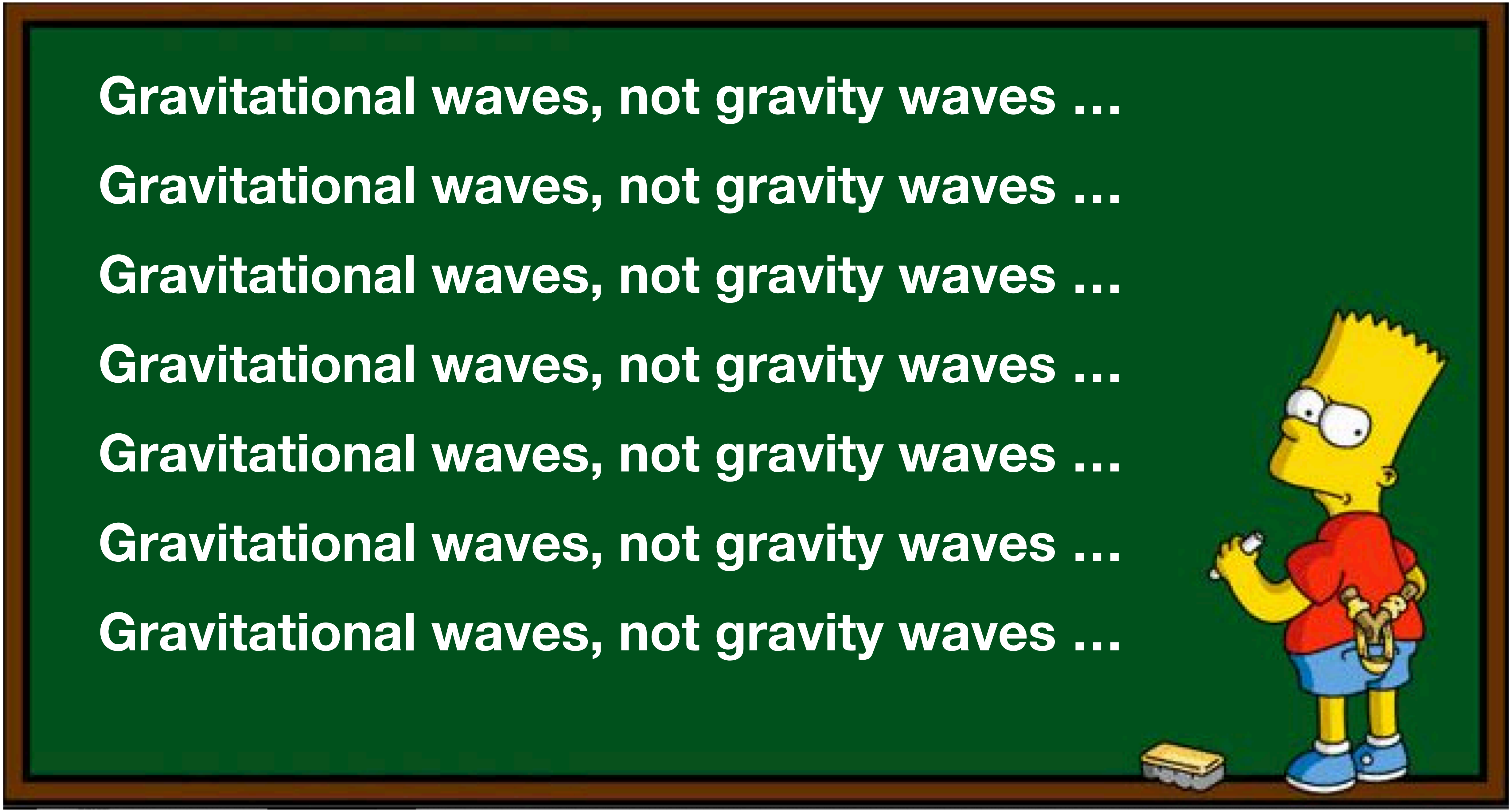
# Gravitational Waves

**DANIEL G. FIGUEROA**
IFIC, Valencia (UV/CSIC)

MITP Summer School - CrossLinks of Early Universe Cosmology, 15 July - August 2, 2024



Gravitational waves, not gravity waves …

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# Gravitational Waves (GW)

