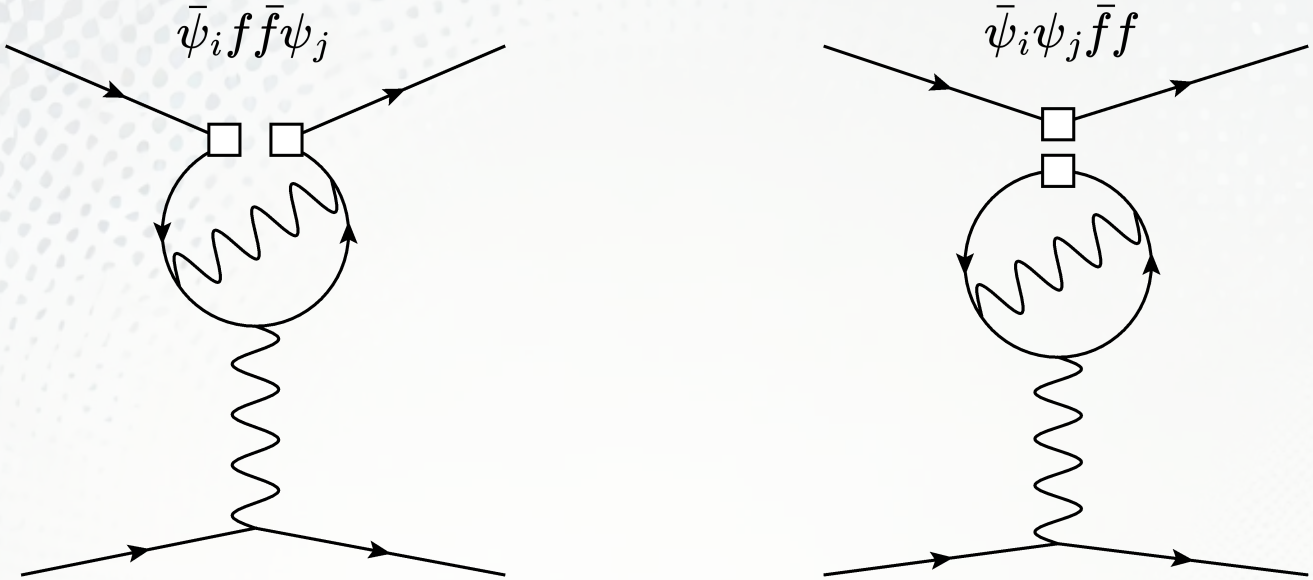
Given that $\hat{\gamma}_{cc}$ does not mix operators with different fermion content, these results are readily usable for either LEFT(n_q).

However, we still need to obtain the penguin contributions, relevant for the $\Delta F = 1^{\bar{f}f}$ and $\Delta F = 0$ sectors...

Given that they mix operators with different fermion content, changing n_q must be handled more carefully in the case of $\hat{\gamma}_p$.

Tables of Diagram Poles: Penguin Contributions

Two types of penguin diagrams contribute to the $\Delta F = 1^{\bar{f}f}$ ADM:



The tables provide the $T^{(m,n;1)}_{D,Q_iQ_j}$ in the LEFT(5) for the open-penguin diagrams $(\bar{\psi}_i f \bar{f} \psi_j)$, with $\psi_{i,j} = d_{i,j}, e_{i,j}$ or $\nu_{i,j}$ (no $u_{i,j}$), for operators:

$$\mathcal{O}_i \sim \bar{\psi}_i f \bar{f} \psi_j \quad \text{mixing onto some} \quad P_i \sim \sum_f \bar{\psi}_i \psi_j \bar{f} f$$

However, the closed-penguins $(\bar{\psi}_i \psi_j \bar{f} f)$ **cannot be calculated in NDR**.

Tables of Diagram Poles: Penguin Contributions

The P_i operators receive all penguin contributions, and thus the ADM for these operators looks roughly like:

$$\hat{\gamma_p}^{(n,m)} \sim \begin{pmatrix} 0 & \hat{O}_P^{(n,m)} \\ 0 & n_f \hat{C}_P^{(n,m)} + 2\hat{O}_P^{(n,m)} \end{pmatrix} ,$$

However, a change of basis like $\mathcal{O}_i \to \widetilde{\mathcal{O}}_i \sim \sum_f \bar{\psi}_i \psi_j \bar{f} f$ is possible, and leads to:

$$\hat{\gamma_p}^{(n,m)} \sim \begin{pmatrix} 0 & \hat{C}_P^{(n,m)} \\ 0 & n_f \hat{C}_P^{(n,m)} + 2\hat{O}_P^{(n,m)} \end{pmatrix} ,$$

which allows for the determination of the $\hat{C}_P^{(n,m)}$.

Tables of Diagram Poles: Penguin Contributions

We can now complete the set of operators into a full basis of the $\Delta F = 1^{\bar{f}f}$ sector in each case ($d_{i,j}$, $e_{i,j}$ and $\nu_{i,j}$), such that the ADM is built out of the $\hat{O}_P^{(n,m)}$ and $\hat{C}_P^{(n,m)}$.

