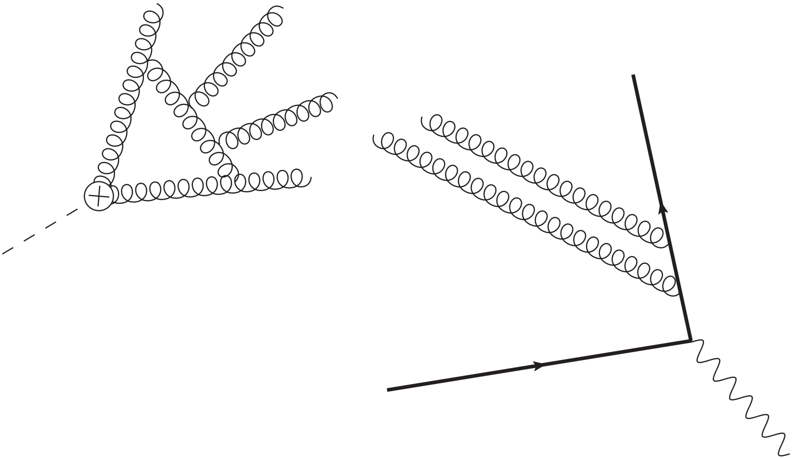
Multi-Loop Squared Matrix Elements Of Massless Soft Wilson Lines, Massive Soft Wilson Lines, And A New Relation Between Them

Robert M. Schabinger

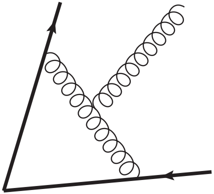
based on Phys. Rev. D**90**, 053006 (2014) with Ye Li, Andreas von Manteuffel, and Hua Xing Zhu
and arXiv:1408.5134 with Andreas von Manteuffel and Hua Xing Zhu

The PRISMA Cluster of Excellence and Institute of Physics

The Current State-Of-The-Art For Soft Functions



Soft Wilson Lines Are Unexpectedly Simple



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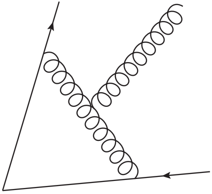
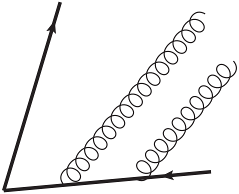
$\ln(x) \to \frac{1}{2\epsilon}$





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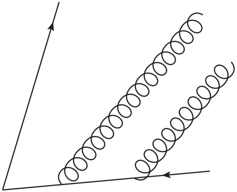


Outline

The Threshold Factorization Formula For The Partonic Cross Section For Higgs Boson Production @ LHC

Collins, Soper, and Sterman, Nucl. Phys. B**261**, 104 (1985)

Matsuura, van der Marck, and van Neerven, Nucl. Phys. B**319**, 570 (1989)

The ratio $z = M_H^2/\hat{s}$ is a scale in the partonic cross section which is then convolved with the proton PDFs to obtain a prediction for the total production cross section. The result is dominated by the contribution from the singular region $z \to 1$:

$$\hat{\sigma}_{gg}^{\mathrm{H}}(z) = \sigma_0^{\mathrm{H}}\, H\Sigma(1-z) + \mathcal{O}\left(1-z\right)$$

In this limit, we say that the partonic cross section *factorizes* into a product of a *hard function*, H, and a *soft function*, $\Sigma(1-z)$.

