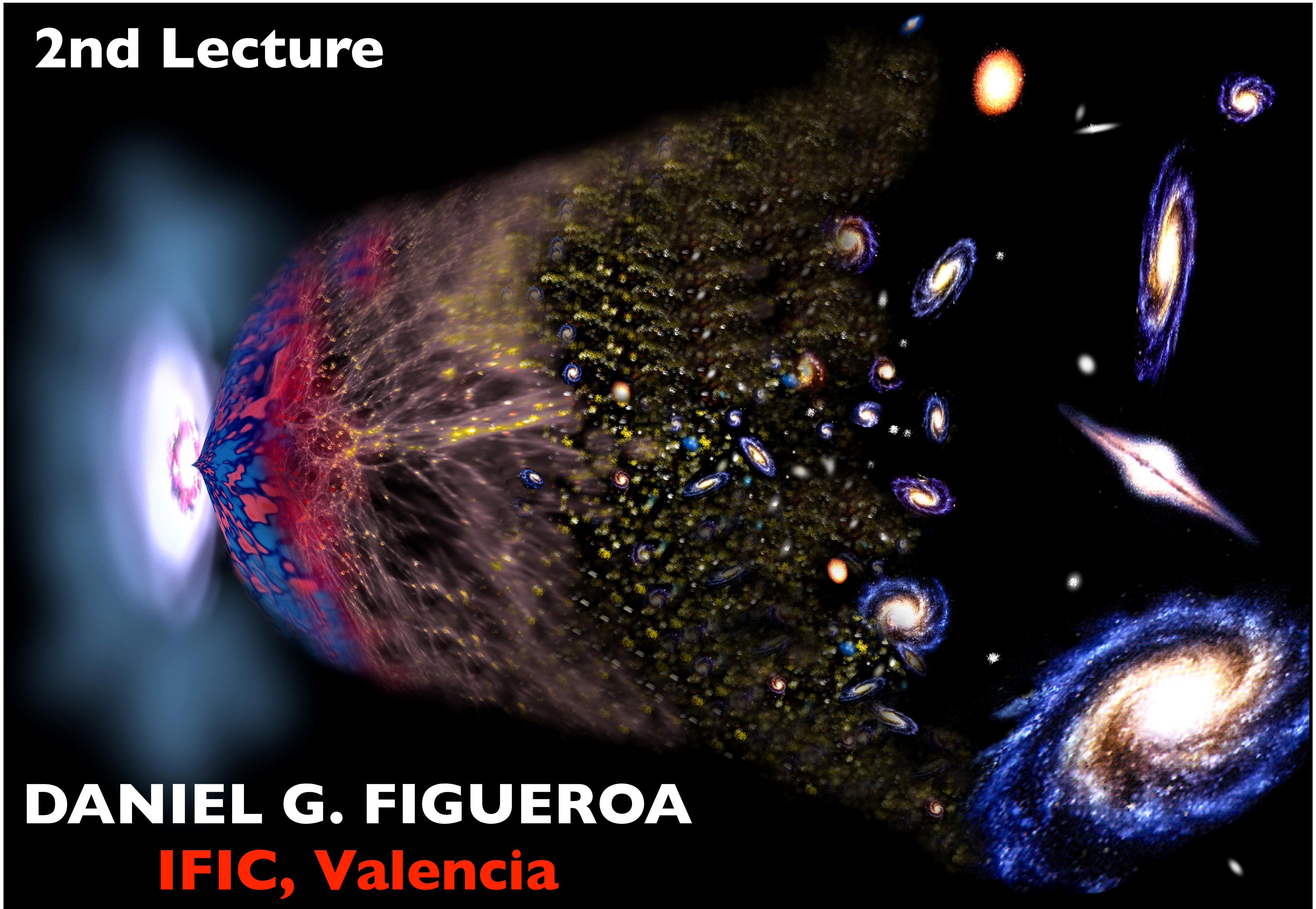
GRAVITATIONAL WAVE
— BACKGROUNDS —



MITP Summer School - CrossLinks of Early Universe Cosmology, 15 July - August 2, 2024

Gravitational Wave Backgrounds

OUTLINE

1) Grav. Waves (GWs) — 1st Topic (Formal Th.)

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

(Briefly)

Gravitational Wave Backgrounds

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1) Grav. Waves (GWs)

1st Topic
(Formal Th.)

Early Universe Sources

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

Core Topics
(Pheno Th.)

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

(Briefly)

The Gravity of the Situation …

GW Propagation/Creation in Cosmology

FLRW: $ds^2 = a^2(-dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j),$ $\quad \mathrm{TT}: \begin{cases} h_{ii} = 0 \\ h_{ij,j} = 0 \end{cases}$

(conformal time)

GW Propagation/Creation in Cosmology

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(conformal time)

Creation/Propagation GWs in FLRW

Eom: $\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi^{\mathrm{TT}}_{ij},}$

Source: Anisotropic Stress

$\boxed{\Pi_{ij} = T_{ij} - \langle T_{ij} \rangle_{\text{FLRW}}}$

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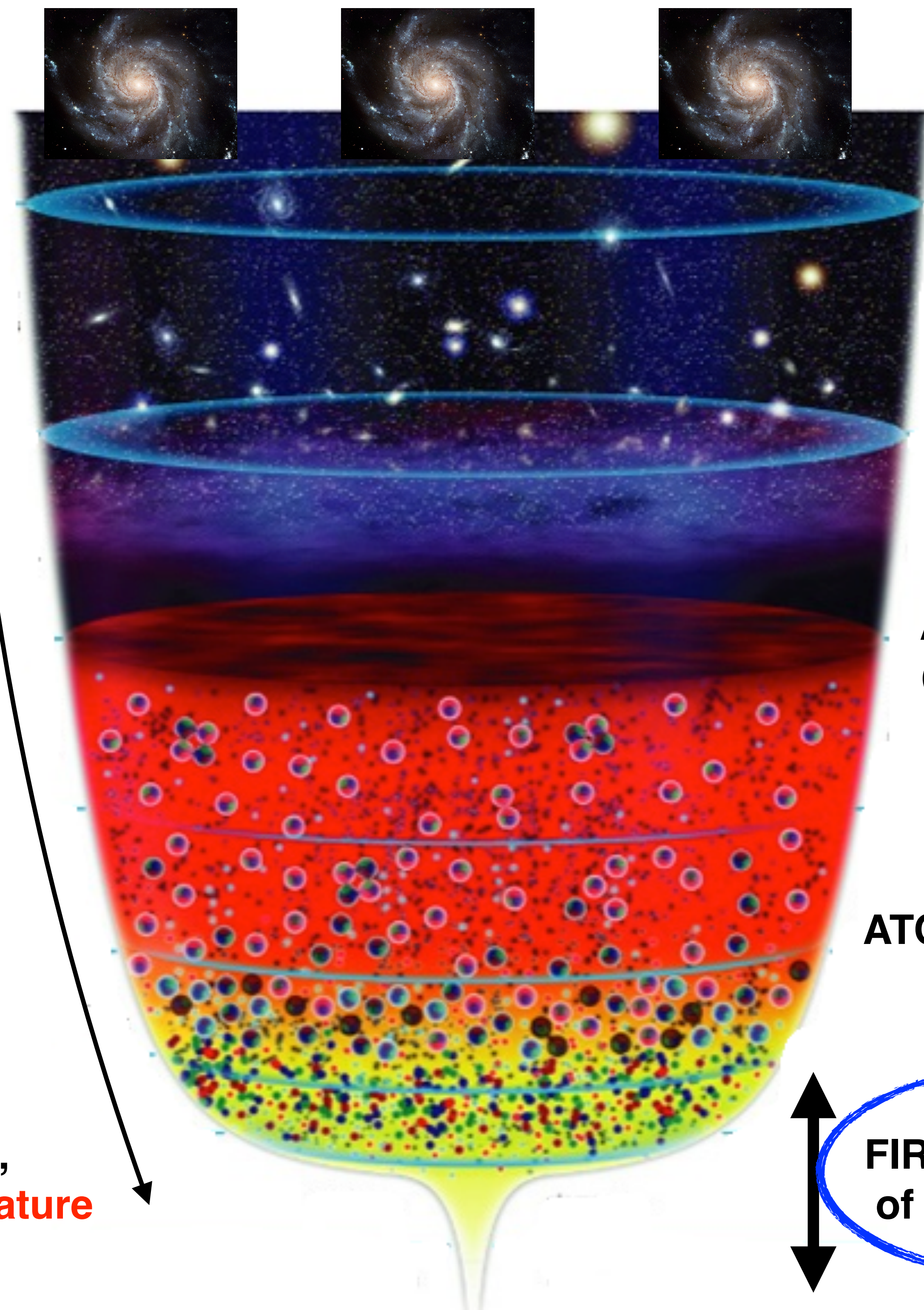
GW Source(s): (SCALARS , VECTOR , FERMIONS)

$$\Pi^{TT}_{ij} \propto \{\partial_i \chi^a \partial_j \chi^a\}^{TT}, \quad \{E_i E_j + B_i B_j\}^{TT}, \quad \{\bar{\psi}\gamma_i D_j \psi\}^{TT}$$

Cosmic History

BiGGER size,
SMALLER Temp



TODAY [Galaxies, Clusters, ...]
(13.700 Million years)

FIRST GALAXIES
(500 Millions years)

ATOMS CREATION
(300.000-400.000 years)

ATOMIC NUCLEI CREATION
(3 minutes !)

FIRST SECOND
of the UNIVERSE !

SMALLER SIZE,
LARGER Temperature

Cosmic History

BiGGER size,
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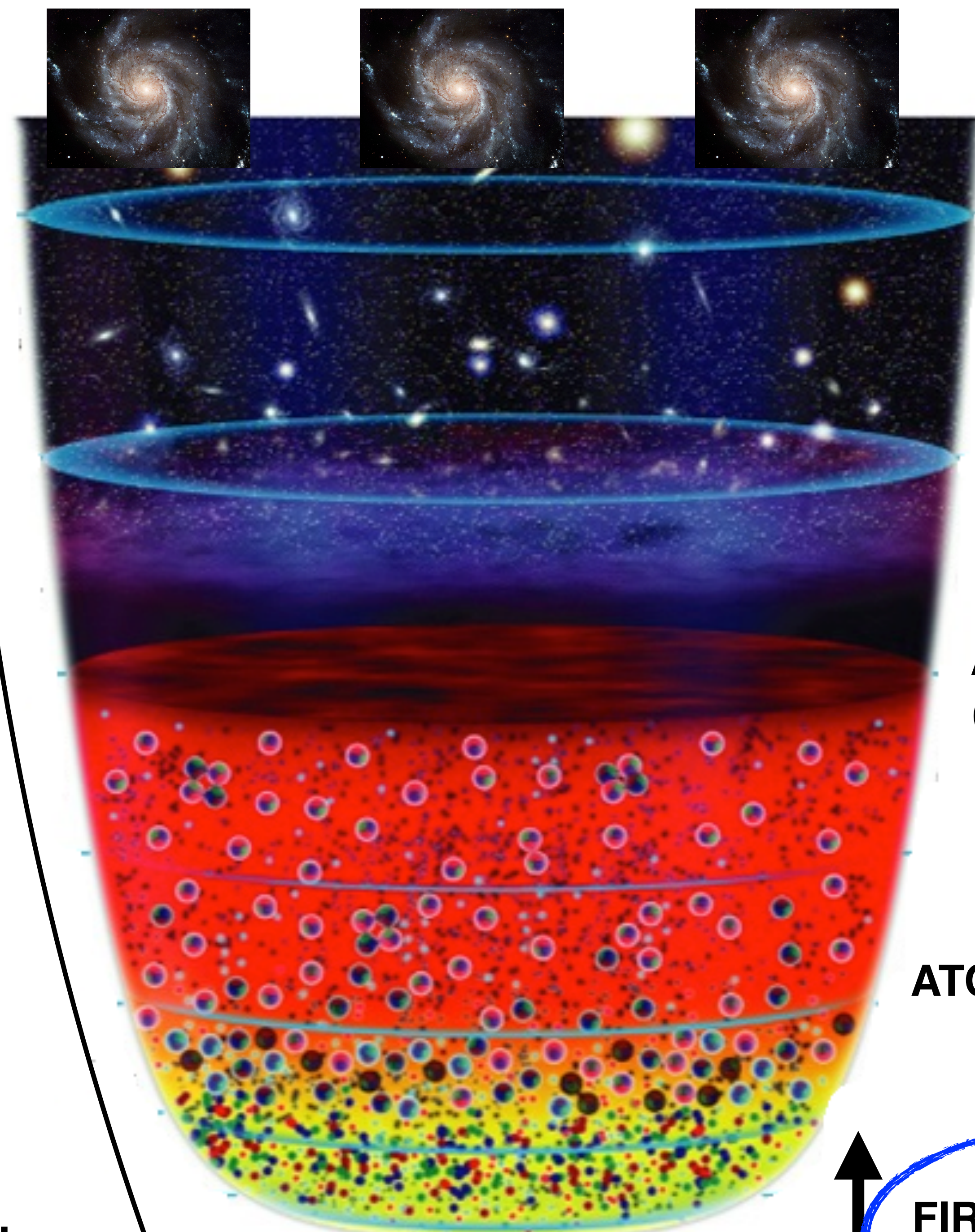

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SMALLER SIZE,
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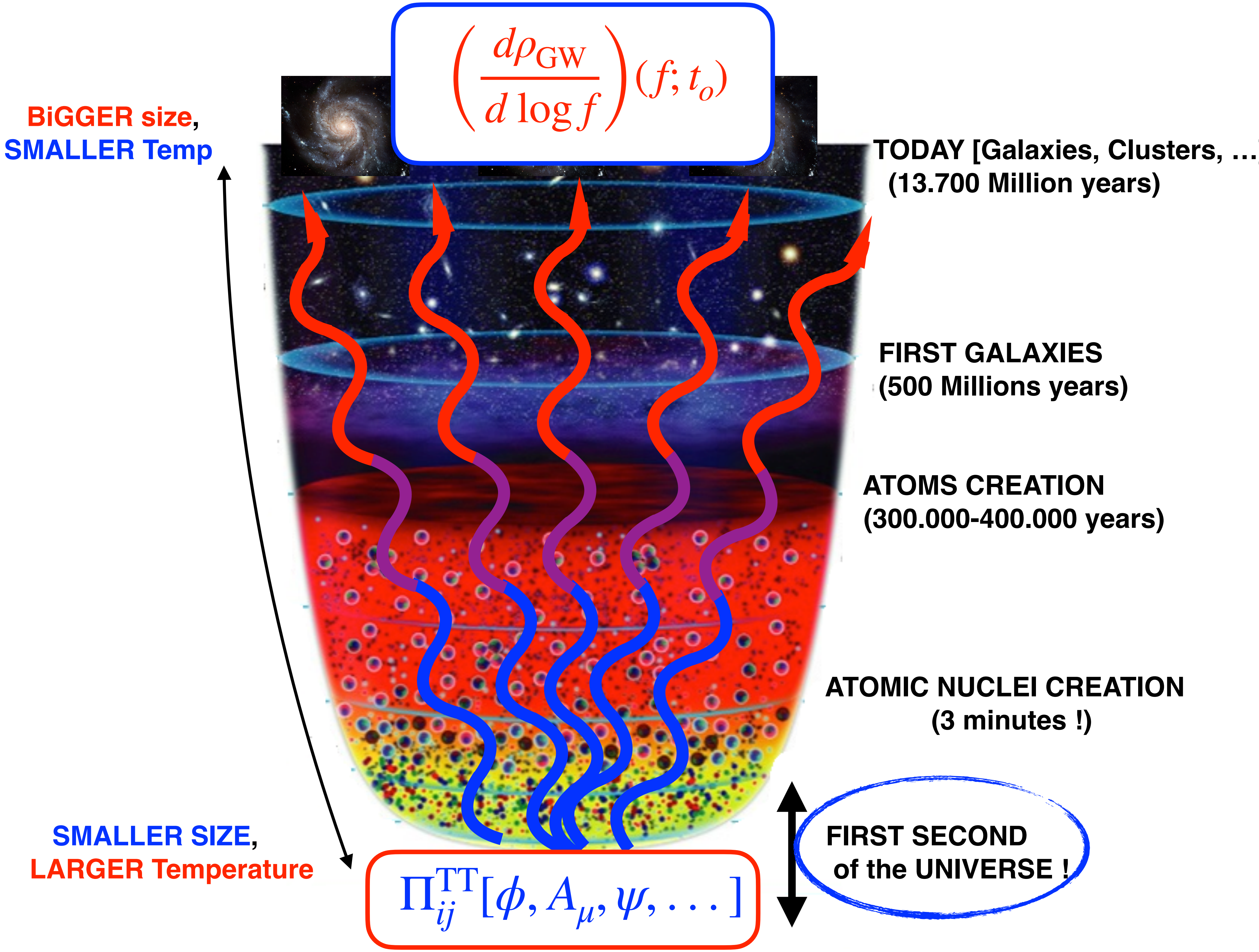
$$\boxed{\Pi_{ij}^{\mathrm{TT}}[\phi, A_\mu, \psi, \ldots]}$$

FIRST SECOND
of the UNIVERSE !

Cosmic History



GW energy density spectrum

Recall, from previous lecture on

the energy-momentum of GW

GW energy density spectrum

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$$\rho_{\rm GW}(t) = \frac{1}{32\pi G a^2(t)}\left\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\right\rangle_V$$

$$\equiv \frac{1}{32\pi G a^2(t)}\frac{1}{V}\int_V d\mathbf{x}\,\dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)$$

t : **conformal time**

GW energy density spectrum

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$\boxed{(V^{1/3} \gg \lambda)}$

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$$\begin{aligned}
\rho_{\rm GW}(t) &= \frac{1}{32\pi G a^2(t)} \left\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\right\rangle_V \\
&\equiv \frac{1}{32\pi G a^2(t)} \frac{1}{V}\int_V d\mathbf{x}\, \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t) \\
&= \frac{1}{32\pi G a^2(t)} \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{d\mathbf{k}'}{(2\pi)^3}\, \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k}',t) \\
&\qquad \times \frac{1}{V}\int_V d\mathbf{x}\; e^{-i\mathbf{x}(\mathbf{k}-\mathbf{k}')}\,,
\end{aligned}$$

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&= \frac{1}{32\pi G a^2(t)} \int \frac{d\mathbf{k}}{(2\pi)^3} \cancel{\frac{d\mathbf{k}'}{(2\pi)^3}}\, \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k}',t) \\
&\qquad \times \frac{1}{V}\int \ \boxed{\longrightarrow (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}
\end{aligned}
$$

$(kV^{1/3} \longrightarrow \infty)$

$\boxed{(V^{1/3} \gg \lambda)}$

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&= \frac{1}{32\pi G a^2(t) V} \int \frac{d\mathbf{k}}{(2\pi)^3}\, \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)
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&= \frac{1}{32\pi G a^2(t) V} \int \frac{d\mathbf{k}}{(2\pi)^3}\, \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) \\
&= \int d\log k \left(\frac{1}{(4\pi)^3 G a^2(t) V} \left\langle \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) \right\rangle_{\Omega_k} \right)
\end{aligned}$$

$$\boxed{(V^{1/3} \gg \lambda)}$$

$$\left[\left\langle |f(\mathbf{k})|^2 \right\rangle_{\Omega_k} \equiv \frac{1}{4\pi} \int_{|\mathbf{k}'|=k} d\Omega_k\, |f(\mathbf{k}')|^2 \right]$$

GW energy density spectrum

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$$\boxed{\rho_{\rm GW}(t)} = \frac{1}{32\pi G a^2(t)}\left\langle \dot h_{ij}(\mathbf{x},t)\dot h_{ij}(\mathbf{x},t)\right\rangle_V$$

$$(V^{1/3} \gg \lambda)$$

$$\equiv \frac{1}{32\pi G a^2(t)}\frac{1}{V}\int_V d\mathbf{x}\,\dot h_{ij}(\mathbf{x},t)\dot h_{ij}(\mathbf{x},t)$$

$$= \frac{1}{32\pi G a^2(t) V}\int \frac{d\mathbf{k}}{(2\pi)^3}\,\dot h_{ij}(\mathbf{k},t)\dot h^*_{ij}(\mathbf{k},t)$$

$$= \int d\log k\,\boxed{\left(\frac{1}{(4\pi)^3 G a^2(t) V}\left\langle \dot h_{ij}(\mathbf{k},t)\dot h^*_{ij}(\mathbf{k},t)\right\rangle_{\Omega_k}\right)}$$

Energy density Spectrum

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$$\equiv \left(\frac{d\rho_{\rm GW}}{d\log k}\right)(k,t)$$

Primer on Inflation

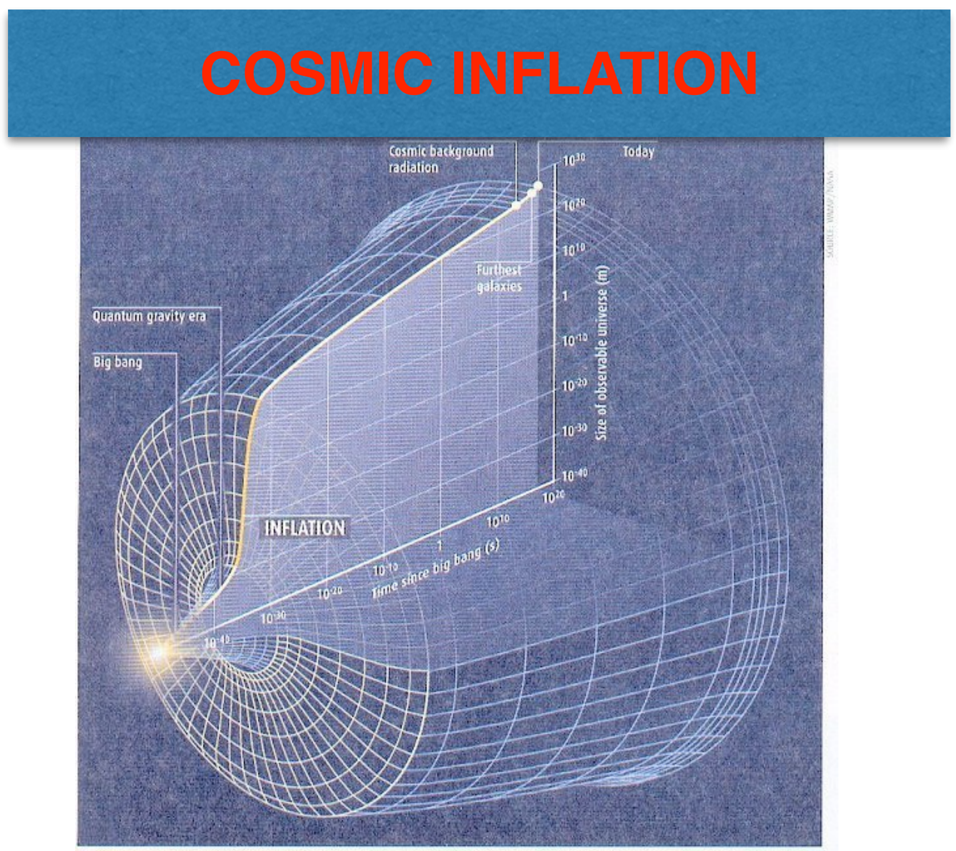
(Brief review)

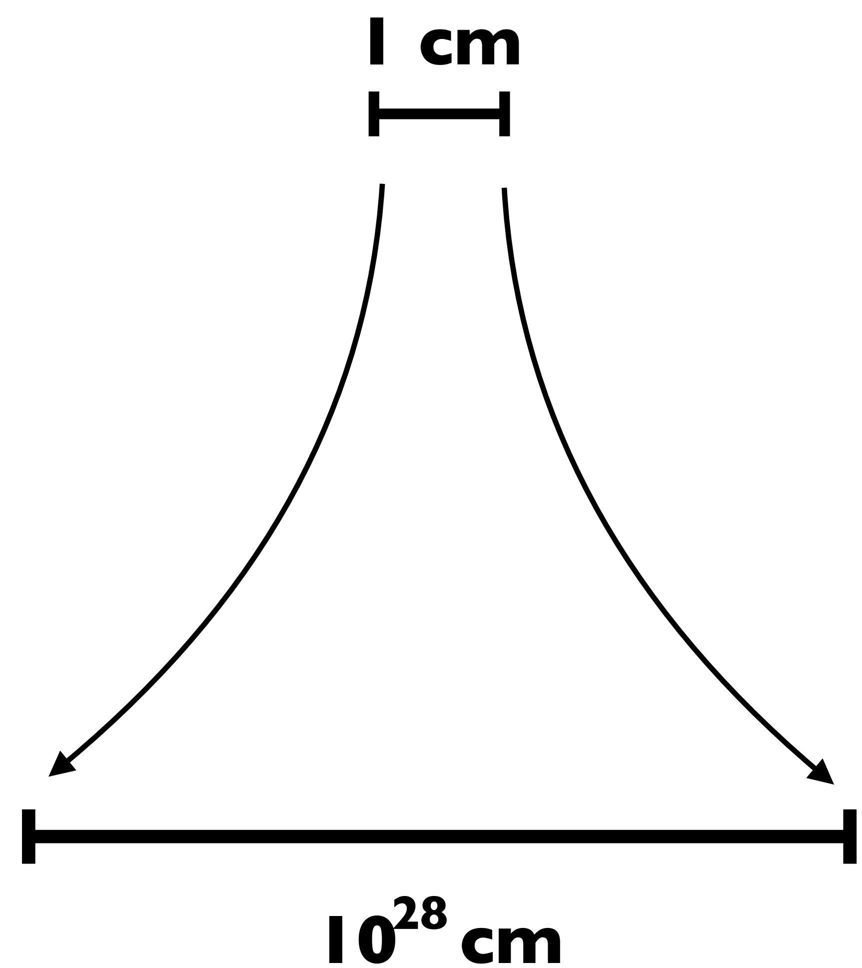
Inflation

INFLATIONARY COSMOLOGY

'cures' hBB ⟹ Cosmological Pple

Inflation

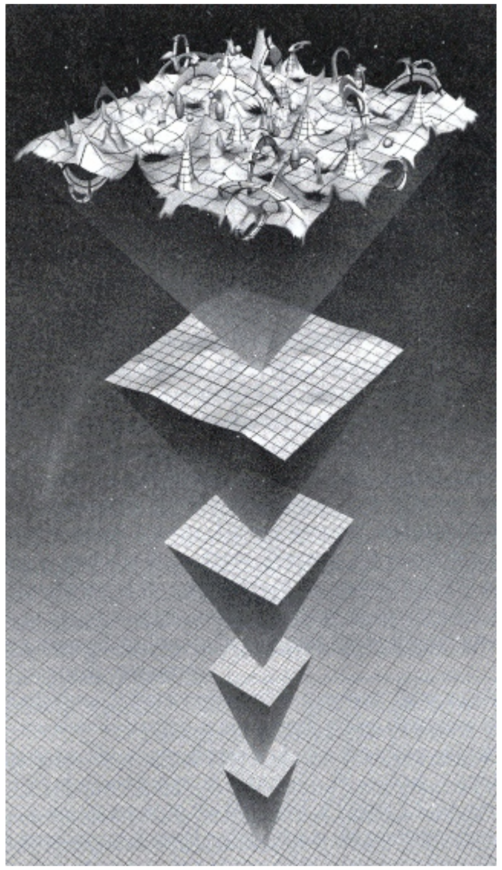




INFLATIONARY COSMOLOGY

'cures' hBB $\Longrightarrow$ Cosmological Pple

Inflation $\rightarrow$ $\begin{pmatrix}\text{initial}\\ \text{cond.}\end{pmatrix}$ = Primordial perturbations $\begin{cases}\text{Scalar } \mathcal{R} \\ \text{Tensor (GWs) } h_{ij}\end{cases}$



INFLATIONARY COSMOLOGY

'cures' hBB $\Longrightarrow$ Cosmological Pple

Inflation

$\begin{pmatrix}\text{initial}\\ \text{cond.}\end{pmatrix}$ = Primordial perturbations

- Scalar $\mathcal{R}$
- Tensor (GWs) h_{ij}

Lectures by Y. Wong ?

INFLATIONARY COSMOLOGY

Inflation: Generator of Primordial Fluctuations

INFLATIONARY COSMOLOGY

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~~Minkowski~~ $\rightarrow$ Curved Space: (quasi)dS

INFLATIONARY COSMOLOGY

Inflation: Generator of Primordial Fluctuations

$$ds^2 = -(1+2\Phi)dt^2 + 2B_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j$$

$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

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$$B_i = \partial_i B - S_i$$

$$E_{ij} = 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}$$

INFLATIONARY COSMOLOGY

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Expanding U. $\longrightarrow$ Vector Perturbations $\quad S_i, F_i \propto \dfrac{1}{a}$

INFLATIONARY COSMOLOGY

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$$\partial_i h_{ij} = h_{ii} = 0$$

(tensors = GWs)

INFLATIONARY COSMOLOGY

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dof $\xrightarrow{\text{scalars}}$ $\delta\phi + \{\Phi, B, \Psi, E\}$

dof $\xrightarrow{\text{tensors}}$ h_{ij}

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$$\text{dof}\;\begin{cases} \xrightarrow{\text{scalars}} \delta\phi + \{\Phi, B, \Psi, E\} \xrightarrow{\text{Diff.}} \delta\phi' + \{\Phi', B', \Psi', E'\} \quad \cancel{\text{Gauge Inv.}} \\ \xrightarrow{\text{tensors}} h_{ij} \xrightarrow{\text{Diff.}} h_{ij} \quad \boxed{\text{Gauge Inv. !}} \end{cases}$$

Diff.: $\boxed{x^\mu \;\rightarrow\; x^\mu + \xi^\mu}$

INFLATIONARY COSMOLOGY

Inflation: Generator of Primordial Fluctuations

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dof $\xrightarrow{\text{scalars}}$ $\delta\phi + \{\Phi, B, \Psi, E\}$

dof $\xrightarrow{\text{tensors}}$ h_{ij}

$$\zeta \equiv -[\Psi + (H/\dot{\rho})\delta\rho_\phi] \xrightarrow{\text{Diff.}} \zeta$$

$$\mathcal{R} \equiv [\Psi + (H/\dot{\phi})\delta\phi] \xrightarrow{\text{Diff.}} \mathcal{R}$$

$$Q \equiv [\delta\phi + (\dot{\phi}/H)\Psi] \xrightarrow{\text{Diff.}} Q$$

Gauge Inv. !

INFLATIONARY COSMOLOGY

Inflation: Generator of Primordial Fluctuations

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dof

scalars

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tensors

h_{ij}

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$$Q \equiv [\delta\phi + (\dot\phi/H)\Psi] \xrightarrow{\text{Diff.}} Q$$

Gauge Inv. !

Curvature Pert.

Tensor Pert. (GW)

Fixing Gauge: e.g. $E, \delta\phi = 0 \Rightarrow g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij} + h_{ij}]$

Inflation & Primordial Perturbations

H & I

INFLATION

$$\boxed{g_{\mu\nu} = g_{\mu\nu}^{FRW} + \delta g_{\mu\nu}[\mathcal{R}, h_{ij}]}$$

Inflation & Primordial Perturbations

INFLATION

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Scalar

$$\langle \mathcal{R}\mathcal{R}^* \rangle \equiv (2\pi)^3 \frac{2\pi^2}{k^3} \Delta^2_{\mathcal{R}}(k)$$

Tensor

$$\langle h_{ij} h^*_{ij} \rangle \equiv (2\pi)^3 \frac{2\pi^2}{k^3} \Delta^2_{h}(k)$$

Inflation & Primordial Perturbations

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Tensor

$$\langle h_{ij} h^*_{ij} \rangle \equiv (2\pi)^3 \frac{2\pi^2}{k^3} \Delta^2_h(k)$$

$$\left[\ \langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle \equiv (2\pi)^3 \frac{2\pi^2}{k^3} \Delta^2_f(k) \delta(\mathbf{k} - \mathbf{k}') \ \right]$$

Quantum fluctuations !

Inflation & Primordial Perturbations

INFLATION

$$g_{\mu\nu} = g_{\mu\nu}^{FRW} + \delta g_{\mu\nu}[\mathcal{R}, h_{ij}]$$

$$\Delta^2_{\mathcal{R}}(k) = \frac{H^4}{(2\pi)^2\dot{\phi}^2}\left(\frac{k}{aH}\right)^{n_s-1}$$

$$n_s - 1 \equiv 2(\eta - 2\epsilon)$$

$$\Delta^2_h(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}$$

$$n_t \equiv -2\epsilon$$

Inflation & Primordial Perturbations

INFLATION → $g_{\mu\nu} = g^{FRW}_{\mu\nu} + \delta g_{\mu\nu}[\mathcal{R}, h_{ij}]$

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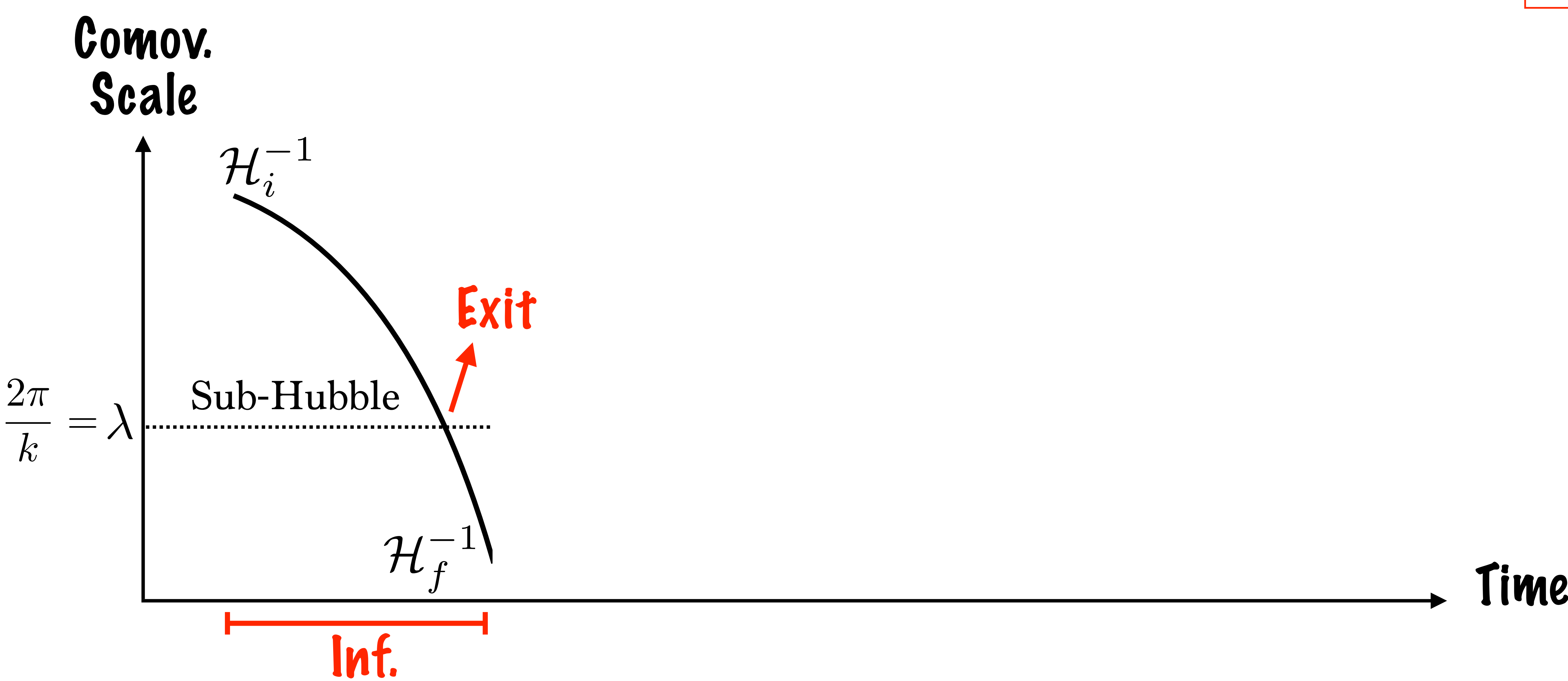

Inflation & Primordial Perturbations

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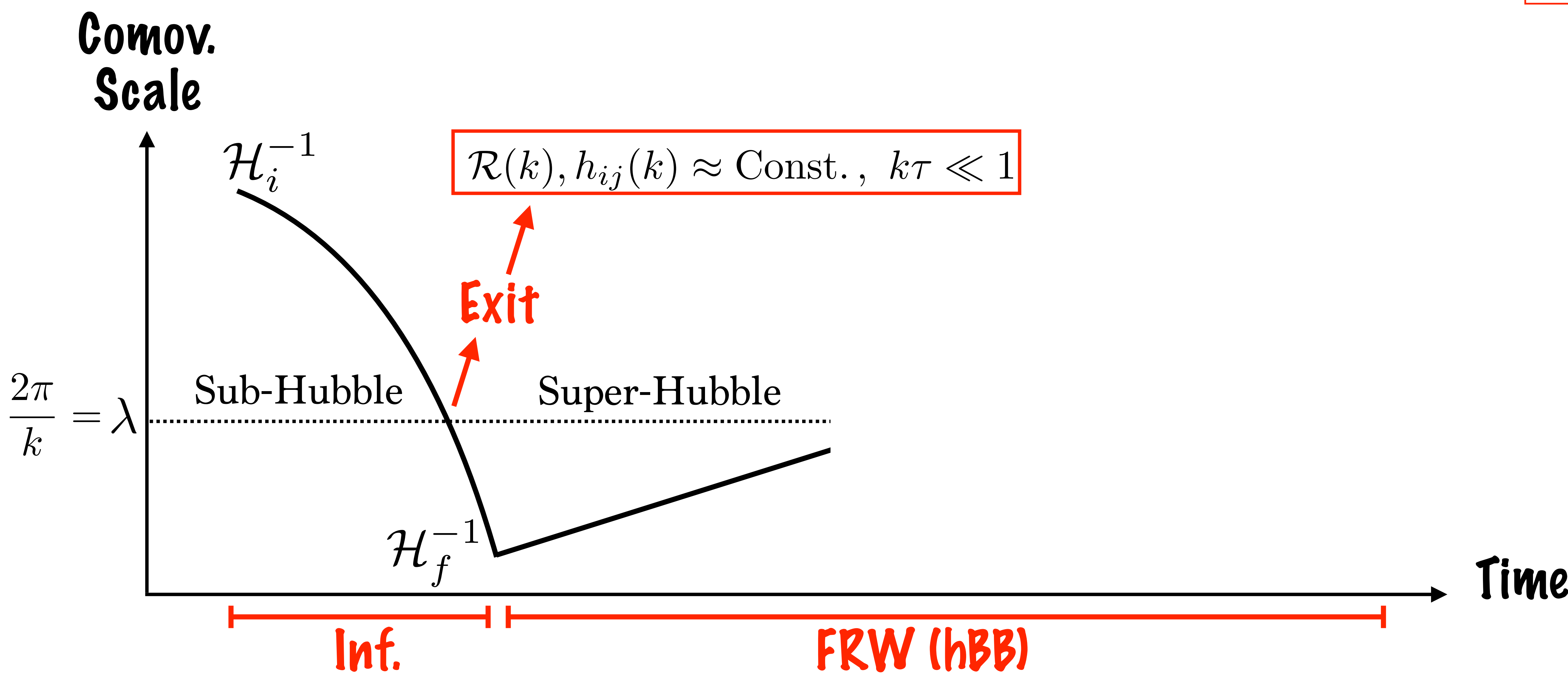