The method thus far is a generalization of the one carried out in [3]. However, the completion of the basis must be done more carefully than they did, as we found recently [4] through a test based on flavor symmetry.

[3] A. J. Buras, M. Misiak and J. Urban, 2000 (hep-ph/0005183 [hep-ph]).

[4] P. Morell, J. Virto, 2024 (2402.00249 [hep-ph]).

Tables of Diagram Poles: Penguin Contributions

We can now complete the set of operators into a full basis of the $\Delta F = 1^{\bar{f}f}$ sector in each case ($d_{i,j}$, $e_{i,j}$ and $\nu_{i,j}$), such that the ADM is built out of the $\hat{O}_P^{(n,m)}$ and $\hat{C}_P^{(n,m)}$.

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After completing the basis, we can perform a change of basis into JMS.

Tables of Diagram Poles: Penguin Contributions

The penguin ADMs obtained thus far are still missing the case $\psi_{i,j} = u_{i,j}$, and they are strictly valid only for the LEFT(5).

We would then need to recompute all the tables once again, but we opt for an alternative procedure.

We identify the following crucial property for an operator basis:

Tables of Diagram Poles: Penguin Contributions

The penguin ADMs obtained thus far are still missing the case $\psi_{i,j} = u_{i,j}$, and they are strictly valid only for the LEFT(5).

We would then need to recompute all the tables once again, but we opt for an alternative procedure.

We identify the following crucial property for an operator basis:

A **flavor-decoupled** basis is ready for any flavor $f \neq \psi_{i,j}$ to be decoupled from the EFT:

- The operators that remain after removing f define a basis in the decoupled EFT.
- The ADM in that basis is identical to the original one, barring the removal of rows/columns.

And analogously, for any extra flavor to be introduced in the theory!

Tables of Diagram Poles: Penguin Contributions

The two-loop ADMs for the u-type $(\bar{u}_i u_j \bar{f} f)$ $\Delta F = 1^{\bar{f}f}$ sector can then be retrieved from the one for the from the d-type $(\bar{d}_i d_j \bar{f} f)$ sectors.

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We then change into a **flavor-decoupled** basis, and perform a series of manipulations in the fermion content until it is functionally equivalent to the JMS u-type basis.

The changes in the $\hat{\gamma}_p^{(m,n)}$ are trivially tracked thanks to the unique properties of the basis!

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The changes in the $\hat{\gamma}_p^{(m,n)}$ are trivially tracked thanks to the unique properties of the basis!

These results are still only valid for the LEFT(5)...
However, in the **flavor-decoupled** bases, going down to the LEFT(4) and the LEFT(3) is trivial!

Tables of Diagram Poles: Penguin Contributions

The $\Delta F = 0$ sector is also affected by penguin diagrams. The operators look like $\mathcal{O}_{\Delta F=0} \sim (\bar{\psi}_1 \Gamma \psi_1)(\bar{\psi}_2 \Gamma \psi_2)$, so now both fields can become part of fermion loops.

The beta function $\dot{L}_{iijj}$ will receive separate penguin contributions proportional to L_{iikk} and L_{kkjj} for any k. Separately, each of these two looks like the penguin contributions to $\Delta F = 1^{\bar{f}f}$.

Therefore we propose to reconstruct the two-loop ADMs for $\Delta F = 0$ starting from $\Delta F = 1^{\bar{f}f}$.

Tables of Diagram Poles: Penguin Contributions

To reconstruct the two-loop ADMs for $\Delta F = 0$ starting from $\Delta F = 1^{\bar{f}f}$:

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1. We start with the flavor-decoupled bases (e-,ν-,d- and u-type).

Tables of Diagram Poles: Penguin Contributions

To reconstruct the two-loop ADMs for $\Delta F = 0$ starting from $\Delta F = 1^{\bar{f}f}$:

1. We start with the flavor-decoupled bases (e-,ν-,d- and u-type).
2. We extend each basis with an extra fermion: e_4, ν_4, d_4 and u_4, respectively. The effect on the penguin ADM is clear.

Tables of Diagram Poles: Penguin Contributions

To reconstruct the two-loop ADMs for $\Delta F = 0$ starting from $\Delta F = 1^{\bar{f}f}$:

1. We start with the flavor-decoupled bases (e-,ν-,d- and u-type).
2. We extend each basis with an extra fermion: e_4, ν_4, d_4 and u_4, respectively. The effect on the penguin ADM is clear.
3. Focusing on $f_4 \to f_i$ transitions (for $f = e, \nu, d, u$), we apply:

$$\left[\dot{L}_{iijj}\right]_p = \left[\dot{L}_{i4jj}\right]_p\Big|_{4\to i} + \left[\dot{L}_{iij4}\right]_p\Big|_{4\to j} ,$$

$$\left[\dot{L}_{ijji}\right]_p = \left[\dot{L}_{ijj4}\right]_p\Big|_{4\to i} + \left[\dot{L}_{i4ji}\right]_p\Big|_{4\to j} ,$$

with some minor corrections to avoid double counting.