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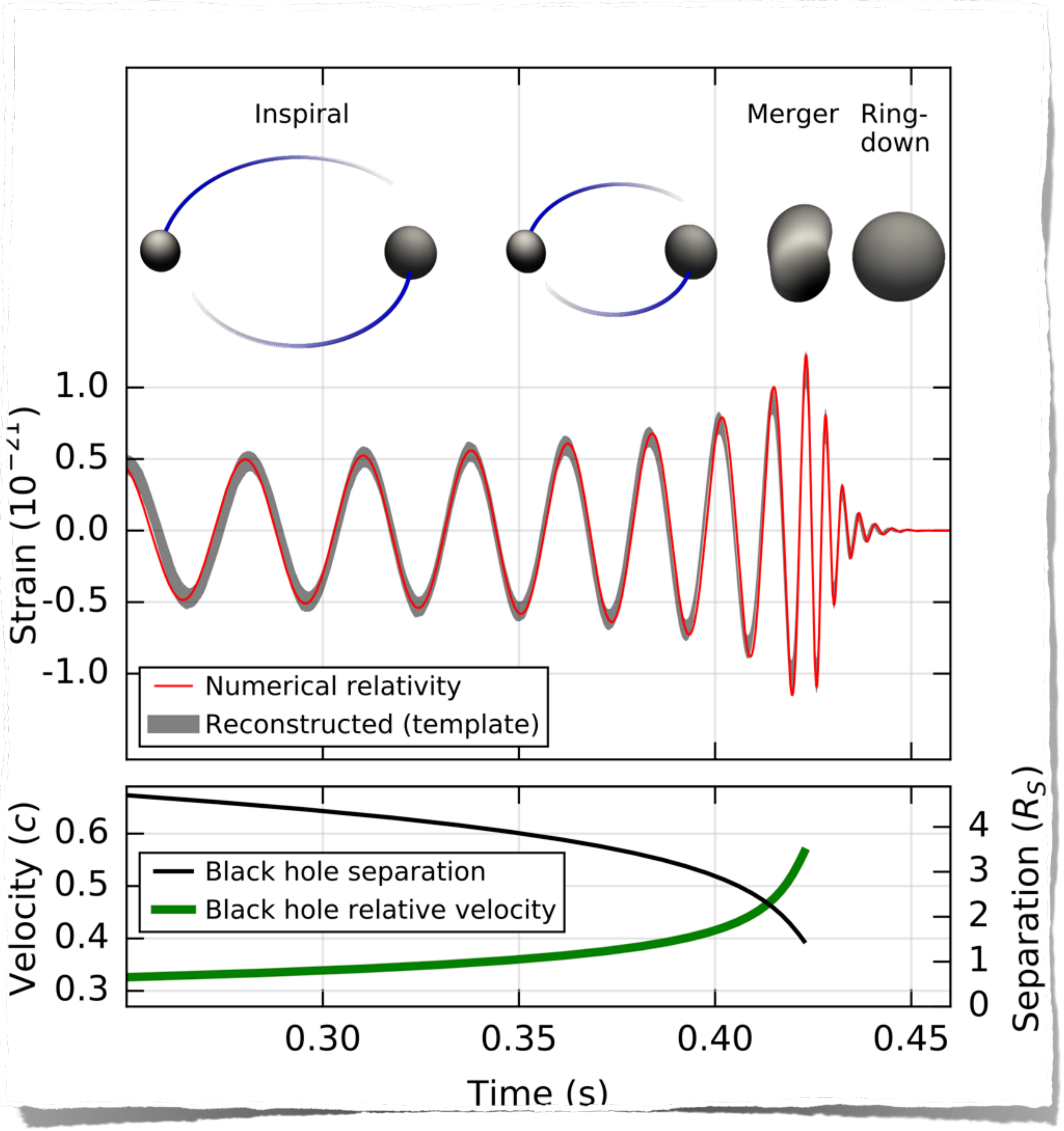
# MOTIVATION

(just to warm up)

# Let us celebrate …

These are special times !

# Let us celebrate …

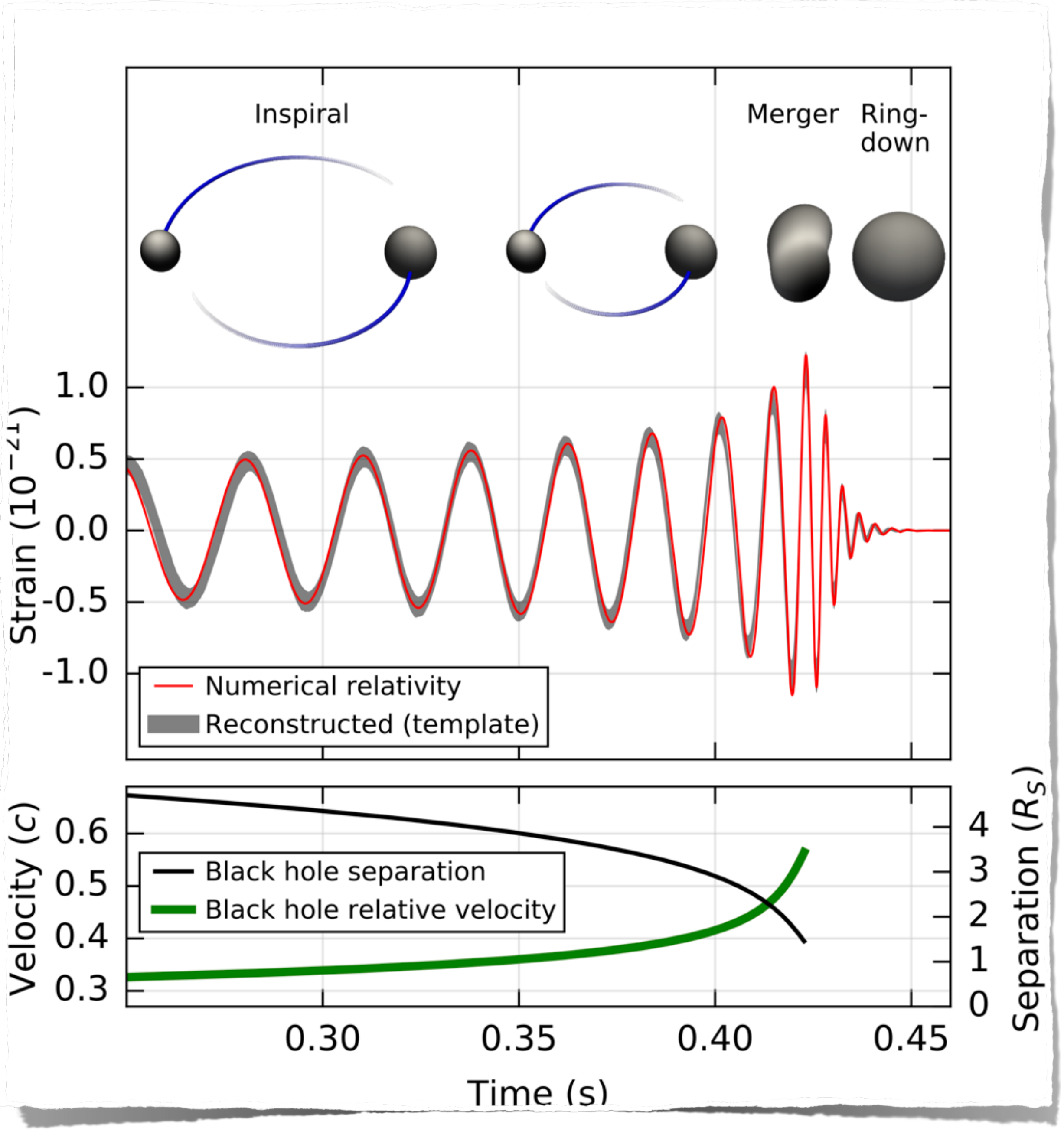


[LIGO & Virgo Scientific Collaborations (arXiv:1602.03841)]