The Threshold Factorization Formula For The Heavy Quark Forward-Backward Asymmetry @ ILC

Eichten and Hill, Phys. Lett. B**234**, 511 (1990); Grinstein, Nucl. Phys. B**339**, 253 (1990);

Isgur and Wise, Phys. Lett. B**237**, 527 (1990); Georgi, Phys. Lett. B**240**, 447 (1990)

In the threshold region where the energy of the QCD radiation off of the top quarks is small, heavy quark effective theory (HQET) implies that $e^+e^- \to t\bar{t}$ differential distributions factorize, *e.g.*

$$\frac{d\sigma^{t\bar{t}}}{d\cos\theta} = \frac{d\sigma_0^{t\bar{t}}}{d\cos\theta} H^{t\bar{t}}\left(x, \ln\left(\frac{m_t}{\mu}\right)\right) \Sigma^{t\bar{t}}\left(x, \ln\left(\frac{2E_{cut}}{\mu}\right)\right) + \mathcal{O}\left(E_{cut}/m_t\right)$$

In the above, $$x = \frac{1-\sqrt{1-\frac{4m_t^2}{s}}}{1+\sqrt{1-\frac{4m_t^2}{s}}}$$

The One-Loop, Two-Emission Part Of The Three-Loop Soft Function For Higgs Production

$$\Sigma\left(\ln\left(\frac{2E_{cut}}{\mu}\right)\right) = \int_0^{E_{cut}} d\lambda\, S(\lambda,\mu)$$

$$S(\lambda,\mu) = \frac{1}{N_c}\sum_{X_s}\delta\Big(\lambda - E_{X_s}\Big)\langle 0|Y_n Y_{\bar{n}}|X_s\rangle\langle X_s|Y_{\bar{n}}^\dagger Y_n^\dagger|0\rangle$$

$$n^2 = \bar{n}^2 = 0 \qquad n\cdot\bar{n} = 2$$

This computation was carried out by a different group as well but never separately published. Rather, they used the result implicitly together with other known results to derive the full N^{3}LO correction to the partonic Higgs boson production cross section at LHC.

Anastasiou *et. al.*, Phys. Lett. B**737**, 325 (2014)

The Complete Two-Loop $t\bar{t}$ Soft Function

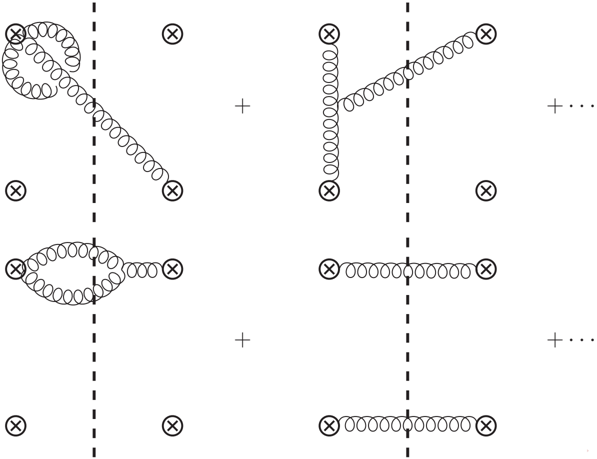
$$\Sigma^{t\bar{t}}\left(x, \ln\left(\frac{2E_{cut}}{\mu}\right)\right) = \int_0^{E_{cut}} d\lambda\, S^{t\bar{t}}(x, \lambda, \mu)$$

$$S^{t\bar{t}}(x, \lambda, \mu) = \frac{1}{N_c}\sum_{X_s} \delta\left(\lambda - E_{X_s}\right)\langle 0|Y_n Y_{\bar{n}}|X_s\rangle\langle X_s|Y_{\bar{n}}^\dagger Y_n^\dagger|0\rangle$$

$$n^2 = \bar{n}^2 = \frac{4m_t^2}{s} \qquad n\cdot\bar{n} = 2 - \frac{4m_t^2}{s}$$

Note that the hard function is known to two-loop order (Bernreuther *et. al.*, Nucl. Phys. B**706**, 245 (2005), Nucl. Phys. B**712**, 229 (2005), and Nucl. Phys. B**723**, 91 (2005); Gluza *et. al.* JHEP **0907**, 001 (2009)) and a two-loop, fully differential, full QCD program is now available. (see Gao and Zhu, arXiv:1410.3165)