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   with some minor corrections to avoid double counting.
4. Finally, a change of basis into JMS is due.

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   with some minor corrections to avoid double counting.
4. Finally, a change of basis into JMS is due.

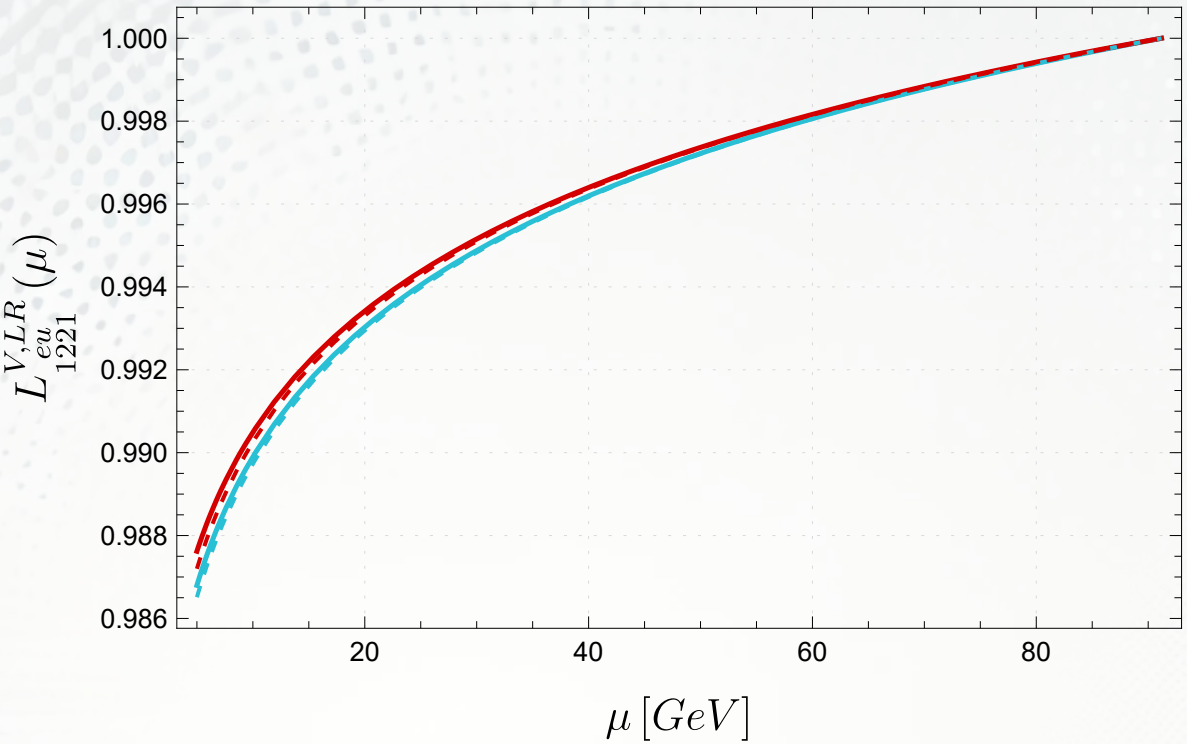
Step 3 provides the penguin ADM in a **flavor-decoupled** basis of $\Delta F = 0$, with which going down to LEFT(4) or LEFT(3) becomes straightforward.

Results

We provide the results for all four-fermion beta functions as an ancillary file: a 2^9-page beta function file Beta-Functions-at-NLO.pdf; and as a Mathematica file implemented in a secondary branch of `DsixTools`, BetaLEFT.m.

Here we present a couple of uneducated examples of the running, with arbitrary matching conditions, just to showcase the potential impact of the RGE to NLL.