**Gravitational Waves (GW) detected !**

**[LIGO/VIRGO]**

# Let us celebrate …



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Gravit al

)

[ VIRGO]

milestone in physics

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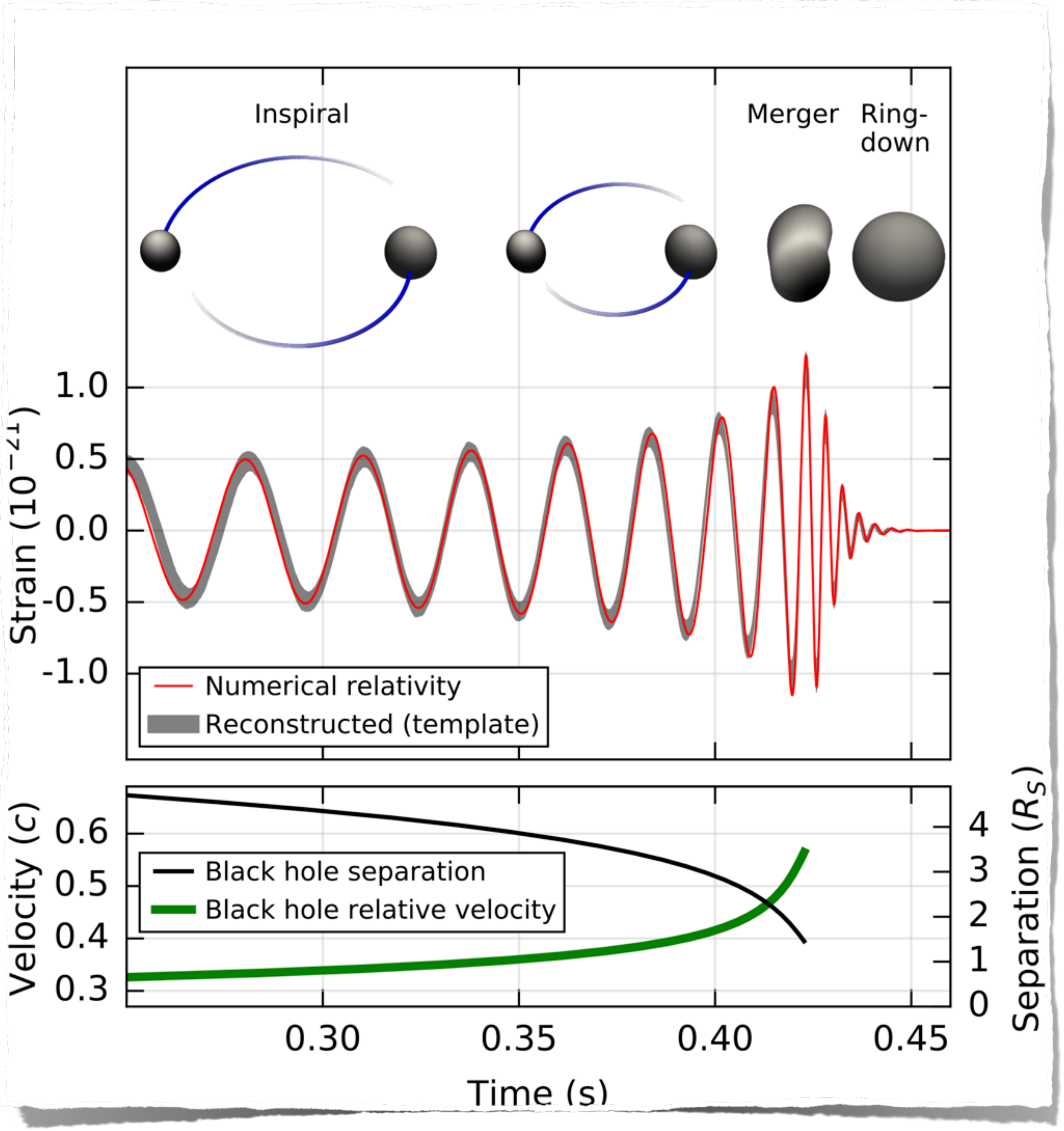

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Gravi  al

)

VIRGO]

milestone in physics

100 yrs !

Einstein 1915 … LIGO/VIRGO 2015-2024

# What have we learnt ?

Grav al

)

[ VIRGO]

**milestone in physics**

# What have we learnt ?

* O(10) Solar mass
Black Holes (BH) exist

* We can test the
BH's paradigm, and
Neutron Star physics

Grav al
)
[ /VIRGO]

**milestone
in physics**

# What have we learnt ?

* O(10) Solar mass
Black Holes (BH) exist

* We can test the
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* We can further test
General Relativity (GR)
[so far no deviation]

* We can observe the
Universe through GWs

* ...

Gravi al
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milestone
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# How did we learn this?

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[so far <u>no</u> deviation]

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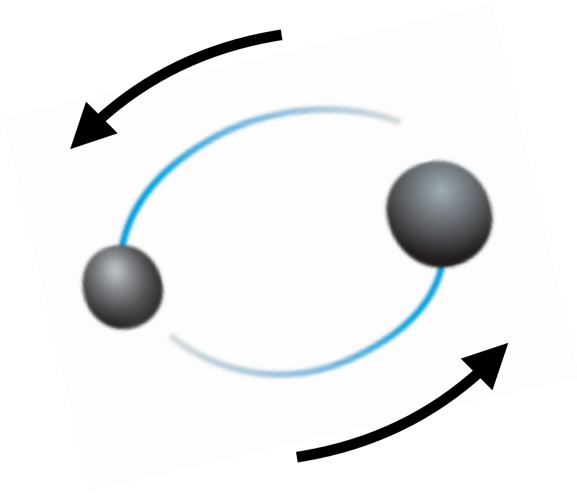
* ...