(Carefully) Evaluate The Appropriate Squared Sum of Cut Eikonal Feynman Diagrams



Integration By Parts Reduction

Tkachov, Phys. Lett. B**100**, 65 (1981); Chetyrkin and Tkachov, Nucl. Phys. B**192**, 159 (1981)

It is well-known that one can generate recurrence relations by considering families of Feynman integrals and then integrating by parts in d spacetime dimensions, *e.g.*

$$\begin{aligned} 0 &= \int \frac{d^d\ell}{(2\pi)^d} \frac{\partial}{\partial \ell_\mu}\left(\frac{\ell_\mu}{(\ell^2 - m^2)^a}\right) \\ &= \int \frac{d^d\ell}{(2\pi)^d}\left(\frac{d}{(\ell^2 - m^2)^a} - \frac{2a\ell^2}{(\ell^2 - m^2)^{a+1}}\right) \\ &= (d-2a)I(a) - 2am^2 I(a+1) \end{aligned}$$

In this case, the recurrence relation can be solved explicitly but it is one of the few known examples where one can proceed directly.

Apply the Reduze 2 Integration By Parts Identity Solver To Reduce The Integrand

- In all but the simplest examples, the strategy used (Laporta, Int. J. Mod. Phys. A**15**, 5087 (2000)) to solve integration by parts identities is to build a linear system of equations for the Feynman integrals in the calculation by explicitly substituting particular values of the indices into the recurrence relations.
- The Reduze 2 (von Manteuffel and Studerus, arXiv:1201.4330) implementation of Laporta's algorithm is robust and well-tested.
- However, the public version of the code was written with virtual corrections in mind and does not support phase space integrals such as those which arise in the calculation under discussion.

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The functionality of the code is straightforward to appropriately extend and we find that there are just 14 master integrals which need to be calculated!

The Method Of Differential Equations

- The method of differential equations for multi-loop Feynman integrals (Remiddi, Nuovo Cim. A**110**, 1435 (1997); Gehrmann and Remiddi, Comput. Phys. Commun. **141**, 296 (2001)) involves first deriving a system of first-order differential equations by differentiating the integrals of interest with respect to the available parameters (in this case, x) and then using integration by parts identities to rewrite the derivatives obtained in terms of master integrals.
- The system of differential equations obtained can be solved order-by-order in ϵ up to constants. In practice, a large percentage of the master integrals are actually completely determined in this approach because many of the integration constants are completely determined by the physics.
- Unfortunately, the method is cumbersome to apply because an order-by-order solution is complicated by the fact that the systems obtained are typically coupled in a non-trivial way.

Normal Form Bases For Systems Of PDEs

- Recently, Henn suggested a novel approach to the decoupling of first-order systems of differential equations for Feynman integrals (Henn, Phys. Rev. Lett. **110**, 251601 (2013)).
- When the method applies, it provides a clean prescription for the computation which is transparent and in many cases usable even by non-experts to obtain results to arbitrarily high orders in ϵ.
- Proceed by finding a basis of integrals $\mathbf{f}(\epsilon, x) = \{f_1(\epsilon, x), \ldots, f_7(\epsilon, x)\}$ with ϵ expansions of the form $f_i(\epsilon, x) = \sum_{n=0}^{\infty} c_i^{(n)}(x)\epsilon^n$ such that:

$$\mathbf{I}(\epsilon, x) = \underset{\sim}{\mathbf{B}}(\epsilon, x)\mathbf{f}(\epsilon, x) \qquad \Longrightarrow$$

$$\frac{\partial}{\partial x}\mathbf{I}(\epsilon, x) = \underset{\sim}{\mathbf{S}}(\epsilon, x)\mathbf{I}(\epsilon, x) \longrightarrow \frac{\partial}{\partial x}\mathbf{f}(\epsilon, x) = \epsilon\underset{\sim}{\mathbf{A}}(x)\mathbf{f}(\epsilon, x)$$

What Is Special About Such A Normal Form?