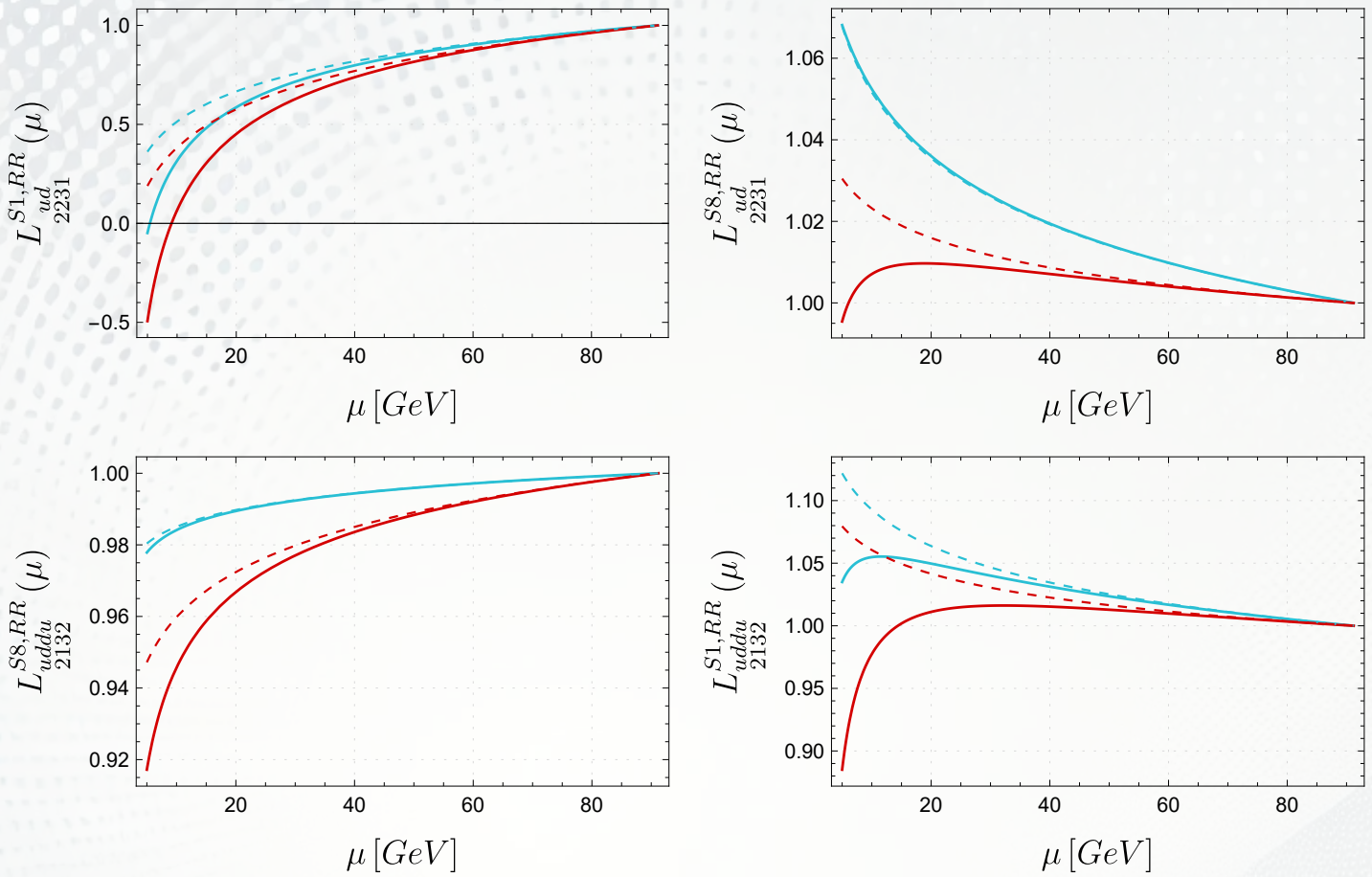
Results



$$L^{V,LR}_{\substack{eu\\1221}}(\Lambda_{EW}) = 5.80 \times 10^{-8}\ \text{GeV}^{-2}.$$

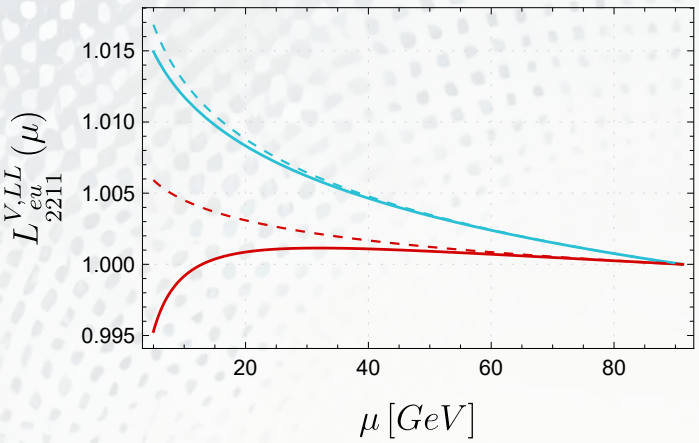
[One-Loop] [Two-Loop] [Solid: RGE] [Dashed: Leading Logs]