**Binaries**



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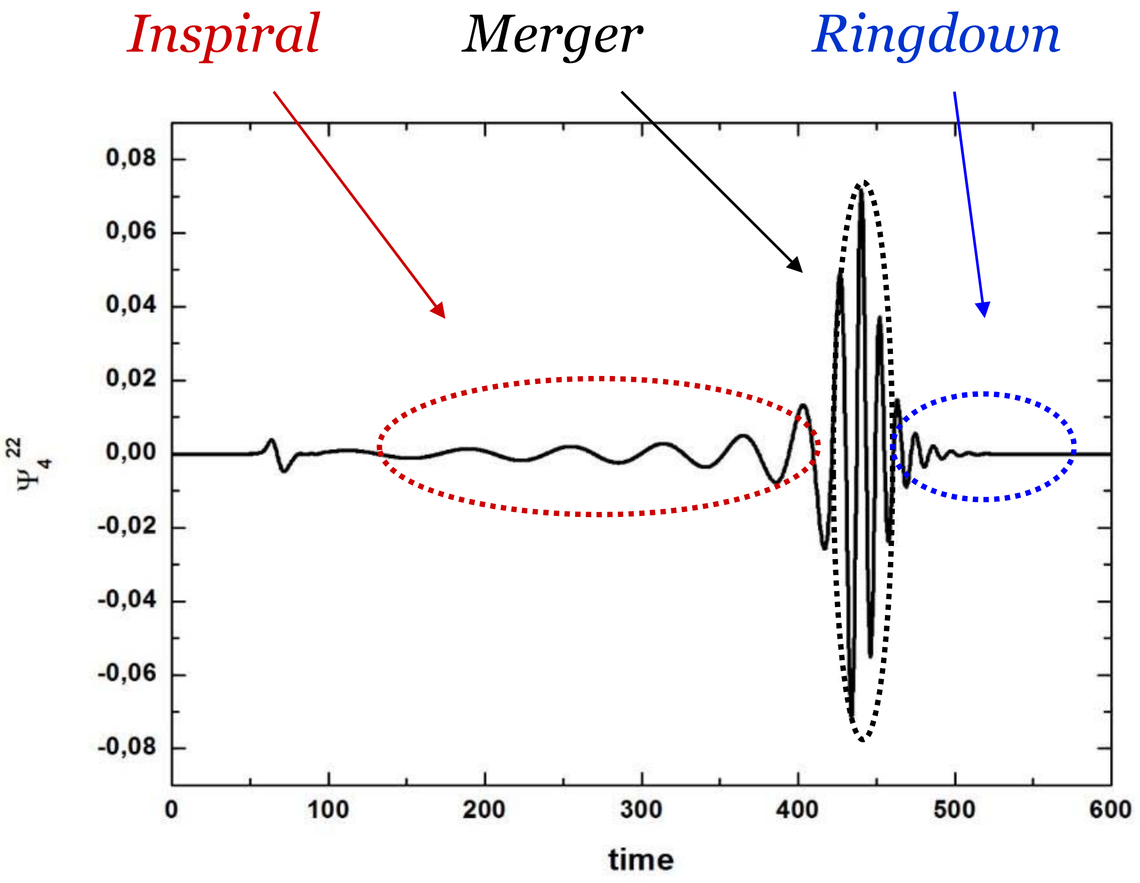
*** ...**

## Binary wave functions



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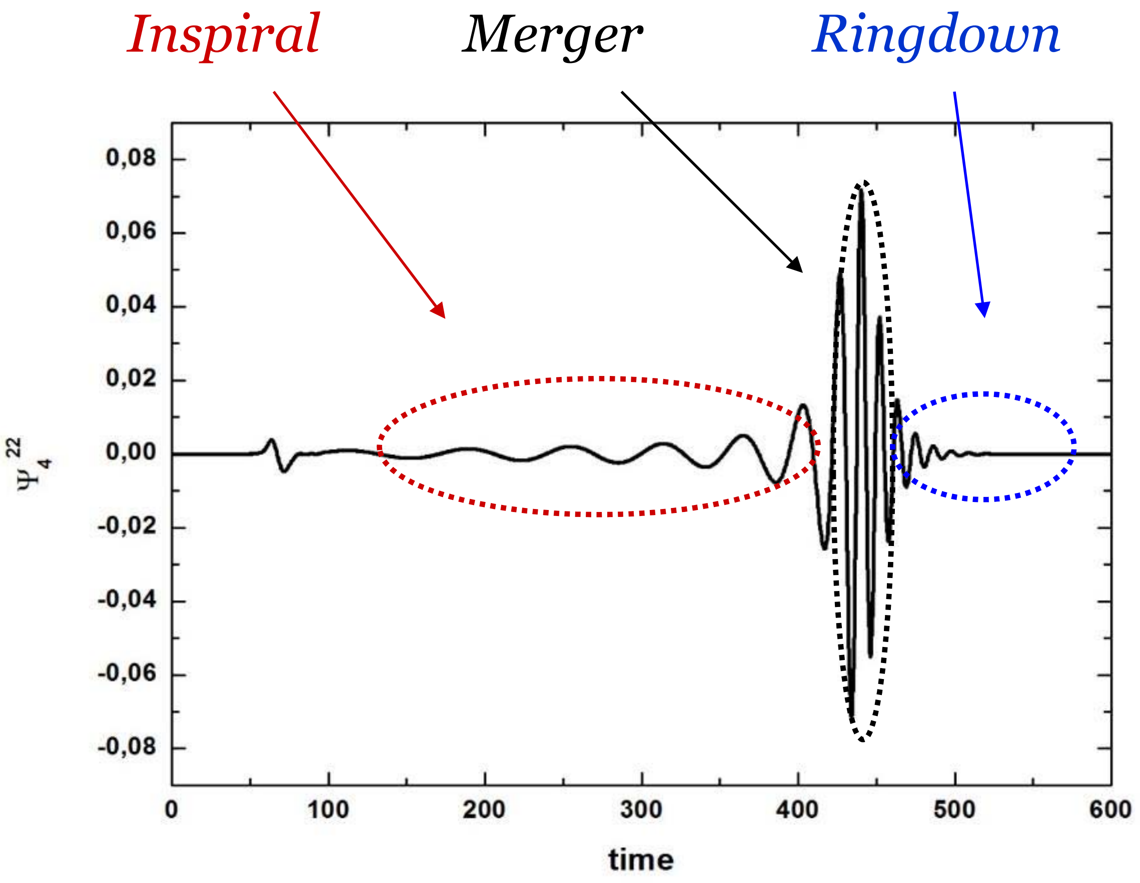

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(binaries)

**Extremely interesting !**

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… We will focus on something else !