Inflation & Primordial Perturbations

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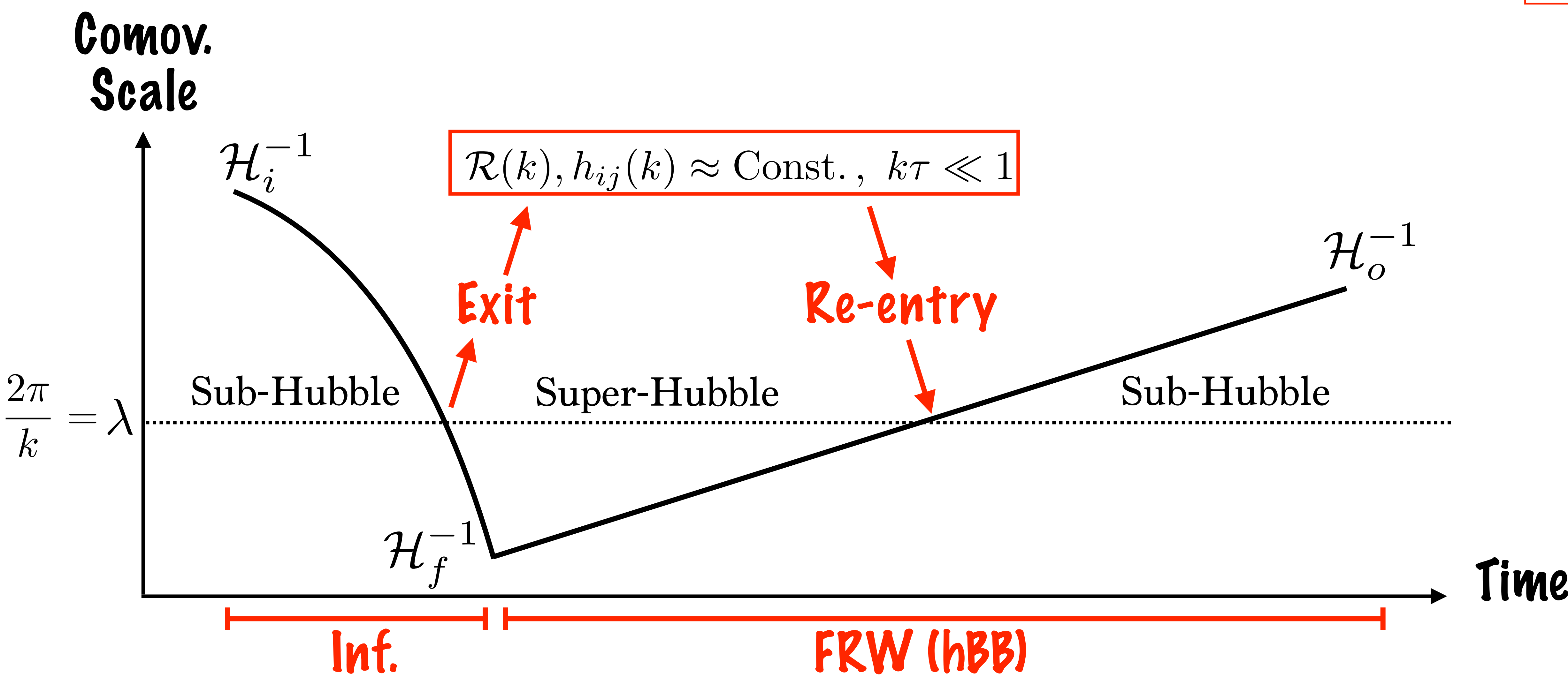

INFLATIONARY COSMOLOGY

'cures' hBB $\Longrightarrow$ Cosmological Pple

Inflation $\rightarrow$ $\begin{pmatrix}\text{initial}\\ \text{cond.}\end{pmatrix}$ = Primordial perturbations $\begin{cases}\text{Scalar}\\ \text{Tensor} \Longrightarrow \text{Stochastic GW Background}\end{cases}$

Irreducible GW background from Inflation

Tensors = GWs

$$\boxed{\hat{h}_{ij}(\mathbf{x},\underset{\substack{\downarrow\\ \text{conformal time}}}{t}) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(h_k(t)\, e^{i\mathbf{k}\mathbf{x}}\, \hat{a}_{\mathbf{k}r} + h_k^*(t) e^{-i\mathbf{k}\mathbf{x}}\, \hat{a}^+_{\mathbf{k}r}\right)\; e^{r}_{ij}(\hat{\mathbf{k}})}$$

conformal
time

Irreducible GW background from Inflation

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quantum operators

Polarizations: +, x

Irreducible GW background from Inflation

Tensors = GWs

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$$\rho_{\rm GW}(t) \;=\; \frac{1}{32\pi G a^2(t)} \left\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\right\rangle_V \longrightarrow$$

Volume/Time Average

Irreducible GW background from Inflation

Tensors = GWs

$$\boxed{\hat{h}_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(h_k(t)\, e^{i\mathbf{kx}}\, \hat{a}_{\mathbf{k}r} + h_k^*(t) e^{-i\mathbf{kx}}\, \hat{a}_{\mathbf{k}r}^{+}\right)\, e_{ij}^r(\hat{\mathbf{k}})}$$

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G a^2(t)} \left\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\right\rangle_{\rm QM} \longrightarrow \text{ensemble average}$$

Irreducible GW background from Inflation

Tensors = GWs

$$\boxed{\hat{h}_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(h_k(t)\, e^{i\mathbf{k}\mathbf{x}}\, \hat{a}_{\mathbf{k}r} + h_k^*(t) e^{-i\mathbf{k}\mathbf{x}}\, \hat{a}^+_{\mathbf{k}r} \right)\, e^r_{ij}(\hat{\mathbf{k}})}$$

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G a^2(t)} \left\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t) \right\rangle_{\rm QM} \longrightarrow \text{ensemble average}$$

$$= \frac{1}{32\pi G a^2(t)} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{d\mathbf{k}'}{(2\pi)^3}\, e^{i\mathbf{x}(\mathbf{k}-\mathbf{k}')} \left\langle \dot{h}_{ij}\,(\mathbf{k},t)\, \dot{h}^*_{ij}\,(\mathbf{k}',t) \right\rangle$$

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→ ensemble average

$$= \frac{1}{32\pi G a^2(t)} \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{d\mathbf{k}'}{(2\pi)^3}\; e^{i\mathbf{x}(\mathbf{k}-\mathbf{k}')}\left\langle \dot{h}_{ij}\,(\mathbf{k},t)\,\dot{h}^*_{ij}\,(\mathbf{k}',t)\right\rangle$$

$$\boxed{\left\langle \dot{h}_{ij}\,(\mathbf{k},t)\,\dot{h}^*_{ij}\,(\mathbf{k}',t)\right\rangle \equiv (2\pi)^3\,\mathcal{P}_{\dot{h}}(k,t)\delta^{(3)}(\mathbf{k}-\mathbf{k}')}$$

Irreducible GW background from Inflation

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$$\rho_{\rm GW}(t) = \frac{1}{(4\pi)^3 G a^2(t)} \int \frac{dk}{k}\, k^3\, \mathcal{P}_{\dot{h}}(k,t) \;= \int \frac{d\rho_{\rm GW}}{d\log k}\, d\log k$$

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Irreducible GW background from Inflation

Tensors = GWs

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$$\rho_{\rm GW}(t) = \frac{1}{(4\pi)^3 G a^2(t)} \int \frac{dk}{k}\; k^3\, \mathcal{P}_{\dot{h}}(k,t) \;\; = \int \frac{d\rho_{\rm GW}}{d\log k}\, d\log k$$

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$$\boxed{\left\langle \dot{h}_{ij}\,(\mathbf{k},t)\, \dot{h}_{ij}^*\,(\mathbf{k}',t) \right\rangle \equiv (2\pi)^3\, \mathcal{P}_{\dot{h}}(k,t) \delta^{(3)}(\mathbf{k}-\mathbf{k}')}$$

Irreducible GW background from Inflation

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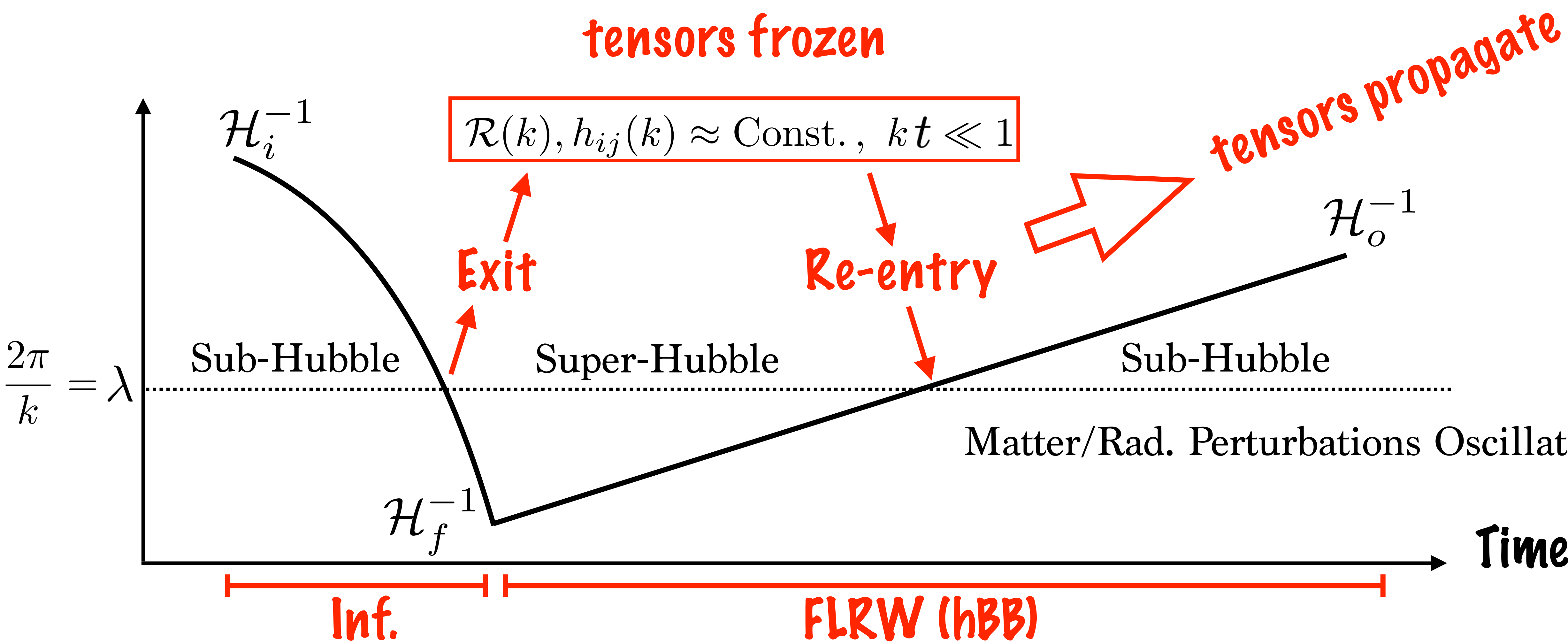

Irreducible GW background from Inflation

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Horizon Re-entry $\Rightarrow$ tensors propagate

Rad Dom: $h_r(\mathbf{k},t) = \frac{A_r(\mathbf{k})}{a(t)}\, e^{ikt} + \frac{B_r(\mathbf{k})}{a(t)}\, e^{-ikt}$

Irreducible GW background from Inflation

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$$\boxed{\begin{array}{c} @\ \text{Horizon}: \left\{ \begin{array}{l} h = h_* \\ \dot h_* = 0 \end{array}\right. \\[1em] A = B = \dfrac{1}{2} a_* h_* \end{array}}$$

Irreducible GW background from Inflation

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$$\boxed{\text{@ Horizon : } \begin{cases} h = h_* \\ \dot h_* = 0 \end{cases} \qquad A = B = \frac{1}{2}a_* h_*}$$

$$\left\langle \dot h \dot h \right\rangle = k^2 \langle hh\rangle$$

After horizon re-entry

Irreducible GW background from Inflation

$$\boxed{\frac{d\rho_{\rm GW}}{d\log k}(k,t) = \frac{1}{(4\pi)^3 G\, a^2(t)}\, k^3\, \mathcal{P}_{\dot h}(k,t)}$$

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$$\left\langle \dot h \dot h\right\rangle = k^2 \langle h h\rangle = \left(\frac{a_*}{a}\right)^2 \frac{k^2}{2}\langle |h_*|^2\rangle$$

After horizon re-entry

Irreducible GW background from Inflation

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After horizon re-entry

Irreducible GW background from Inflation

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After horizon re-entry

Redshift (re-entry)

Inflationary Tensor Spectrum !

Irreducible GW background from Inflation

$$\boxed{\frac{d\rho_{\rm GW}}{d\log k}(k,t) = \frac{1}{(4\pi)^3 G\, a^2(t)}\, k^3\, \mathcal{P}_{\dot h}(k,t)}$$

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Irreducible GW background from Inflation

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Scales as
Radiation !

(This happens for any GWB, once freely propagating @ sub-H scales)

Irreducible GW background from Inflation

$$\boxed{\frac{d\rho_{\rm GW}}{d\log k}(k,t) = \frac{1}{(4\pi)^3 G\, a^2(t)}\, k^3\, \mathcal{P}_{\dot h}(k,t)}$$

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$$(1+z_*)^{-2}_{\rm RD} = \Omega^{(o)}_{\rm Rad}\frac{a_o^2 H_o^2}{k^2} \Rightarrow \frac{d\rho_{\rm GW}}{d\log k} = \frac{\Omega^{(o)}_{\rm Rad}}{24}\left(\frac{a_o}{a}\right)^4 3m_p^2 H_o^2 \Delta^2_{h_*}$$

Irreducible GW background from Inflation

$$\boxed{\frac{d\rho_{\rm GW}}{d\log k}(k,t) = \frac{1}{(4\pi)^3 G\, a^2(t)}\, k^3\, \mathcal{P}_{\dot h}(k,t)}$$

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RD: $(1+z_*)^{-2}_{\rm RD} = \Omega^{(o)}_{\rm Rad}\dfrac{a_o^2 H_o^2}{k^2}$ $\;\Rightarrow\;$ $\dfrac{d\rho_{\rm GW}}{d\log k} = \dfrac{\Omega^{(o)}_{\rm Rad}}{24}\left(\dfrac{a_o}{a}\right)^4 \overbrace{3m_p^2 H_o^2}^{\rho_c^{(o)}}\Delta^2_{h_*}$

$$\Omega^{(o)}_{\rm GW} \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o$$

Irreducible GW background from Inflation

$$\boxed{\frac{d\rho_{\rm GW}}{d\log k}(k,t) = \frac{1}{(4\pi)^3 G\, a^2(t)}\, k^3\, \mathcal{P}_{\dot h}(k,t)}$$

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$$\boxed{\Omega^{(o)}_{\rm GW} \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \frac{\Omega^{(o)}_{\rm Rad}}{24}\Delta^2_{h_*}}$$

(RD)

Irreducible GW background from Inflation

$$\boxed{\frac{d\rho_{\rm GW}}{d\log k}(k,t) = \frac{1}{(4\pi)^3 G\, a^2(t)}\, k^3\, \mathcal{P}_{\dot h}(k,t)}$$

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$$\boxed{\Omega^{(o)}_{\rm GW}(f) \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \frac{\Omega^{(o)}_{\rm Rad}}{24}\Delta^2_{h_*}(k)} \qquad (k = 2\pi f)$$

(RD)

GW normalized energy density spectrum (today)

Inflationary tensor spectrum

Irreducible GW background from Inflation

$$\boxed{\frac{d\rho_{\rm GW}}{d\log k}(k,t) = \frac{1}{(4\pi)^3 G\, a^2(t)}\, k^3\, \mathcal{P}_{\dot h}(k,t)}$$

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(RD)

Transfer Funct

$$T(k) \equiv \frac{\Omega^{(o)}_{\rm GW}(k)}{\Delta^2_{h_*}(k)} \propto k^0\,(\mathrm{RD})$$

Irreducible GW background from Inflation

energy scale

$$\boxed{\Delta_h^2(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}}$$

$$\boxed{n_t \equiv -2\epsilon}$$

Small red-tilt

(almost-) scale-invariant

Irreducible GW background from Inflation

$$\boxed{\Omega_{\rm GW}^{(o)}(f) \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \underbrace{\frac{\Omega_{\rm Rad}^{(o)}}{24}}\Delta_{h_*}^2(k)}$$

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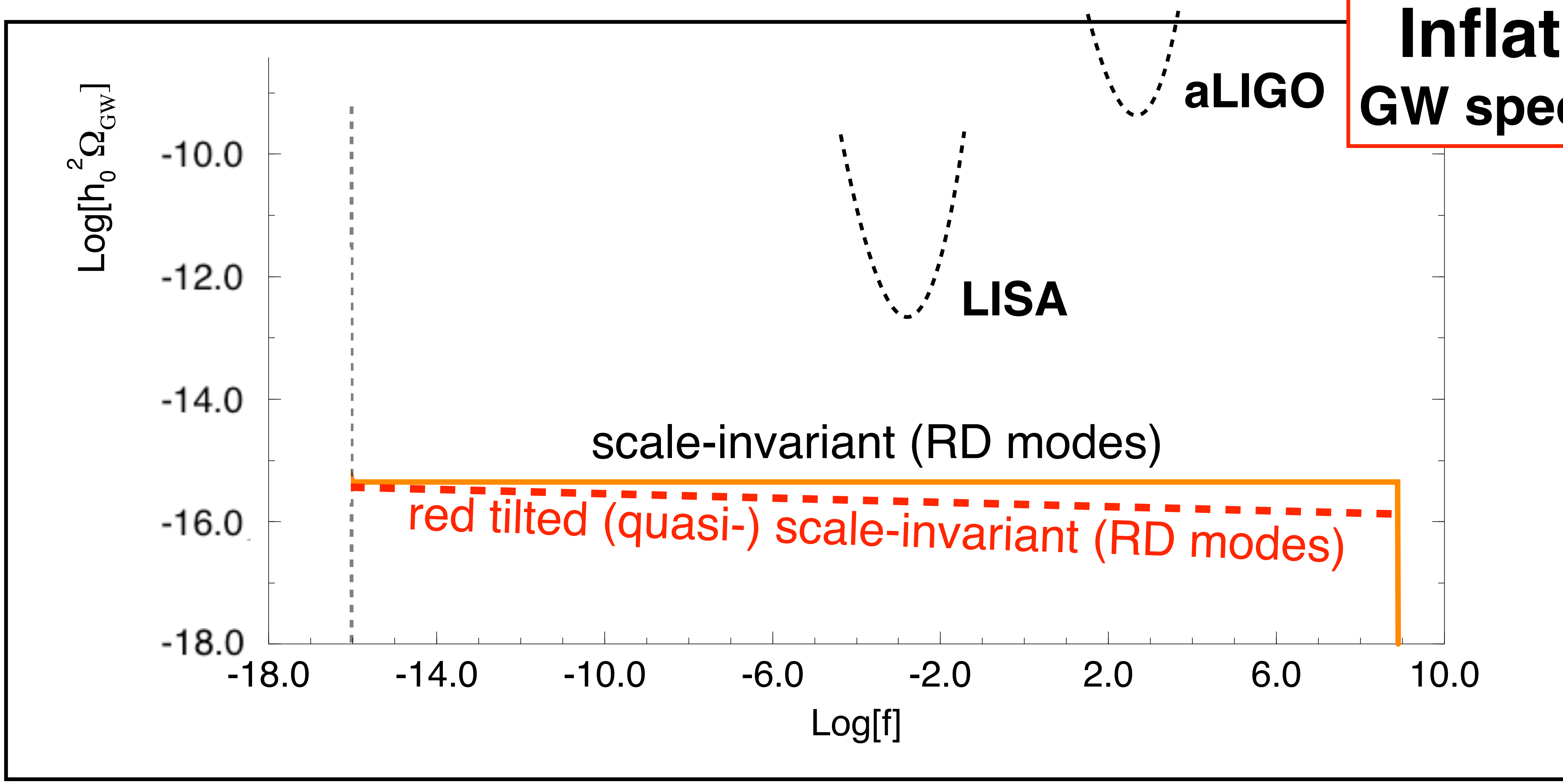


Inflation GW spectrum

Irreducible GW background from Inflation

$$\frac{\Omega_{\rm RD}}{24}\left(\frac{k_{\rm eq}}{k}\right)^2$$

$$\Omega^{(o)}_{\rm GW}(f) \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \underbrace{\frac{\Omega^{(o)}_{\rm Rad}}{24}}\,\Delta^2_{h_*}(k)$$

energy scale

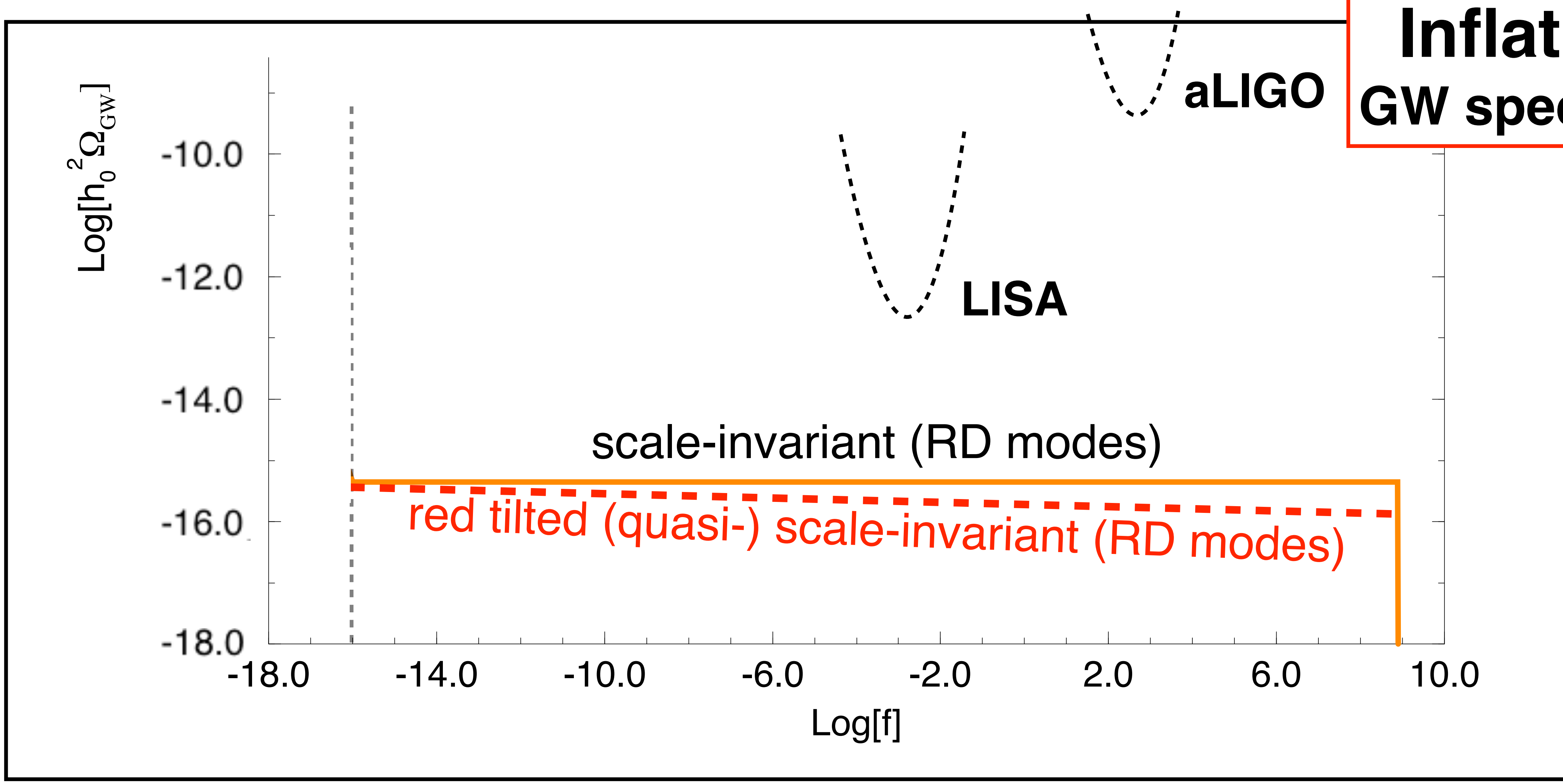
$$\Delta^2_h(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}$$

$$n_t \equiv -2\epsilon$$

Small red-tilt

Transfer Funct.: $T(k) \propto k^{-2}(\mathrm{MD})$

Inflation GW spectrum



Irreducible GW background from Inflation

$$\Omega_{\rm GW}^{(o)}(f) \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \underbrace{\frac{\Omega_{\rm Rad}^{(o)}}{24}}\Delta_{h_*}^2(k)$$

$$\frac{\Omega_{\rm RD}}{24}\left(\frac{k_{\rm eq}}{k}\right)^2$$

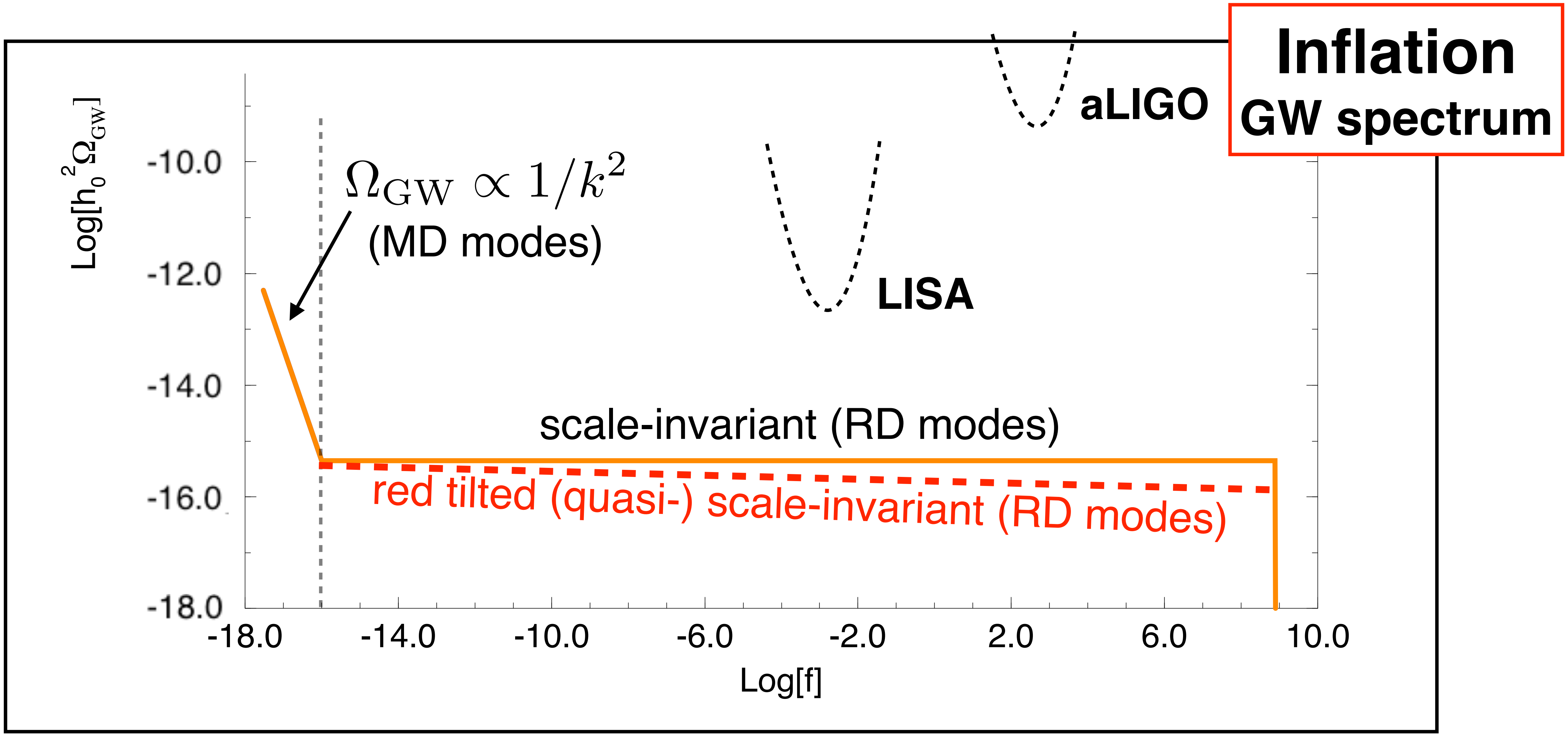
Transfer Funct.: $T(k) \propto k^{-2}(\rm MD)$

energy scale

$$\Delta_h^2(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}$$

$n_t \equiv -2\epsilon$

Small red-tilt



Irreducible GW background from Inflation

$$\boxed{\Omega^{(o)}_{\rm GW}(f) \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \frac{\Omega^{(o)}_{\rm rad}}{24}\,T(k)\,\Delta^2_{h*}(k)}$$

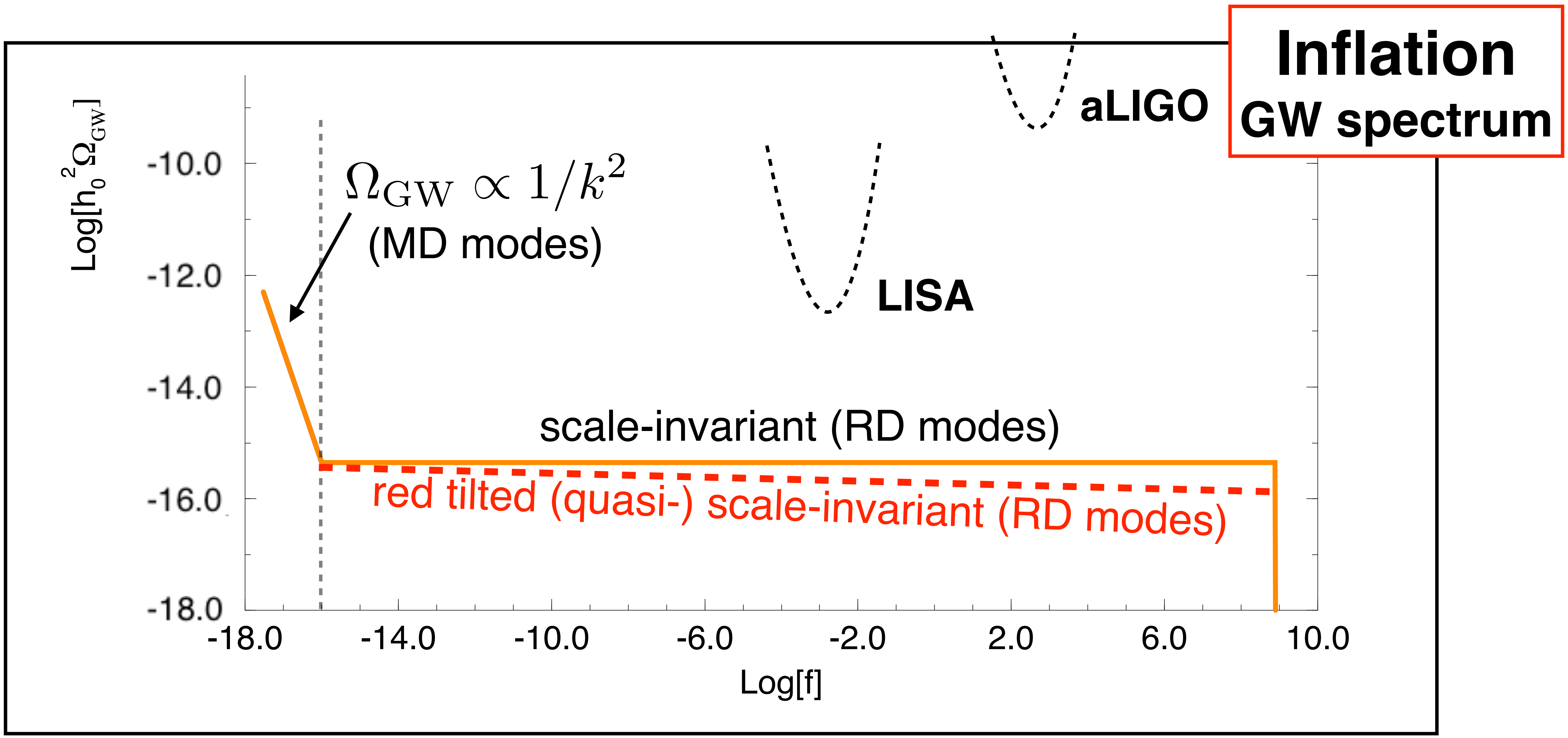
energy scale

$$\boxed{\Delta_h^2(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}}$$

$\boxed{n_t \equiv -2\epsilon}$

Small red-tilt

Transfer Funct.: $T(k) = \begin{cases} 1 & ,\mathrm{RD} \\ (k_{\rm eq}/k)^2 & ,\mathrm{MD} \end{cases}$



Irreducible GW background from Inflation

$$\boxed{\Omega_{\rm GW}^{(o)}(f) \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \frac{\Omega_{\rm rad}^{(o)}}{24} T(k)\Delta_{h_*}^2(k)}$$

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$$\boxed{\Delta_h^2(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}}$$

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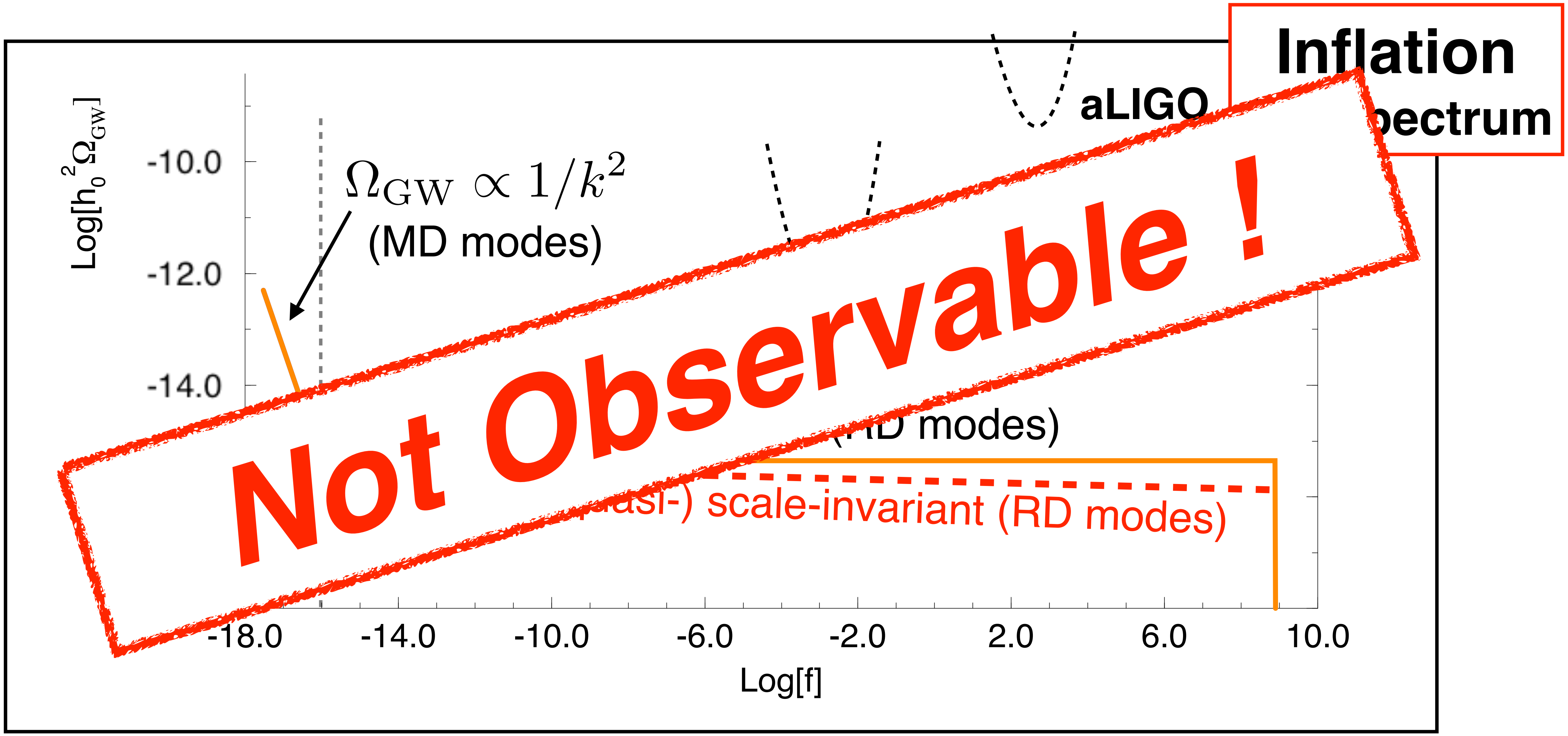


Irreducible GW background from Inflation

$$\Omega_{\rm GW}^{(o)}(f) \equiv \frac{1}{\rho_c^{(o)}}\left(\frac{d\rho_{\rm GW}}{d\log k}\right)_o = \frac{\Omega_{\rm rad}^{(o)}}{24}T(k)\Delta_{h*}^2(k)$$

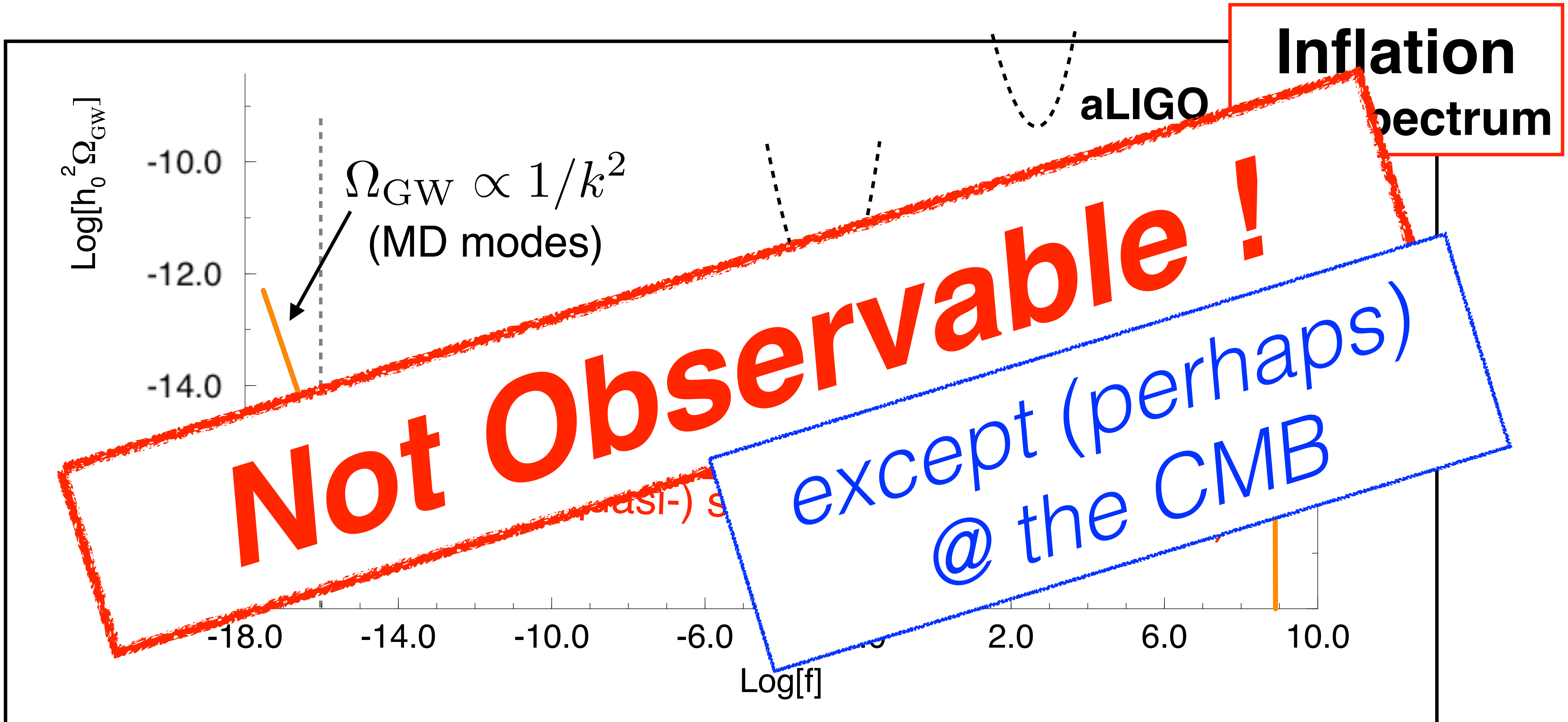
energy scale

$$\Delta_h^2(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}$$

$$\boxed{n_t \equiv -2\epsilon}$$

Small red-tilt

Transfer Funct.: $T(k) = \begin{cases} 1 & , \mathrm{RD} \\ (k_{\rm eq}/k)^2 & , \mathrm{MD} \end{cases}$



Not Observable !