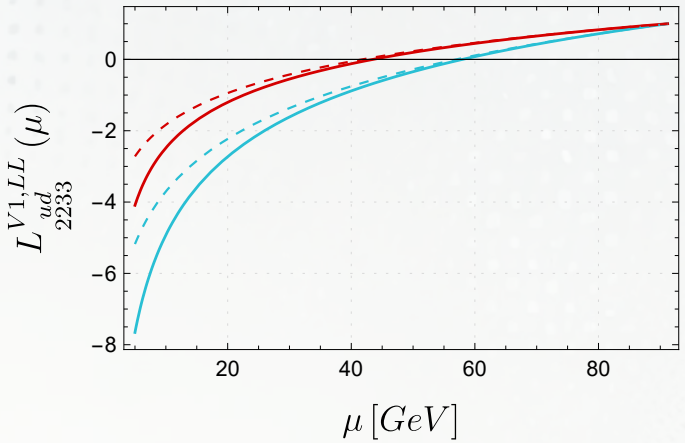
Results

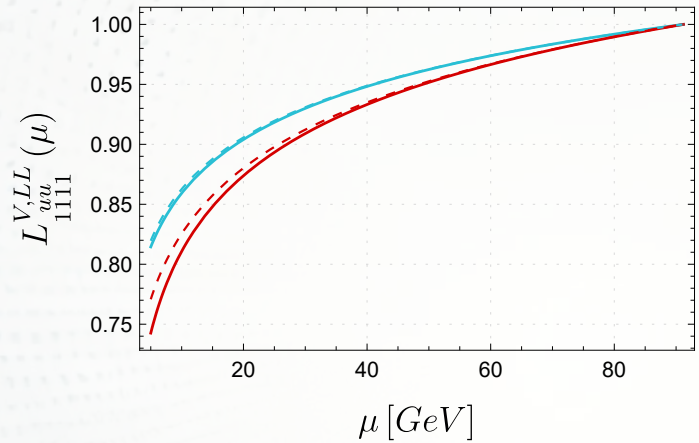


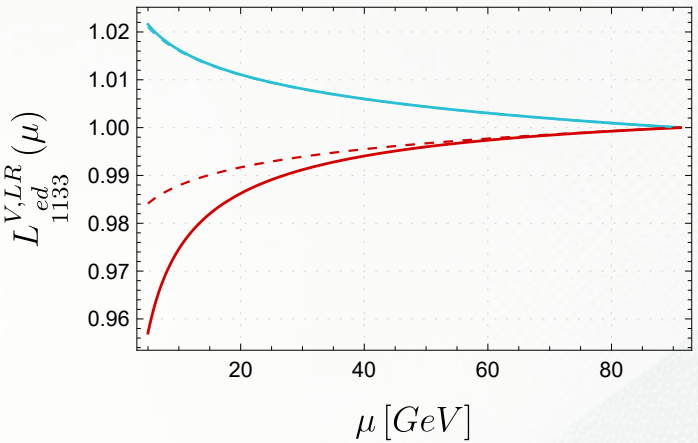
$$L^{S1,RR}_{ud\,ud\,2231}(\Lambda_{EW}) = -5.50 \times 10^{-7}\ \text{GeV}^{-2},\ L^{S1,RR}_{ud\,ud\,2231}(\Lambda_{EW}) = 7.60 \times 10^{-6}\ \text{GeV}^{-2},$$
$$L^{S1,RR}_{ud\,ud\,2231}(\Lambda_{EW}) = -9.90 \times 10^{-6}\ \text{GeV}^{-2},\ L^{S1,RR}_{ud\,ud\,2231}(\Lambda_{EW}) = 8.20 \times 10^{-7}\ \text{GeV}^{-2}.$$

Results









$$L^{V,LL}_{eu\,2211}(\Lambda_{EW}) = -7.60 \times 10^{-6}\ \text{GeV}^{-2},\ L^{V1,LL}_{ud\,2233}(\Lambda_{EW}) = 4.10 \times 10^{-8}\ \text{GeV}^{-2},$$

$$L^{V,LL}_{uu\,1111}(\Lambda_{EW}) = -2.50 \times 10^{-6}\ \text{GeV}^{-2},\ L^{V,LR}_{ed\,1133}(\Lambda_{EW}) = 1.60 \times 10^{-6}\ \text{GeV}^{-2}.$$

Thank You!

BACKUP SLIDES

Renormalization Group Evolution: Expanded Formulas

The expressions for the different LO and NLO contributions to the ADM in terms of the matrix elements $\hat{a}^{(n,m;\ell)}$ read:

$$\gamma_{ij}^{(1,0)} = -2\hat{a}_{Q_iQ_j}^{(1,0;1)} - 2Z_{Q_i}^{(1,0;1)}\delta_{Q_iQ_j} \ ,$$

$$\gamma_{ij}^{(0,1)} = -2\hat{a}_{Q_iQ_j}^{(0,1;1)} - 2Z_{Q_i}^{(0,1;1)}\delta_{Q_iQ_j} \ ,$$

$$\gamma_{ij}^{(2,0)} = -4\hat{a}_{Q_iQ_j}^{(2,0;1)} + 4\hat{a}_{Q_iQ_k}^{(1,0;1)}\hat{a}_{Q_kQ_j}^{(1,0;0)} + 2\hat{a}_{Q_iE_k}^{(1,0;1)}\hat{a}_{E_kQ_j}^{(1,0;0)} - 4Z_{Q_i}^{(2,0;1)}\delta_{Q_iQ_j} \ ,$$

$$\gamma_{ij}^{(0,2)} = -4\hat{a}_{Q_iQ_j}^{(0,2;1)} + 4\hat{a}_{Q_iQ_k}^{(0,1;1)}\hat{a}_{Q_kQ_j}^{(0,1;0)} + 2\hat{a}_{Q_iE_k}^{(0,1;1)}\hat{a}_{E_kQ_j}^{(0,1;0)} - 4Z_{Q_i}^{(0,2;1)}\delta_{Q_iQ_j} \ ,$$