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- We can observe the Universe through GWs

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→ (binaries)

**Extremely interesting !**

**however …**

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*** We can observe the Universe through GWs**

* We can observe the Universe through GWs

* We can observe the Universe through GWs

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* Cosmology with GWs

* We can observe the Universe through GWs

# * Cosmology with GWs

* Late Universe:

* Early Universe:

*** We can observe the Universe through GWs**

# * Cosmology with GWs

*** Late Universe:** Standard sirens: distances in cosmology;
Measuring H0 and EoS dark energy;
cosmological parameters;
modify gravity, lensing, …

*** We can observe the Universe through GWs**

## * Cosmology with GWs

- **Late Universe:**

- **Early Universe:** High Energy Particle Physics

* **We can observe the Universe through GWs**

# * Cosmology with GWs

* **Late Universe:** Are we going to forget about this ?

* **Early Universe:** High Energy Particle Physics

*** We can observe the Universe through GWs**

# * Cosmology with GWs

*** Late Universe:** Nope, we simply postpone …

*** Early Universe:** High Energy Particle Physics

* **We can observe the Universe through GWs**

# * Cosmology with GWs

* **Early Universe:** High Energy Particle Physics

*** We can observe the Universe through GWs**

*** Cosmology with GWs**

* **Early Universe:** High Energy Particle Physics

**Can we really probe High Energy Physics using Gravitational Waves (GWs) ? How ?**

# Can we really probe High Energy Physics using Gravitational Waves (GWs) ? How ?

## Can we really probe High Energy Physics using Gravitational Waves (GWs) ? How ?

**Answering these questions lies at the heart of what these lectures are about !**

# Can we really probe High Energy Physics using Gravitational Waves (GWs) ? How ?

**Before answering …**

**… let us ask another question**

# GWs: probe of the early Universe

**WHY ??**

# GWs: probe of the early Universe

**WHY ??**

**ONE & ONLY ONE reason**

# GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

   **ADVANTAGE**: GW DECOUPLE upon Production
   **DISADVANTAGE**: DIFFICULT DETECTION

# GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

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2. **ADVANTAGE**: GW → Probe for Early Universe

$$\rightarrow \begin{cases} \textbf{Decouple} \rightarrow \text{Spectral Form Retained} \\ \textbf{Specific HEP} \;\Leftrightarrow\; \text{Specific GW} \end{cases}$$

# GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

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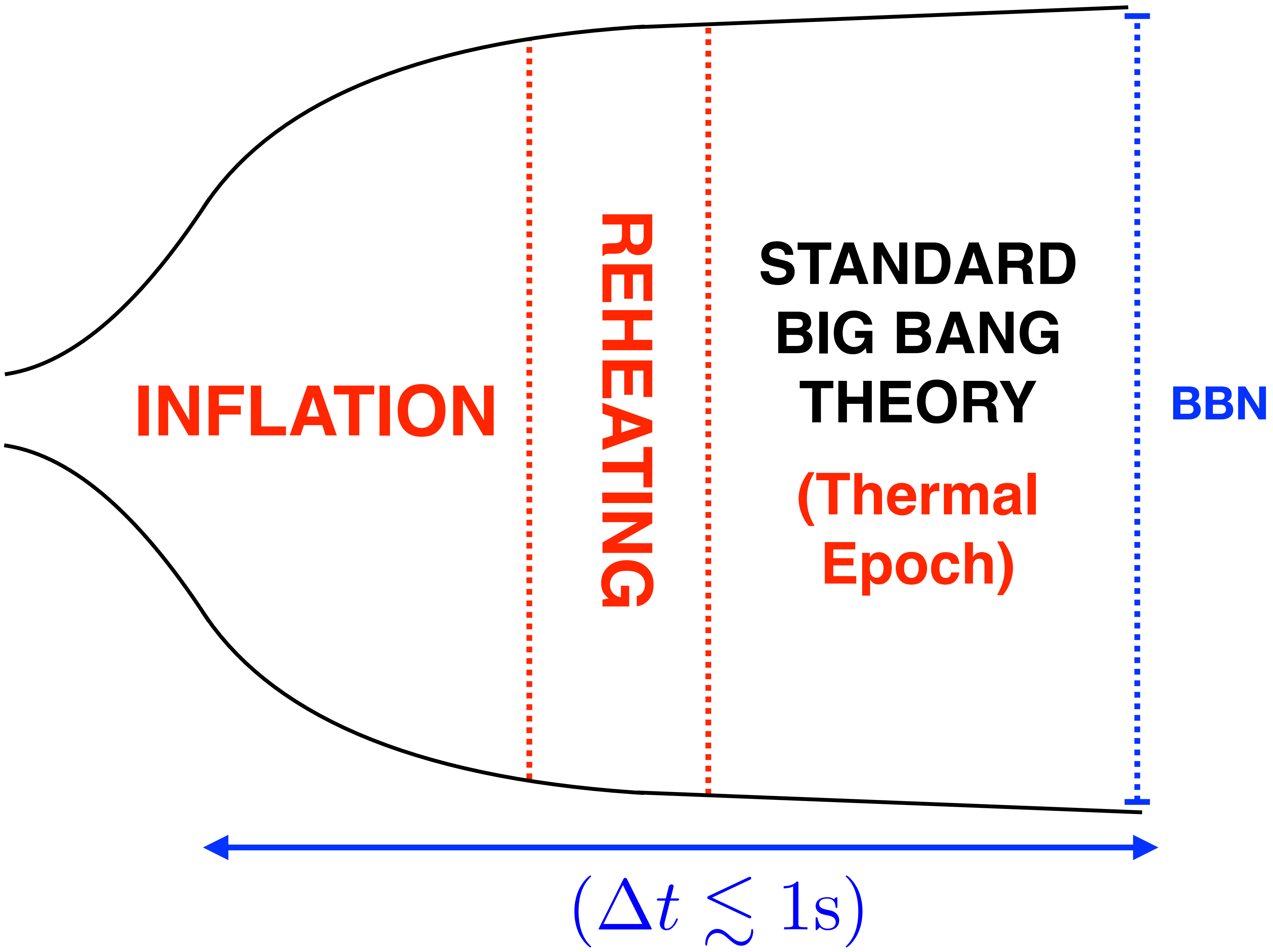
$$\rightarrow \begin{cases} \textbf{Decouple} \rightarrow \text{Spectral Form Retained} \\ \textbf{Specific HEP} \;\Leftrightarrow\; \text{Specific GW} \end{cases}$$