except (perhaps) @ the CMB

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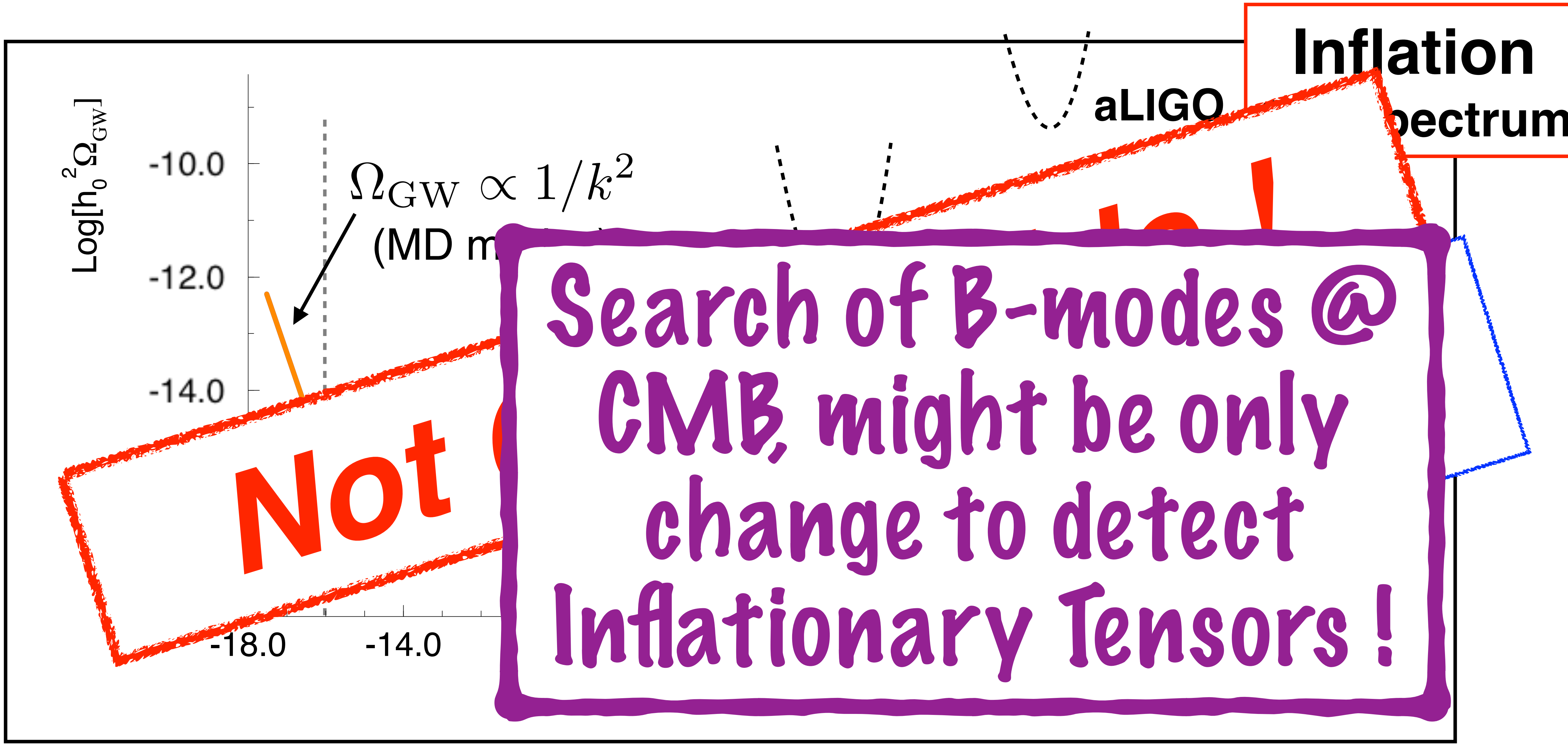

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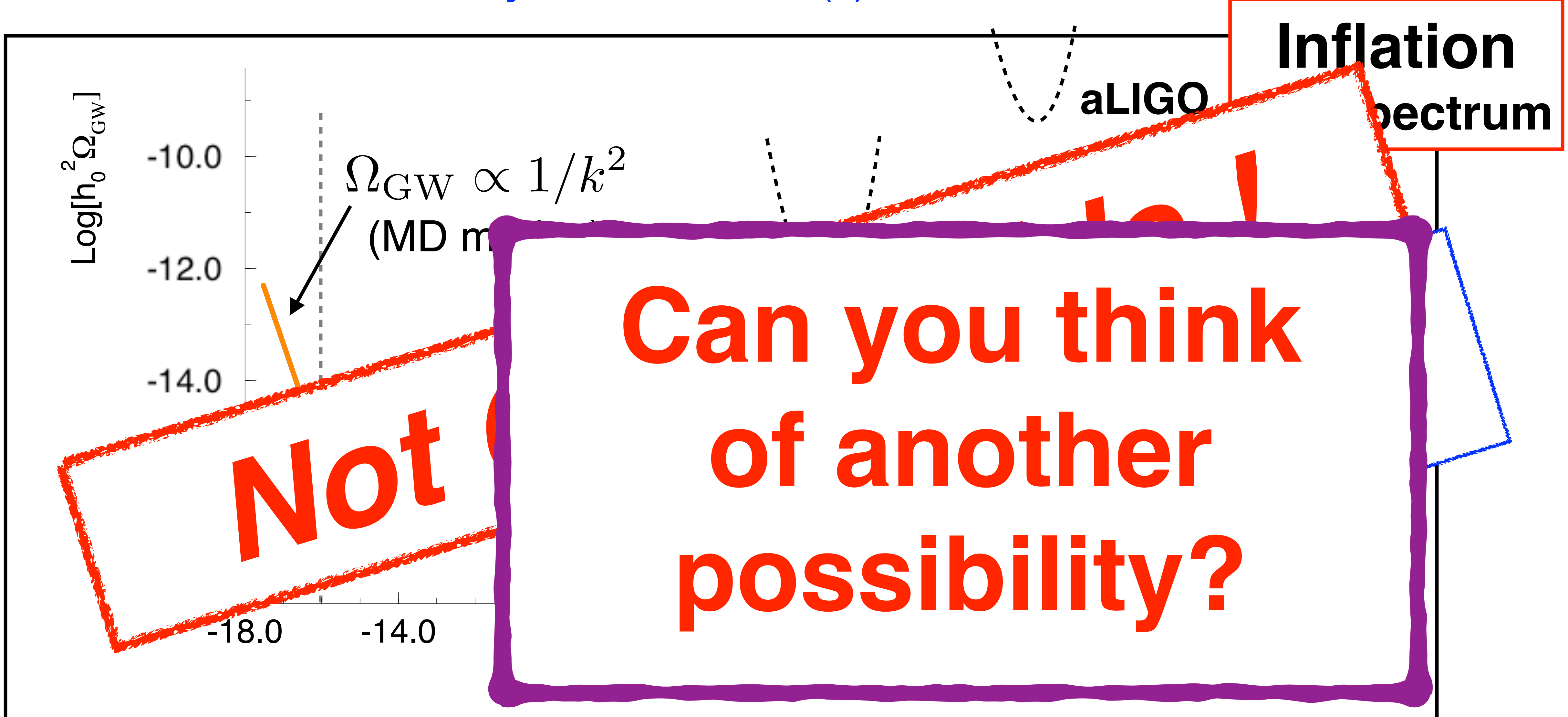
Key, think about T(k)



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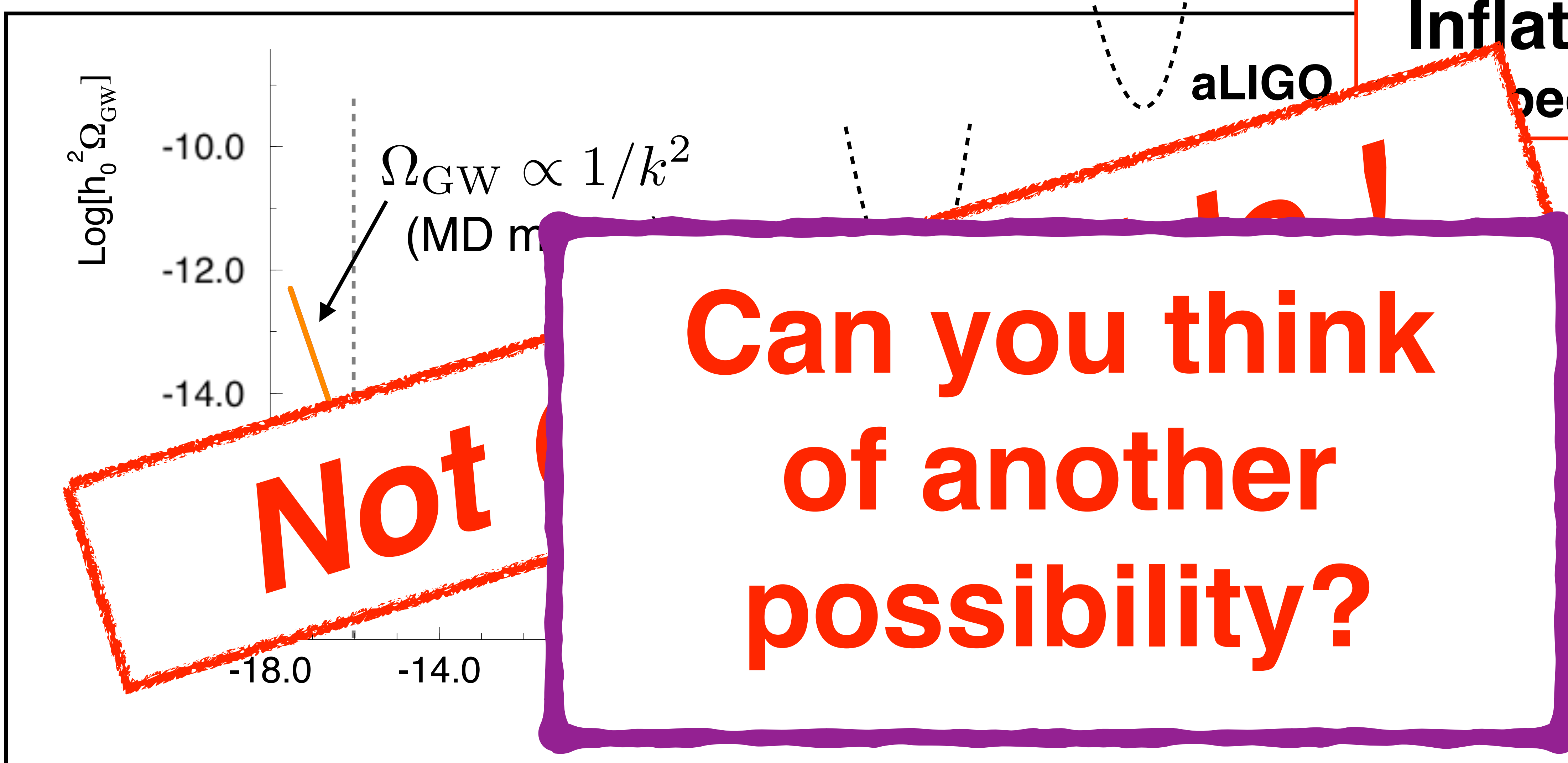

Inflation spectrum

Not G... !

Can you think of another possibility?

Irreducible GW background from Inflation

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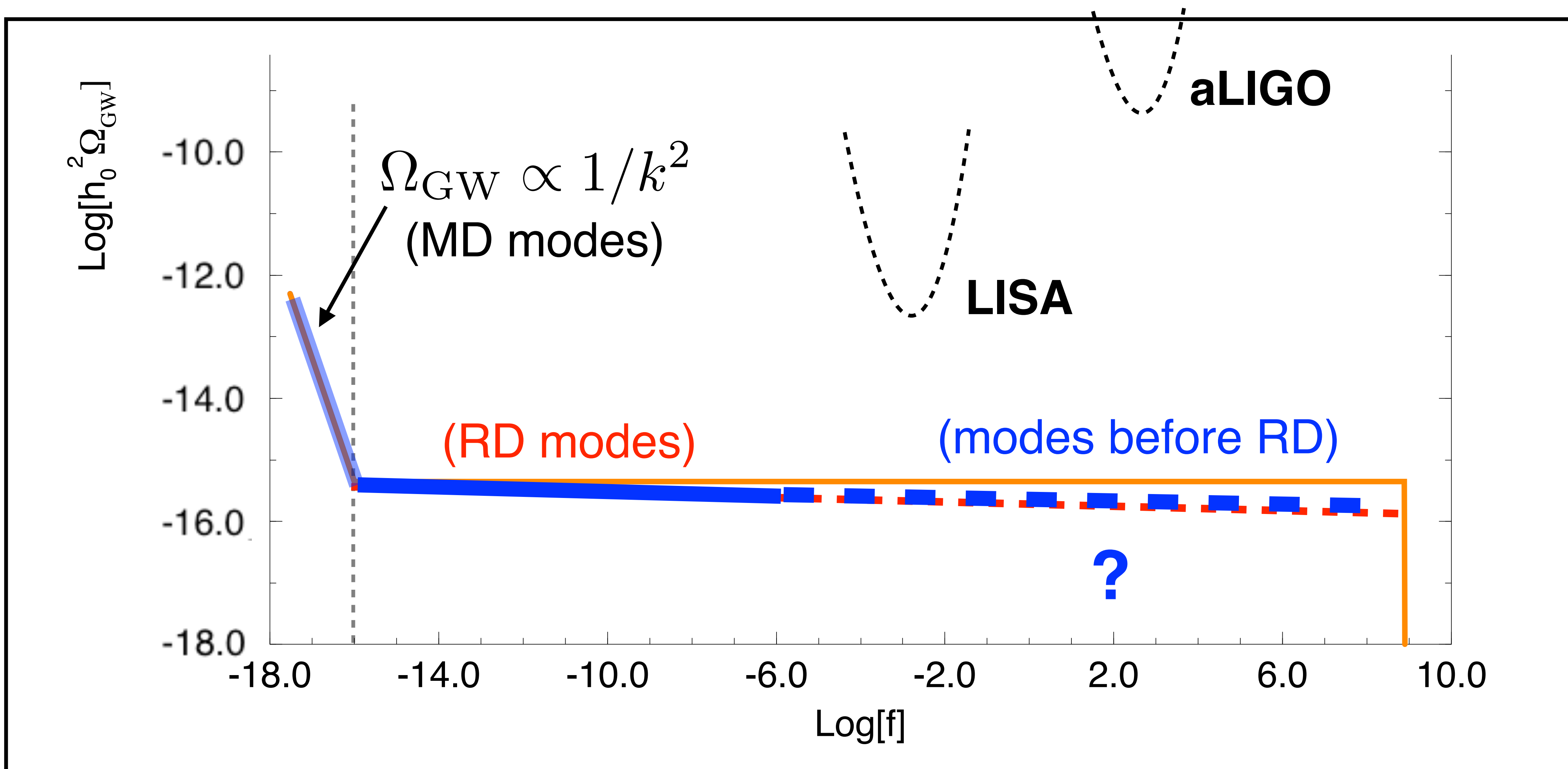

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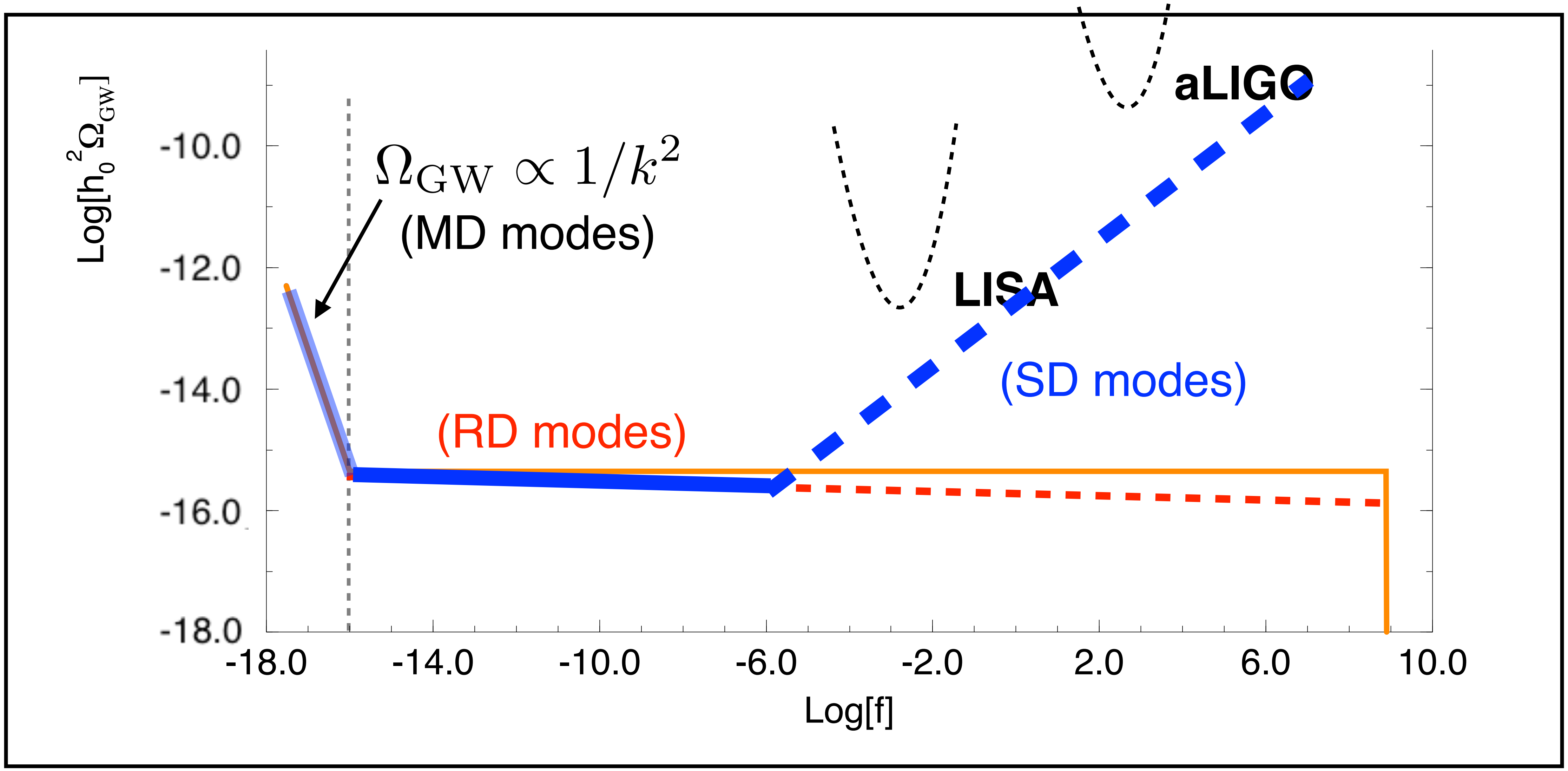

(

Realistic computation of Transfer function @ Stiff Domination —> —> Radiation Dom.

Inflationary GW background

$$\boxed{\Omega_{\rm GW}^{(0)}(f) = \underbrace{\Omega_{\rm GW}^{(0)}\Big|_{\rm plateau}}_{\text{Rad. Plateau}} \times \underbrace{\mathcal{W}(f/f_{\rm RD}) \times \mathcal{A}_{\rm s}\left(\frac{f}{f_{\rm RD}}\right)^{n_t(w_s)}}_{\text{Transfer Funct. Stiff Period}}}\,, \quad \boxed{\Omega_{\rm GW}^{(0)}\Big|_{\rm plateau} \simeq 2\cdot 10^{-16}\left(\frac{H_*}{H_{\rm max}}\right)^2}$$

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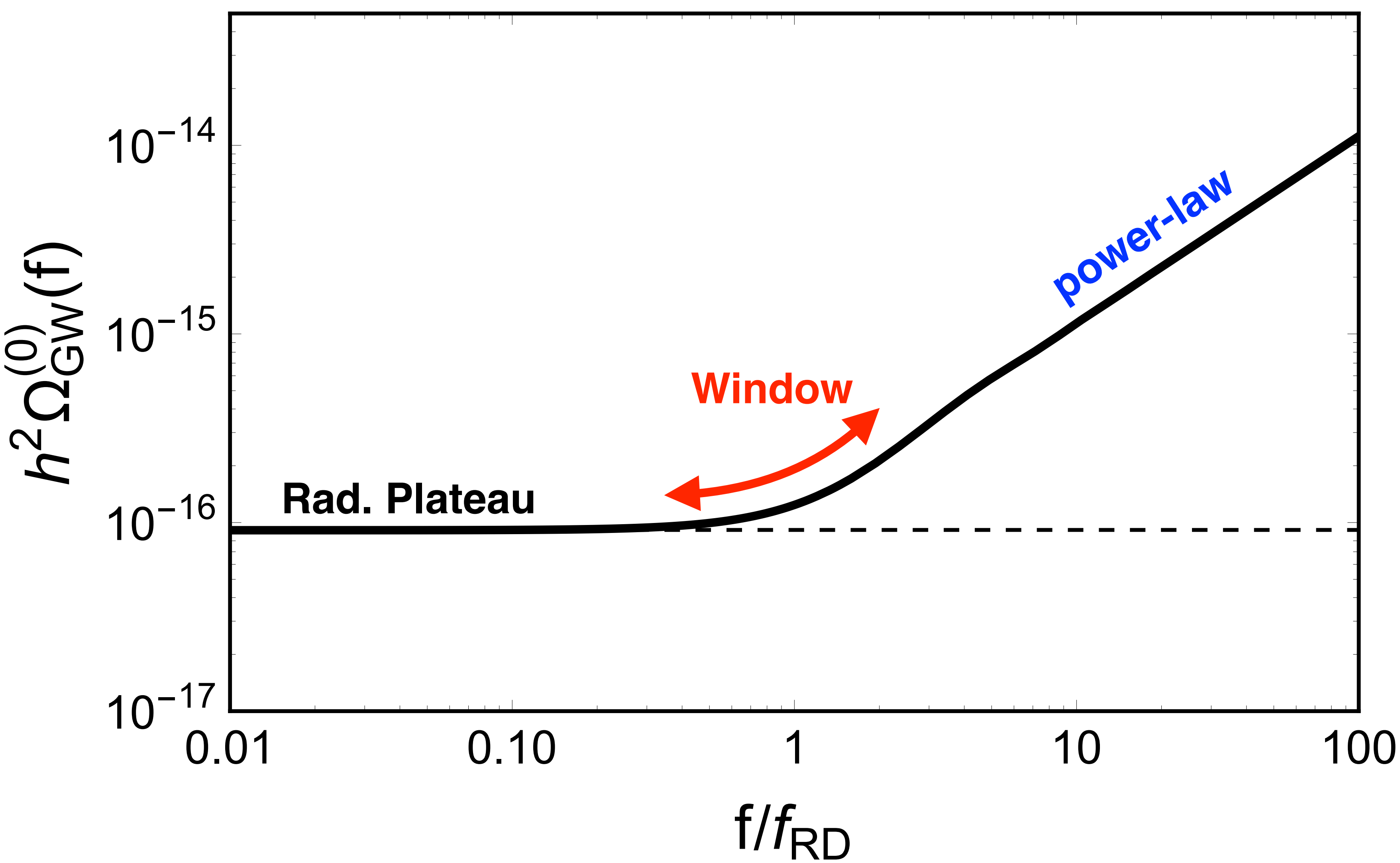
Window x *power-law*



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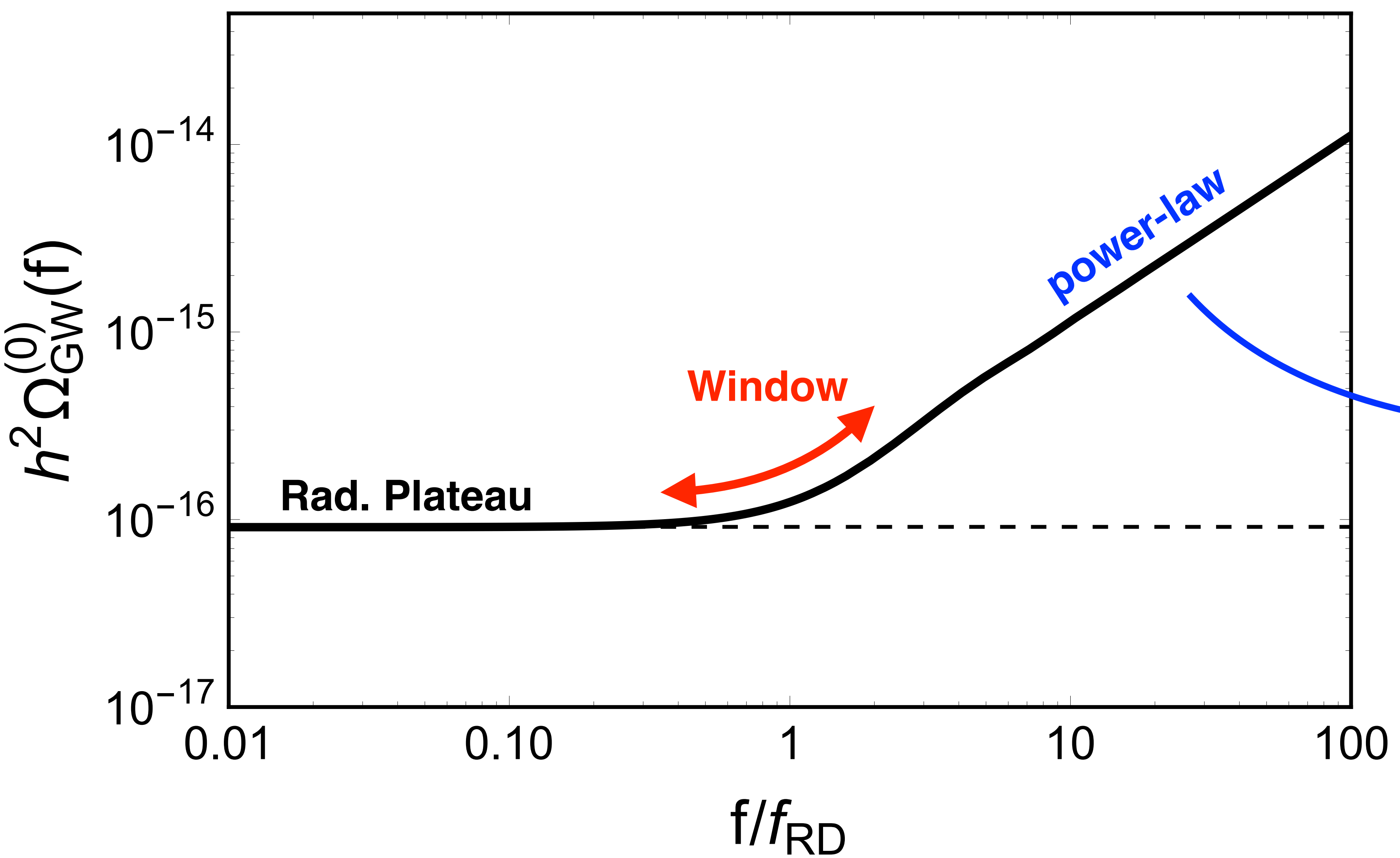

modes crossing during Stiff era

Inflationary GW background

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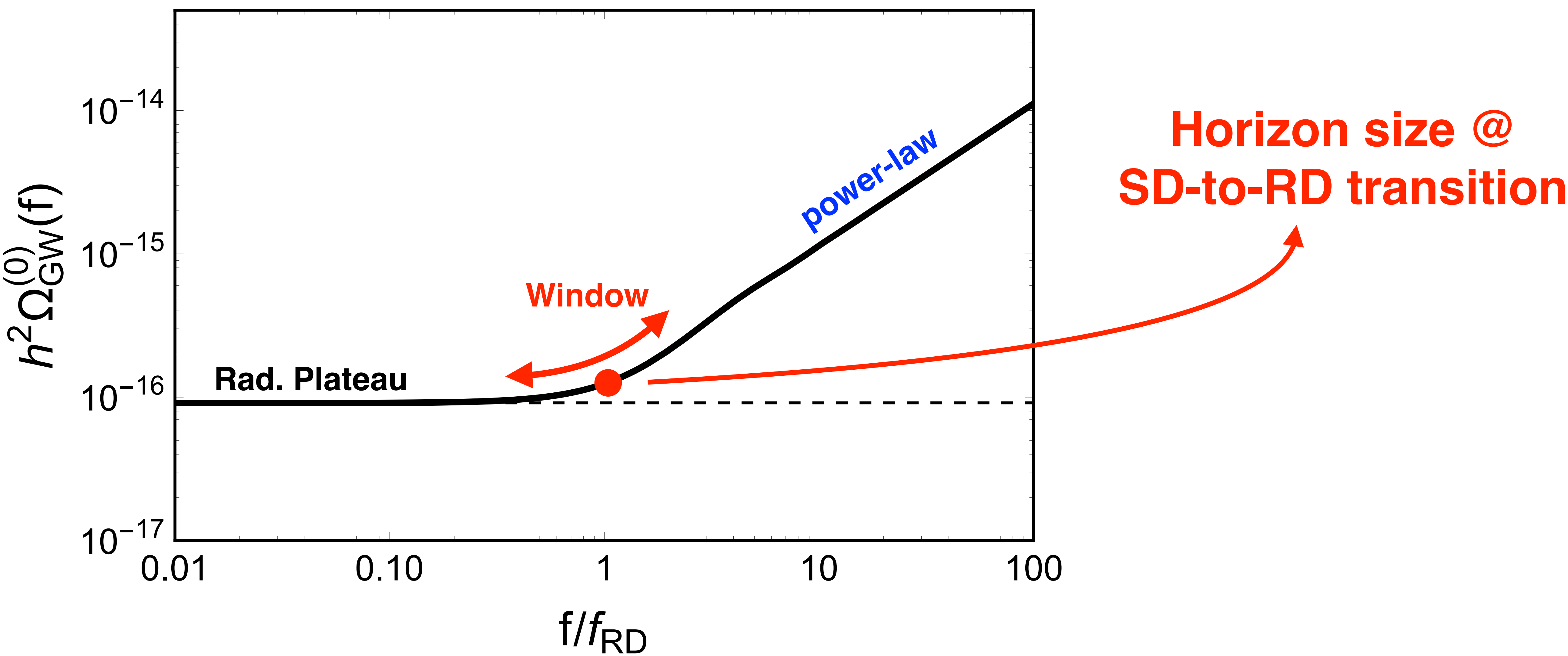

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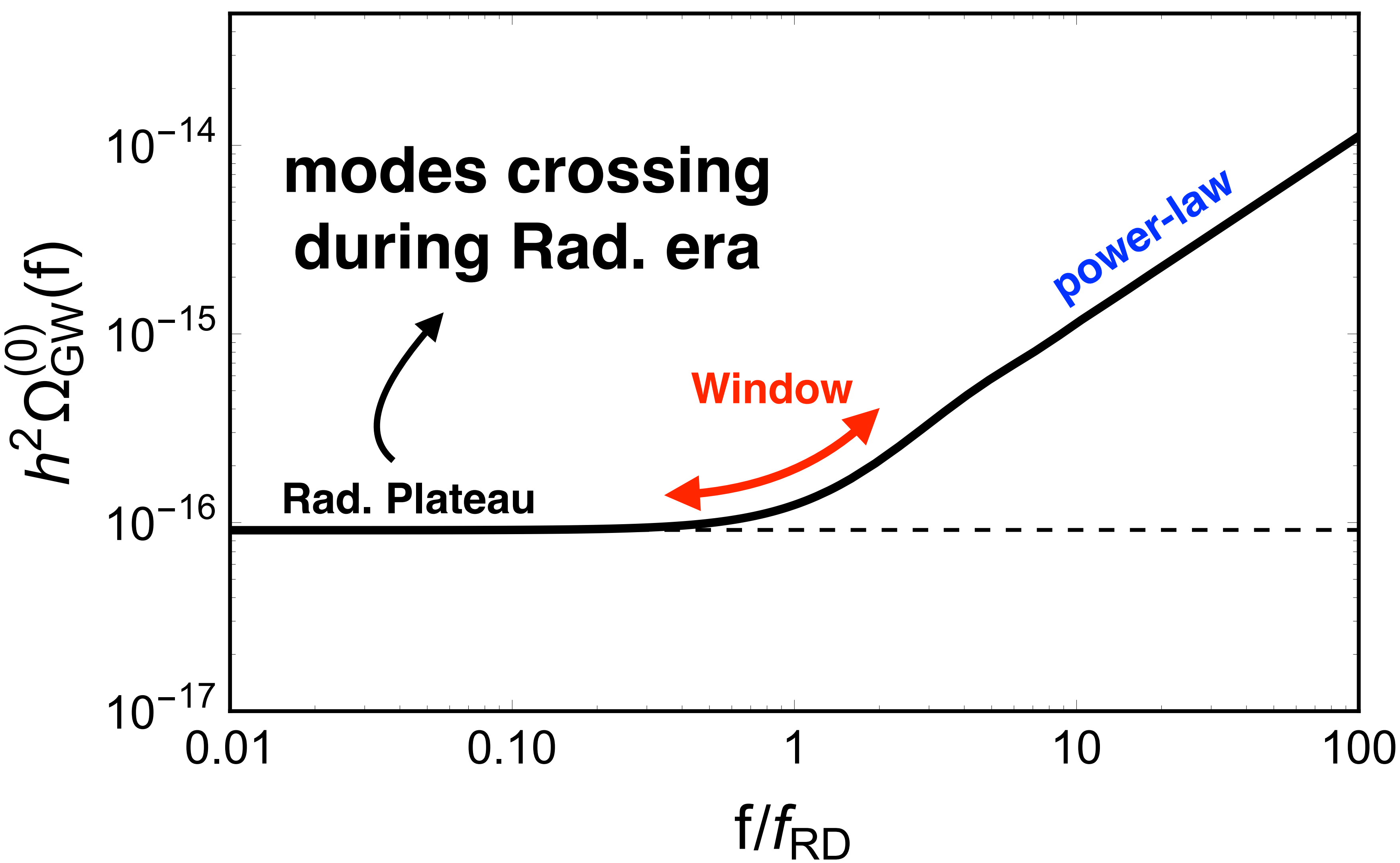

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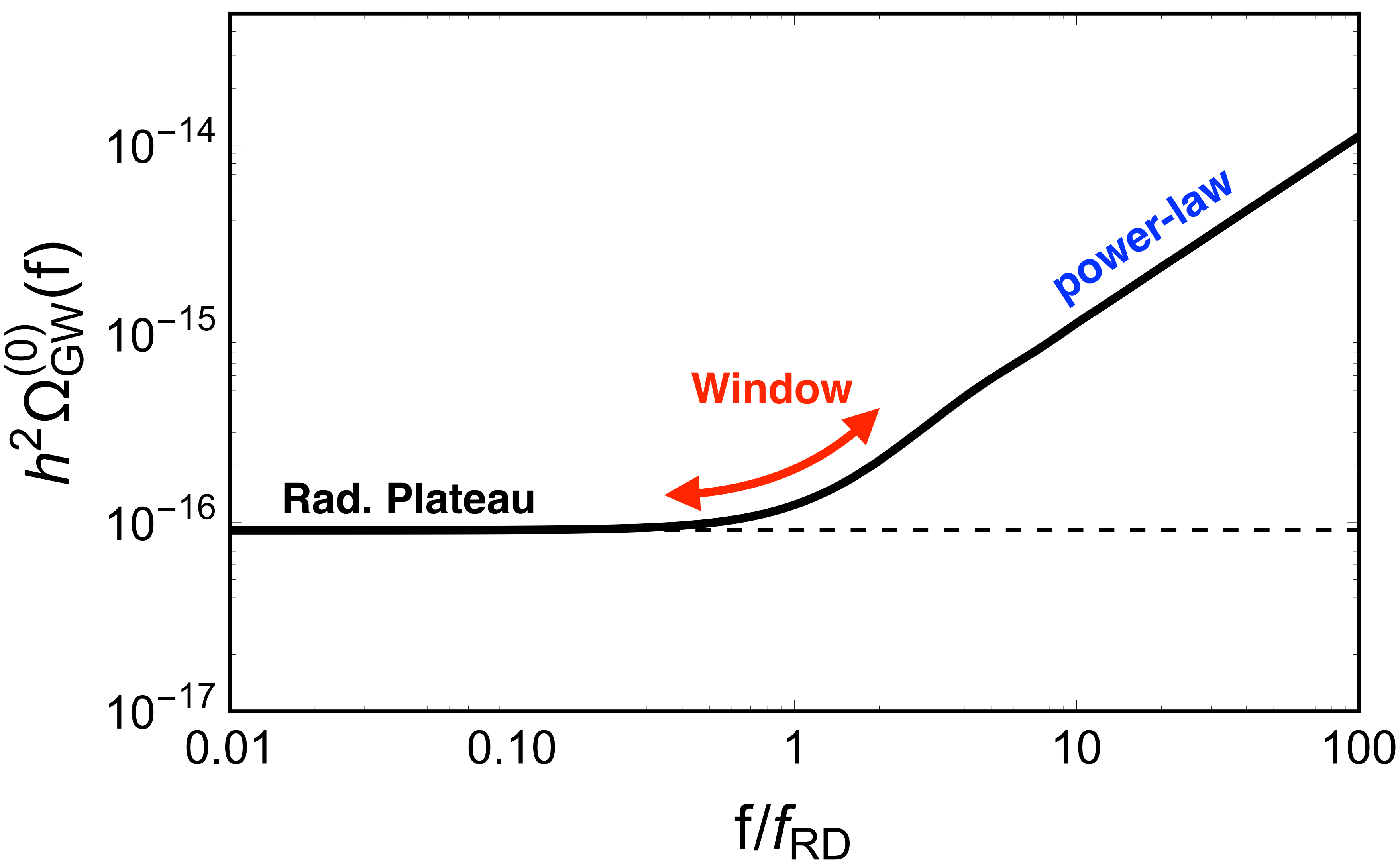

Is this the real GW spectrum ?