$$\begin{aligned}\gamma_{ij}^{(1,1)} = &-4\hat{a}_{Q_iQ_j}^{(1,1;1)} + 4\hat{a}_{Q_iQ_k}^{(1,0;1)}\hat{a}_{Q_kQ_j}^{(0,1;0)} + 4\hat{a}_{Q_iQ_k}^{(0,1;1)}\hat{a}_{Q_kQ_j}^{(1,0;0)} + 2\hat{a}_{Q_iE_k}^{(1,0;1)}\hat{a}_{E_kQ_j}^{(0,1;0)} \\ &+ 2\hat{a}_{Q_iE_k}^{(0,1;1)}\hat{a}_{E_kQ_j}^{(1,0;0)} - 4Z_{Q_i}^{(1,1;1)}\delta_{Q_iQ_j} \ .\end{aligned}$$

The $Z_{Q_i}^{(n,m;1)}$ are the WFR factors due to the fields contained in Q_i.

Change of Basis: Expanded Formulas

Two-loop change of basis:

$$\begin{aligned}
\hat{\gamma}'^{(2,0)} =& \hat{R}\hat{\gamma}^{(2,0)}\hat{R}^{-1} + 4Z_{g_s}^{(1,0;1)}\Delta\hat{r}^{(1,0)} - \left[\Delta\hat{r}^{(1,0)}, \hat{\gamma}^{(1,0)}\right] , \\
\hat{\gamma}'^{(0,2)} =& \hat{R}\hat{\gamma}^{(0,2)}\hat{R}^{-1} + 4Z_{e}^{(0,1;1)}\Delta\hat{r}^{(0,1)} - \left[\Delta\hat{r}^{(0,1)}, \hat{\gamma}^{(0,1)}\right] , \\
\hat{\gamma}'^{(1,1)} =& \hat{R}\hat{\gamma}^{(1,1)}\hat{R}^{-1} - \left[\Delta\hat{r}^{(1,0)}, \hat{\gamma}^{(0,1)}\right] - \left[\Delta\hat{r}^{(0,1)}, \hat{\gamma}^{(1,0)}\right] ,
\end{aligned}$$

where

$$Z_{g_s}^{(1,0;1)} = \frac{1}{3}n_q - \frac{11}{6}N_c , \quad Z_{e}^{(0,1;1)} = \frac{2}{3}\left(Q_u^2 n_u N_c + Q_d^2 n_d N_c + Q_e^2 n_e\right) ,$$

are the one-loop renormalization constants of g_s and e, respectively,

$$\frac{d\tilde{\alpha}_s}{d\log\mu} = 4\tilde{\alpha}_s^2 Z_{g_s}^{(1,0;1)} + \cdots , \quad \frac{d\tilde{\alpha}}{d\log\mu} = 4\tilde{\alpha}^2 Z_{e}^{(0,1;1)} + \cdots .$$

The one-loop matrices $\Delta\hat{r}$ contain the shift in the renormalization scheme

$$\begin{aligned}
\Delta r_{ij}^{(1,0)} \equiv& - R_{Q_i'Q_k} W_{Q_k E_l}\, a_{E_l Q_r}^{(1,0;0)} R^{-1}_{Q_r Q_j'} = R_{Q_i'Q_k}\, a_{Q_k Q_l}^{(1,0;0)} R^{-1}_{Q_l Q_j'} - a_{Q_i' Q_j'}^{(1,0;0)} , \\
\Delta r_{ij}^{(0,1)} \equiv& - R_{Q_i'Q_k} W_{Q_k E_l}\, a_{E_l Q_r}^{(0,1;0)} R^{-1}_{Q_r Q_j'} = R_{Q_i'Q_k}\, a_{Q_k Q_l}^{(0,1;0)} R^{-1}_{Q_l Q_j'} - a_{Q_i' Q_j'}^{(0,1;0)} .
\end{aligned}$$

Tables of Diagram Poles: General Formalism

For that, we separate the contributions to the two-loop ADM in **current-current** and **penguin**, depending on the diagrams involved:

$$\hat{\gamma}^{(m,n)} = \hat{\gamma}_{cc}^{(m,n)} + \hat{\gamma}_{p}^{(m,n)} \ ,$$

where $\hat{\gamma}_{cc}^{(m,n)}$ includes contributions from current-current diagrams alone*, $\hat{a}^{(m,n;\ell)}$, while $\hat{\gamma}_{p}^{(m,n)}$ contains the rest.

The two contributions transform differently under a change of basis:

$$\begin{aligned}
\hat{\gamma}_{cc}^{\prime(2,0)} =& \hat{R}\hat{\gamma}_{cc}^{(2,0)}\hat{R}^{-1} + 4Z_g^{(1,0;1)}\Delta\hat{r}_{cc}^{(1,0)} - \left[\Delta\hat{r}_{cc}^{(1,0)}, \hat{\gamma}_{cc}^{(1,0)}\right] \ , \\
\hat{\gamma}_{p}^{\prime(2,0)} =& \hat{R}\hat{\gamma}_{p}^{(2,0)}\hat{R}^{-1} + 4Z_g^{(1,0;1)}\Delta\hat{r}_{p}^{(1,0)} - \left[\Delta\hat{r}_{p}^{(1,0)}, \hat{\gamma}^{(1,0)}\right] \ .
\end{aligned}$$

In the second eq. we ignore a (vanishing) term: $-\left[\Delta\hat{r}_{cc}^{(1,0)}, \hat{\gamma}_{p}^{(1,0)}\right]$.