**What processes of the early Universe ?**

# The Early Universe



INFLATION

REHEATING

STANDARD
BIG BANG
THEORY

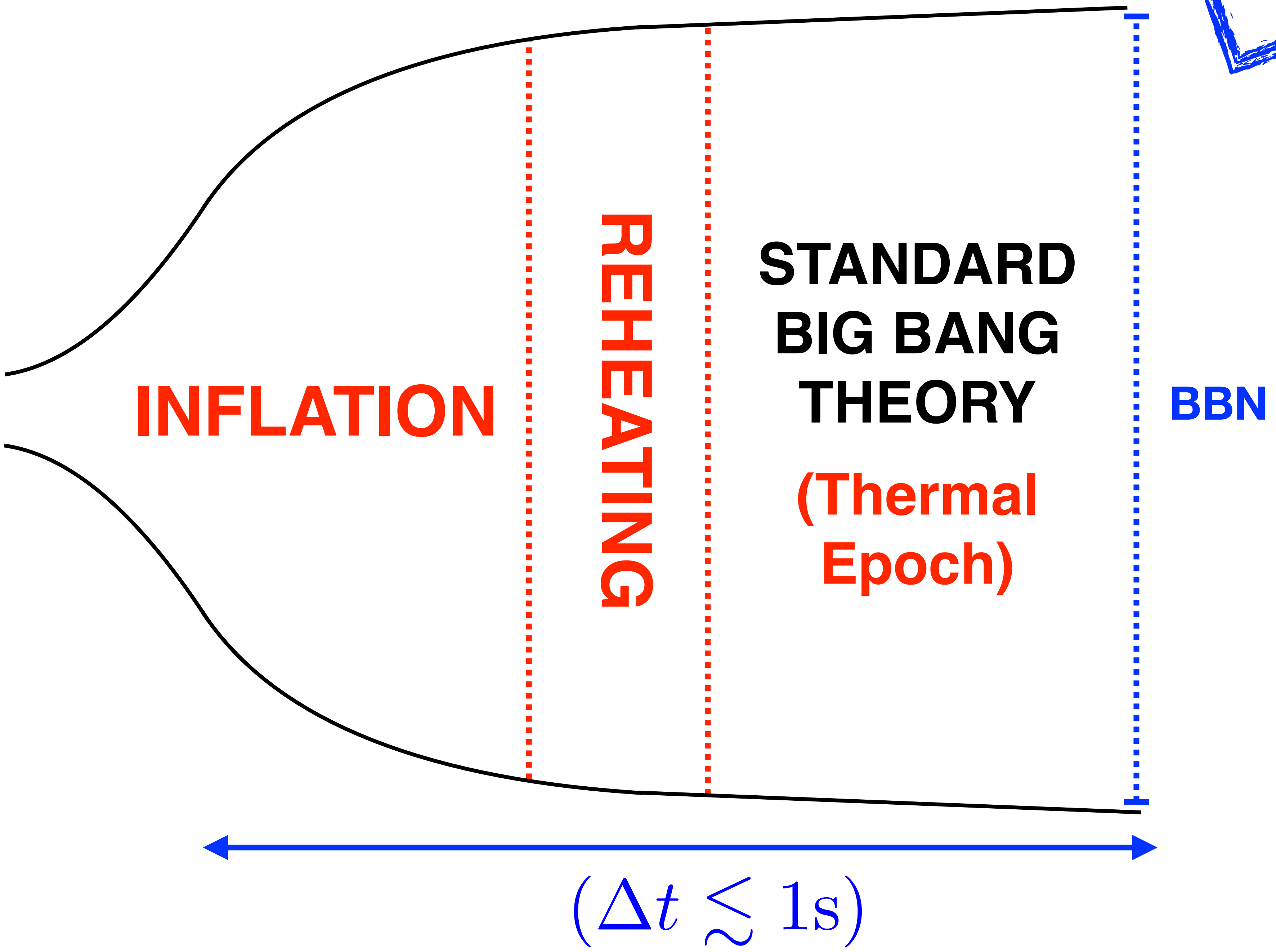
(Thermal
Epoch)