Inflationary GW background

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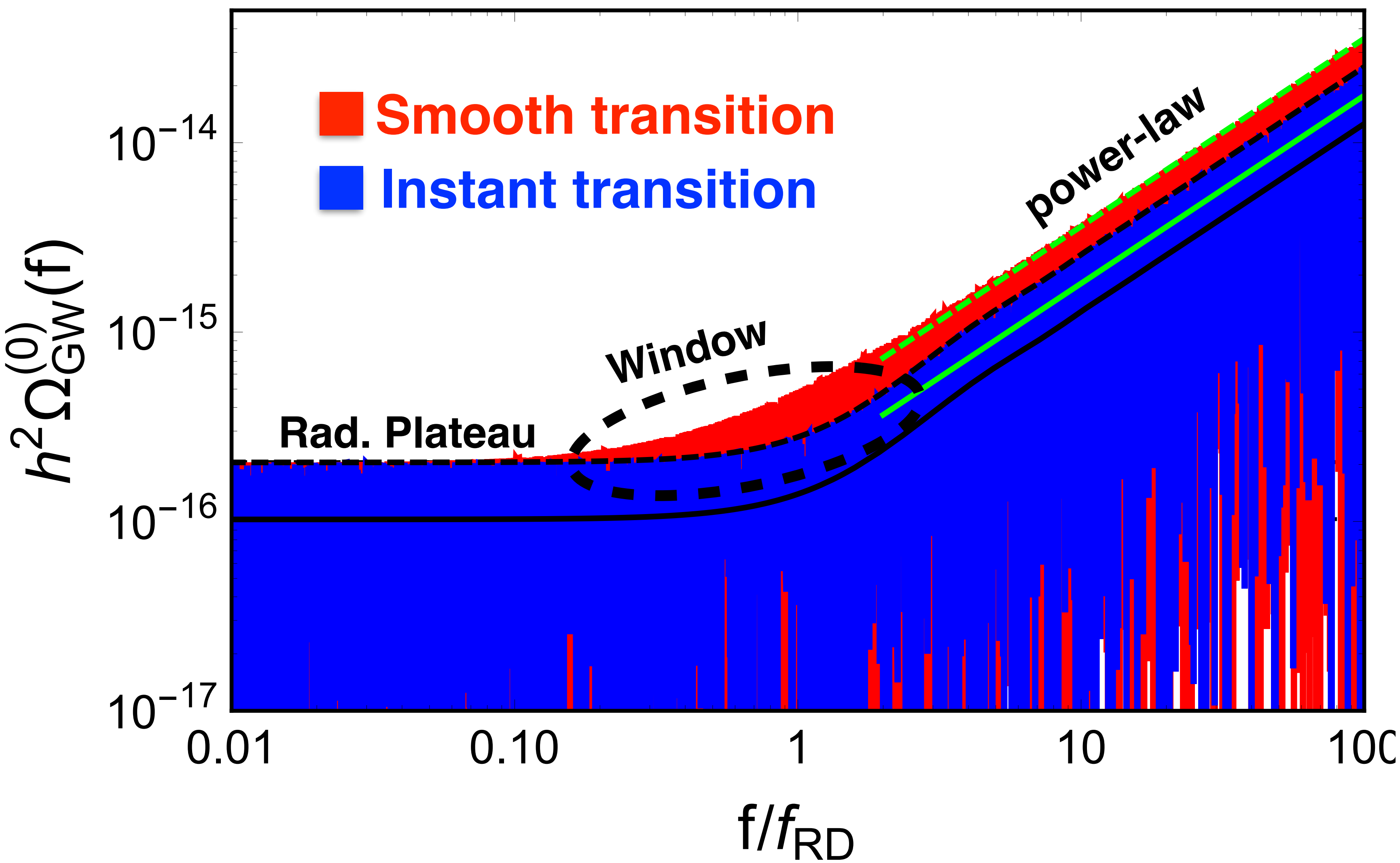

Real signal: highly oscillatory

Stochastic Signal: average measurement

$$\langle \dot h_{ij}(f)\dot h_{ij}(f)\rangle = \mathcal{P}_h(f)$$

Inflationary GW background

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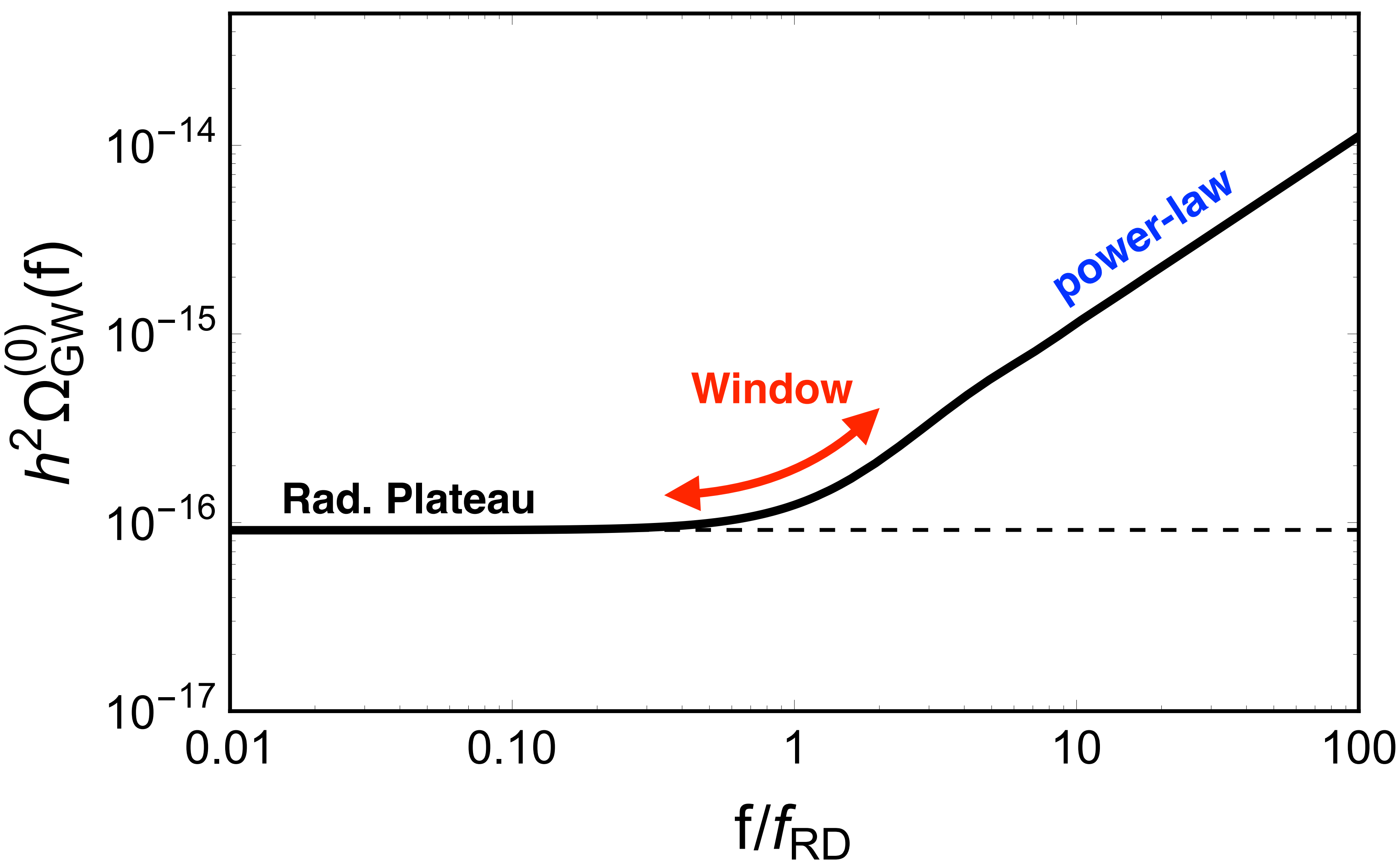

This is the oscillation averaged spectrum !

Inflationary GW background

$$\Omega^{(0)}_{\rm GW}(f) = \underbrace{\Omega^{(0)}_{\rm GW}\Big|_{\rm plateau}}_{\text{Rad. Plateau}} \times \underbrace{\mathcal{W}(f/f_{\rm RD}) \times \mathcal{A}_{\rm s}\left(\frac{f}{f_{\rm RD}}\right)^{n_t(w_s)}}_{\text{Transfer Funct. Stiff Period}}\,, \qquad \Omega^{(0)}_{\rm GW}\Big|_{\rm plateau} \simeq 2\cdot 10^{-16}\left(\frac{H_*}{H_{\rm max}}\right)^2$$

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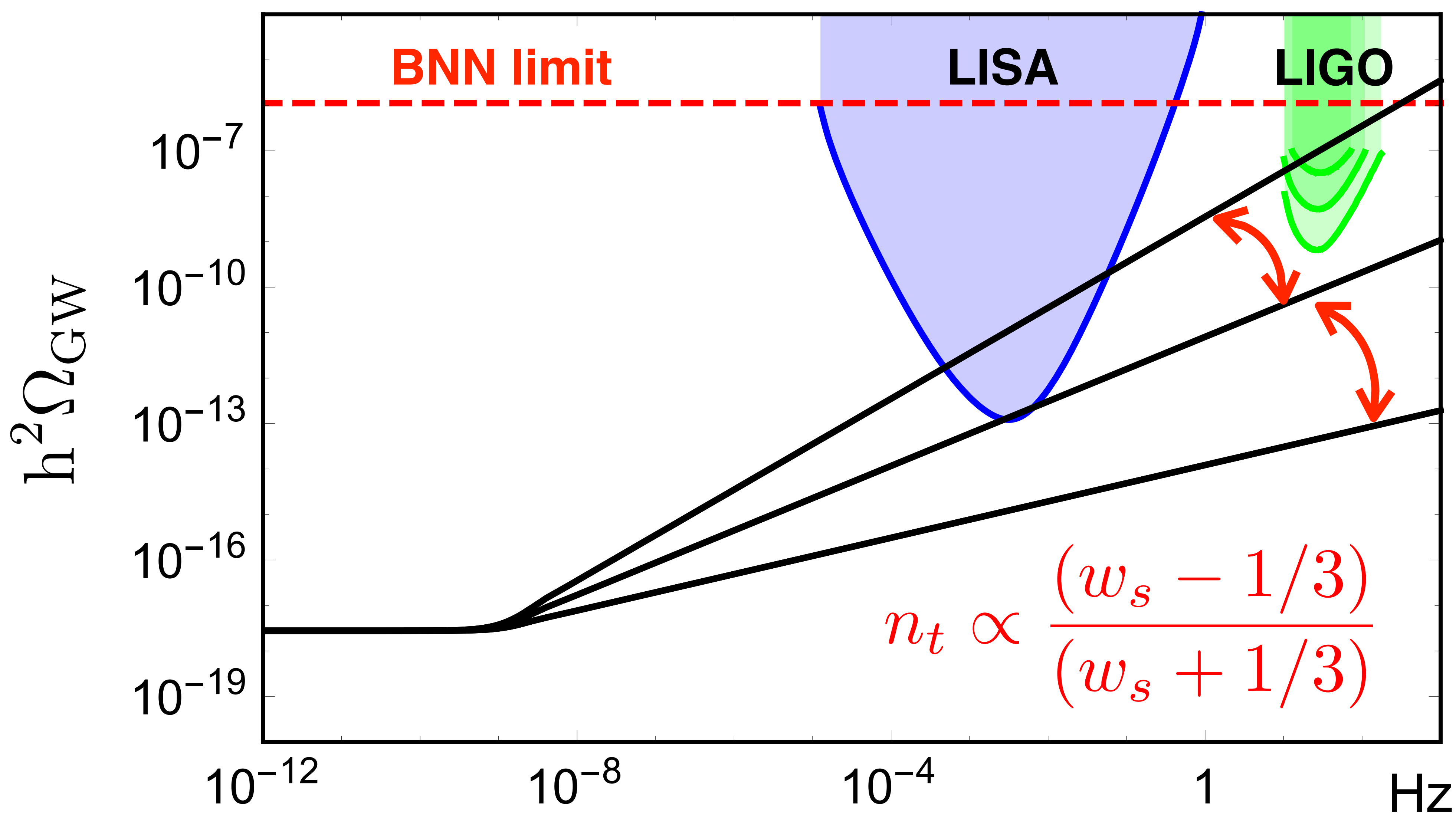

Slope/Tilt
(EoS Stiff Period)

Inflationary GW background

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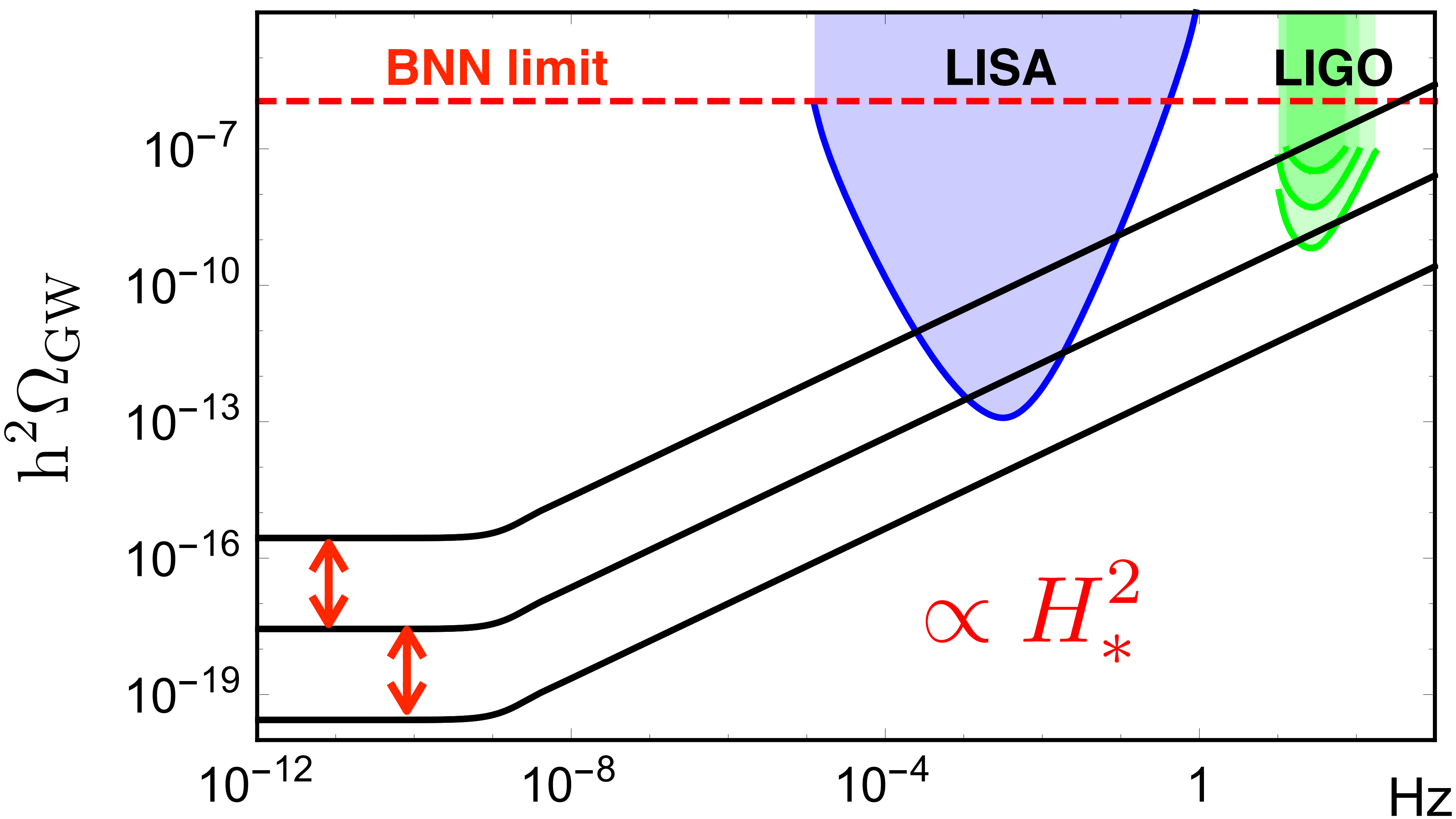

Overall Amplitude
(Energy Scale Inflation)

Inflationary GW background

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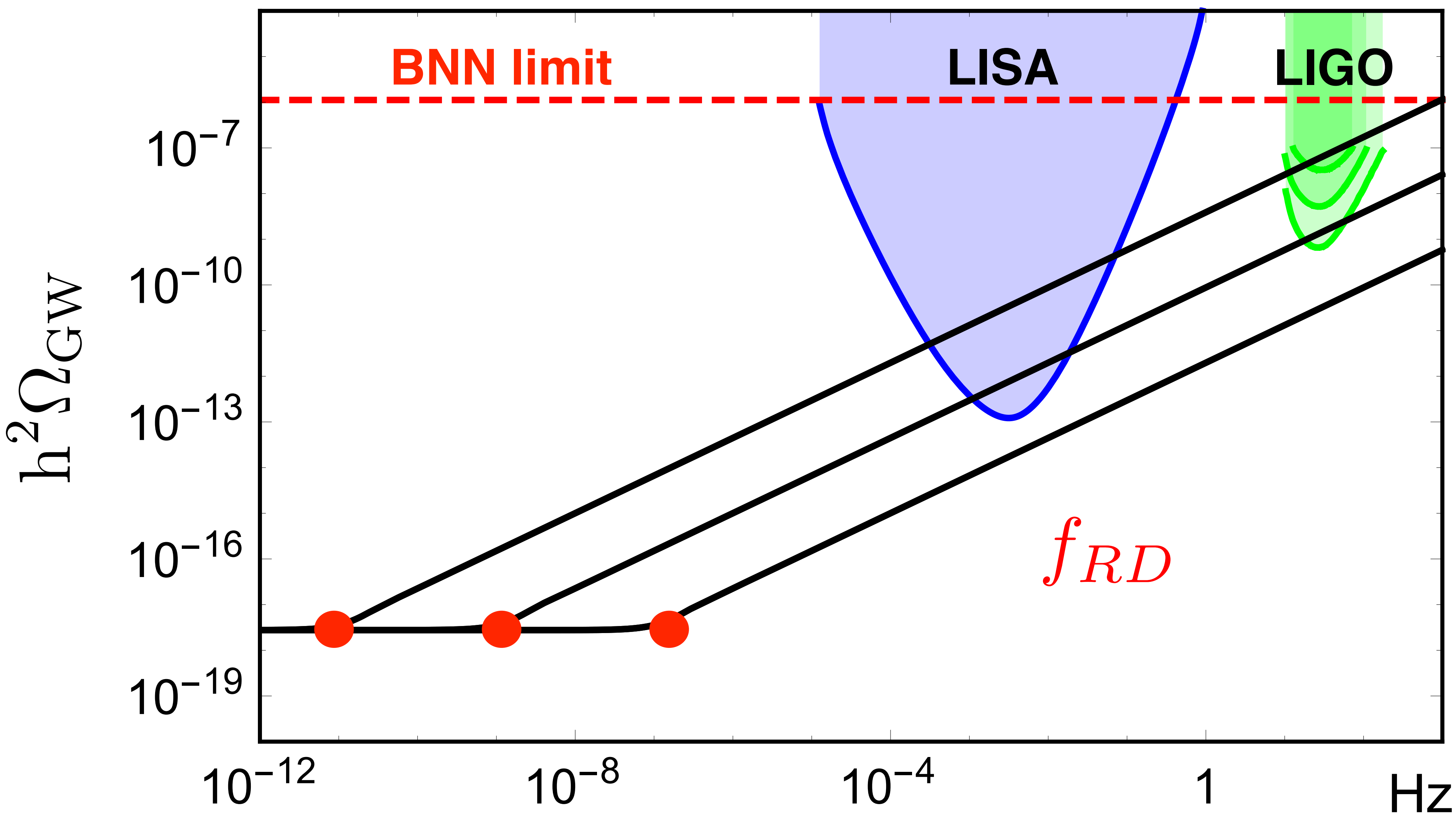


Freq. RD

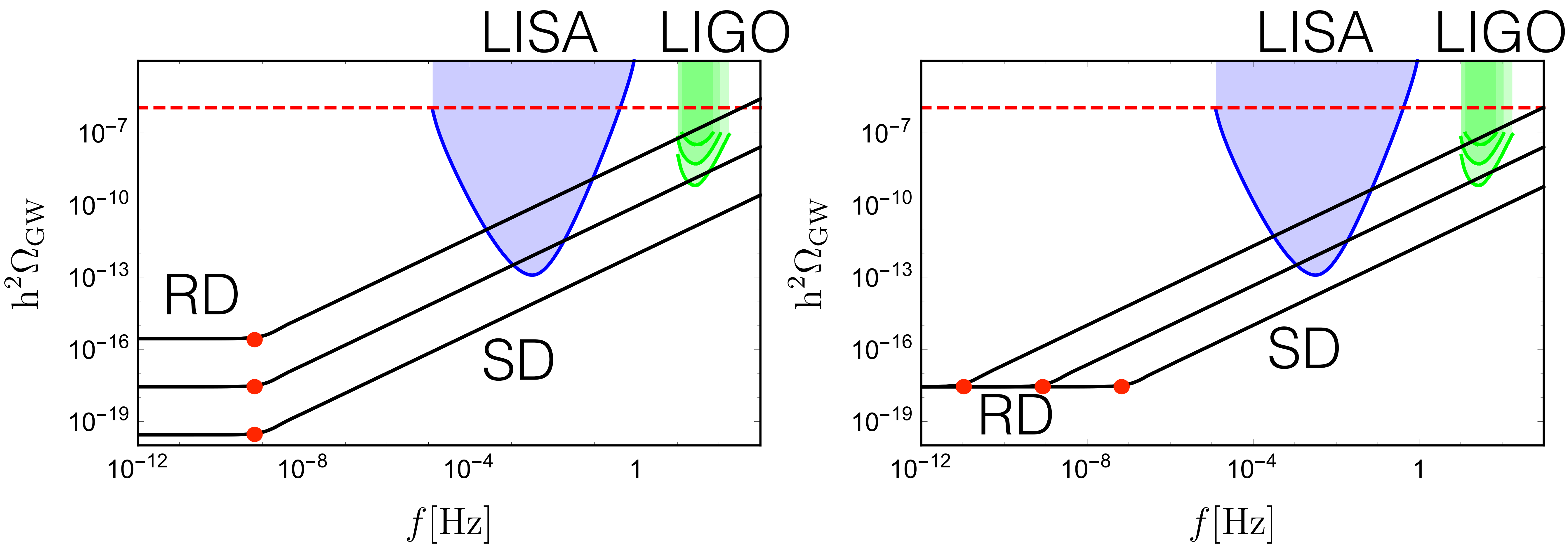
$$k_{\rm RD} = a_{\rm RD} H_{\rm RD}$$

$$f_{\rm RD} \equiv k_{\rm RD}/(2\pi a_0)$$

SD-to-RD transition

)

STIFF EQ of STATE $(1/3 < \omega_s < 1)$



$$\Omega_{\rm GW}(f) \propto H_{\rm inf}^2 \left(\frac{f}{f_{\rm RD}}\right)^{\frac{2(w-1/3)}{(w+1/3)}}$$

Not Scale Invariant !

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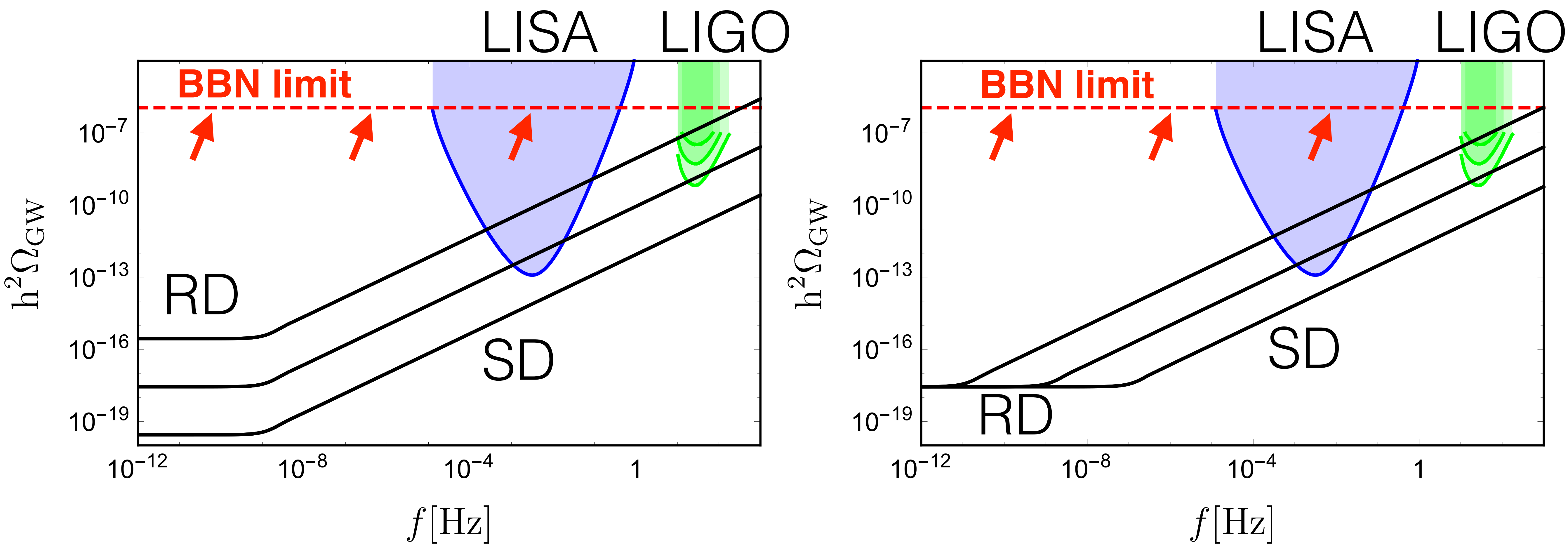

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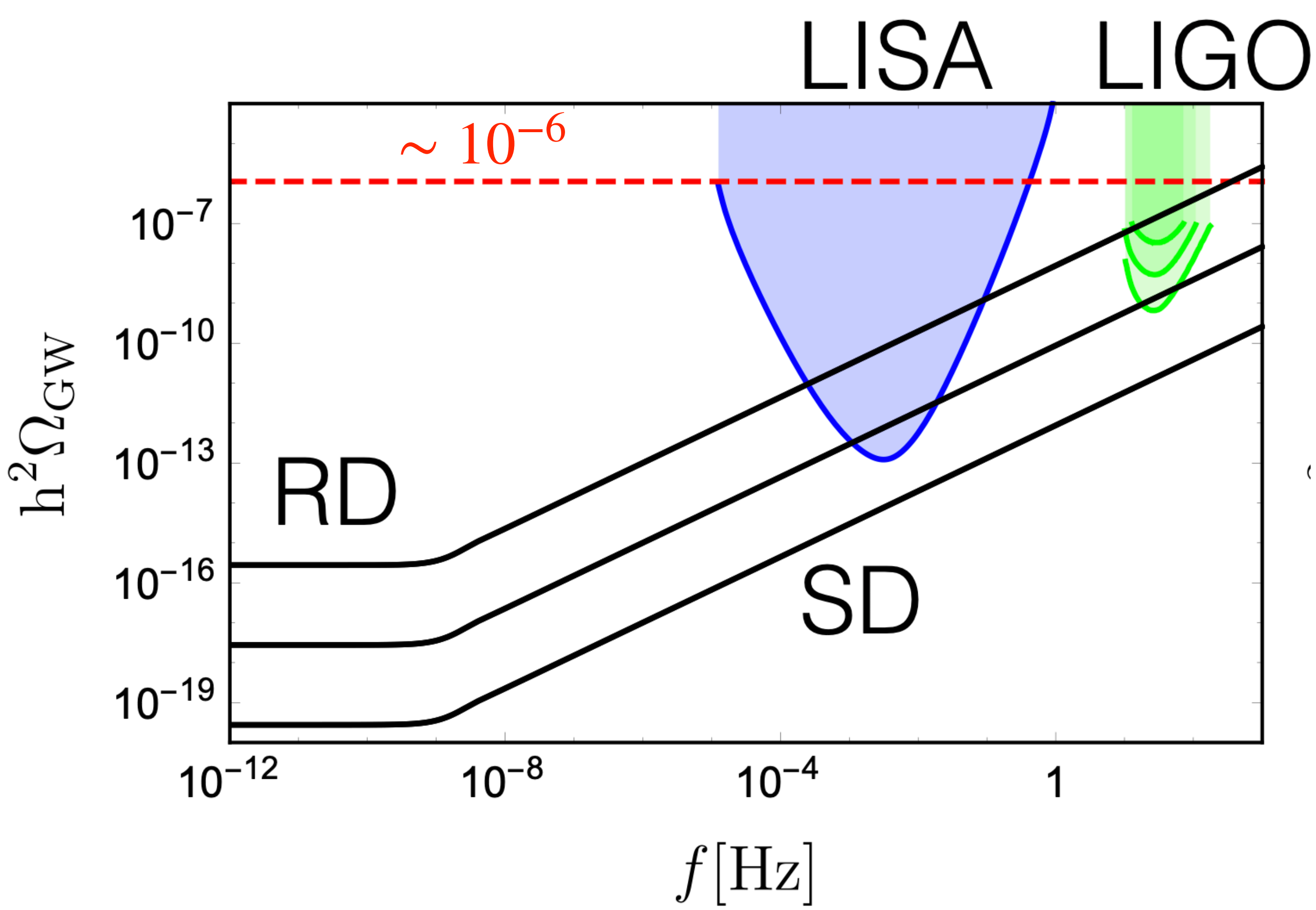

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DGF, Tanin '19, 1905.11960

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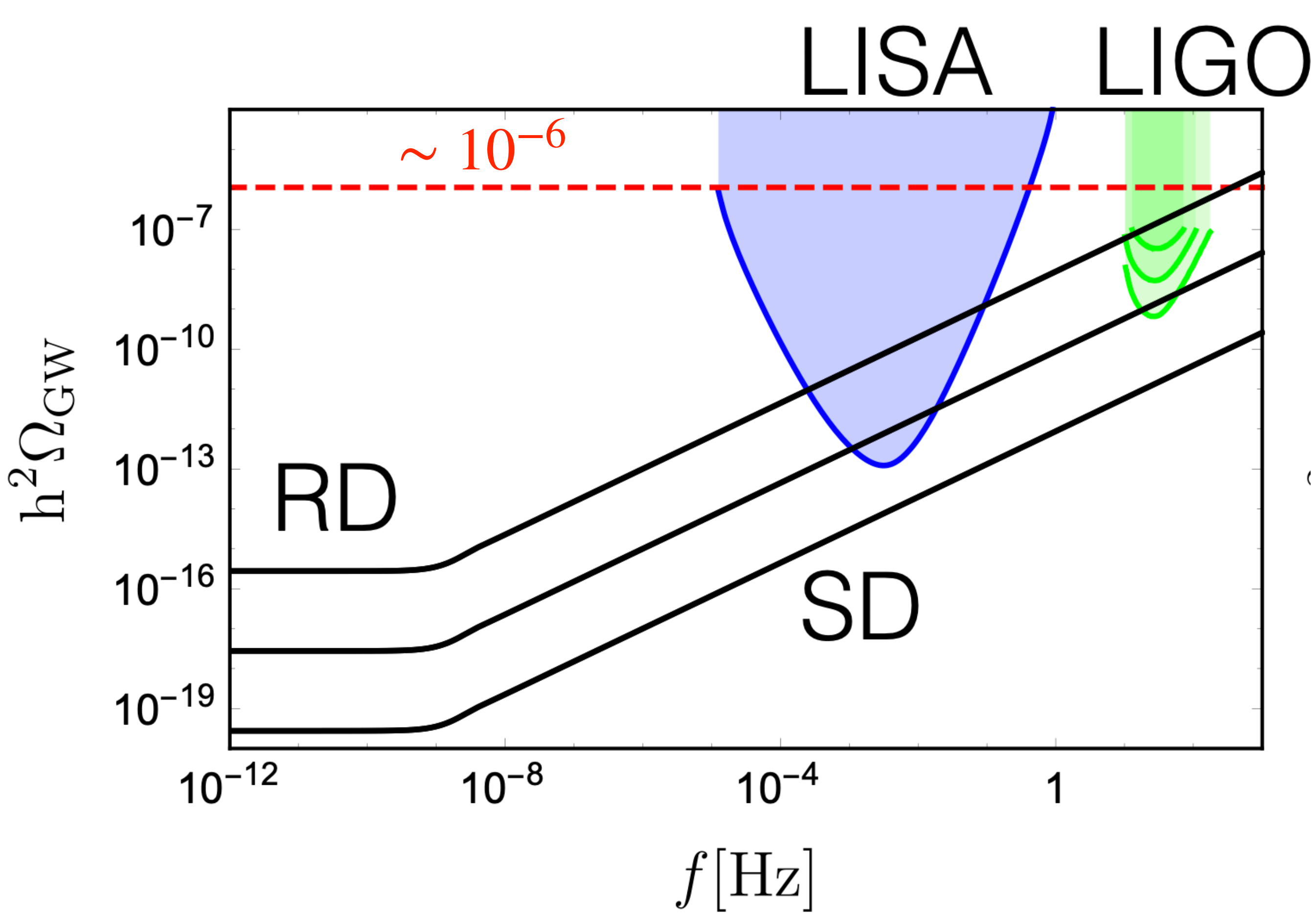

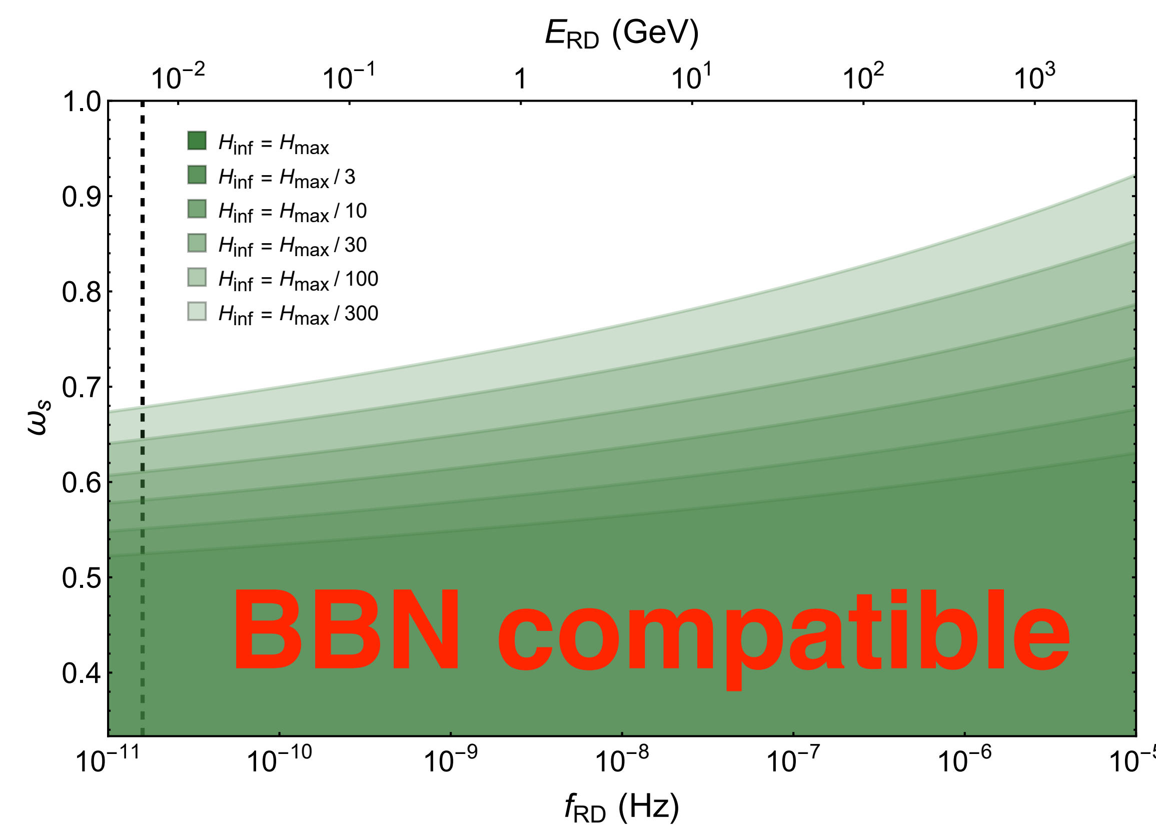

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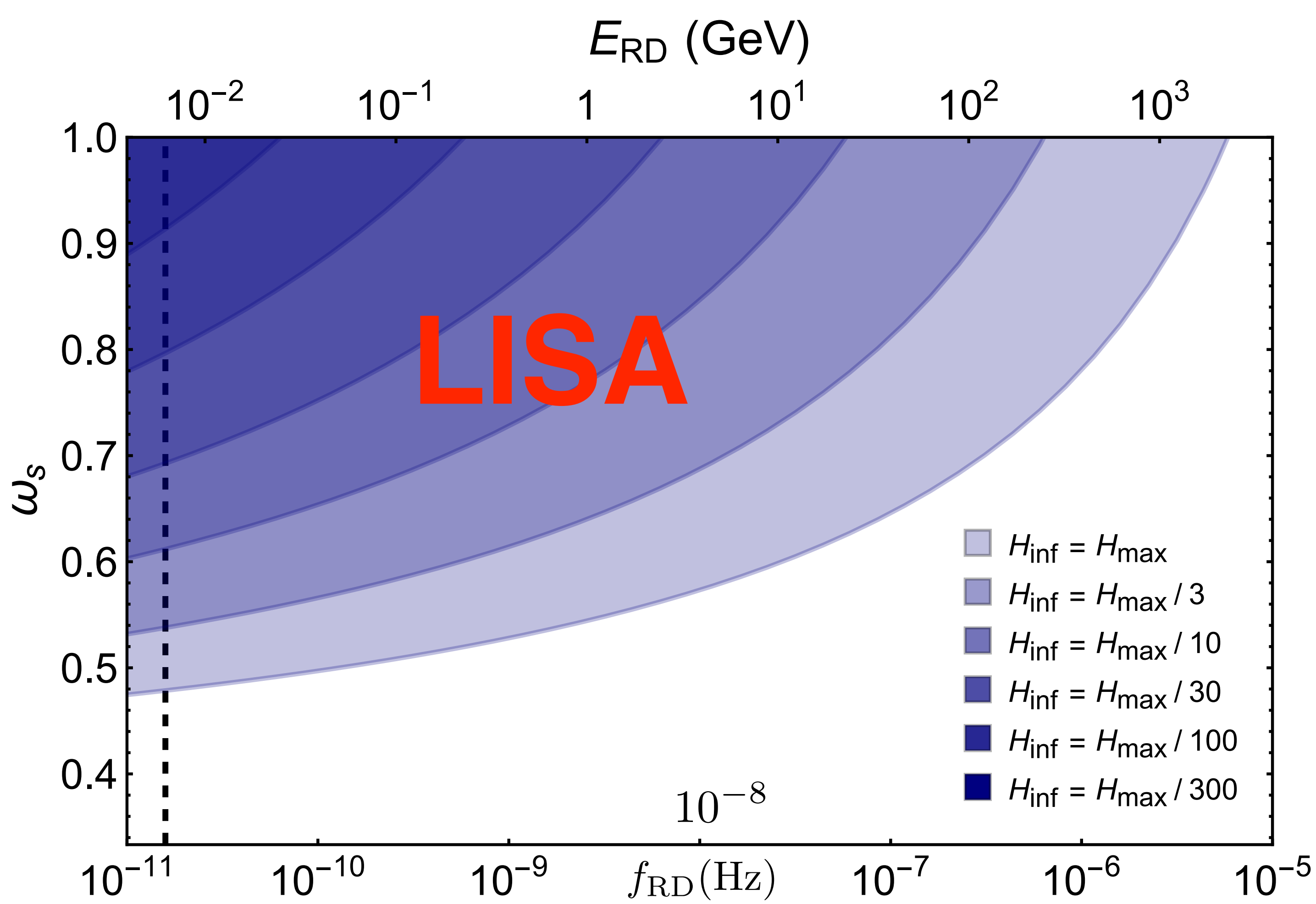