One obtains PDEs (here ODEs) such that the functional form of the term of $\mathcal{O}\left(\epsilon^{n+1}\right)$ is completely determined by the term of $\mathcal{O}\left(\epsilon^{n}\right)$:

$$\frac{\partial}{\partial x}\mathbf{c}^{(n+1)}(x) = \underset{\sim}{\mathbf{A}}(x)\mathbf{c}^{(n)}(x)$$

Here, the $\underset{\sim}{\mathbf{A}}_{ij}(x)$ are rational linear combinations of $\frac{1}{x}$, $\frac{1}{1-x}$, and $\frac{1}{1+x}$.

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For the problem at hand, the generation of solutions to arbitrarily high orders in the ϵ expansion becomes an almost trivial exercise once the integration constants are fixed. Among other things, this requires explicit integrations of some of the simpler master integrals.

A Typical Input Integral

The only real-real input integral is the x-independent phase space volume. However, there are four real-virtual input integrals, *e.g.*

$$-i\pi^{2\epsilon-3}\int \mathrm{d}^d k \int \mathrm{d}^d q \frac{\delta\left(1-k\cdot(n+\bar{n})\right)\delta\left(k^2\right)}{k\cdot n\,(q\cdot\bar{n}+i0)\,((k-q)\cdot n+i0)\,((k-q)^2+i0)} =$$

$$e^{2\pi i\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2\epsilon)}{2\Gamma(2-2\epsilon)\Gamma(1+\epsilon)}F_1\left(1-\epsilon;2\epsilon,1;2-2\epsilon;1-x,\frac{x-1}{x}\right)$$

$$\times\Big(2x^{-1-\epsilon}(1-x)^{-1+2\epsilon}(1+x)^{2+2\epsilon}\Gamma(-2\epsilon)\Gamma^2(1+\epsilon)$$

$$-x^{-1+\epsilon}(1+x)^3\Gamma(1-\epsilon)\Gamma(\epsilon)\,{}_2F_1\left(1,1-\epsilon;1+\epsilon;x^2\right)\Big)+e^{i\pi\epsilon}\cos(\pi\epsilon)$$

$$\times\frac{2\Gamma(1-\epsilon)\Gamma(-2\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(2-2\epsilon)}x^{-1-\epsilon}(1+x)^{2+2\epsilon}F_1\left(1-\epsilon;2\epsilon,1;2-2\epsilon;1-x,\frac{x-1}{x}\right)$$

$$\times\Big((1-2\epsilon)\,{}_2F_1\left(2-2\epsilon,-\epsilon;1-\epsilon;x\right)-(1-\epsilon)\,{}_2F_1\left(1-2\epsilon,-\epsilon;1-\epsilon;x\right)\Big)$$

How Do We Know Our Result Is Correct?

- All pole terms in our expression for the two-loop bare soft function coincide with the prediction furnished by renormalization group invariance.
- The threshold limit of the $\mathcal{O}\left(\alpha_s^2\right)$ result $(x \to 1)$ is zero.
- The finite part of the C_F^2 color structure is correctly predicted by the non-Abelian exponentiation theorem.
- We were able to make an explicit comparison to the recent calculation of the single-soft emission contributions by Bierenbaum, Czakon, and Mitov (Nucl. Phys. B**856**, 228 (2012)) and our real-virtual terms are completely consistent with their results.
- Finally, we discovered a direct connection between our bare result in the small x limit and the bare $q\bar{q}$ soft function which actually checks the terms in the expression which are most challenging to correctly compute (more on this below).