*And also the contribution from the wave-function renormalization factor.

Tables of Diagram Poles: Penguin Contributions

We start by defining some operator sets closed under renormalization, describing a $\Delta F = 1$ $\psi_2 \to \psi_1$ transition:

$$\mathcal{O}_1^{(\psi)} \sim (\bar{\psi}_1 \Gamma \ell_i)(\bar{\ell}_i \Gamma \psi_2) \qquad \mathcal{O}_2^{(\psi)} \sim (\bar{\psi}_1 \Gamma q_i)(\bar{q}_i \Gamma \psi_2) \,,$$
$$\mathcal{O}_3^{(\psi)} \sim (\bar{\psi}_1 \Gamma \psi_2) \sum_k (\bar{e}_k \Gamma e_k) \qquad \mathcal{O}_4^{(\psi)} \sim (\bar{\psi}_1 \Gamma \psi_2) \sum_k (\bar{\nu}_k \Gamma \nu_k) \,,$$
$$\mathcal{O}_5^{(\psi)} \sim (\bar{\psi}_1 \Gamma \psi_2) \sum_k (\bar{u}_k \Gamma u_k) \qquad \mathcal{O}_6^{(\psi)} \sim (\bar{\psi}_1 \Gamma \psi_2) \sum_k (\bar{u}_k \Gamma u_k) \,,$$

with $\psi_{1,2} = d_{i,j}, e_{i,j}$ or $\nu_{i,j}$ (no $u_{i,j}$). Their penguin ADM looks like

$$\hat{\gamma}_p^{(m,n)} = \begin{pmatrix} 0_{2\times 2} & \hat{O}_{P_e}^{(n,m)} & \hat{O}_{P_\nu}^{(n,m)} & \hat{O}_{P_u}^{(n,m)} & \hat{O}_{P_d}^{(n,m)} \\ 0_{4\times 2} & \hat{X}_e^{(n,m)} & \hat{X}_\nu^{(n,m)} & \hat{X}_u^{(n,m)} & \hat{X}_d^{(n,m)} \end{pmatrix} \,,$$

with the $\hat{O}_{P_{e,\nu,u,d}}^{(n,m)}$ being provided by the tables for penguin diagrams.

Each $\hat{X}_f^{(n,m)}$ is a combination of the $\hat{O}_{P_{e,\nu,u,d}}^{(n,m)}$ and some $\hat{C}_{P_{e,\nu,u,d}}^{(n,m)}$.

Tables of Diagram Poles: Penguin Contributions

Changing basis into $\widetilde{\mathcal{O}}_{3-6}^{(\psi)} = \mathcal{O}_{3-6}^{(\psi)}$ and

$$\widetilde{\mathcal{O}}_1^{(\psi)} \sim (\psi_1 \Gamma \psi_2)(\bar{\ell}_i \Gamma \ell_i) \quad \widetilde{\mathcal{O}}_2^{(\psi)} \sim (\psi_1 \Gamma \psi_2)(\bar{q}_i \Gamma q_i) \,,$$

their penguin ADM becomes

$$\hat{\gamma}_p^{(m,n)} = \begin{pmatrix} 0_{2\times 2} & \hat{C}_{P_e}^{(n,m)} & \hat{C}_{P_\nu}^{(n,m)} & \hat{C}_{P_u}^{(n,m)} & \hat{C}_{P_d}^{(n,m)} \\ 0_{4\times 2} & \hat{X}_e^{(n,m)} & \hat{X}_\nu^{(n,m)} & \hat{X}_u^{(n,m)} & \hat{X}_d^{(n,m)} \end{pmatrix} \,,$$

which fully determines the $\hat{C}_{P_{e,\nu,u,d}}^{(n,m)}$ and $\hat{X}_{e,\nu,u,d}^{(n,m)}$.

These operators sets can then be completed into a full basis for the relevant block in the $\Delta F = 1^{\bar{f}f}$ sector, such that their ADM can still be written in terms of the $\hat{O}_{P_{e,\nu,u,d}}^{(n,m)}$ and $\hat{C}_{P_{e,\nu,u,d}}^{(n,m)}$.

The procedure thus far is a generalization of the work done in [1-3].

Tables of Diagram Poles: Completion of the Penguin Bases

The completion of the penguin bases can be carried out quite straightforwardly if one simply adds a copy of $\widetilde{\mathcal{O}}_1^{(\psi)}$ and $\widetilde{\mathcal{O}}_2^{(\psi)}$ for each possible ℓ_i and q_i.