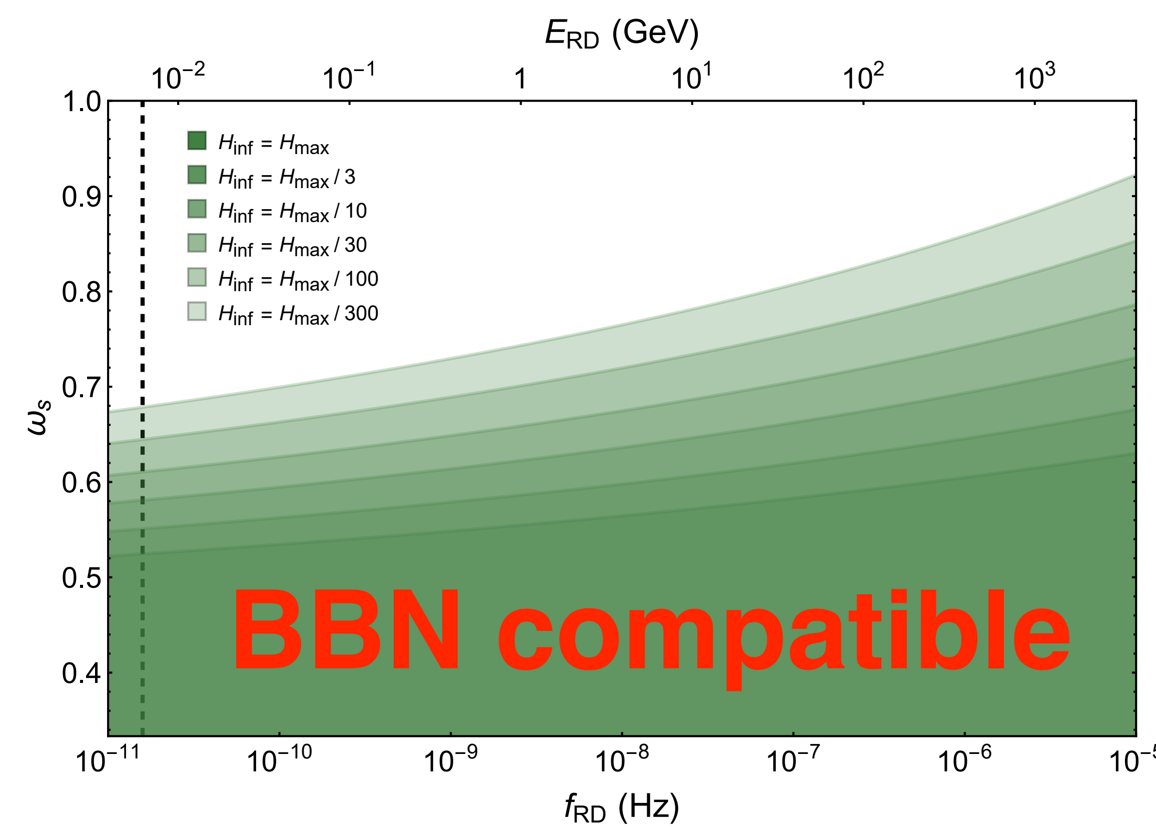

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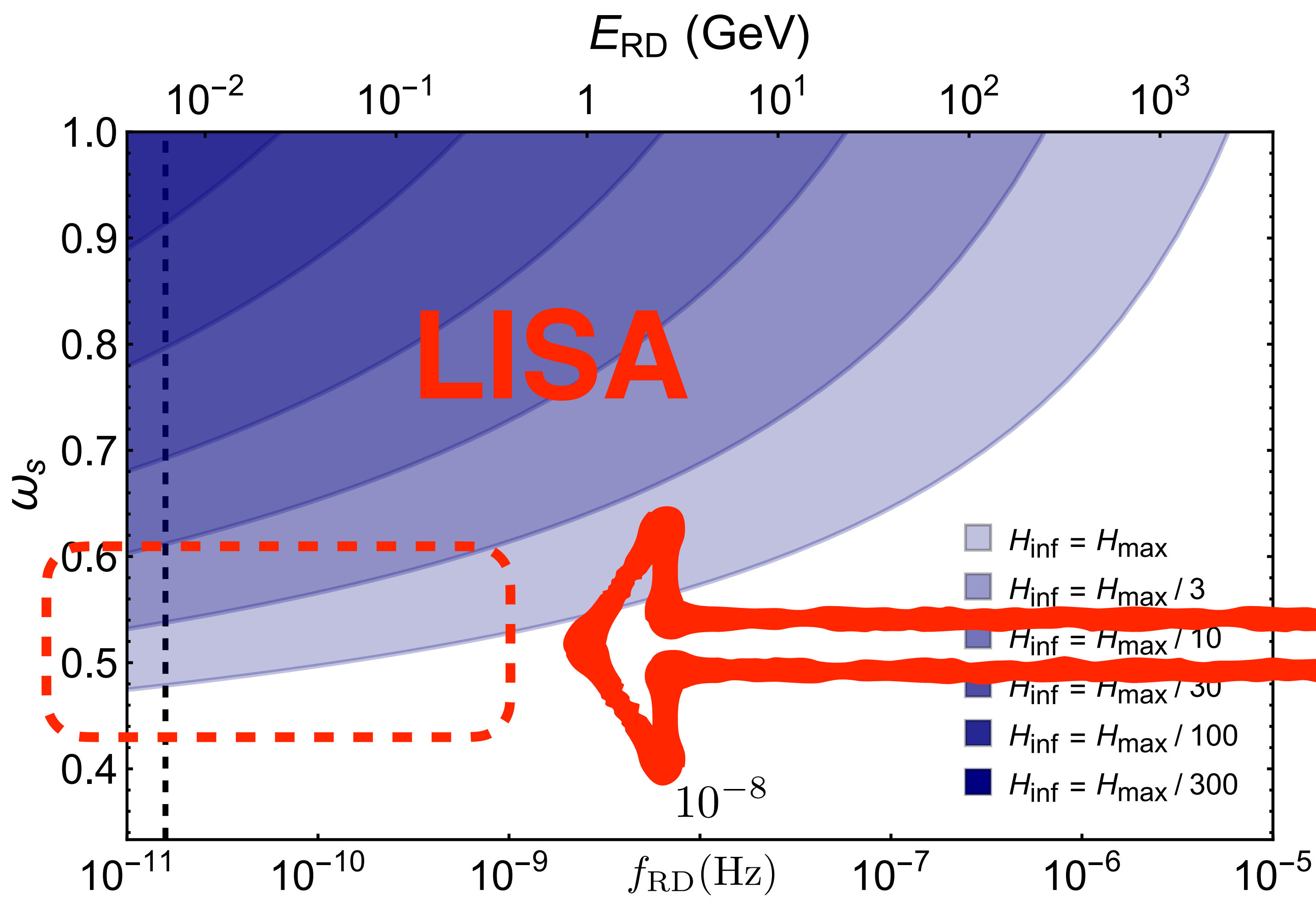

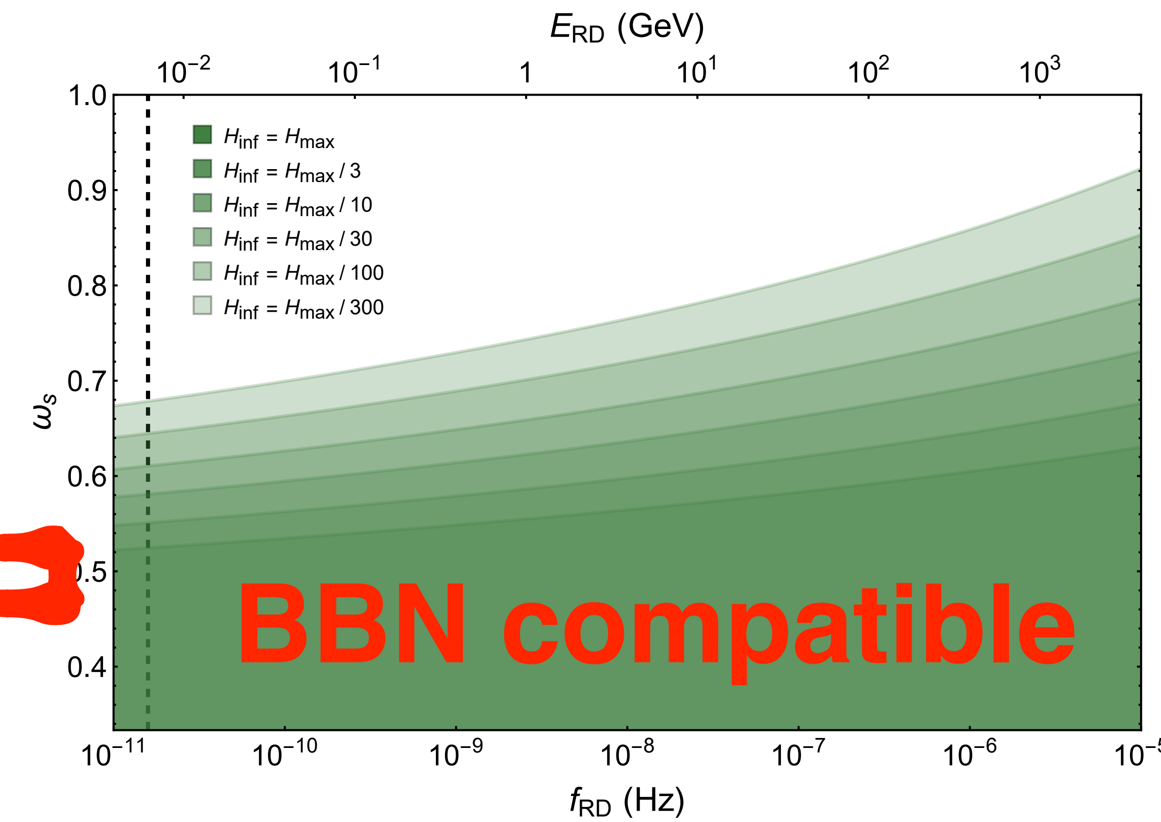

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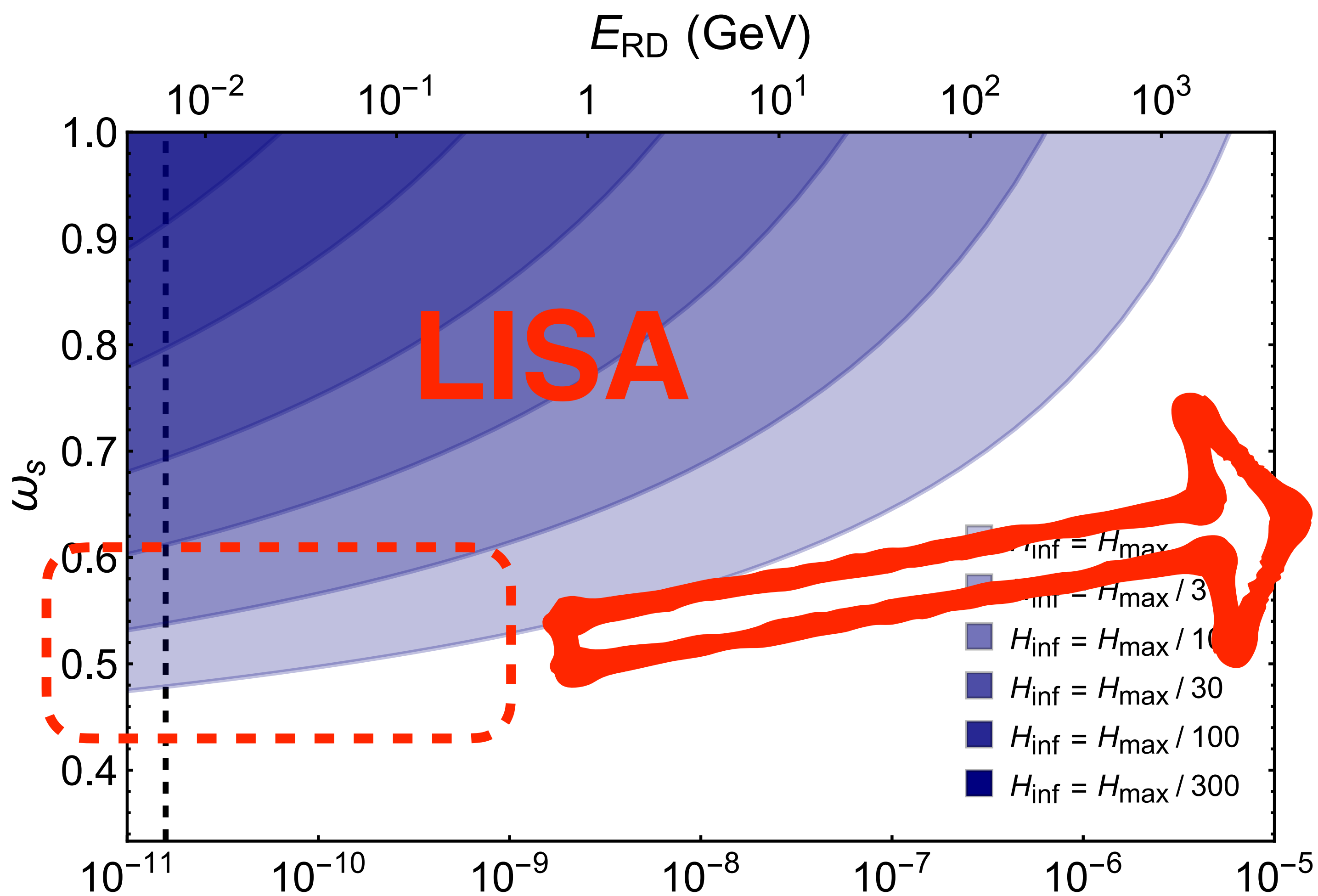
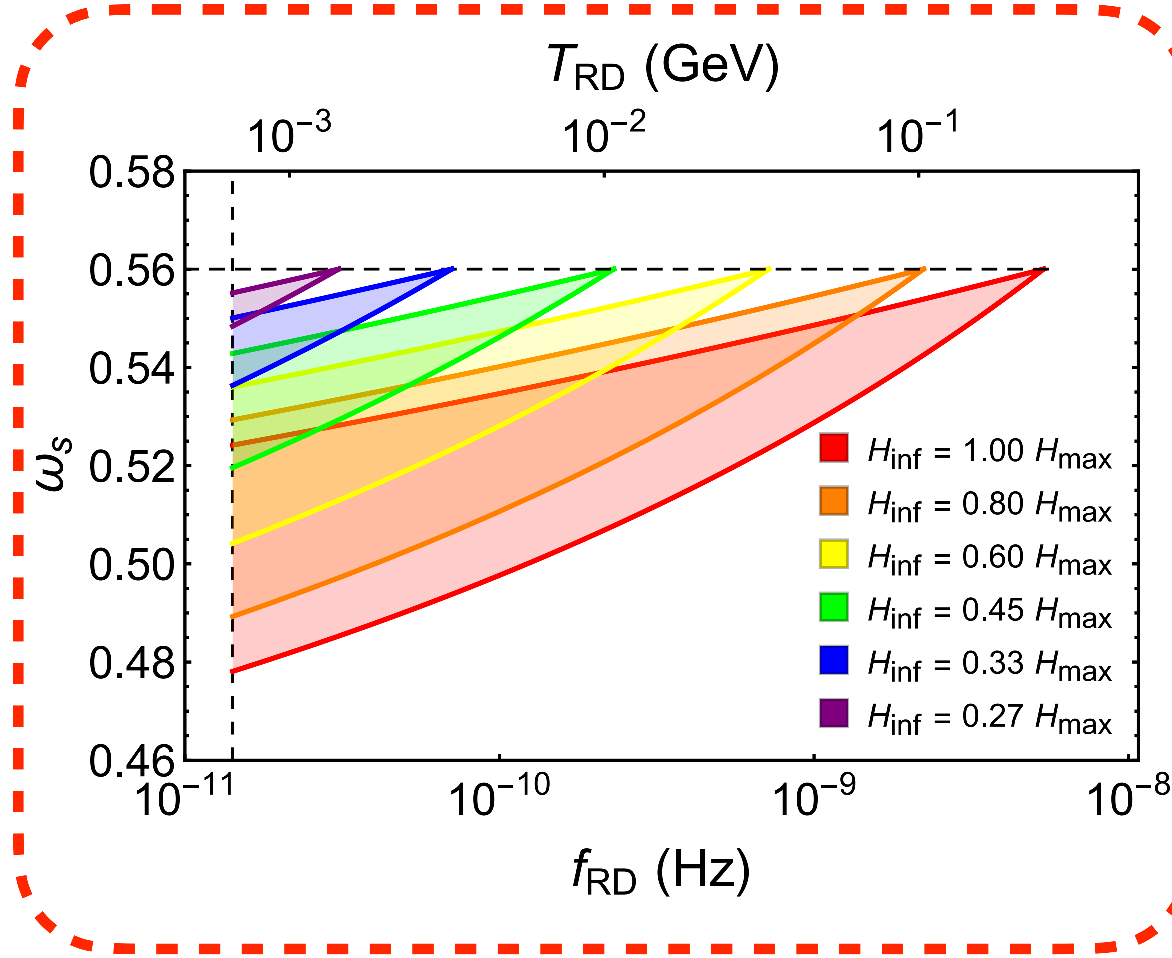


after BBN cut



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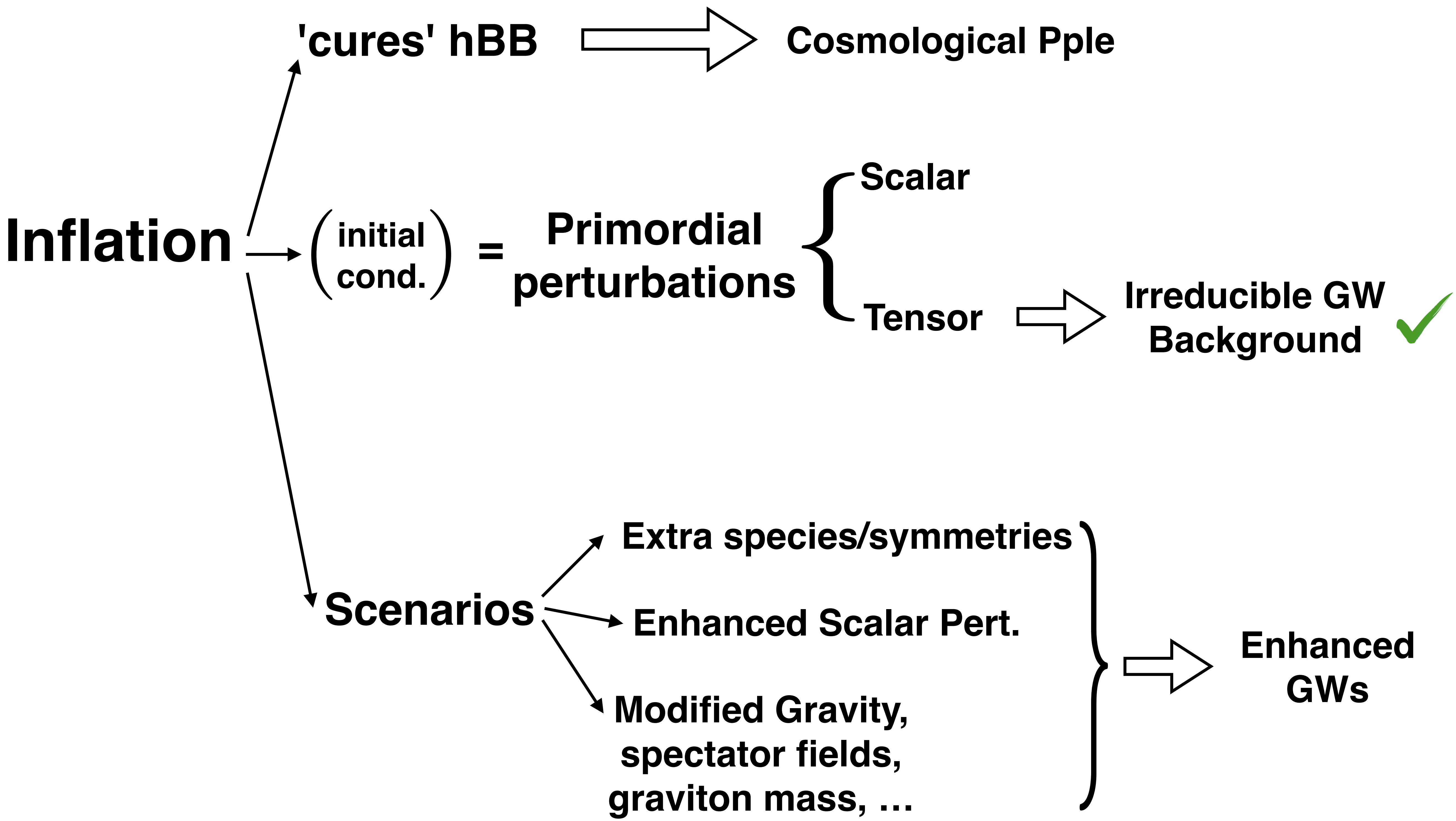
INFLATIONARY COSMOLOGY

'cures' hBB $\Longrightarrow$ Cosmological Pple

Inflation $\rightarrow$ (initial cond.) = Primordial perturbations $\begin{cases} \text{Scalar} \\ \text{Tensor} \Longrightarrow \text{Irreducible GW Background} \end{cases}$

INFLATIONARY COSMOLOGY



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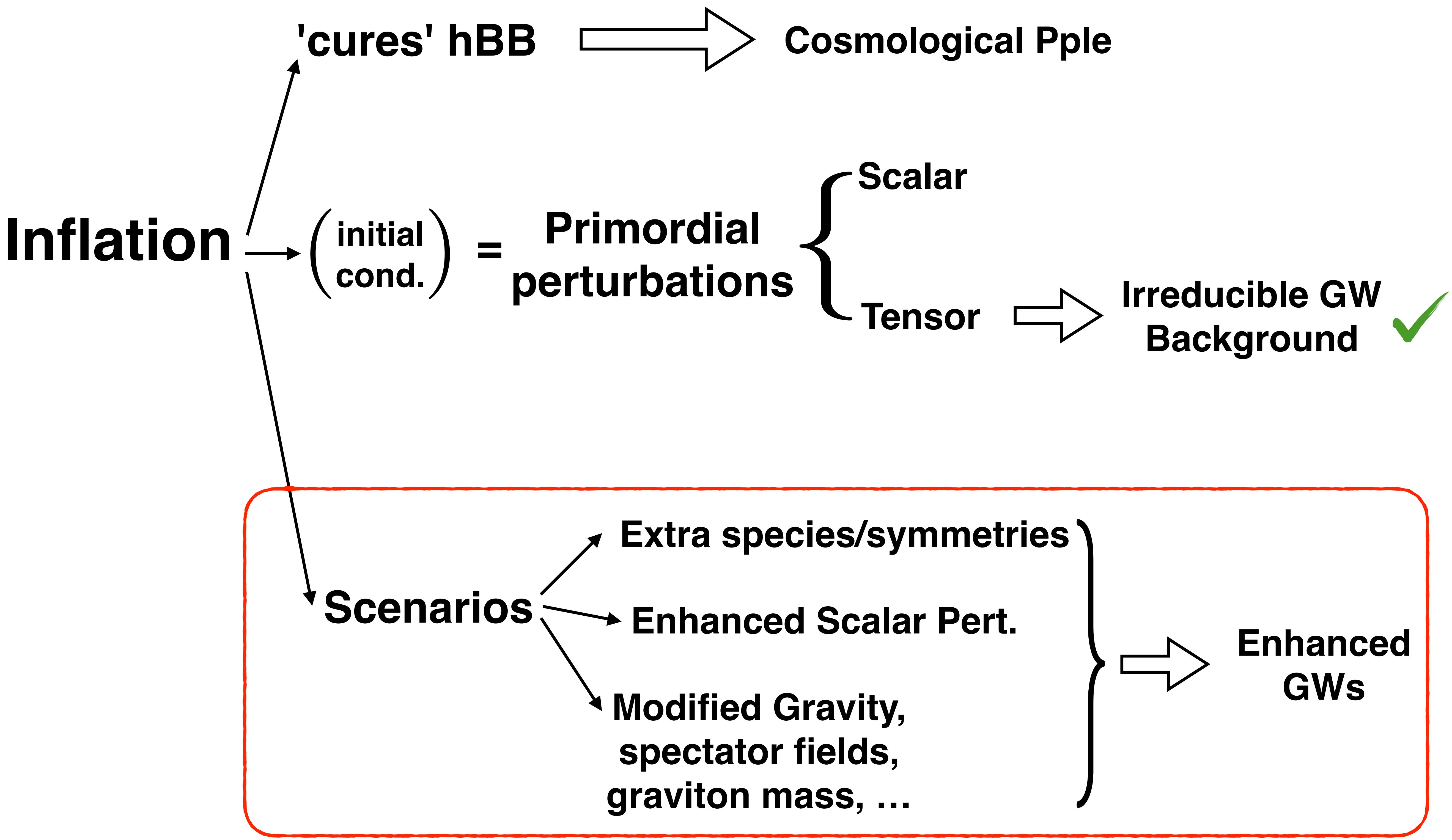

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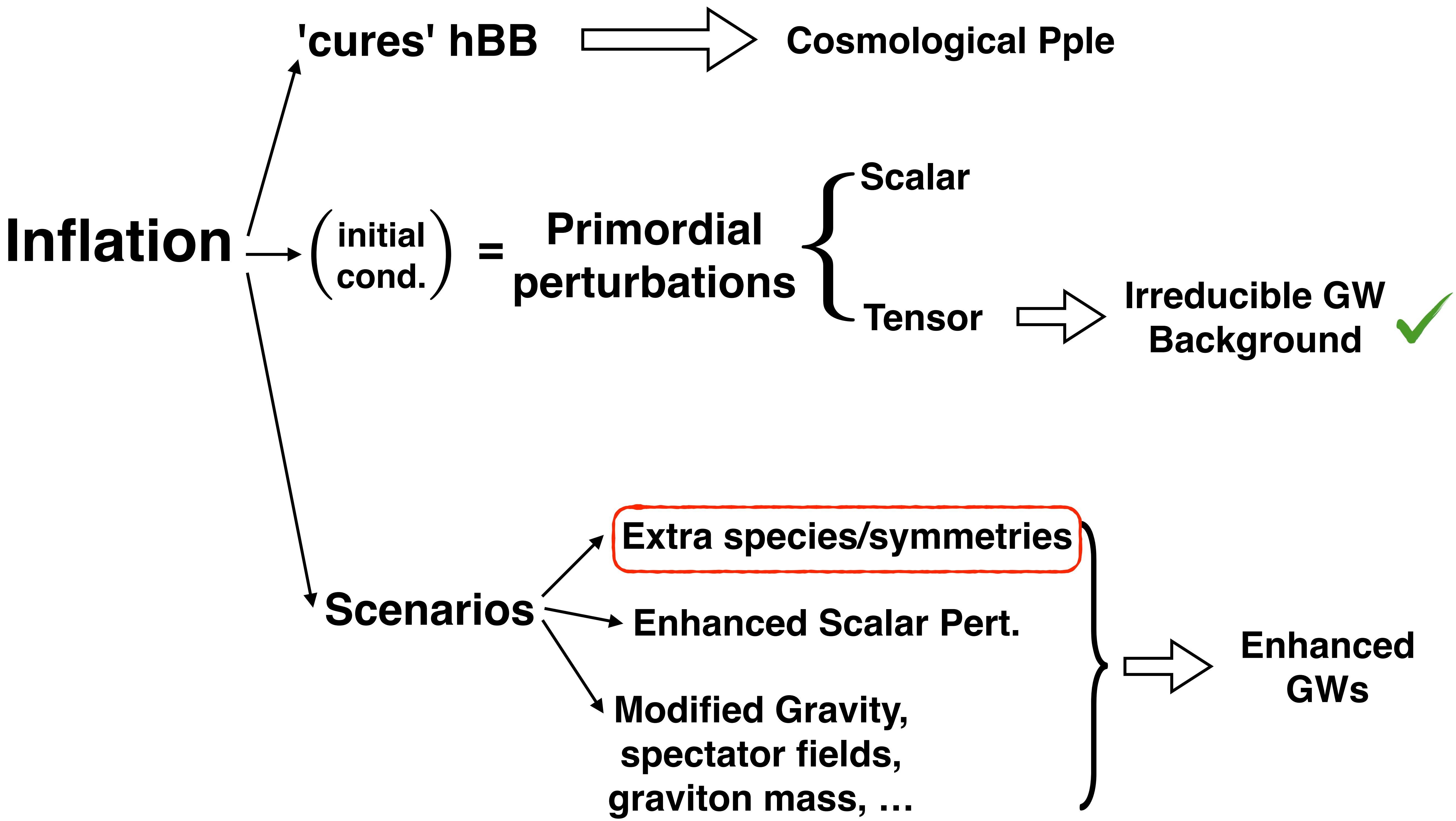

INFLATIONARY MODELS

Axion-Inflation

Freese, Frieman, Olinto '90; ...

Shift symmetry $\phi \to \phi + C$ on couplings to other fields

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Freese, Frieman, Olinto '90; . . .

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$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + V_{\cancel{\text{shift}}}(\phi) + \frac{c_\psi}{f}\,\partial_\mu \phi\,\bar{\psi}\,\gamma^\mu\,\gamma_5\,\psi + \frac{\alpha}{f}\,\phi\,F_{\mu\nu}\,\tilde{F}^{\mu\nu}}$$

derivative couplings to: fermions gauge fields

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derivative couplings to: fermions gauge fields

Not the QCD axion;

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fermions

gauge fields

(derivative couplings)

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fermions

gauge fields

(derivative couplings)

$= \partial_\mu K^\mu$

topological term

$[\phi \partial_\mu K^\mu = K^\mu \partial_\mu \phi]$

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breaks shift-symm

fermions

gauge fields

(derivative couplings)

$= \partial_\mu K^\mu$
topological term
$[\phi\partial_\mu K^\mu = K^\mu\partial_\mu\phi]$

With shift symmetry, $\Delta V \propto V_{\cancel{\text{shift}}}$ $\Rightarrow$ **Protected against radiative corrections !**

INFLATIONARY MODELS

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$

$$V(\varphi) + \frac{\phi}{f} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

inflaton ϕ = pseudo-scalar axion

INFLATIONARY MODELS

Axion-Inflation

Freese, Frieman, Olinto '90; . . .

Shift symmetry $\phi \to \phi + const.$

$$V(\varphi) + \frac{\phi}{f} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

inflaton ϕ = pseudo-scalar axion

Photon: 2 helicities

$$\vec{A}(\tau, \mathbf{x}) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^{3/2}} \left[\vec{\epsilon}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) A_\lambda(\tau, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.} \right]$$

$$\left[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}') \right] = \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

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$$\vec{A}'' - \nabla^2 \vec{A} - \frac{1}{f} \phi' \, \vec{\nabla} \times \vec{A} = 0 \quad \Rightarrow \quad \left[\frac{\partial^2}{\partial \tau^2} + k^2 \pm \frac{2k\xi}{\tau} \right] A_\pm(\tau, k) = 0,$$

$$\boxed{\xi \equiv \frac{\dot{\phi}}{2fH}}$$

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Chiral instability

$$\boxed{A_+(\tau, k) \cong \frac{1}{\sqrt{2k}} \left(\frac{k}{2\xi a H}\right)^{1/4} e^{\pi\xi - 2\sqrt{2\xi k/(aH)}}}$$

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$$A_+ \propto e^{\pi\xi}\,, \quad |A_-| \ll |A_+|$$

A_+ exponentially amplified,

A_- has no amplification

INFLATIONARY MODELS

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Gauge field excitation creates chiral GWs !

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$$h''_{ij} + 2\mathcal{H} h'_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi_{ij}^{\mathrm{TT}} \propto \{E_i E_j + B_i B_j\}^{TT}$$

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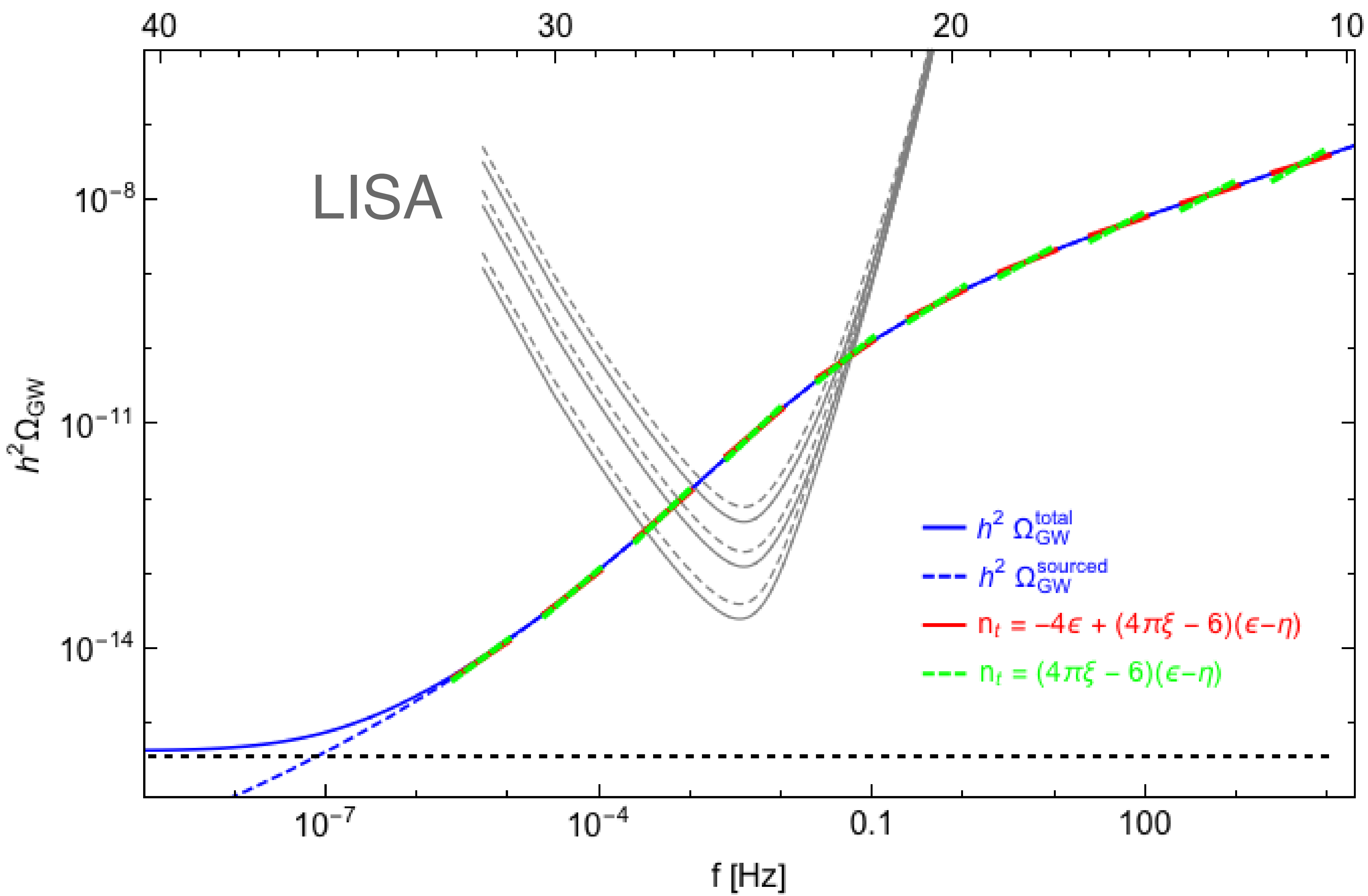
GW mostly* one-chirality ⇐ A_μ **Chiral**

(*why not exactly just one?)

INFLATIONARY MODELS

Axion-Inflation

GW energy spectrum today



Gauge fields
source a
Blue-Tilted
+ Chiral
+ Non-G
GW background

INFLATIONARY MODELS

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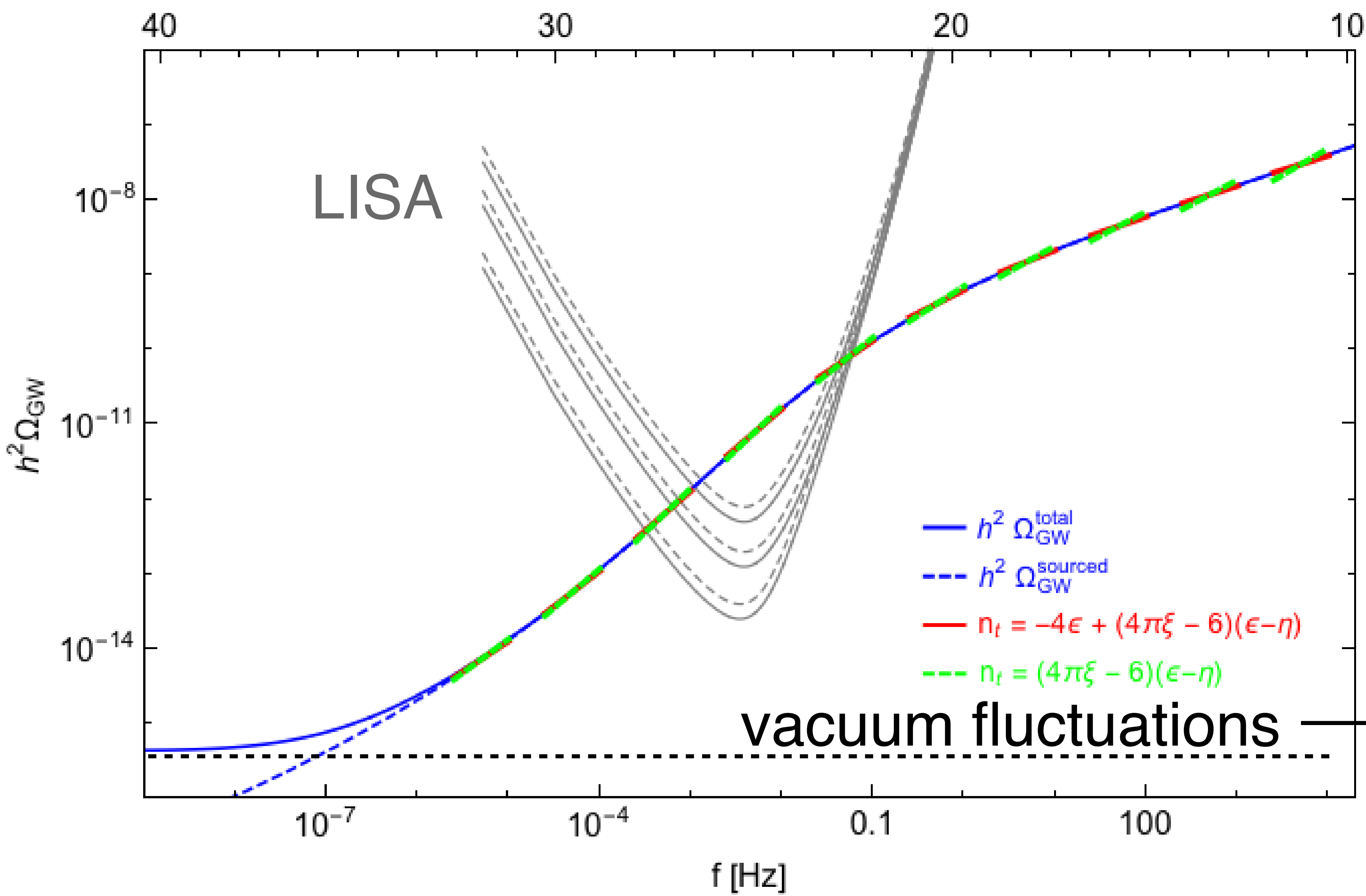

$(\Omega_{\rm Rad}/24)\Delta_h^2$

Gauge fields source a Blue-Tilted + Chiral + Non-G GW background

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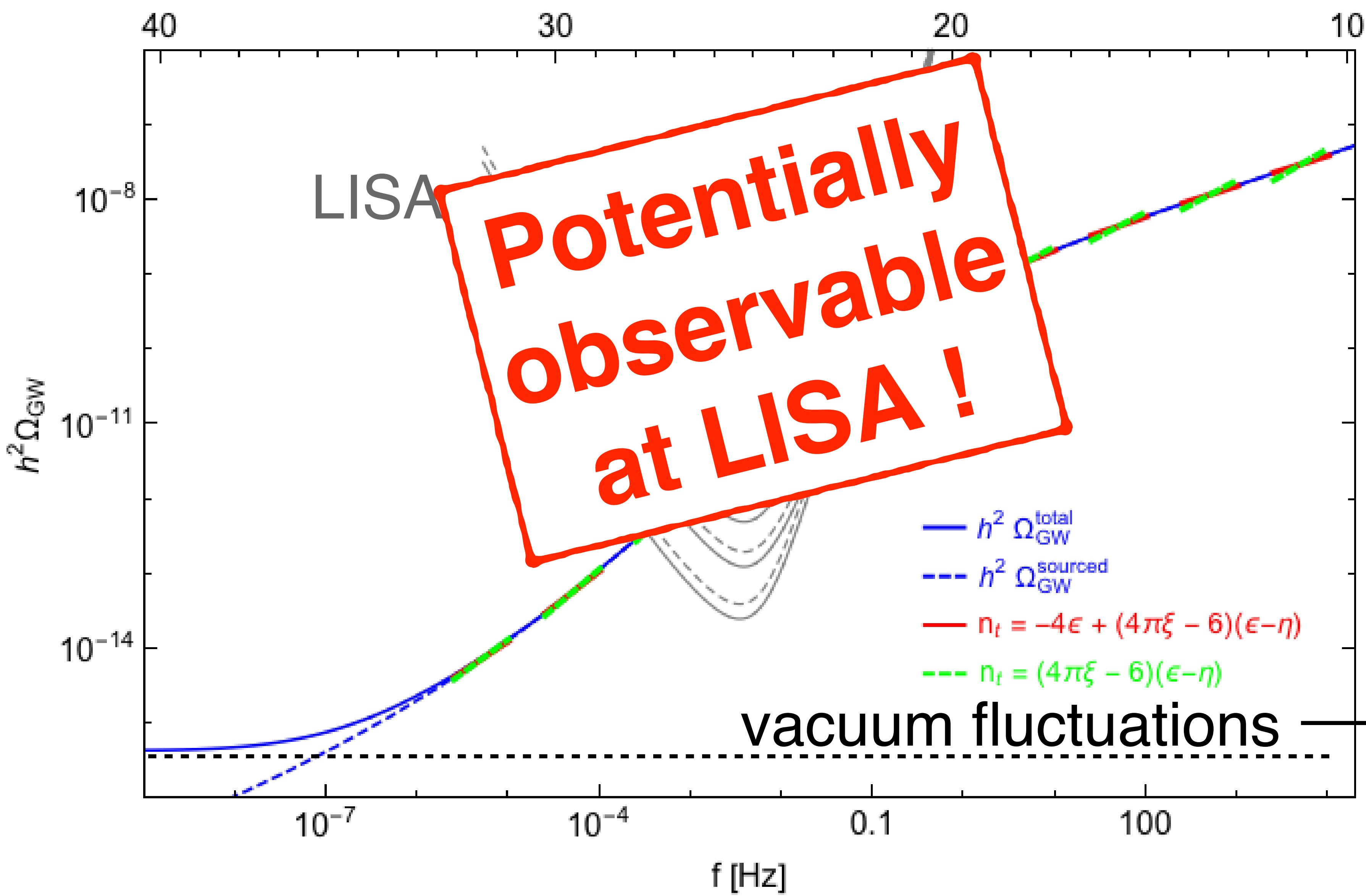