BBN

$(\Delta t \lesssim 1\mathrm{s})$

# The Early Universe

Lectures by
Yvonne Wong



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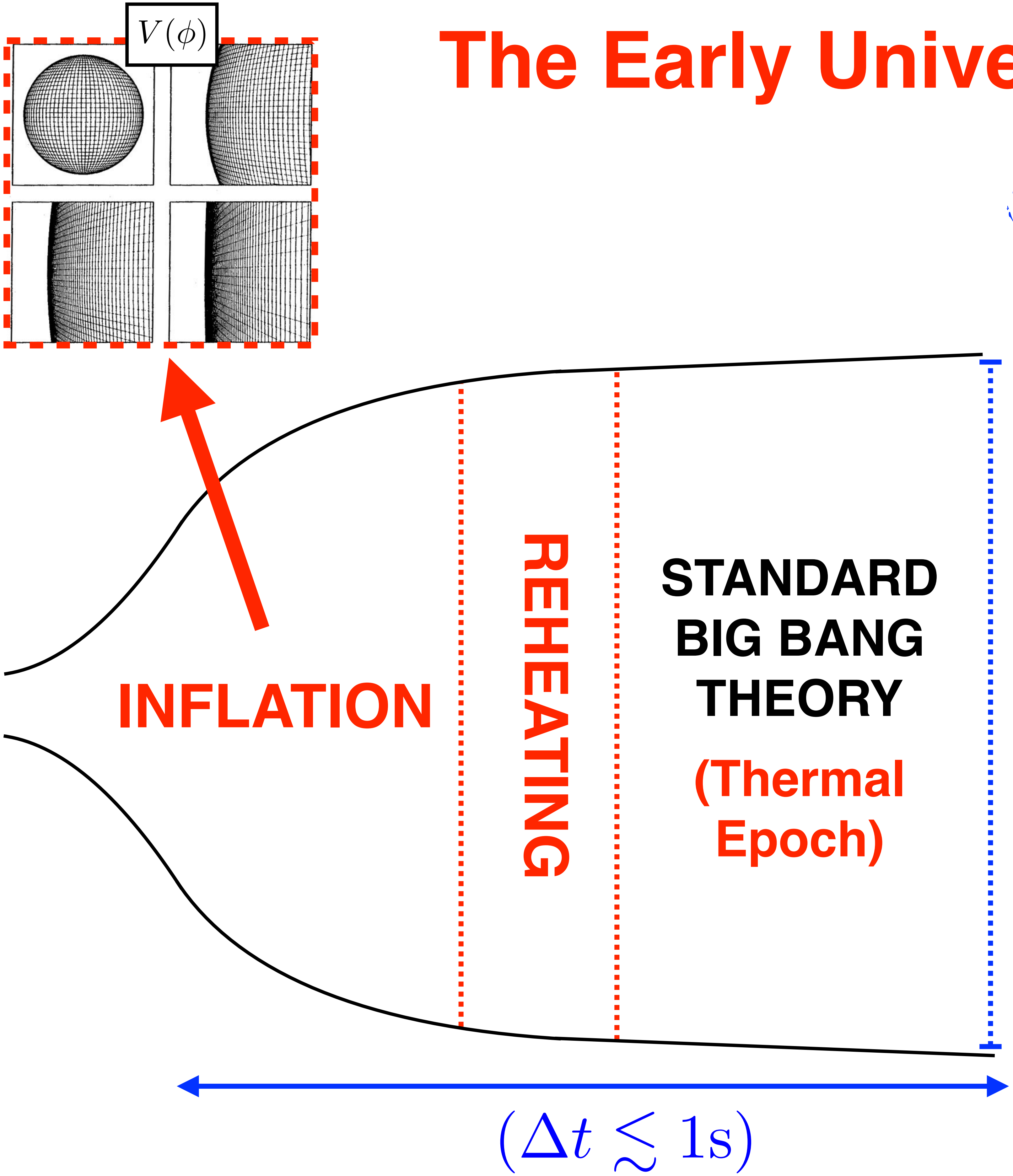

$V(\phi)$

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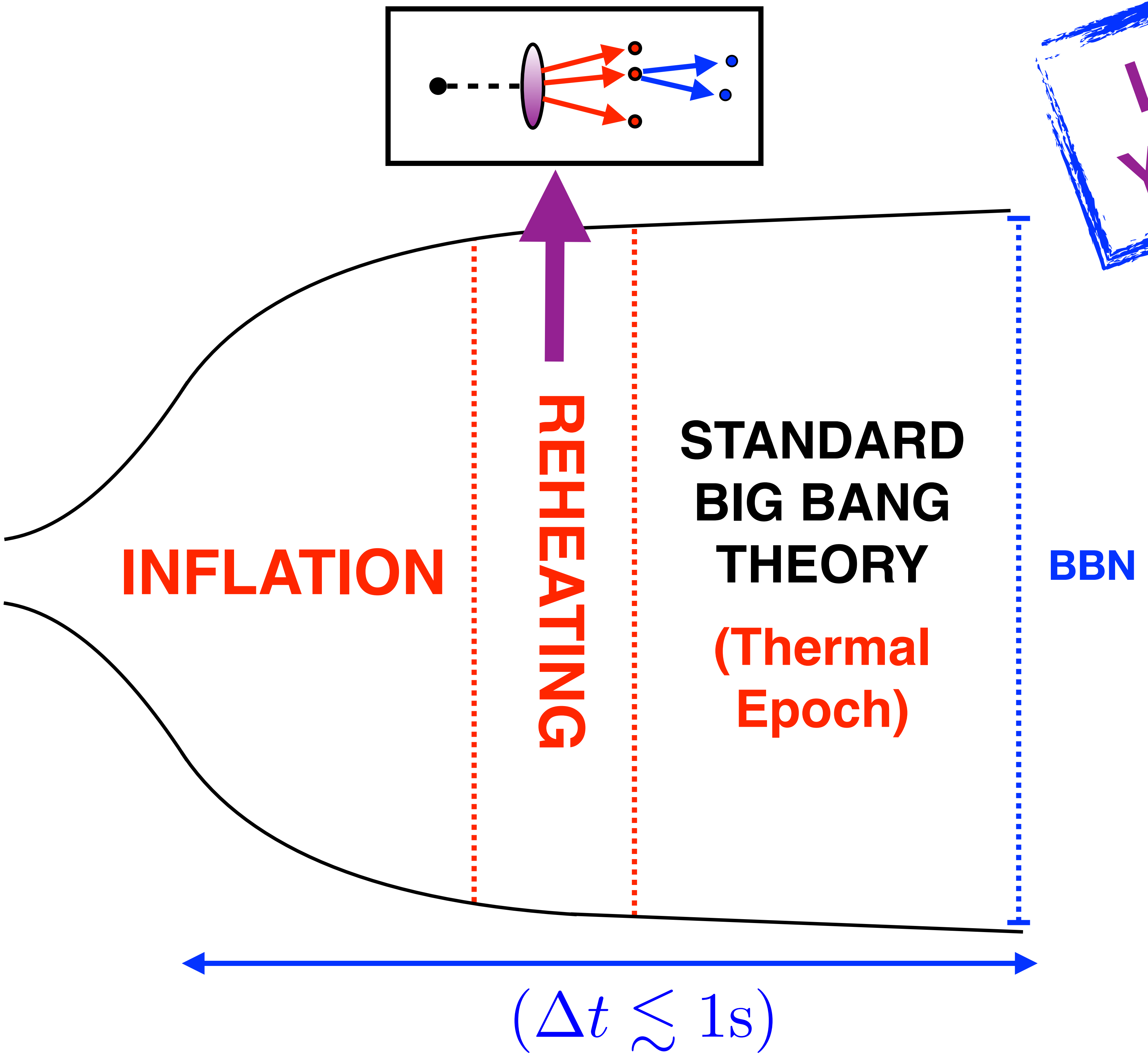