Their contributions to $\hat{\gamma}^{(n,m)}$ are perfectly described by the $\hat{C}_P^{(n,m)}$.

However, one must be careful with the cases $\psi_{1,2} = \ell_i$ and $\psi_{1,2} = q_i$, which also get contributions from the $\hat{O}_P^{(n,m)}$.

Furthermore, the case $\psi_{1,2} = q_i$ needs extra care [4]...

The operators added in this case must contain **all color possibilities**!

[4] P. Morell and J. Virto, 2024 (2402.00249 [hep-ph]).

The operators should be something like:

$$\frac{1}{2}\Big[(\bar{d}_i^\alpha\gamma^\mu P_L d_j^\alpha)(\bar{d}_j^\beta\gamma_\mu P_L d_j^\beta) + (\bar{d}_i^\alpha\gamma^\mu P_L d_j^\beta)(\bar{d}_j^\beta\gamma_\mu P_L d_j^\alpha) + (\bar{d}_i^\alpha\gamma^\mu P_L d_j^\alpha)(\bar{d}_i^\beta\gamma_\mu P_L d_i^\beta) + (\bar{d}_i^\alpha\gamma^\mu P_L d_j^\beta)(\bar{d}_i^\beta\gamma_\mu P_L d_i^\alpha)\Big] .$$

Surface-Level Flavor Symmetry

- A basis $\{Q_i, E_i\}$ is said to be *flavor-symmetric in flavors* f, f' if

$$\{Q_i, E_i\}\big|_{f\leftrightarrow f'} = \{Q_i, E_i\} \quad \text{(up to reordering)}.$$

Note that this can only be true if f, f' are inner flavors. Then:

$$\left.\begin{aligned}\gamma_{ik}^{(n,m)} &= \gamma_{jk}^{(n,m)}\\ \gamma_{ki}^{(n,m)} &= \gamma_{kj}^{(n,m)}\end{aligned}\right. \quad \forall Q_k \quad \text{if} \quad \begin{cases} Q_i \sim (\bar{\psi}_1\Gamma\psi_2)(\bar{f}\Gamma f) & \text{with } f \neq \psi_{1,2}\\ Q_j \sim (\bar{\psi}_1\Gamma\psi_2)(\bar{f}'\Gamma f') & \text{with } f' \neq \psi_{1,2}\end{cases},$$

- A basis is said to be *flavor-decoupled* if, when decoupling any inner flavor f from the EFT, the set of operators obtained by removing from this basis any terms containing the flavor f defines a basis for the sector of the new EFT, whose penguin ADM is given by the old one with the appropriate rows and columns removed.

Surface-Level Flavor Symmetry

For an operator basis to be flavor decoupled it is sufficient that two conditions are simultaneously satisfied:

1. Each operator contains no more than one inner flavor. This guarantees that all operators that are not removed completely in the procedure are left unaltered.
2. When decomposing the penguin ADM into $\hat{a}$ matrices, each entry $(\hat{\gamma}_p)_{ij}$ depends only on flavors common to Q_i and Q_j, whereas any contribution involving other flavors vanishes. This ensures that the ADM entries for all operators not eliminated by the decoupling (i.e. not featuring the decoupled field) retain exactly the same expression. At NLO this second condition reads:

$$\left[-4\,\hat{p}^{(2,0;1)}_{Q_iQ_j} + 4\,\hat{p}^{(1,0;1)}_{Q_iQ_k}\hat{p}^{(1,0;0)}_{Q_kQ_j} + 2\,\hat{p}^{(1,0;1)}_{Q_iE_k}\hat{p}^{(1,0;0)}_{E_kQ_j}\right]_{\text{featuring fields } \notin Q_i,Q_j} = 0\,,$$

where we used $\hat{p} \equiv \hat{a}_p$ for notational simplicity.

Tables of Diagram Poles: Penguin Contributions

The two-loop ADMs for the u-type $(\bar{u}_i u_j \bar{f} f)$ $\Delta F = 1^{\bar{f}f}$ sector can then be retrieved from the one for the from the d-type $(\bar{d}_i d_j \bar{f} f)$ sectors:

1. We start with the penguin ADM for the d-type sector.
2. We change into a **flavor-decoupled** basis.
3. We remove the third d_k flavor and extend the basis with an extra u_3, with the corresponding effects on the ADM being trivial.
4. This modified basis is functionally equivalent to a u-type symmetric basis up a relabeling of field names, $u \leftrightarrow d$.
5. Swapping $Q_u \leftrightarrow Q_d$ in the ADM already provides the u-type ADM in the JMS basis.

This algorithm provides the correct penguin ADM for the u-type sectors in all three cases: $O(\alpha_s^2), O(\alpha_s \alpha)$ and $O(\alpha^2)$.

Tables of Diagram Poles: Penguin Contributions

This flowchart represents the complete method we have followed to obtain all two-loop penguin contributions to the ADM in the JMS basis of the LEFT.

The dashed arrows represent the steps that explicitly require the basis to be **flavor-symmetric** and **flavor-decoupled**.