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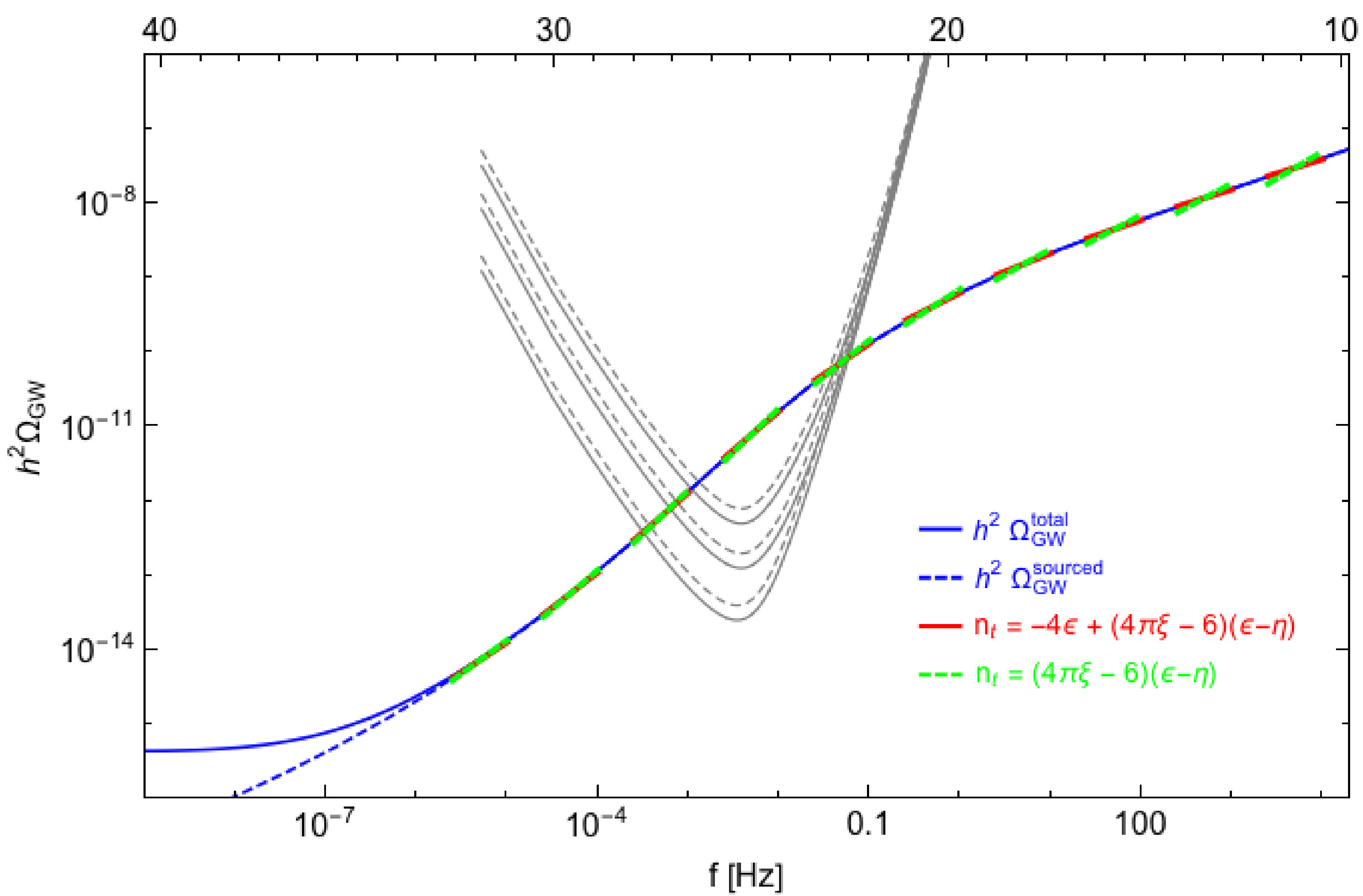
INFLATIONARY MODELS

Axion-Inflation

Bartolo et al ‘16

$$h^2\Omega_{\rm gw} = A_* \left(\frac{f}{f_*}\right)^{n_T}$$



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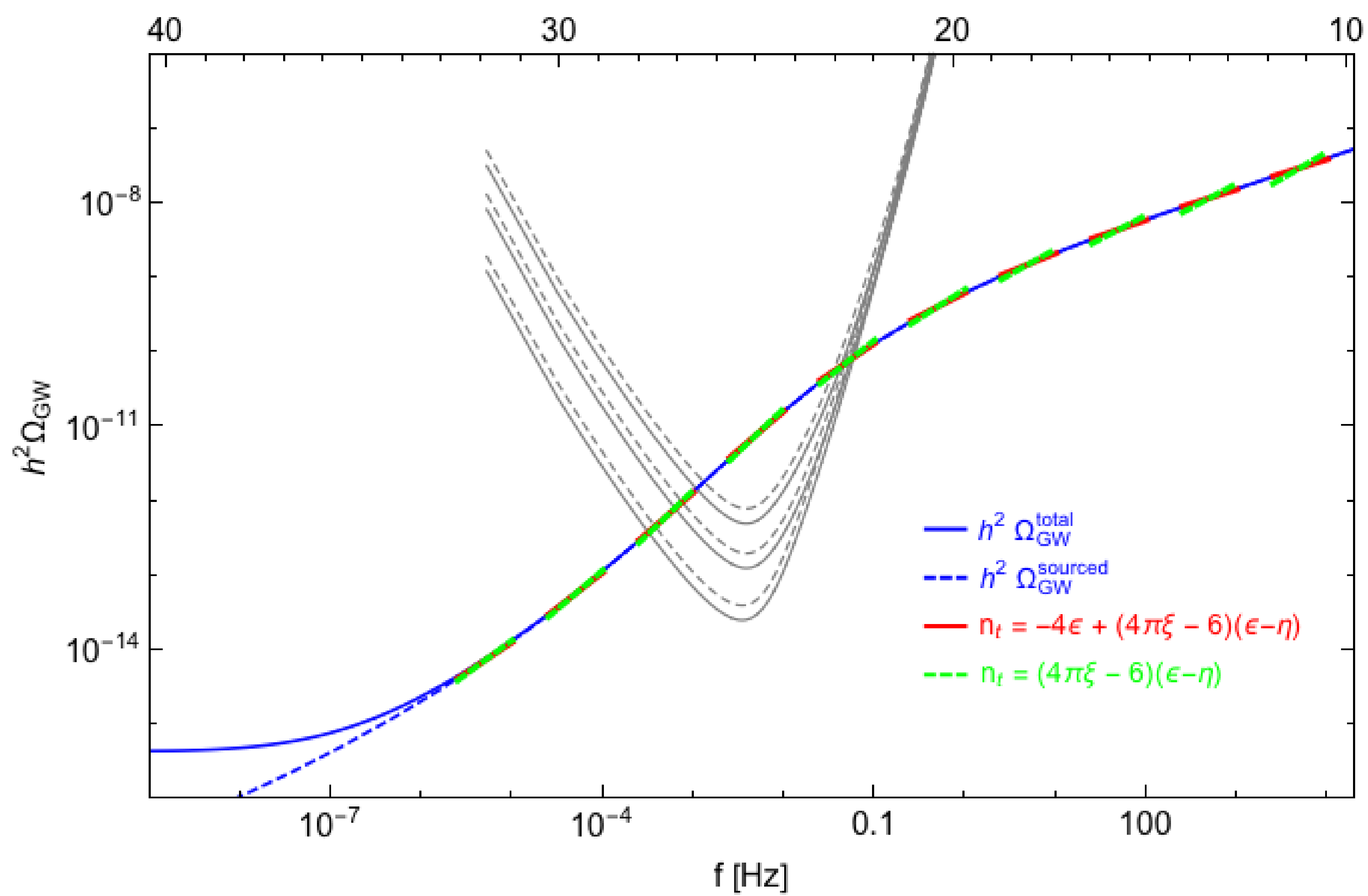
$$h^2\Omega_{\rm gw} = A_* \left(\frac{f}{f_*}\right)^{n_T} \rightarrow \boxed{n_T \simeq (4\pi\xi - 6)(\epsilon_H - \eta)}$$

$$\boxed{1.5 \cdot 10^{-13} \frac{H^4}{M_{Pl}^4} \frac{e^{4\pi\xi}}{\xi^6}},$$

$$\boxed{\xi \equiv \frac{\alpha\dot{\phi}}{2fH}}$$

$\boxed{H,\, \xi,\, \epsilon_H - \eta}$ **(3 parameters!)**



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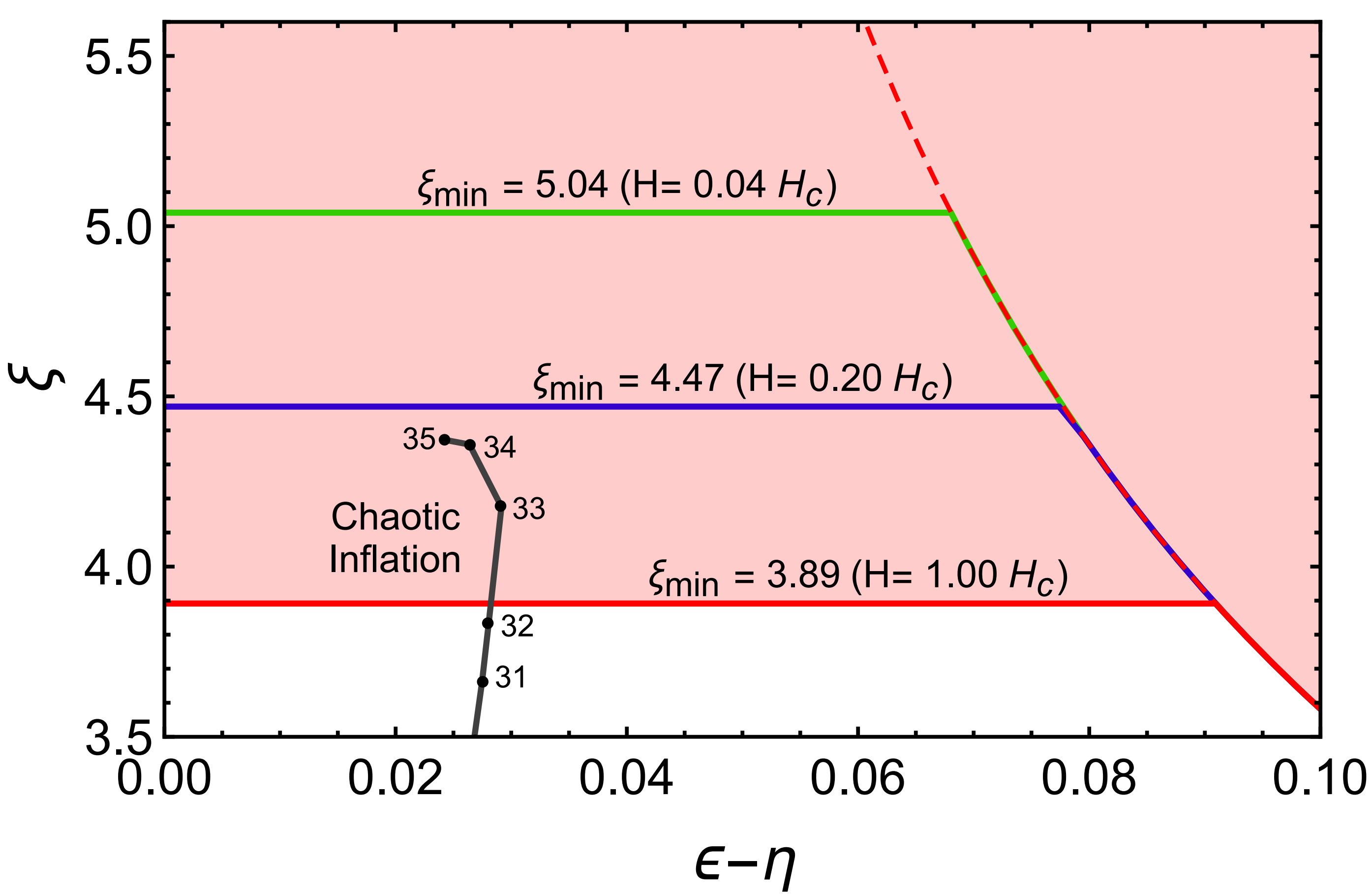
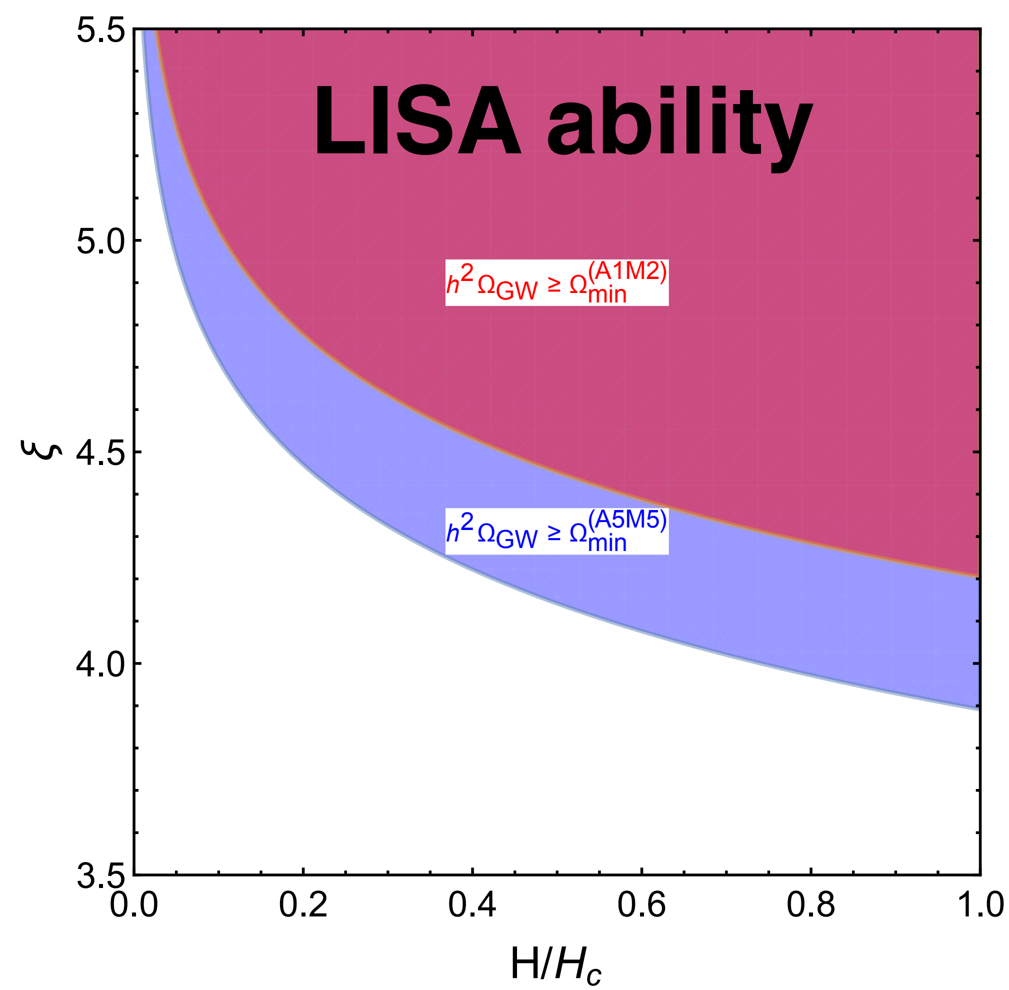
LISA ability





INFLATIONARY MODELS

Axion-Inflation

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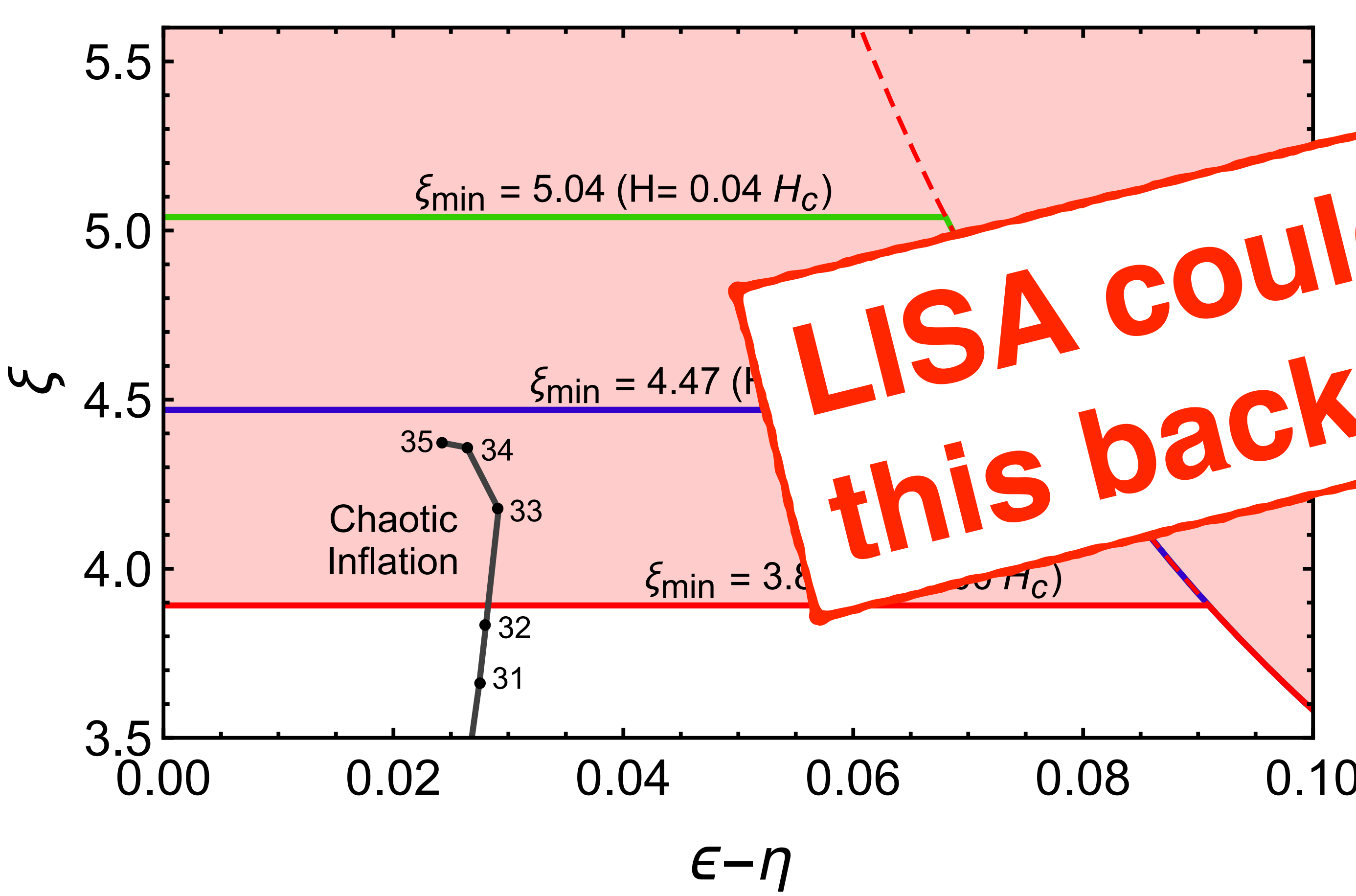

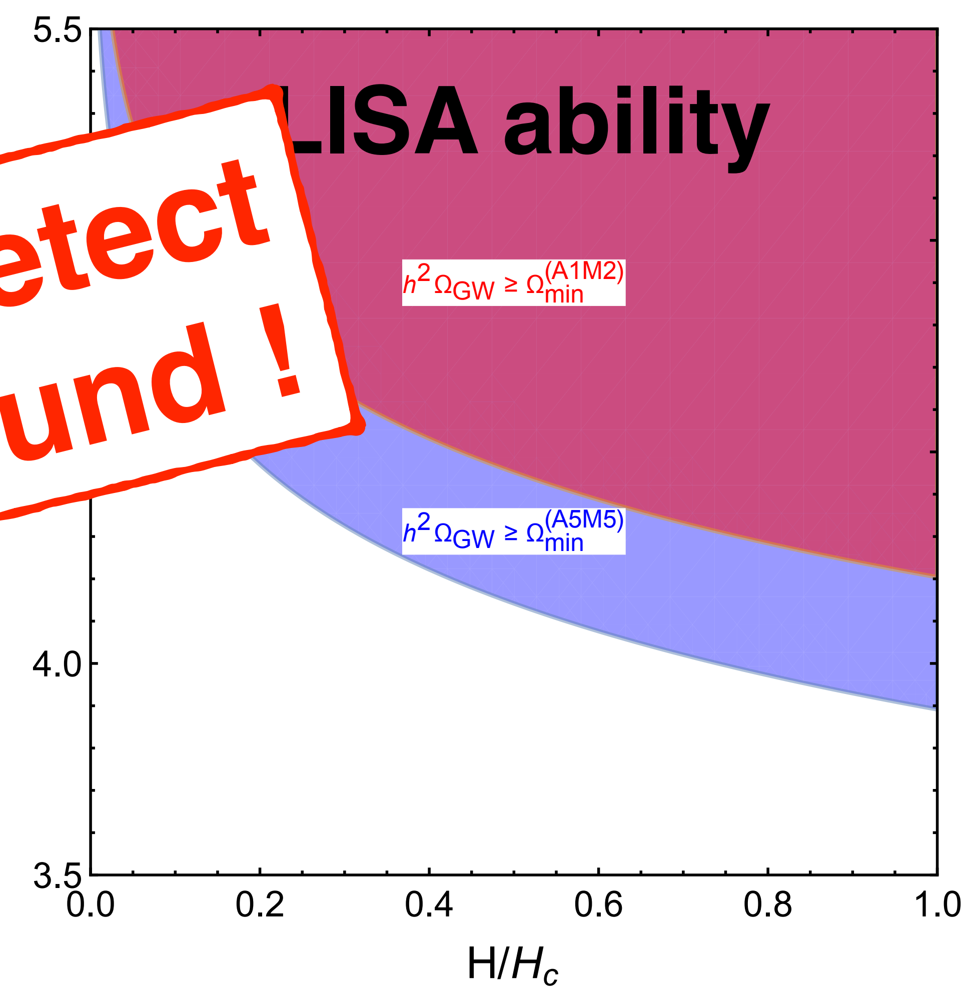

LISA could detect this background !

INFLATIONARY MODELS

Axion-Inflation: ***Shift*** **symmetry** $\longrightarrow$ Natural (chiral) coupling to A_μ

huge excitation of fields ! (photons)

INFLATIONARY MODELS

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(i.e. no need of extra symmetry)

INFLATIONARY MODELS

Axion-Inflation: *Shift* symmetry $\longrightarrow$ Natural (chiral) coupling to A_μ

huge excitation of fields ! (photons)

What if there are arbitrary fields coupled to the inflaton ? $\rightarrow$ large excitation of these fields !?

(i.e. no need of extra symmetry)

will they create GWs?

INFLATIONARY MODELS

fields coupled to the inflaton ? → large excitation ?

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$$-\mathcal{L}_\chi = (\partial\chi)^2/2 + g^2(\phi - \phi_0)^2\chi^2/2$$ **Scalar Fld**

$$-\mathcal{L}_\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi + g(\phi - \phi_0)\bar{\psi}\psi$$ **Fermion Fld**

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - gA_\mu)\Phi)|^2 - V(\Phi^\dagger\Phi)$$ **Gauge Fld ($\Phi = \phi e^{i\theta}$)**

INFLATIONARY MODELS

fields coupled to the inflaton ? → **large excitation ?**

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All 3 cases:

$$m = g(\phi(t) - \phi_0) \xRightarrow{\text{non-adiabatic}} \dot{m} \gg m^2 \quad \text{during} \quad \Delta t_{\rm na} \sim 1/\mu\,, \qquad \boxed{\mu^2 \equiv g\dot{\phi}_0}$$

$$\boxed{n_k = \mathrm{Exp}\{-\pi(k/\mu)^2\}}$$ **Non-adiabatic field excitation (particle creation)**

INFLATIONARY MODELS

fields coupled to the inflaton ? → large excitation ✓

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Non-adiabatic field excitation (particle creation !)
(spin-independent)

In all three cases (scalars, fermions, and vectors)

GWs generated by anisotropic distribution of the created species

(Only $k \ll \mu$ long-wave modes excited)

INFLATIONARY MODELS

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GW generation !?

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GWs power spectrum: $\mathcal{P}_h^{(\mathrm{tot})}(k) = \mathcal{P}_h^{(\mathrm{vac})}(k) + \mathcal{P}_h^{(\mathrm{pp})}(k)$

from particle production

GW Source(s): (SCALARS , VECTOR , FERMIONS)

$$\Pi_{ij}^{TT} \propto \{\partial_i\chi^a\partial_j\chi^a\}^{TT}, \; \{E_iE_j + B_iB_j\}^{TT}, \; \{\bar{\psi}\gamma_i D_j\psi\}^{TT}$$

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$(W \lesssim 0.5)$

N. Barnaby *et al.*, Phys. Rev. **D86**, 103508 (2012), [1206.6117].
J. L. Cook and L. Sorbo, Phys. Rev. **D85**, 023534 (2012), [1109.0022].

INFLATIONARY MODELS

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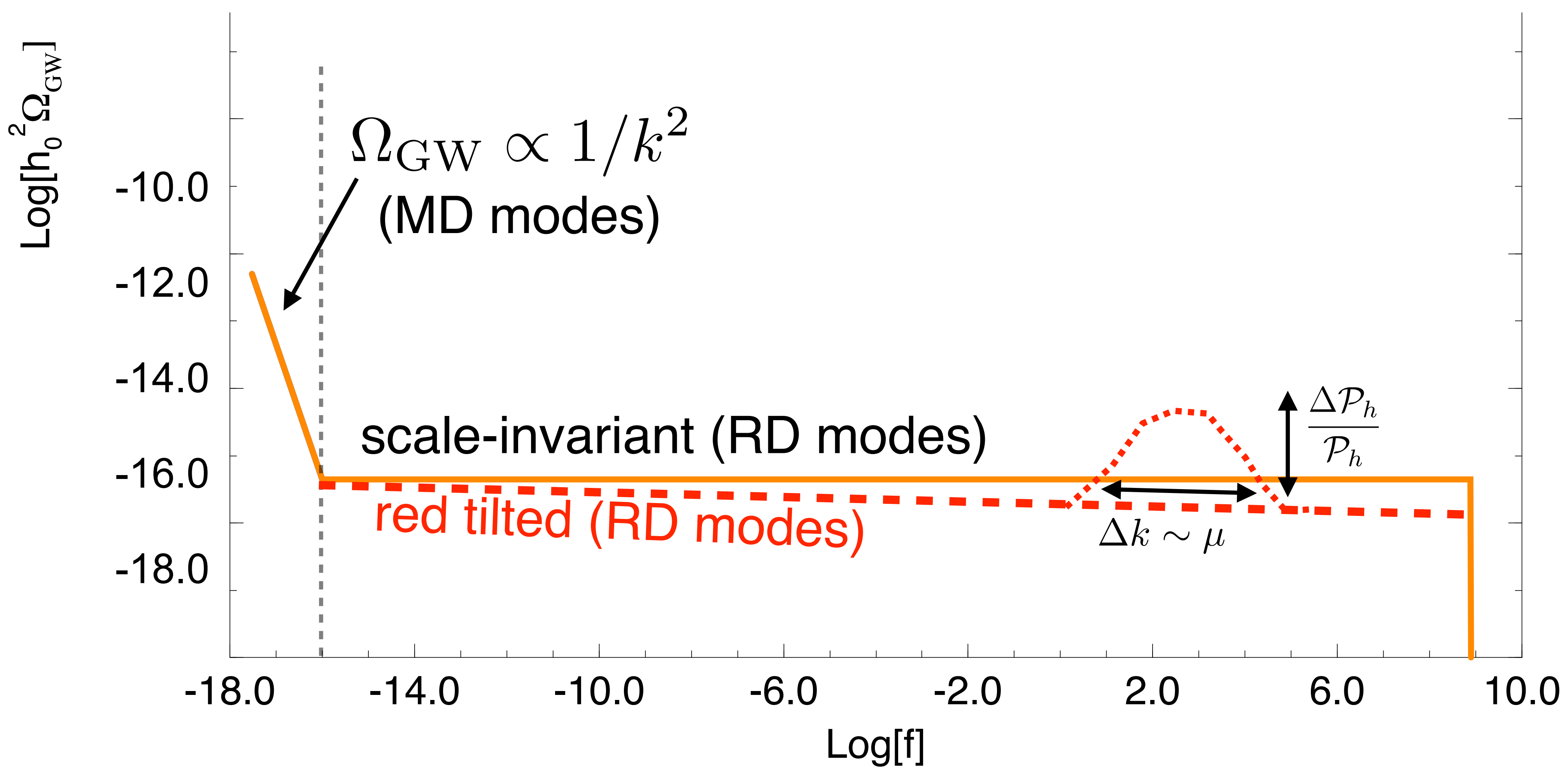
$$\boxed{\mu^2 \equiv g\dot{\phi}_0}$$

(Sorbo et al 2011, Peloso et al 2012-2013, Caprini & DGF 2018)



INFLATIONARY MODELS

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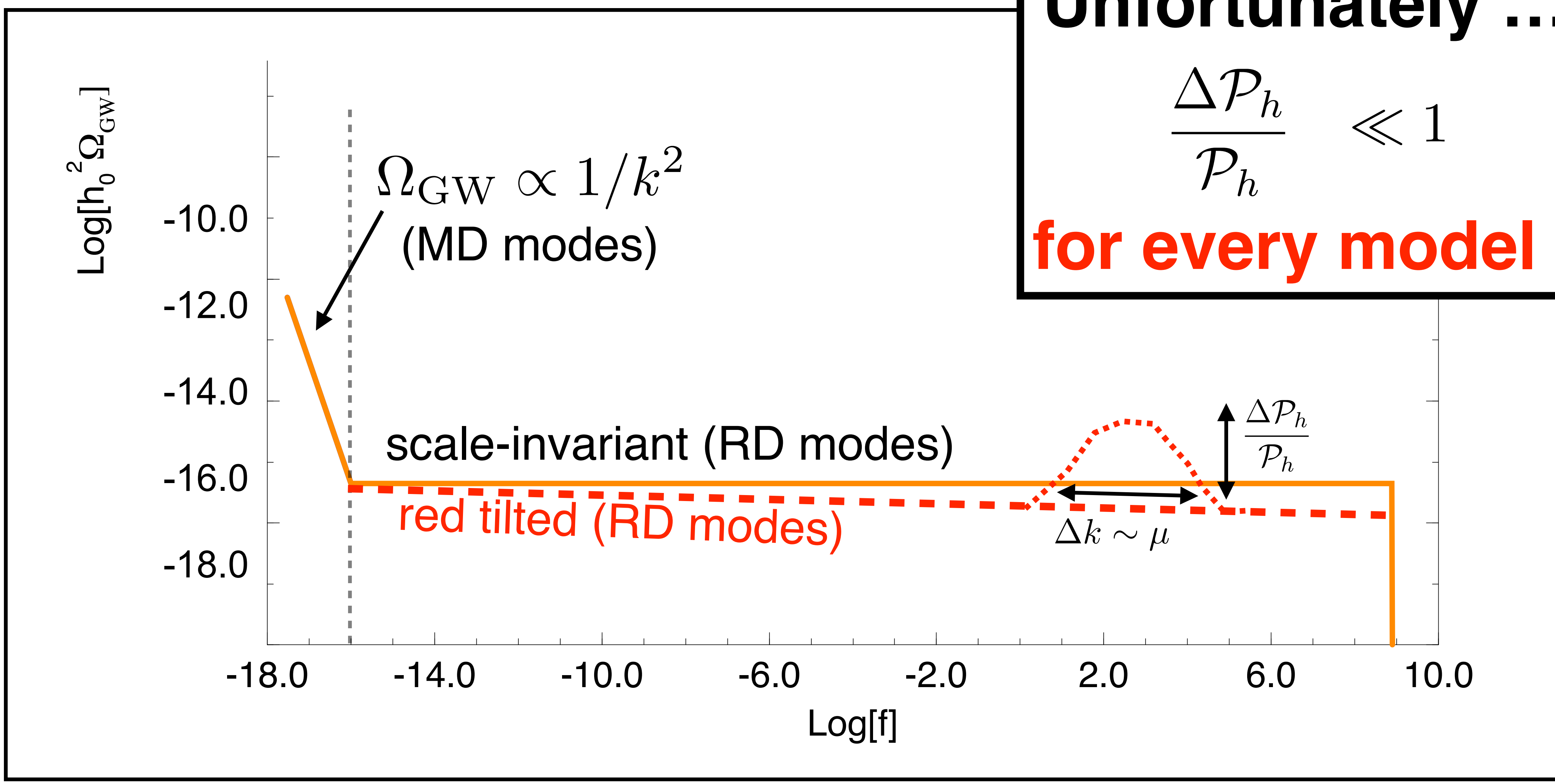
Unfortunately …

$$\frac{\Delta\mathcal{P}_h}{\mathcal{P}_h} \ll 1$$

for every model !



INFLATIONARY COSMOLOGY

Inflation

'cures' hBB ⇒ Cosmological Pple

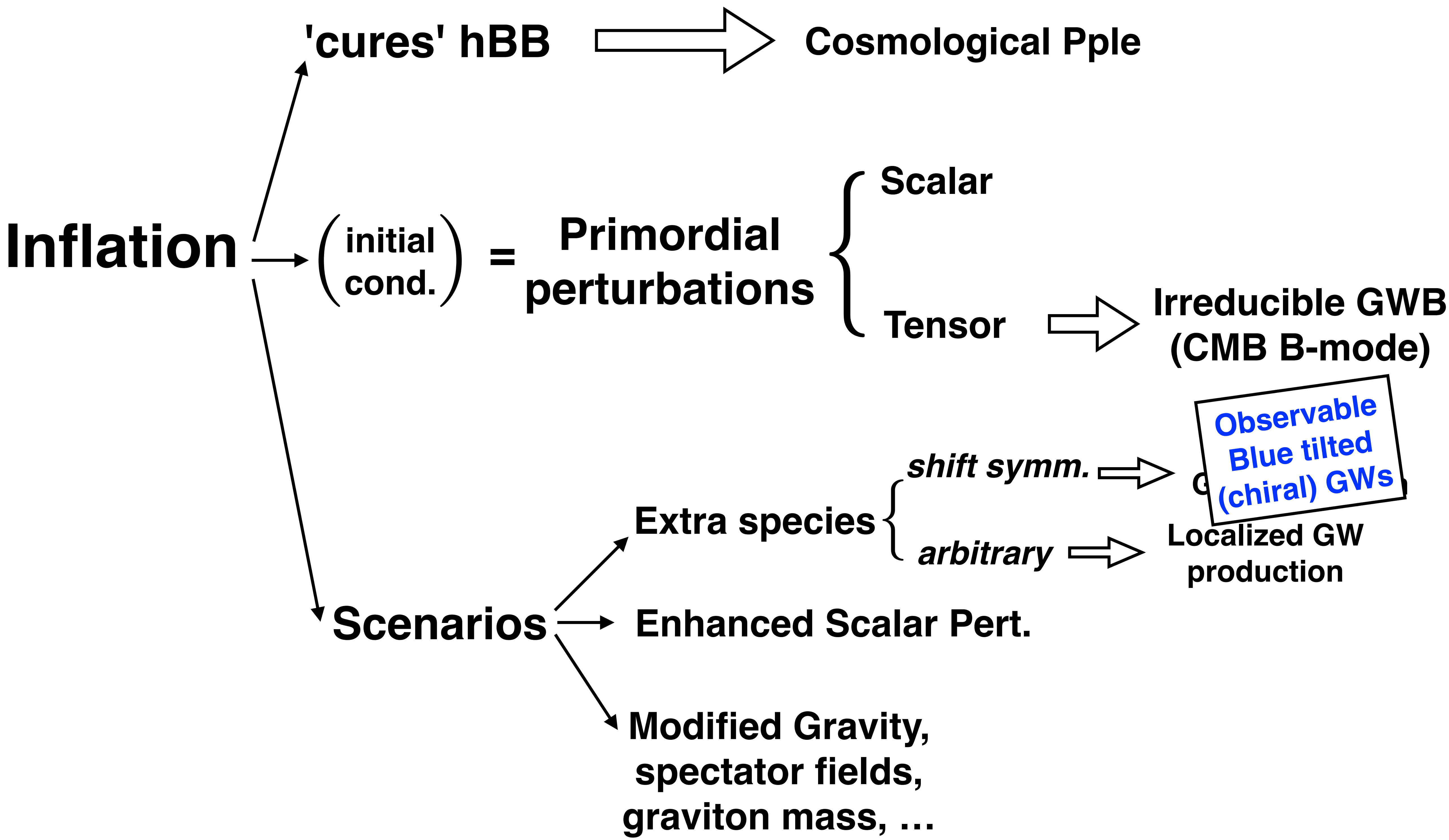
(initial cond.) = Primordial perturbations

- Scalar
- Tensor ⇒ Irreducible GWB (CMB B-mode)

Scenarios

- Extra species
  - *shift symm.* ⇒ Continuous GW production
  - *arbitrary* ⇒ Localized GW production
- Enhanced Scalar Pert.
- Modified Gravity, spectator fields, graviton mass, …

INFLATIONARY COSMOLOGY



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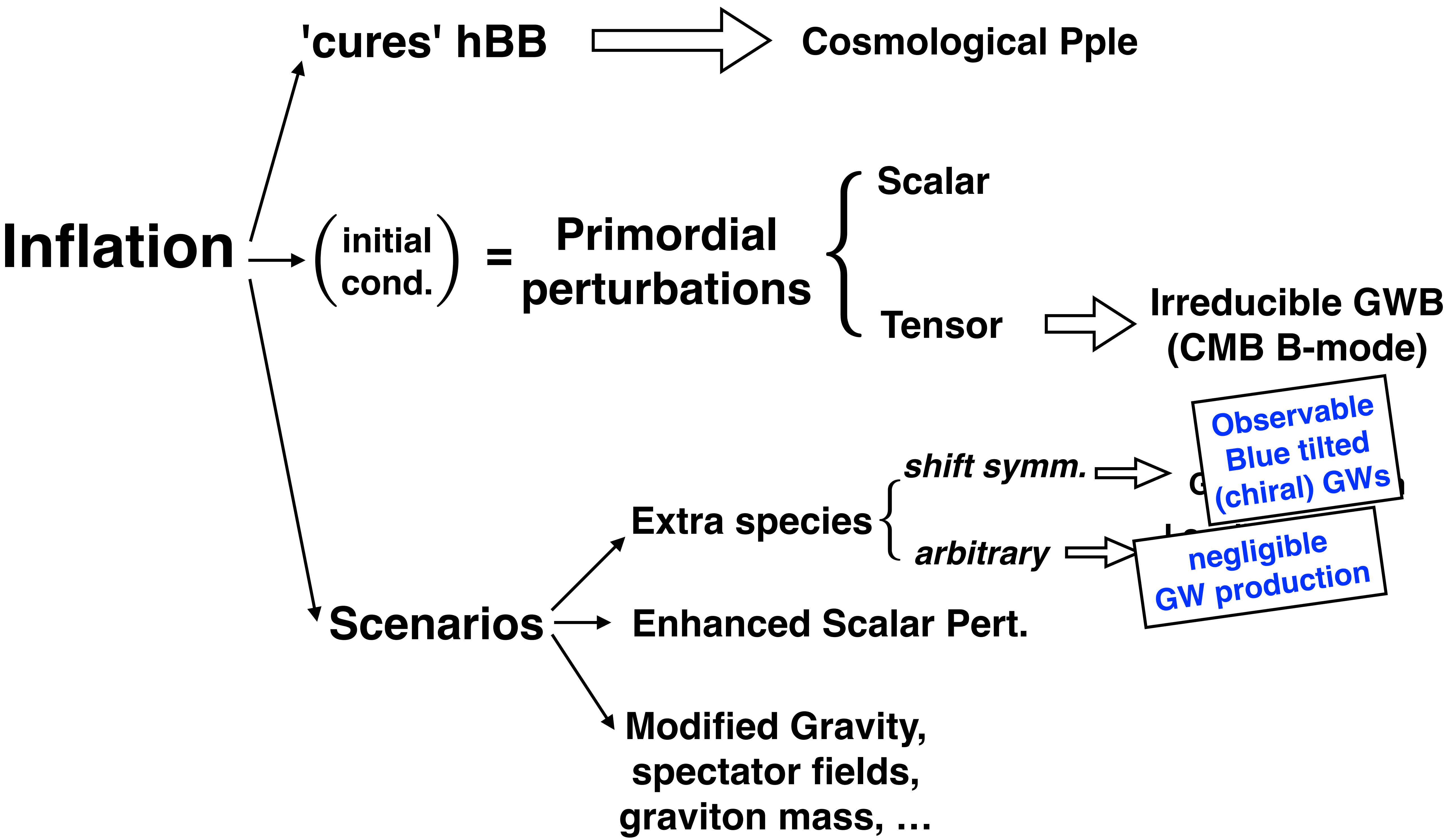