The Small x Asymptotics Of The Bare Soft Function

$$
\begin{aligned}
S^{t\bar{t}}_{\text{bare}}(x \to 0, \lambda, \mu) &= \delta(\lambda) + \left(\frac{\alpha_s}{4\pi}\right) \frac{\mu^{2\epsilon}}{\lambda^{1+2\epsilon}} \Bigg[-8 - 8\ln(x) + \epsilon\left(\frac{8\pi^2}{3} + 8\ln(x)\right. \\
&\left. + 4\ln^2(x)\right) + \epsilon^2\left(-\frac{2\pi^2}{3} + 16\zeta(3) - \frac{2\pi^2}{3}\ln(x) - 4\ln^2(x) - \frac{4}{3}\ln^3(x)\right) + \mathcal{O}\left(\epsilon^3\right) \Bigg] C_F \\
&+ \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{\mu^{4\epsilon}}{\lambda^{1+4\epsilon}} \Bigg\{ \Bigg[\frac{1}{\epsilon}\left(\frac{32}{3} + \frac{32}{3}\ln(x)\right) + \frac{160}{9} - \frac{64\pi^2}{9} - \frac{32}{9}\ln(x) - \frac{32}{3}\ln^2(x) \\
&+ \epsilon\left(\frac{896}{27} - \frac{272\pi^2}{27} - \frac{256\zeta(3)}{3} + \left(-\frac{64}{27} + \frac{16\pi^2}{9}\right)\ln(x) + \frac{32}{9}\ln^2(x) + \frac{64}{9}\ln^3(x)\right) \\
&+ \epsilon^2\left(\frac{5248}{81} - \frac{1552\pi^2}{81} - 192\zeta(3) - \frac{32\pi^4}{135} + \left(-\frac{128}{81} - \frac{16\pi^2}{27} - \frac{448\zeta(3)}{9}\right)\ln(x)\right. \\
&\left. + \left(\frac{64}{27} - \frac{16\pi^2}{9}\right)\ln^2(x) - \frac{64}{27}\ln^3(x) - \frac{32}{9}\ln^4(x)\right) + \mathcal{O}\left(\epsilon^3\right) \Bigg] C_F n_f T_F + \cdots \Bigg\} + \cdots
\end{aligned}
$$

Magic Connection To The Bare $q\bar{q}$ Soft Function

Surprisingly, we find that one can produce the bare $q\bar{q}$ soft function for all non-trivial color structures to all orders in ϵ by making simple replacements!

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For example, if we take $\ln^n(x) \to 0$ for all $n > 1$ and $\ln(x) \to \frac{1}{2\epsilon}$ in $S^{t\bar{t}}_{\text{bare}}(x \to 0, \lambda, \mu)\Big|_{C_F n_f T_F}$, we reproduce

Belitsky, Phys. Lett. B**442**, 307 (1998)

$$S^{q\bar{q}}_{\text{bare}}(\lambda, \mu)\Big|_{C_F n_f T_F} = \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{\mu^{4\epsilon}}{\lambda^{1+4\epsilon}} \Bigg[\frac{16}{3\epsilon^2} + \frac{80}{9\epsilon} + \frac{448}{27} - \frac{56\pi^2}{9} + \epsilon\left(\frac{2624}{81} - \frac{280\pi^2}{27} - \frac{992\zeta(3)}{9}\right) + \mathcal{O}\left(\epsilon^2\right)\Bigg]$$

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We conjecture that, for all non-trivial color structures, we can obtain the massless result via $\ln^n(x) \to 0$ for all $n > 1$ and $\ln(x) \to \frac{1}{L\epsilon}$ at L loop order by expanding to one order higher in ϵ than normal.

Outlook

As usual, there is much more work to do:

- Test our conjecture at higher loops in the back-to-back case and consider it for more general Wilson line configurations.
- Understand better the connection between our mysterious relation and well-known relations between massive and massless soft functions, *e.g.* Fleming *et. al.* Phys. Rev. D**77**, 074010 (2008); Ferroglia *et. al.* Phys. Rev. D**86**, 034010 (2012).
- Make the automated code developed by our colleagues for $e^+e^- \to t\bar{t}$ observables accessible (usable?) to experimentalists.
- Extend the functionality of Reduze further still.