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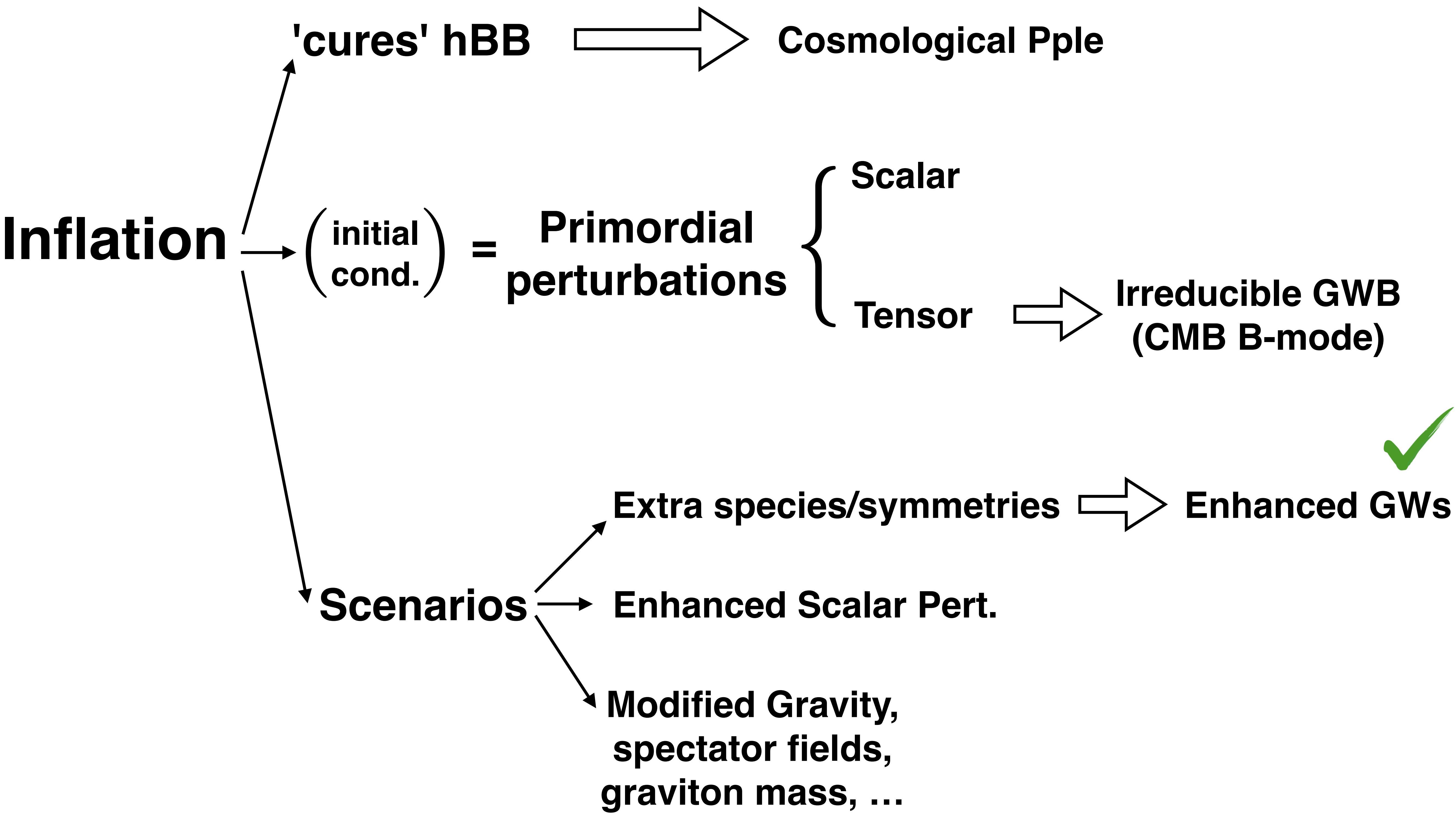

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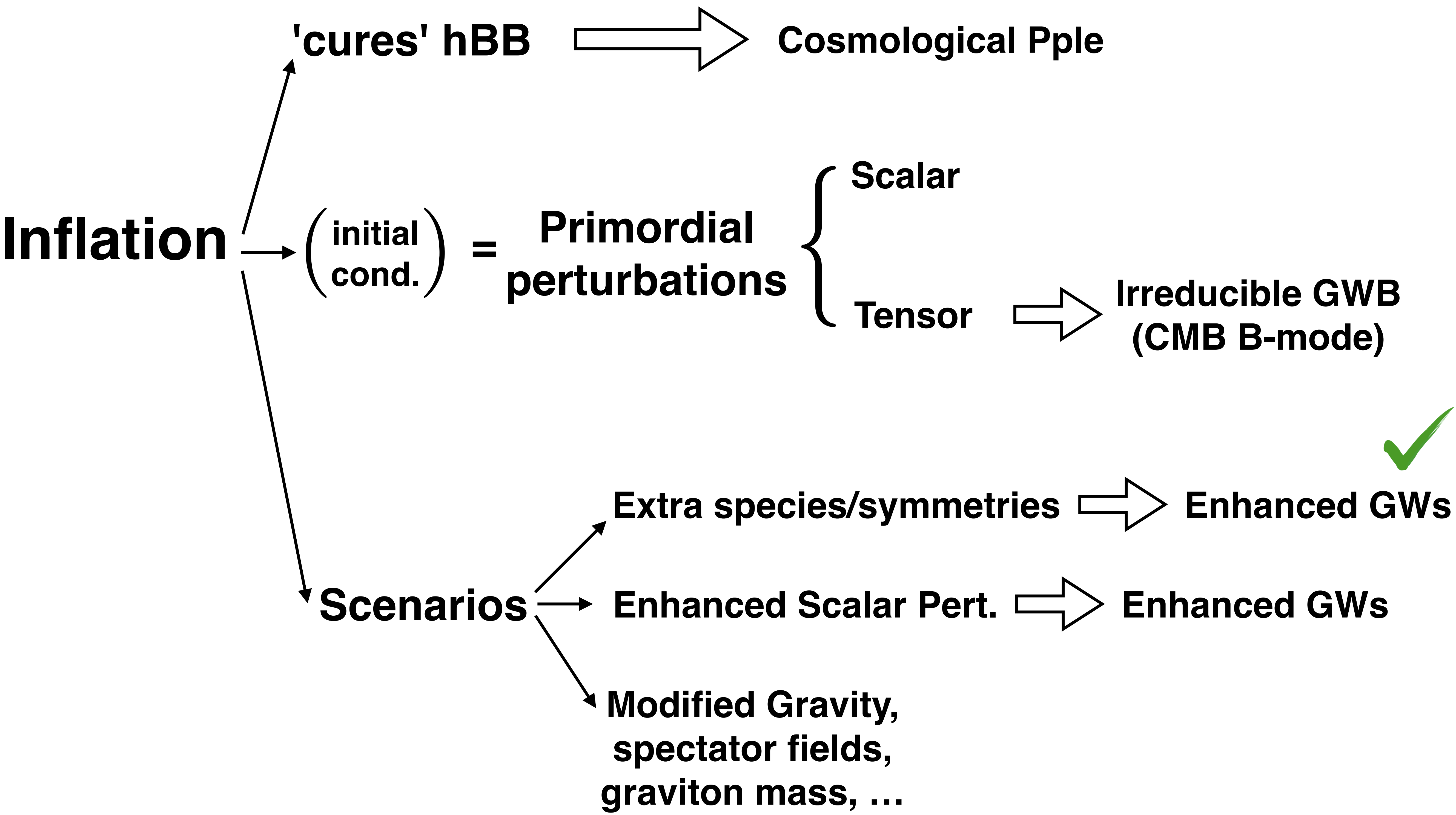

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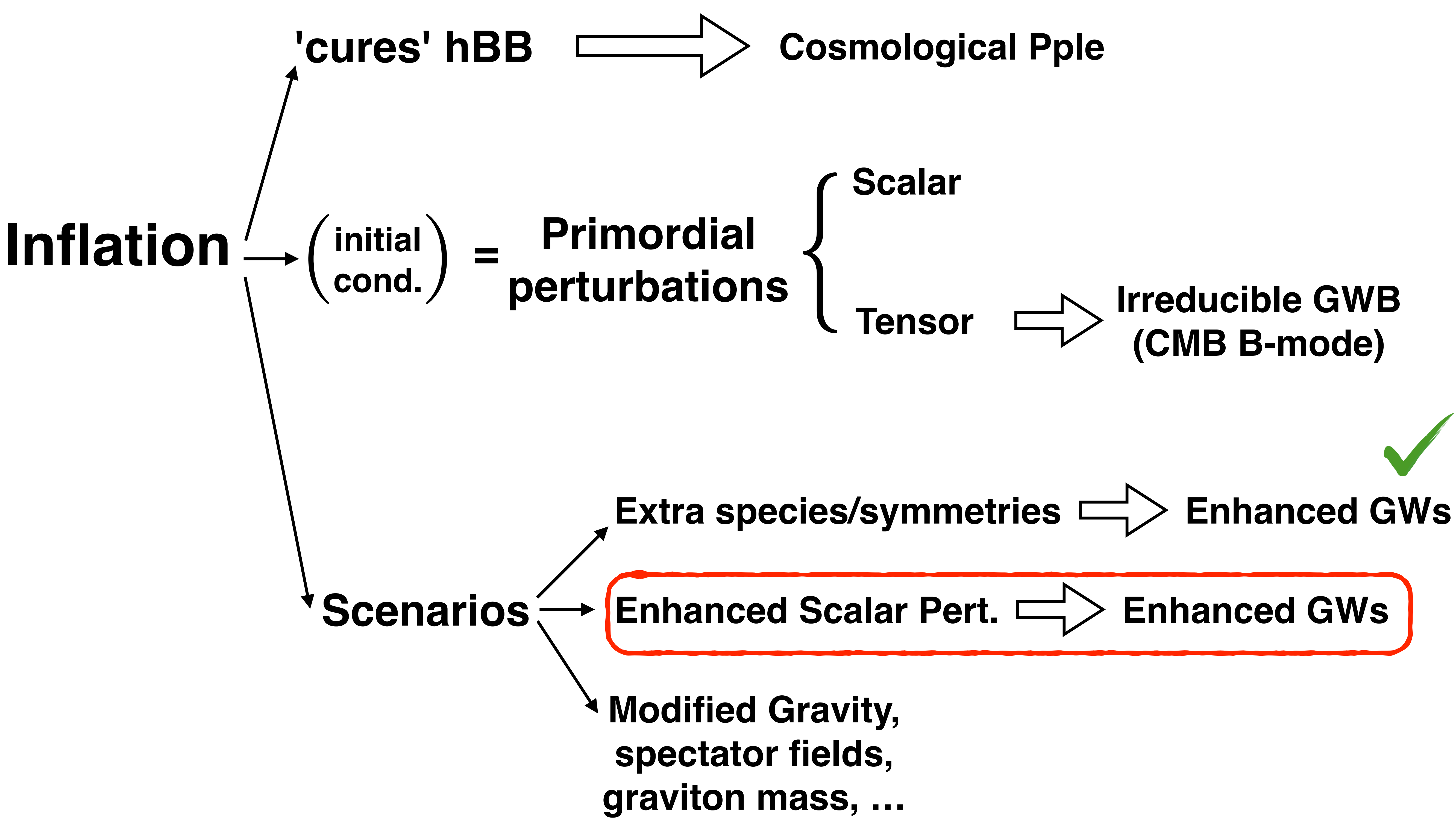

INFLATIONARY MODELS

INFLATION

$$g_{\mu\nu} = g_{\mu\nu}^{FRW} + \delta g_{\mu\nu}[\mathcal{R}, h_{ij}]$$

$$\boxed{\Delta^2_{\mathcal{R}}(k) = \frac{H^4}{(2\pi)^2\dot{\phi}^2}\left(\frac{k}{aH}\right)^{n_s-1}}$$

$$\boxed{n_s - 1 \equiv 2(\eta - 2\epsilon)}$$

$$\boxed{\Delta^2_h(k) = \frac{2}{\pi^2}\left(\frac{H}{m_p}\right)^2\left(\frac{k}{aH}\right)^{n_t}}$$

$$\boxed{n_t \equiv -2\epsilon}$$

(quasi-)scale invariance $\longleftrightarrow$ Slow roll monotonic potentials

INFLATIONARY MODELS

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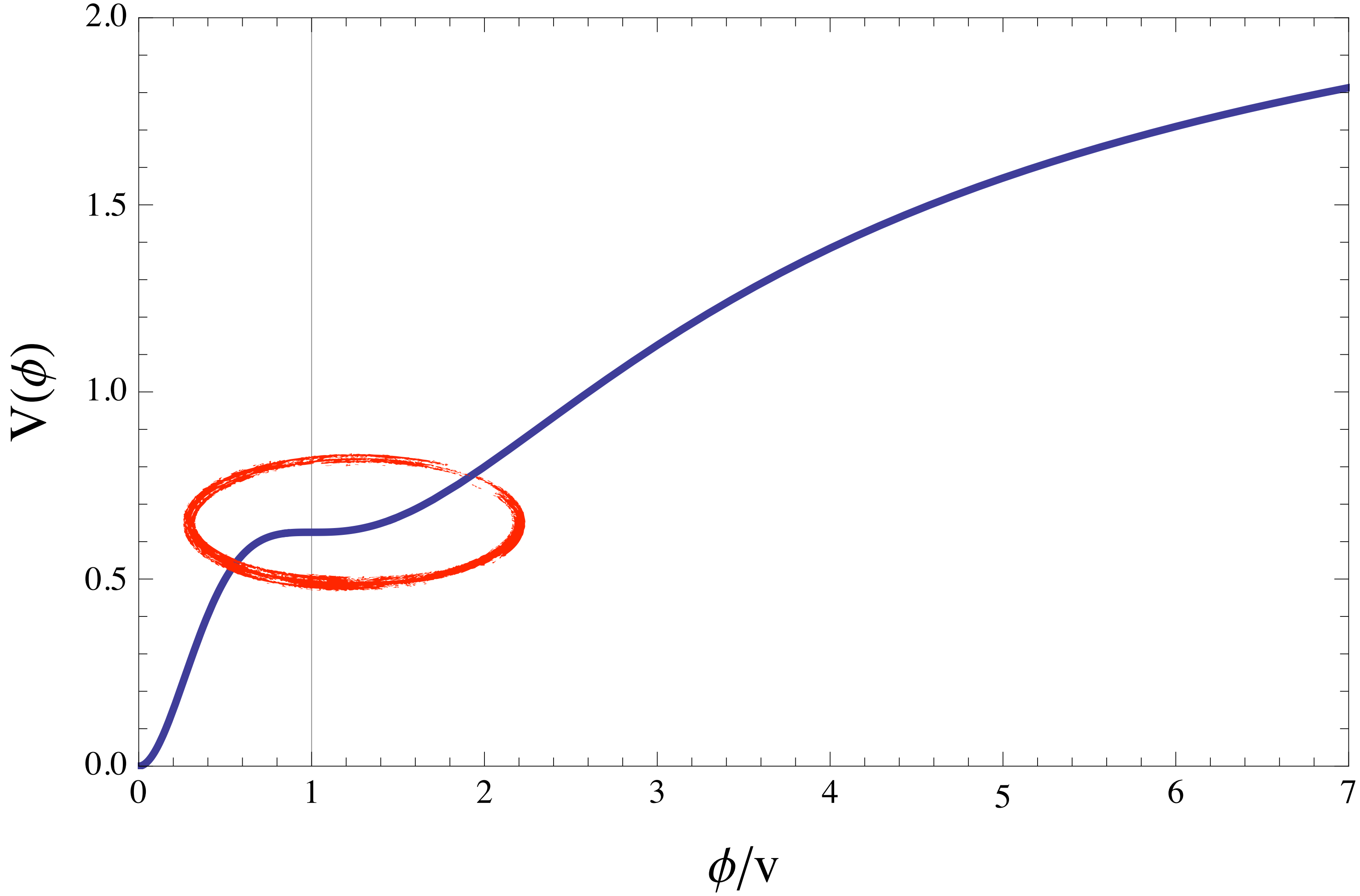
$$g_{\mu\nu} = g_{\mu\nu}^{FRW} + \delta g_{\mu\nu}[\mathcal{R}, h_{ij}]$$

Monotonic
Single Field
Slow Roll

$n_t \equiv -2\epsilon$

(quasi-)scale invariance $\longleftrightarrow$ Slow roll monotonic potentials

INFLATIONARY MODELS

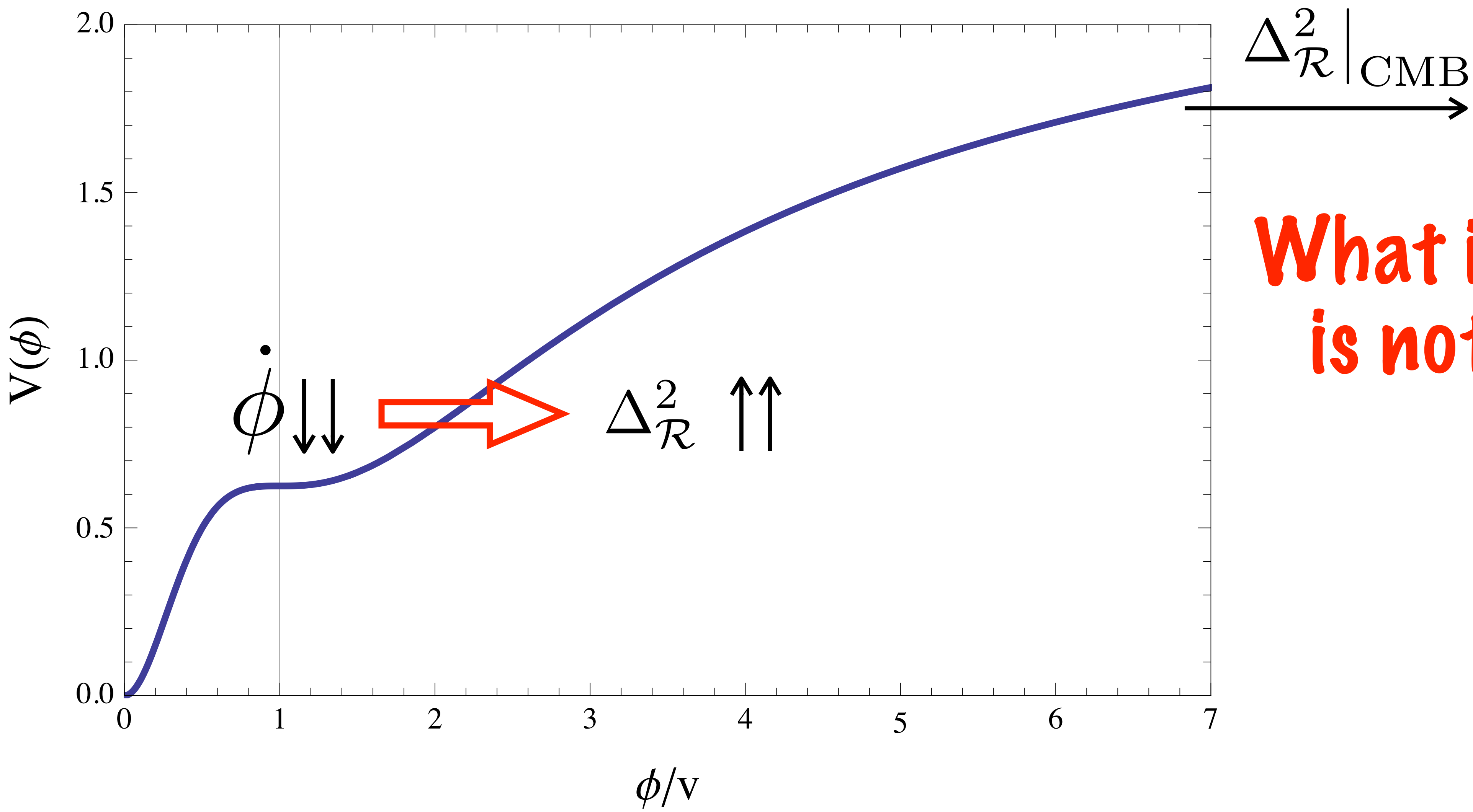


What if the potential is not monotonic ?

INFLATIONARY MODELS

Ultra Slow-Roll Regime

$\Delta^2_{\mathcal{R}}\big|_{\rm CMB} \sim 3 \cdot 10^{-9}$



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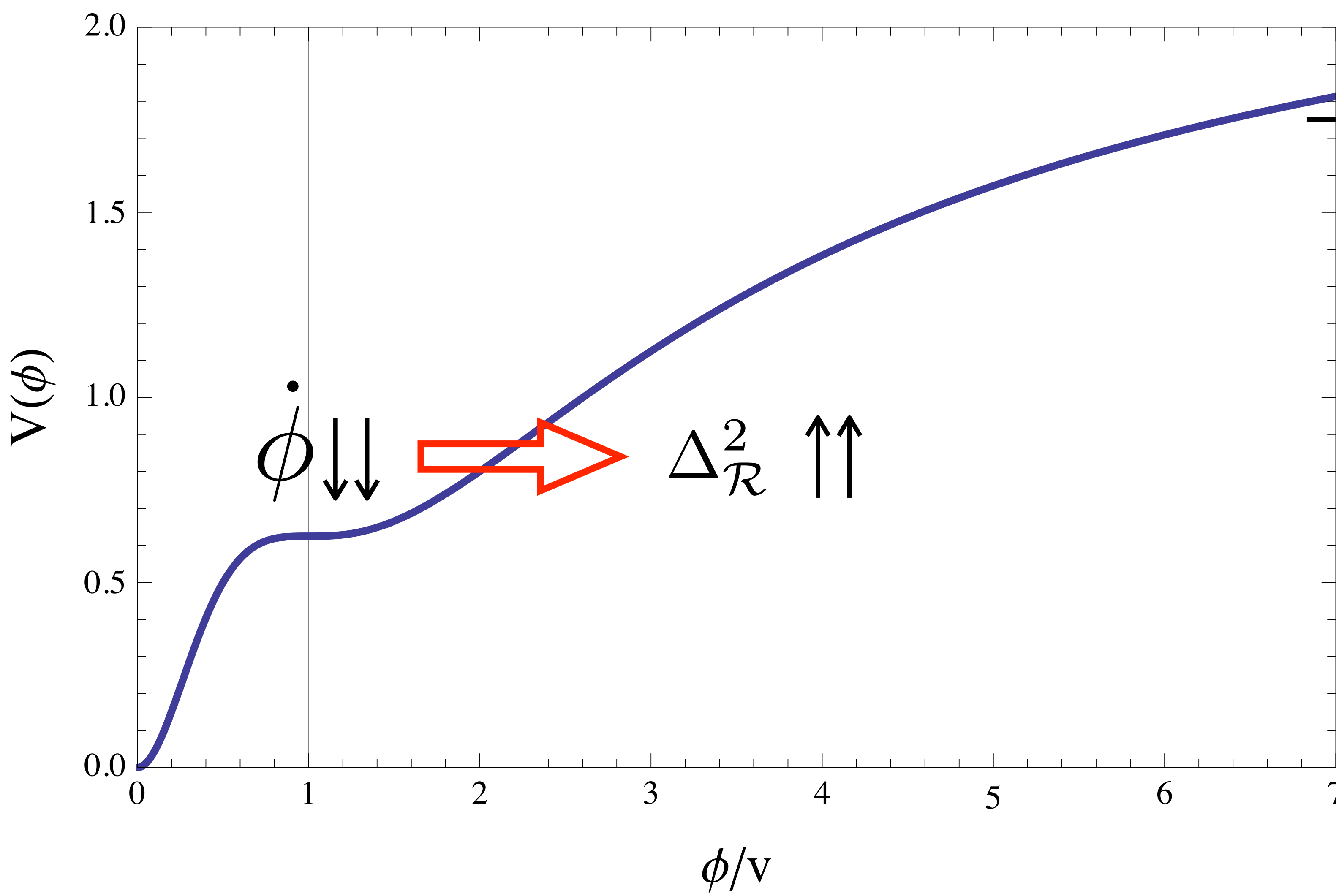


$\Delta^2_{\mathcal{R}}\big|_{\mathrm{CMB}} \sim 3 \cdot 10^{-9}$

What if the potential is not monotonic ?

$\Delta^2_{\mathcal{R}}$ **greatly enhanced!**
(at small scales)

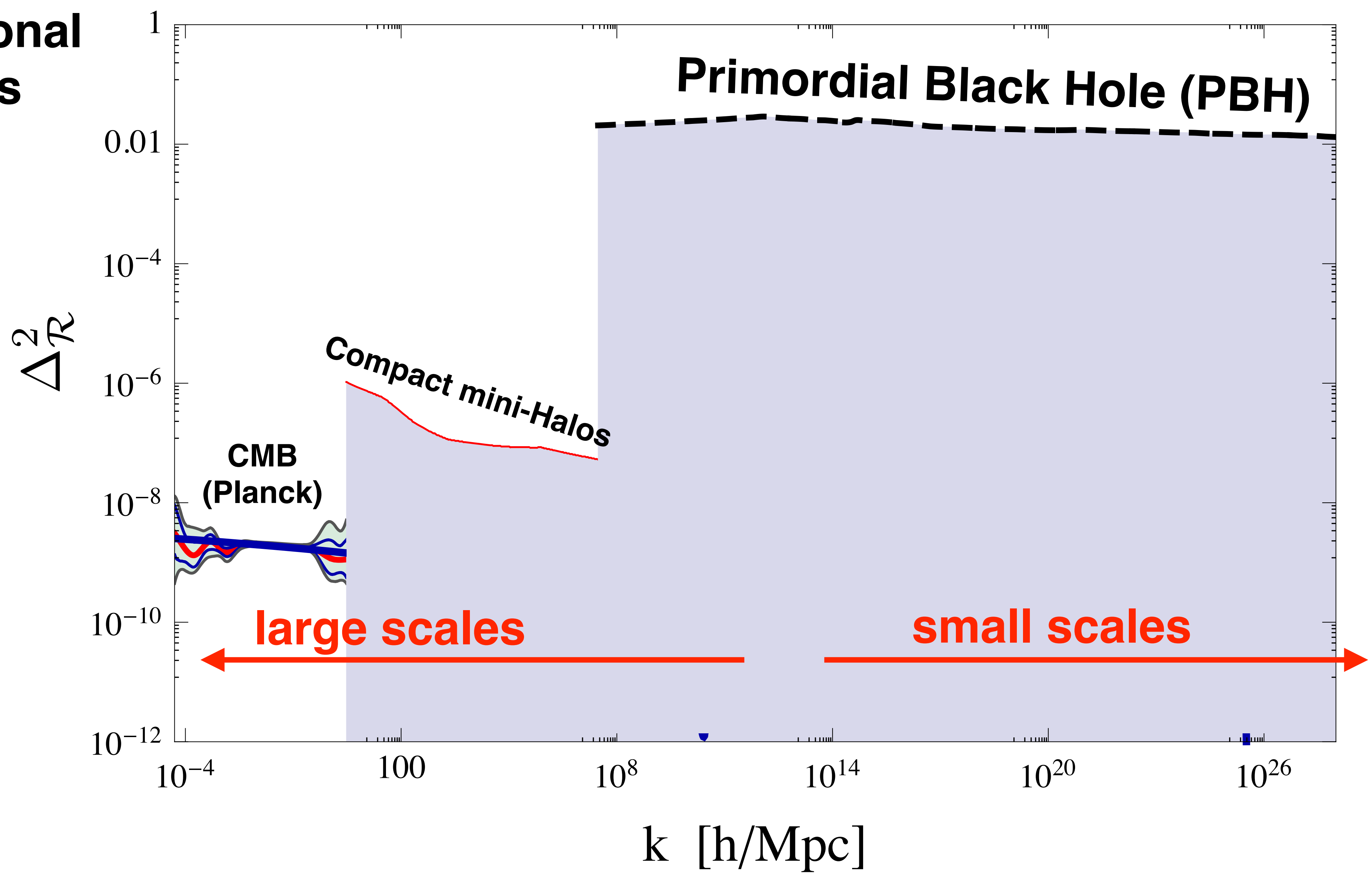
INFLATIONARY MODELS

INFLATION ➡ IF $\left\{ \begin{array}{l} \text{non-monotonic} \\ \text{multi-field} \end{array} \right\}$ ⇨ possible to enhance $\Delta^2_{\mathcal{R}}$ (at small scales)

INFLATIONARY MODELS

INFLATION → IF { non-monotonic, multi-field } ⇨ possible to enhance $\Delta^2_{\mathcal{R}}$ (at small scales)

Observational constraints



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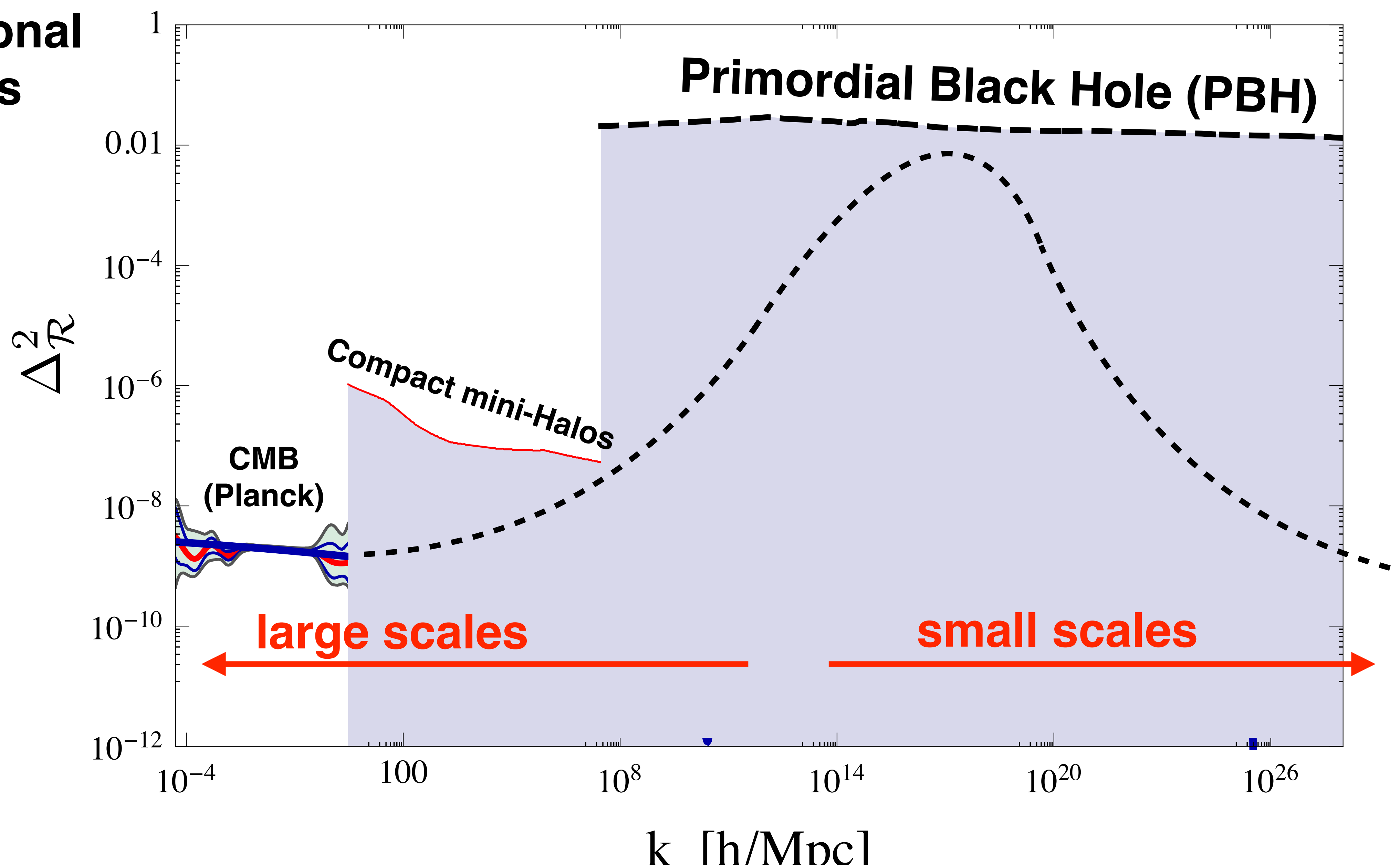

INFLATIONARY MODELS

INFLATION $\Rightarrow$ IF $\left\{ \begin{array}{l} \text{non-monotonic} \\ \text{multi-field} \end{array} \right\}$ $\Rightarrow$ possible to enhance $\Delta^2_{\mathcal{R}}$ (at small scales)

Let us suppose $\boxed{\Delta^2_{\mathcal{R}} \gg \Delta^2_{\mathcal{R}}\big|_{\rm CMB} \sim 3 \cdot 10^{-9}}$, @ small scales

$$ds^2 = a^2(\eta)[-(1+2\Phi)d\eta^2 + [(1-2\Psi)\delta_{ij} + 2F_{(i,j)} + h_{ij}]dx^i dx^j]$$

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$$\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = S^{TT}_{ij}} \sim \Phi * \Phi$$

(2nd Order Pert.)

$$\begin{aligned} S_{ij} = {} & 2\Phi\partial_i\partial_j\Phi - 2\Psi\partial_i\partial_j\Phi + 4\Psi\partial_i\partial_j\Psi + \partial_i\Phi\partial_j\Phi - \partial^i\Phi\partial_j\Psi - \partial^i\Psi\partial_j\Phi + 3\partial^i\Psi\partial_j\Psi \\ & - \frac{4}{3(1+w)\mathcal{H}^2}\partial_i(\Psi' + \mathcal{H}\Phi)\partial_j(\Psi' + \mathcal{H}\Phi) \\ & - \frac{2c_s^2}{3w\mathcal{H}}\left[3\mathcal{H}(\mathcal{H}\Phi - \Psi') + \nabla^2\Psi\right]\partial_i\partial_j(\Phi - \Psi) \end{aligned}$$

D. Wands et al, 2006-2010
Baumann et al, 2007
Peloso et al, 2018

INFLATIONARY MODELS

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$$\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = S^{TT}_{ij}} \sim \Phi * \Phi \quad \textbf{(2nd Order Pert.)}$$

$$\Omega^{(0)}_{\rm GW}(f) = \frac{\Omega^{(0)}_{\rm rad}\mathcal{G}(\eta_c)}{24}\left(\frac{2\pi f}{a(\eta_c)H(\eta_c)}\right)^2 \overline{\mathcal{P}^{\rm ind}_h(\eta_c, 2\pi f)}$$

INFLATIONARY MODELS

INFLATION → **IF** $\left\{ \begin{array}{l} \text{non-monotonic} \\ \text{multi-field} \end{array} \right\}$ ⇨ **possible to enhance $\Delta^2_{\mathcal{R}}$ (at small scales)**

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$$\overline{\mathcal{P}^{\rm ind}_h(\eta, k)} = 2\int_0^\infty dt \int_{-1}^1 ds \left[\frac{t(2+t)(s^2-1)}{(1-s+t)(1+s+t)}\right]^2 \times \overline{I^2(u,v,k,\eta)}\, \Delta^2_{\mathcal{R}}(ku) \cdot \Delta^2_{\mathcal{R}}(kv),$$

INFLATIONARY MODELS

INFLATION $\rightarrow$ IF $\left\{ \begin{array}{l} \text{non-monotonic} \\ \text{multi-field} \end{array} \right\} \Rightarrow$ possible to enhance $\Delta^2_{\mathcal{R}}$ (at small scales)

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INFLATIONARY MODELS

INFLATION $\rightarrow$ IF $\left\{ \begin{array}{l} \text{non-monotonic} \\ \text{multi-field} \end{array} \right\} \Rightarrow$ possible to enhance $\Delta^2_{\mathcal{R}}$ (at small scales)

BBN $\Omega_{gw,0} < 1.5 \times 10^{-6} \longrightarrow \triangle^2_{\mathcal{R}} < 0.1$

LIGO $\Omega_{gw,0} < 6.9 \times 10^{-8} \longrightarrow \triangle^2_{\mathcal{R}} < 0.01$

PTA $\Omega_{gw,0} < 1 \times 10^{-9} \longrightarrow \triangle^2_{\mathcal{R}} < 5 \times 10^{-3}$

LISA $\Omega_{gw,0} < 10^{-13} \longrightarrow \triangle^2_{\mathcal{R}} < 1 \times 10^{-5}$

BBO $\Omega_{gw,0} < 10^{-17} \longrightarrow \triangle^2_{\mathcal{R}} < 3 \times 10^{-7}$

(Numbers not updated !)

INFLATIONARY MODELS

INFLATION $\rightarrow$ IF $\left\{ \begin{array}{l} \text{non-monotonic} \\ \text{multi-field} \end{array} \right\}$ $\Rightarrow$ possible to enhance $\Delta^2_{\mathcal{R}}$ (at small scales)

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INFLATIONARY MODELS

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IF $\Delta^2_{\mathcal{R}}$ very enhanced ➡ Primordial Black Holes (PBH) may be produced!

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PBH candidate for DM ? Yes !, for $\sim 10^{-15}\text{–}10^{-11} M_{\odot}$

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* If PBH are the DM, what is the GWB from 2nd O(Φ)? Bartolo et al, '18

Right in the middle of LISA !

INFLATIONARY MODELS

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Has LIGO detected PBH's ?

INFLATIONARY MODELS

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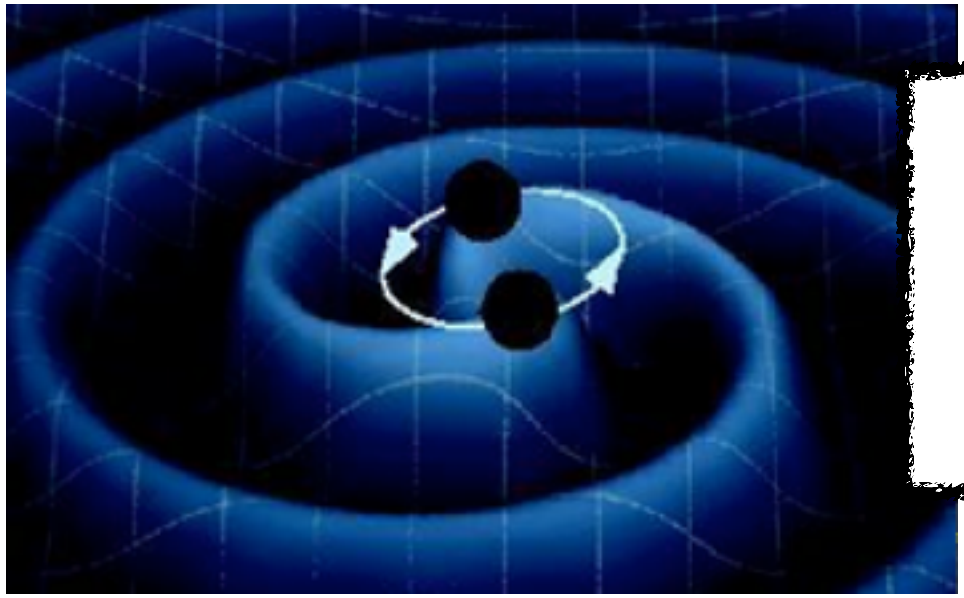

'We will know determining the mass/spin distribution'
(M. Fishbach (LIGO), Moriond'19)

e.g. **2102.03809, 2105.03349, De Luca** *et al*

INFLATIONARY MODELS

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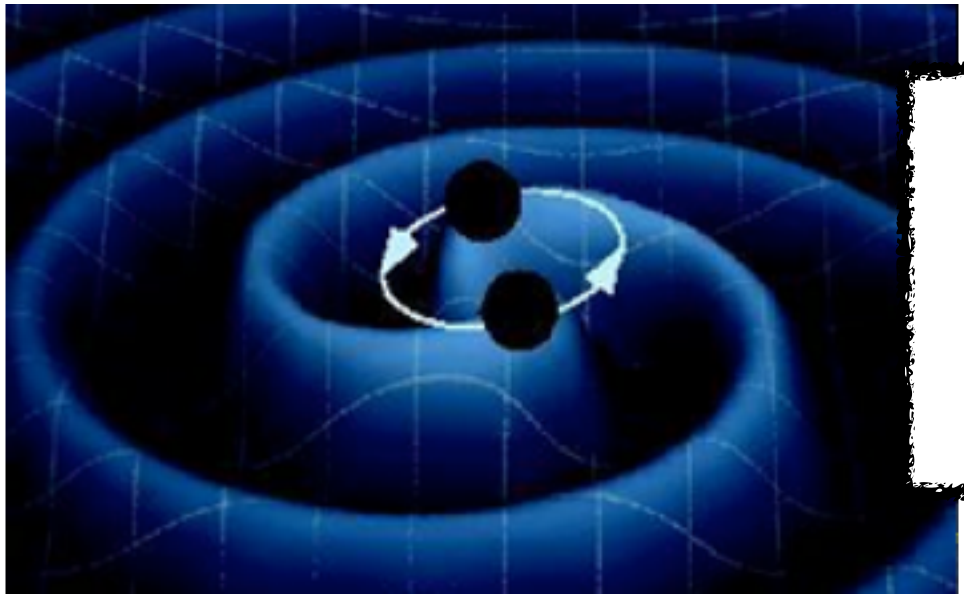
IF $\Delta^2_{\mathcal{R}}$ very enhanced → **Primordial Black Holes (PBH) may be produced!**

Has LIGO detected PBH's ? **it does not look like…**



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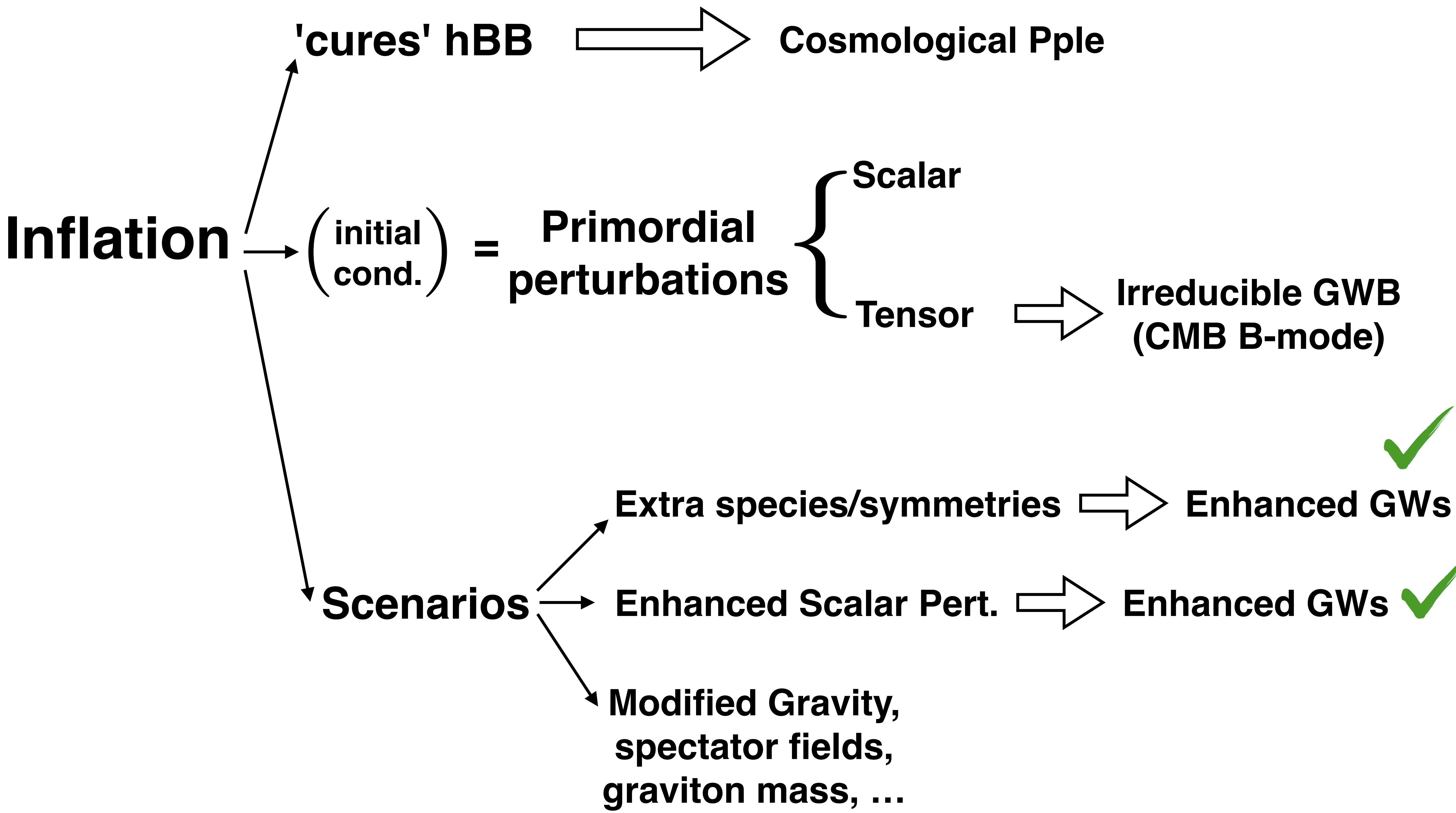
INFLATIONARY COSMOLOGY



INFLATIONARY COSMOLOGY

Inflation → 'cures' hBB ⇒ Cosmological Pple

Inflation → (initial cond.) = Primordial perturbations { Scalar; Tensor ⇒ Irreducible GWB (CMB B-mode)

Inflation → Scenarios

- → Extra species/symmetries ⇒ Enhanced GWs ✓
- → Enhanced Scalar Pert. ⇒ Enhanced GWs ✓
- → Modified Gravity, spectator fields, graviton mass, … ⇒ Enhanced GWs ?

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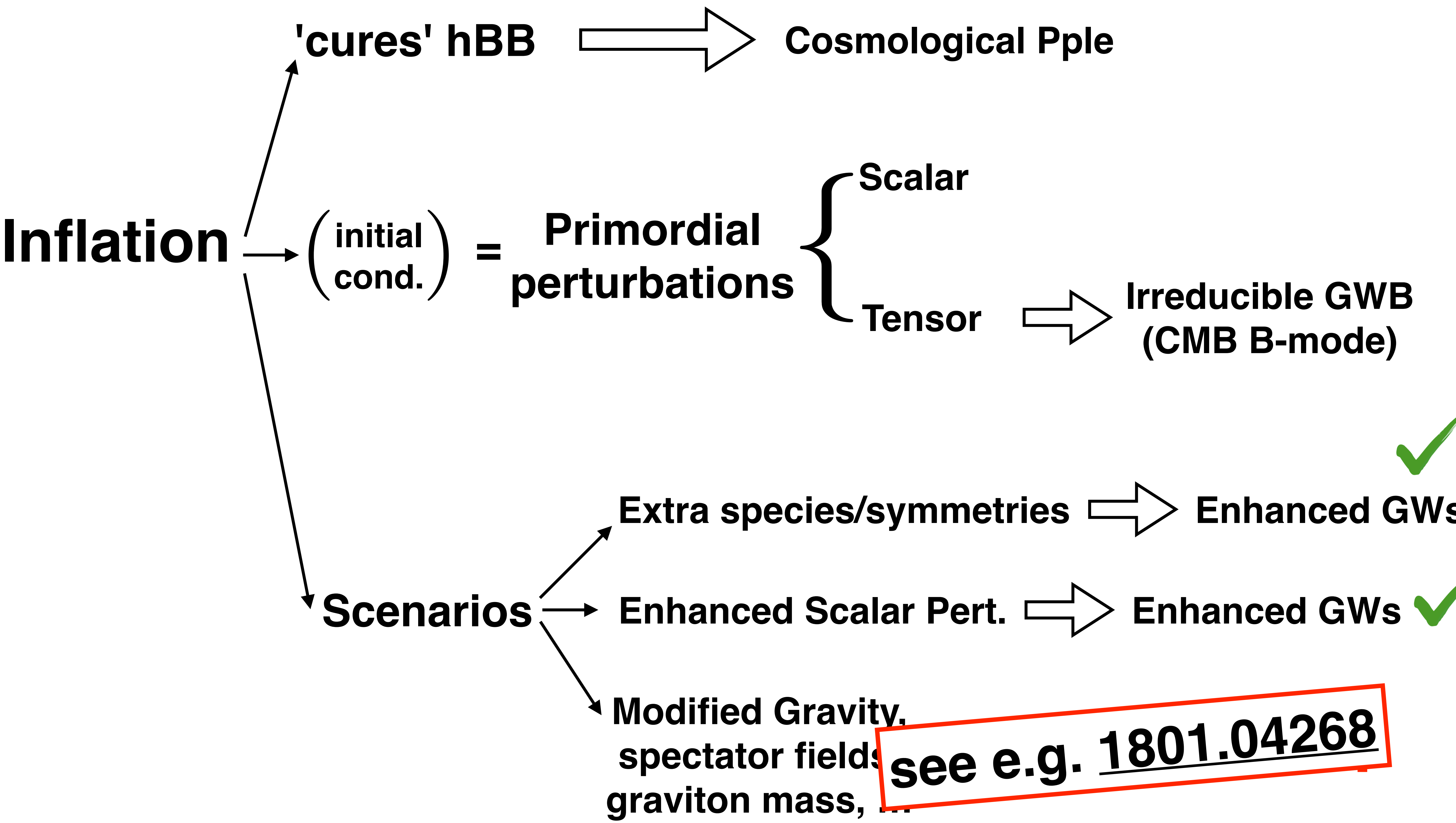
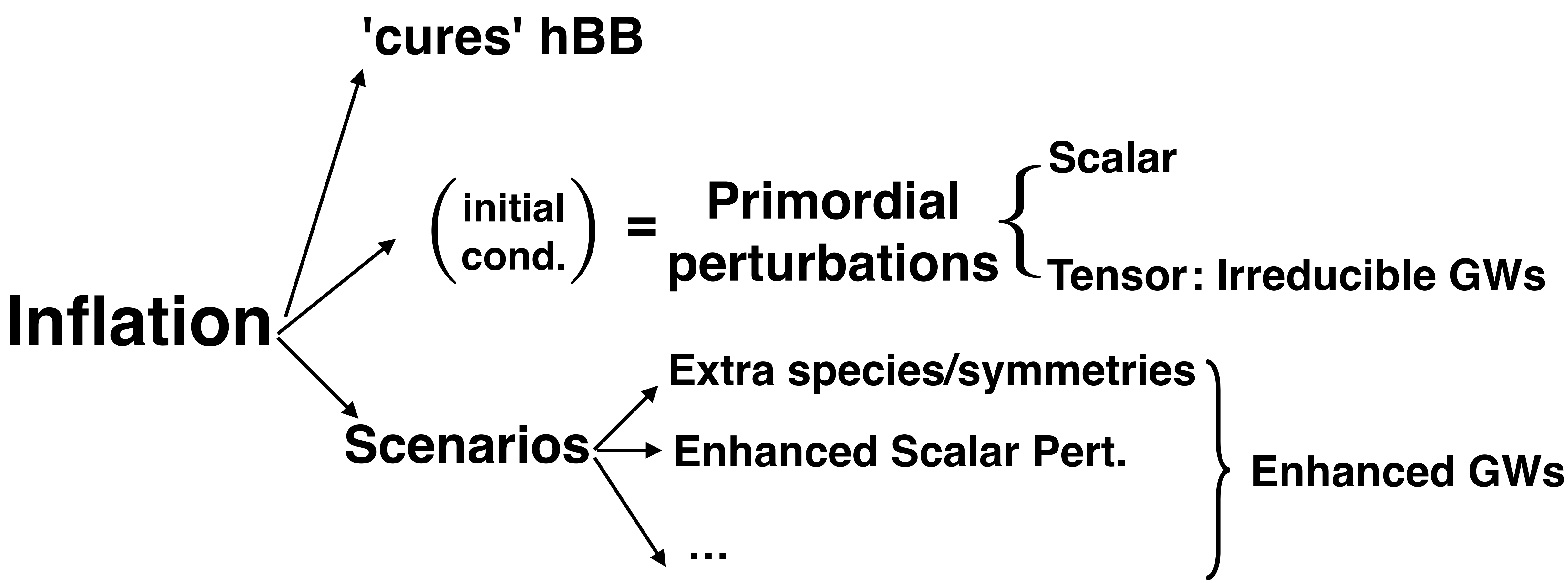
- → Enhanced Scalar Pert. ⇒ Enhanced GWs ✓
- → Modified Gravity, spectator fields, graviton mass, …

Yes! but we leave it for another school

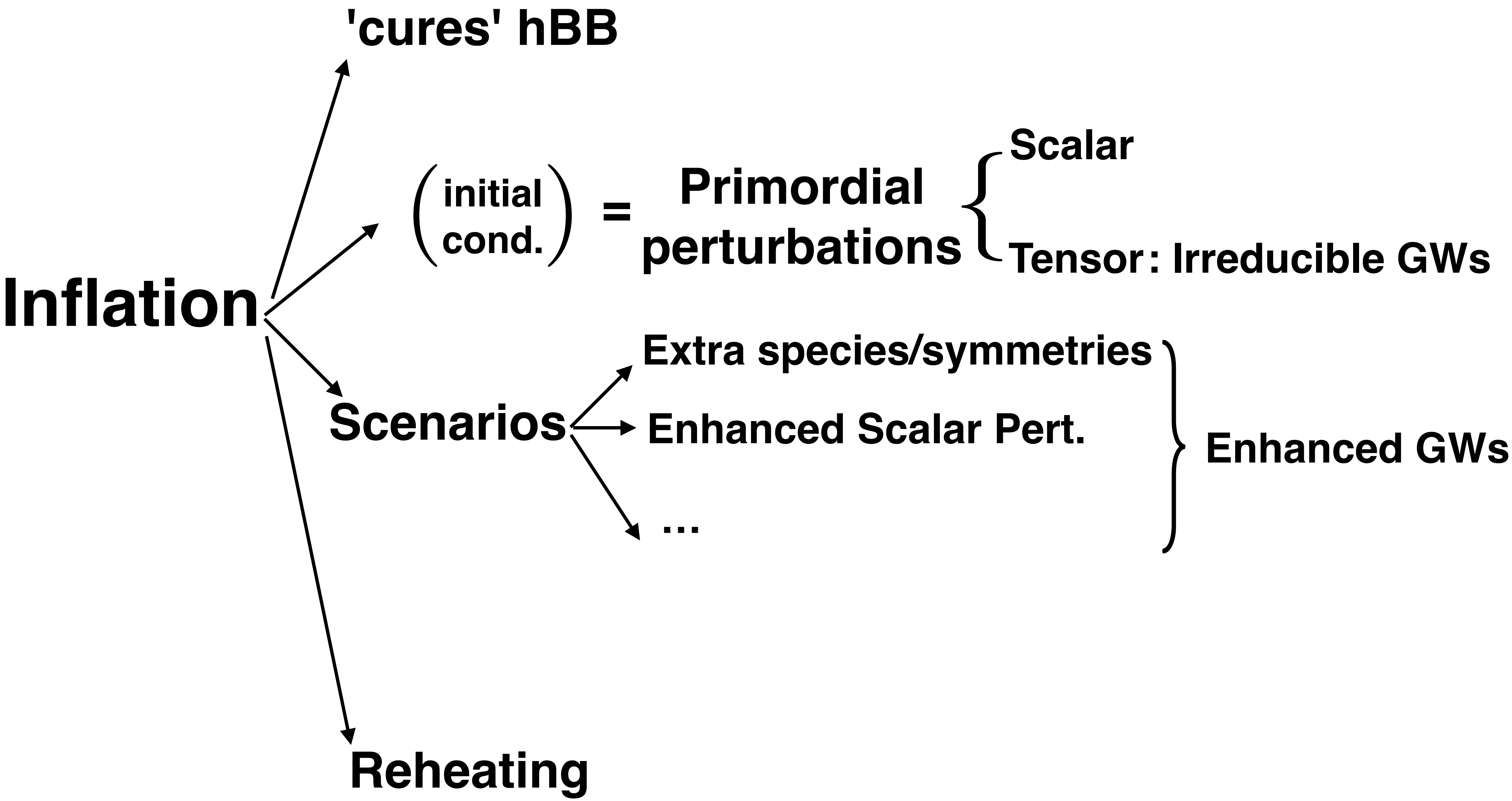
INFLATIONARY COSMOLOGY



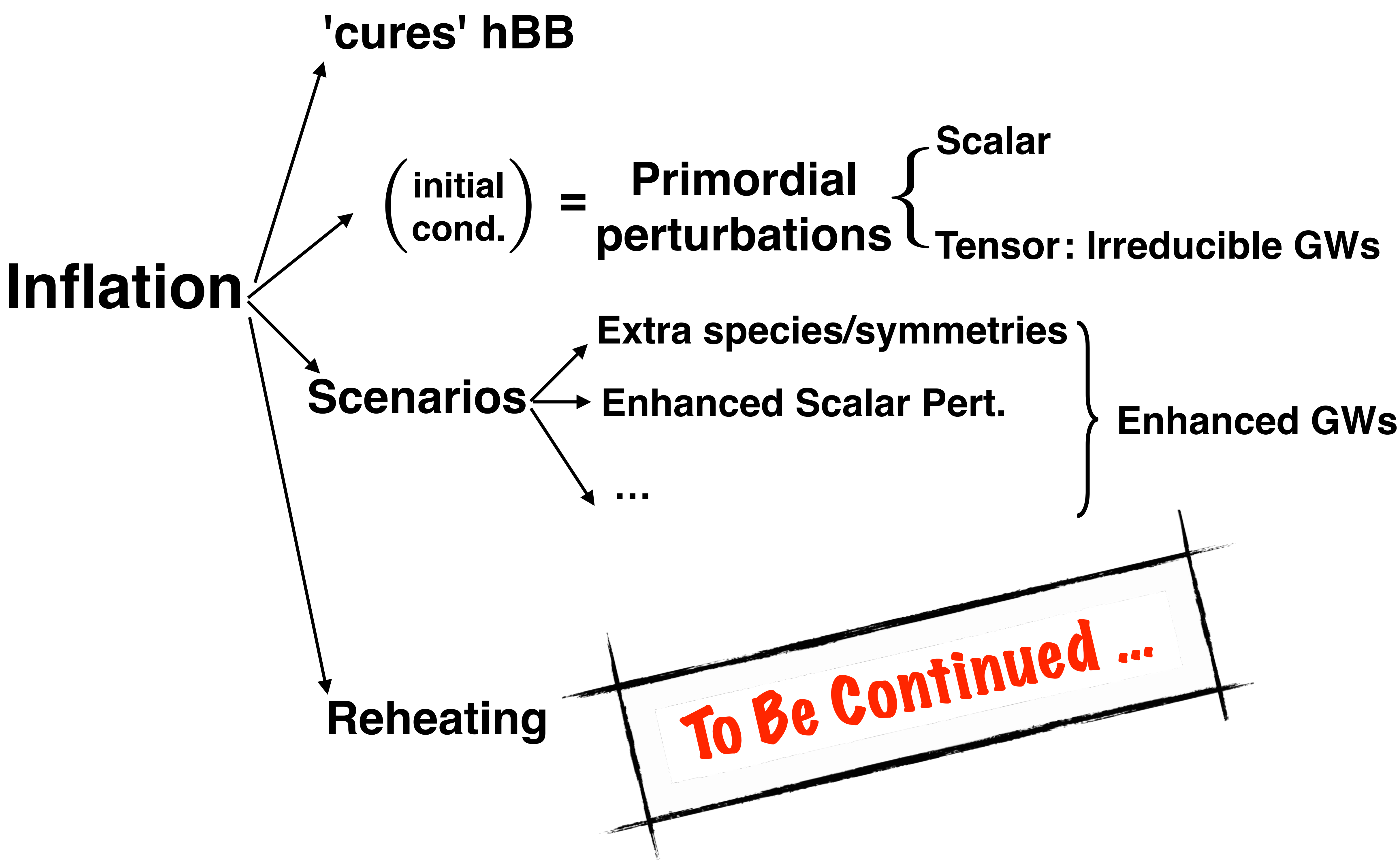
INFLATIONARY COSMOLOGY



INFLATIONARY COSMOLOGY



INFLATIONARY COSMOLOGY



Inflation
& Primordial Perturbations

Inflation & Primordial Perturbations

INF → **SR:** $\epsilon, \eta \ll 1 \rightarrow \epsilon, \eta \simeq 1$

(Start) **(End)**

$\boxed{a \sim e^{\int H dt'} \gtrsim e^{60}}$ (qdS)

No Hor. Problem !

Flat Universe !

Inflation & Primordial Perturbations

INF → **SR:** $\boxed{\epsilon, \eta \ll 1 \to \epsilon, \eta \simeq 1}$

(Start) — **(End)**

$\boxed{a \sim e^{\int H dt'} \gtrsim e^{60}}$ (qdS)

Flat Universe !

No Hor. Problem !

* **Is that ALL?** **NO!**

$$\boxed{\begin{array}{c} \phi(t) \\ \\ g_{\mu\nu}(t) \end{array}}$$

(Background)

Inflation & Primordial Perturbations

$$a \sim e^{\int H dt'} \gtrsim e^{60}$$ (qdS)

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*** Is that ALL? NO!**

(Background)		(Fluctuations)
$\phi(t)$	→	$\phi(t) + \delta\phi(\vec{x}, t)$
$g_{\mu\nu}(t)$	→	$g_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x}, t)$

INF

↓

Primordial fluctuations !

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

$$\phi(t) \rightarrow \phi(t) + \delta\phi(\vec{x}, t)$$

$$g_{\mu\nu}(t) \rightarrow g_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x}, t)$$

(Background) **(Fluctuations)**

but WHY fluctuations ?

Quantum Mechanics !

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

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but WHY fluctuations ?

Quantum Mechanics !

VeV

Vacuum Quam. Fluct.

QM: $\hat{\phi}(\vec{x}, t) \;\rightarrow\; \langle\hat{\phi}(\vec{x}, t)\rangle = \phi(t) \quad \Rightarrow \boxed{\hat{\phi}(\vec{x}, t) = \phi(t) + \hat{\delta\phi}(\vec{x}, t)}$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

$$\phi(t) \longrightarrow \phi(t) + \delta\phi(\vec{x},t)$$

$$g_{\mu\nu}(t) \longrightarrow g_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

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$$\langle\hat{\delta\phi}(\vec{x},t)\rangle = 0 \quad \textbf{but...} \quad \left\langle\, [\hat{\delta\phi}(\vec{x},t)]^2 \,\right\rangle \neq 0$$

Inflation & Primordial Perturbations

$\boxed{\text{Inflation: A generator of Primordial Fluctuations}}$

$$\boxed{\begin{matrix} \phi(t) \\ g_{\mu\nu}(t) \end{matrix}} \;\Rightarrow\; \boxed{\begin{matrix} \phi(t) + \delta\phi(\vec{x},t) \\ g_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t) \end{matrix}}$$

(Background) **(Fluctuations)**

but WHY fluctuations ?

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but...

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Inflation & Primordial Perturbations

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but ... ~~Minkowski~~ → Curved Space: (quasi)dS

Inflation & Primordial Perturbations

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$$S = \frac{m_p^2}{2}\int d^4x\sqrt{-g}\{R - (\partial\phi)^2 - 2V(\phi)\}$$

$$\phi(t) + \delta\phi(\vec{x},t)$$

$$g_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

Inflation & Primordial Perturbations

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$$g_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

$$\begin{aligned} ds^2 &= g^{\rm tot}_{\mu\nu}dx^\mu dx^\nu = (g_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t))dx^\mu dx^\nu \\ &= -(1+2\Phi)dt^2 + 2B_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j \end{aligned}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

$$ds^2 = -(1+2\Phi)dt^2 + 2B_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j$$

$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

Inflation & Primordial Perturbations

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$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

$$B_i = \partial_i B - S_i$$

$$E_{ij} = 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

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$B_i = \partial_i B \cancel{- S_i}$

$E_{ij} = 2\partial_{ij}E + \cancel{2\partial_{(i}F_{j)}} + h_{ij}$

Expanding U. $\longrightarrow$ Vector Perturbations $S_i, F_i \propto \frac{1}{a}$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

$$ds^2 = -(1+2\Phi)dt^2 + 2B_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j$$

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$$B_i = \partial_i B - \cancel{S_i}$$

$$E_{ij} = 2\partial_{ij}E + \cancel{2\partial_{(i}F_{j)}} + h_{ij}$$

$$\partial_i h_{ij} = h_{ii} = 0$$

(tensors = GWs)

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

$$ds^2 = -(1+2\Phi)dt^2 + 2B_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j$$

$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

$$\text{dof} \xrightarrow{\text{scalars}} \delta\phi + \{\Phi, B, \Psi, E\}$$

$$\text{dof} \xrightarrow{\text{tensors}} h_{ij}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

$$ds^2 = -(1+2\Phi)dt^2 + 2B_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j$$

$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

dof

scalars: $\delta\phi + \{\Phi, B, \Psi, E\} \xrightarrow{\text{Diff.}} \delta\phi' + \{\Phi', B', \Psi', E'\}$ ~~Gauge Inv.~~

tensors: $h_{ij} \xrightarrow{\text{Diff.}} h_{ij}$ Gauge Inv. !

Diff.: $\boxed{x^\mu \;\to\; x^\mu + \xi^\mu}$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

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$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

dof

scalars: $\delta\phi + \{\Phi, B, \Psi, E\}$

tensors: h_{ij}

$$\zeta \equiv -[\Psi + (H/\dot{\rho})\delta\rho_\phi] \xrightarrow{\text{Diff.}} \zeta$$

$$\mathcal{R} \equiv [\Psi + (H/\dot{\phi})\delta\phi] \xrightarrow{\text{Diff.}} \mathcal{R}$$

$$Q \equiv [\delta\phi + (\dot{\phi}/H)\Psi] \xrightarrow{\text{Diff.}} Q$$

All Gauge Inv. !

Inflation & Primordial Perturbations

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dof

scalars $\delta\phi + \{\Phi, B, \Psi, E\}$

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$$\mathcal{R} \equiv [\Psi + (H/\dot\phi)\delta\phi] \xrightarrow{\text{Diff.}} \mathcal{R}$$

$$Q \equiv [\delta\phi + (\dot\phi/H)\Psi] \xrightarrow{\text{Diff.}} Q$$

All Gauge Inv. !

Curvature Pert.

Tensor Pert. (GW)

Fixing Gauge: e.g. $E, \delta\phi = 0 \Rightarrow g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij} + h_{ij}]$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

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$$S = \frac{m_p^2}{2}\int d^4x\sqrt{-g}\{R - (\partial\phi)^2 - 2V(\phi)\}$$

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$$S = \frac{m_p^2}{2}\int d^4x\sqrt{-g}\{R - (\partial\phi)^2 - 2V(\phi)\}$$

$$S = S_{(0)} + S^{(s)}_{(2)} + S^{(t)}_{(2)}$$

$$S^{(s)}_{(2)} = \frac{1}{2}\int d^4x\ a^3\frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]$$

$$S^{(t)}_{(2)} = \frac{m_p^2}{8}\int dtdx^3a^3\left[(\dot{h}_{ij})^2 - a^{-2}(\partial_l h_{ij})^2\right]$$

Background
Inflationary dynamics

(UV limit: deep inside Hubble radius)

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluctuations:

$$\boxed{S_{(2)}^{(s)} = \frac{1}{2}\int d^4x \; a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluctuations:

$d\tau \equiv dt/a(t)$ (Conformal time)

$$\boxed{S^{(s)}_{(2)} = \frac{1}{2}\int d^4x \; a^3 \frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]} = \boxed{\frac{1}{2}\int d\tau dx^3 \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]}$$

$$\left[v \equiv z\mathcal{R}\,, \quad z \equiv a\frac{\dot\phi}{H}\right] \text{ (Mukhanov variable)}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluctuations:

$$\boxed{S_{(2)}^{(s)} = \frac{1}{2}\int d^4x\ a^3\frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]} = \boxed{\frac{1}{2}\int d\tau dx^3\left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]} \Rightarrow$$

(F.T.: $v(\mathbf{x},t) = \int d\mathbf{k}\, e^{-i\mathbf{k}\mathbf{x}}\, v_{\mathbf{k}}(t)$)

$$\Rightarrow \boxed{v''_{\vec k} + (k^2 - z''/z)v_{\vec k} = 0} \quad \text{with} \quad \boxed{\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\ ,\quad \nu \equiv \frac{3}{2} + 2\epsilon - \eta}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluctuations:

$$\boxed{S^{(s)}_{(2)} = \tfrac{1}{2}\int d^4x \; a^3 \tfrac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]} = \boxed{\frac{1}{2}\int d\tau dx^3 \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]} \Rightarrow$$

$$\Rightarrow \boxed{v''_{\vec{k}} + (k^2 - z''/z)v_{\vec{k}} = 0} \quad \text{with} \quad \boxed{\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right) \; , \quad \nu \equiv \frac{3}{2} + 2\epsilon - \eta}$$

Quantization: $\boxed{v_{\vec{k}}(t) \to v_k(t)\hat{a}_{\vec{k}} + v^*_k(t)\hat{a}^\dagger_{-\vec{k}} \; , \quad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta(\vec{k} - \vec{k}')}$ $\Rightarrow$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluctuations:

$$\boxed{S^{(s)}_{(2)} = \frac{1}{2}\int d^4x\; a^3\frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]} = \boxed{\frac{1}{2}\int d\tau dx^3\left[(v')^2-(\nabla v)^2+\frac{z''}{z}v^2\right]} \Rightarrow$$

$$\Rightarrow \boxed{v''_{\vec k} + (k^2 - z''/z)v_{\vec k} = 0}$$

with

$$\boxed{\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2-\frac{1}{4}\right)\;,\quad \nu \equiv \frac{3}{2}+2\epsilon-\eta}$$

Quantization: $\boxed{v_{\vec k}(t)\to v_k(t)\hat a_{\vec k} + v_k^*(t)\hat a^\dagger_{-\vec k}\;,\quad [a_{\vec k}, a^\dagger_{\vec k'}] = (2\pi)^3\delta(\vec k-\vec k')}$ $\Rightarrow$

$\Rightarrow$ **2 linearly independent solutions (Hankel functions)**

Inflation & Primordial Perturbations

$$\boxed{\text{Inflation: A generator of Primordial Fluctuations}}$$

Scalar Fluctuations:

$$\boxed{S^{(s)}_{(2)} = \frac{1}{2}\int d^4x \; a^3 \frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]} = \boxed{\frac{1}{2}\int d\tau dx^3 \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]} \Rightarrow$$

$$\Rightarrow \boxed{v''_{\vec{k}} + (k^2 - z''/z)v_{\vec{k}} = 0} \quad \textbf{with} \quad \boxed{\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right) \; , \quad \nu \equiv \frac{3}{2} + 2\epsilon - \eta}$$

Quantization: $\boxed{v_{\vec{k}}(t) \to v_k(t)\hat{a}_{\vec{k}} + v^*_k(t)\hat{a}^\dagger_{-\vec{k}} \quad , \quad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta(\vec{k} - \vec{k}')}$ $\rightarrow$

$$\rightarrow \boxed{v_k(\tau) = \frac{\sqrt{\pi}}{2}e^{i\pi(\nu+1/2)/2}(-\tau)^{1/2}H^{(1)}_\nu(-k\tau)}$$

(we keep only one, $\hat{H}v_k = +kv_k$ **,** $\langle v_k, v_k\rangle > 0$ **)**

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluctuations:

$$\boxed{S^{(s)}_{(2)} = \frac{1}{2}\int d^4x\, a^3 \frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]} = \boxed{\frac{1}{2}\int d\tau dx^3 \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z} v^2\right]} \Rightarrow$$

$$\Rightarrow \boxed{v''_{\vec k} + (k^2 - z''/z) v_{\vec k} = 0} \quad \text{with} \quad \boxed{\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\ ,\quad \nu \equiv \frac{3}{2} + 2\epsilon - \eta}$$

Quantization: $\boxed{v_{\vec k}(t) \to v_k(t)\hat a_{\vec k} + v^*_k(t)\hat a^\dagger_{-\vec k}\ ,\quad [a_{\vec k}, a^\dagger_{\vec k'}] = (2\pi)^3\delta(\vec k - \vec k')}$ $\Rightarrow$

$$\Rightarrow \boxed{v_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\pi(\nu+1/2)/2}(-\tau)^{1/2} H^{(1)}_\nu(-k\tau)} \xrightarrow[\text{(sub-Hubble)}]{-k\tau \gg 1} \frac{1}{\sqrt{2k}} e^{-ik\tau}$$

(we keep only one, $\hat H v_k = +k v_k$, $\langle v_k, v_k\rangle > 0$)

Positive define freq

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluctuations:

$$\boxed{S^{(s)}_{(2)} = \frac{1}{2}\int d^4x\ a^3 \frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]} = \boxed{\frac{1}{2}\int d\tau dx^3 \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]} \Rightarrow$$

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$$\rightarrow \boxed{v_k(\tau) = \frac{\sqrt{\pi}}{2}e^{i\pi(\nu+1/2)/2}(-\tau)^{1/2}H^{(1)}_\nu(-k\tau)}$$

(Bunch-Davies) Vacuum Fluct.

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluct:

$$\boxed{v_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\pi(\nu+1/2)/2} (-\tau)^{1/2} H_\nu^{(1)}(-k\tau)}$$

$$\boxed{\hat{v}_{\vec{k}}(t) \to v_k(t)\hat{a}_{\vec{k}} + v_k^*(t)\hat{a}^\dagger_{-\vec{k}}}$$

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(Bunch-Davies) Vacuum Fluct.

$$\boxed{\left[v \equiv z\mathcal{R}\,, \quad z \equiv a\frac{\dot{\phi}}{H}\right]} \Rightarrow \boxed{\left\langle \hat{\mathcal{R}}_{\vec{k}} \hat{\mathcal{R}}_{\vec{k}'} \right\rangle \equiv \frac{1}{z^2} \left\langle \hat{v}_{\vec{k}} \hat{v}_{\vec{k}'} \right\rangle \equiv (2\pi)^3 \boxed{\frac{H^2}{a^2\dot{\phi}^2} \left| v_k(\eta) \right|^2} \delta(\vec{k} + \vec{k}')}$$

Inflation & Primordial Perturbations

$$\boxed{\text{Inflation: A generator of Primordial Fluctuations}}$$

Scalar Fluct:

$$\boxed{v_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\pi(\nu+1/2)/2}(-\tau)^{1/2} H_\nu^{(1)}(-k\tau)}$$

$$\boxed{\hat{v}_{\vec{k}}(t) \to v_k(t)\hat{a}_{\vec{k}} + v_k^*(t)\hat{a}^\dagger_{-\vec{k}}}$$

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$$\boxed{\equiv P_{\mathcal{R}}(k,\eta)}$$

Scalar Power Spectrum

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Scalar Fluct:

$$v_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\pi(\nu+1/2)/2}(-\tau)^{1/2} H_\nu^{(1)}(-k\tau)$$

$$\hat{v}_{\vec{k}}(t) \to v_k(t)\hat{a}_{\vec{k}} + v_k^*(t)\hat{a}^\dagger_{-\vec{k}}$$

(Bunch-Davies) Vacuum Fluct.

$$\left[v \equiv z\mathcal{R}\,, \quad z \equiv a\frac{\dot{\phi}}{H}\right]$$

$$\left\langle \hat{\mathcal{R}}_{\vec{k}}\hat{\mathcal{R}}_{\vec{k}'}\right\rangle \equiv \frac{1}{z^2}\left\langle \hat{v}_{\vec{k}}\hat{v}_{\vec{k}'}\right\rangle \equiv (2\pi)^3 \boxed{\frac{H^2}{a^2\dot{\phi}^2}|v_k(\tau)|^2}\,\delta(\vec{k}+\vec{k}')$$

$$\equiv P_{\mathcal{R}}(k,\tau)$$

Scalar Power Spectrum

$$\Delta^2_{\mathcal{R}}(k,\tau) \equiv \frac{k^3}{2\pi^2}P_{\mathcal{R}}(k,\tau)$$

Dimensionless Scalar PS

(k << aH)

$$\Delta^2_{\mathcal{R}}(k) = \frac{H^4}{(2\pi)^2\dot{\phi}^2}\left(\frac{k}{aH}\right)^{2\eta-4\epsilon}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Tensor Fluctuations:

$$S_{(2)}^{(t)} = \frac{m_p^2}{8} \int dt dx^3 a^3 \left[(\dot{h}_{ij})^2 - a^{-2} (\partial_l h_{ij})^2 \right]$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Tensor Fluctuations:

$d\tau \equiv dt/a(t)$ (Conformal time)

$$\boxed{S^{(t)}_{(2)} = \frac{m_p^2}{8} \int dt dx^3 a^3 \left[(\dot{h}_{ij})^2 - a^{-2} (\partial_l h_{ij})^2 \right]} = \boxed{\sum_s \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3\mathbf{k} \left[(v^{s\,\prime}_{\mathbf{k}})^2 - \left(k^2 - \frac{a''}{a} \right) (v^s_{\mathbf{k}})^2 \right]}$$

$$h_{ij}(\vec{k}, \tau) = \epsilon^{(s)}_{ij} h^{(s)}_{\vec{k}} \longrightarrow v^{(s)} \equiv \frac{a}{2} m_p h^{(s)}_{\vec{k}}$$

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Tensor Fluctuations:

$$S_{(2)}^{(t)} = \frac{m_p^2}{8}\int dt dx^3 a^3 \left[(\dot{h}_{ij})^2 - a^{-2}(\partial_l h_{ij})^2\right] = \sum_s \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3\mathbf{k}\left[(v_{\mathbf{k}}^{s\,\prime})^2 - \left(k^2 - \frac{a''}{a}\right)(v_{\mathbf{k}}^s)^2\right] \Rightarrow$$

$\Rightarrow$ [Same Procedure as with Scalar Pert.

Quantize → Bunch-Davies → Power Spectrum]

Quantization of Gravity dof !

Inflation & Primordial Perturbations

Inflation: A generator of Primordial Fluctuations

Tensor Fluctuations:

$$\boxed{S_{(2)}^{(t)} = \frac{m_p^2}{8} \int dt dx^3 a^3 \left[(\dot{h}_{ij})^2 - a^{-2} (\partial_l h_{ij})^2 \right]} = \boxed{\sum_s \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3\mathbf{k} \left[(v_{\mathbf{k}}^{s\,\prime})^2 - \left(k^2 - \frac{a''}{a} \right) (v_{\mathbf{k}}^s)^2 \right]} \Rightarrow$$

$\Rightarrow$ [Same Procedure as with Scalar Pert.

Quantize → Bunch-Davies → Power Spectrum]

Quantization of Gravity dof !

$$\boxed{\Delta_h^2(k,\tau) \equiv \frac{k^3}{2\pi^2} P_h(k,\tau)} \xrightarrow[(\,k \ll aH\,)]{} \boxed{\Delta_h^2(k) = \frac{2}{\pi^2} \left(\frac{H}{m_p} \right)^2 \left(\frac{k}{aH} \right)^{-2\epsilon}}$$