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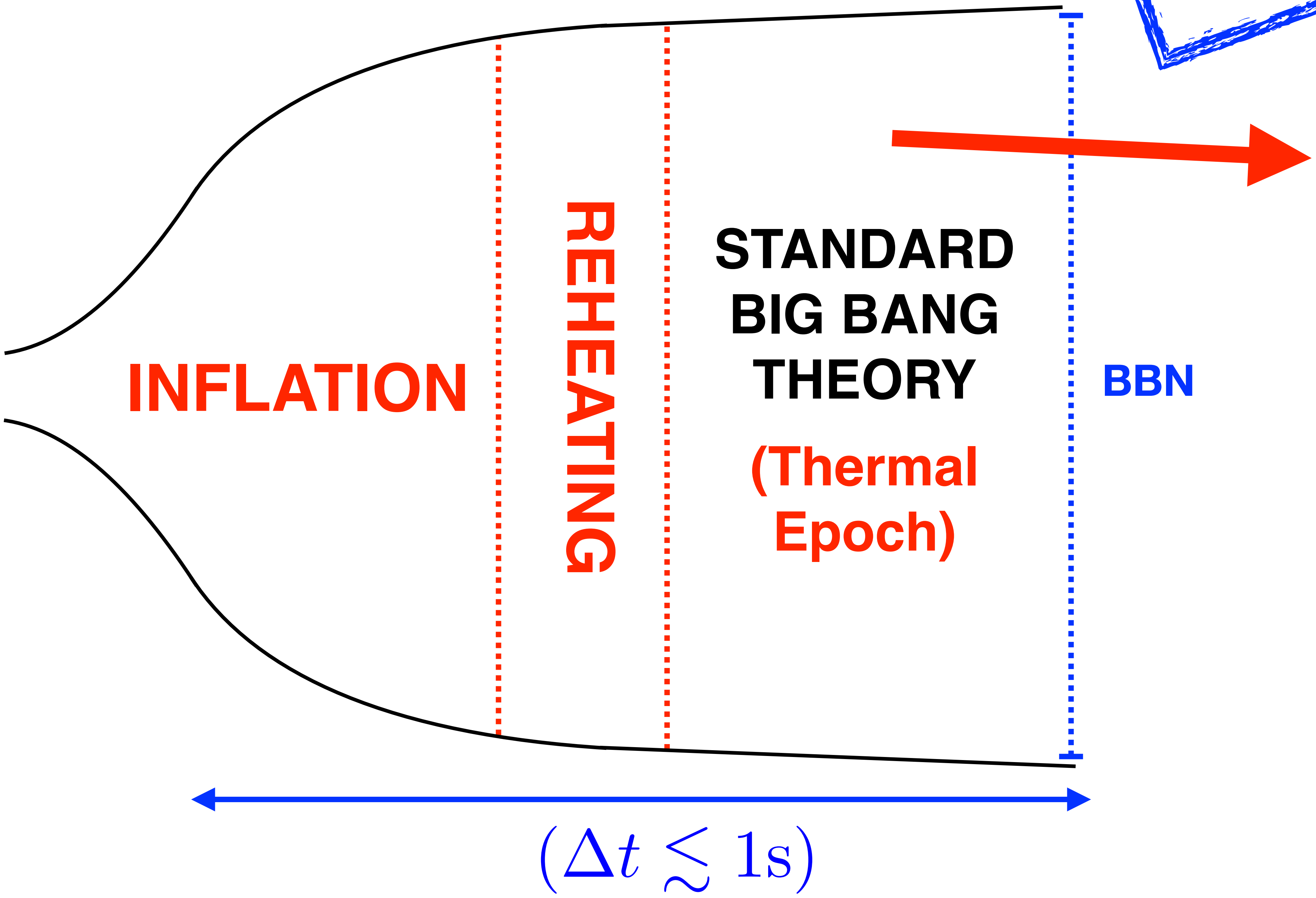

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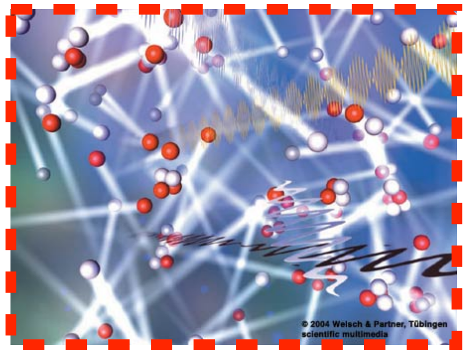

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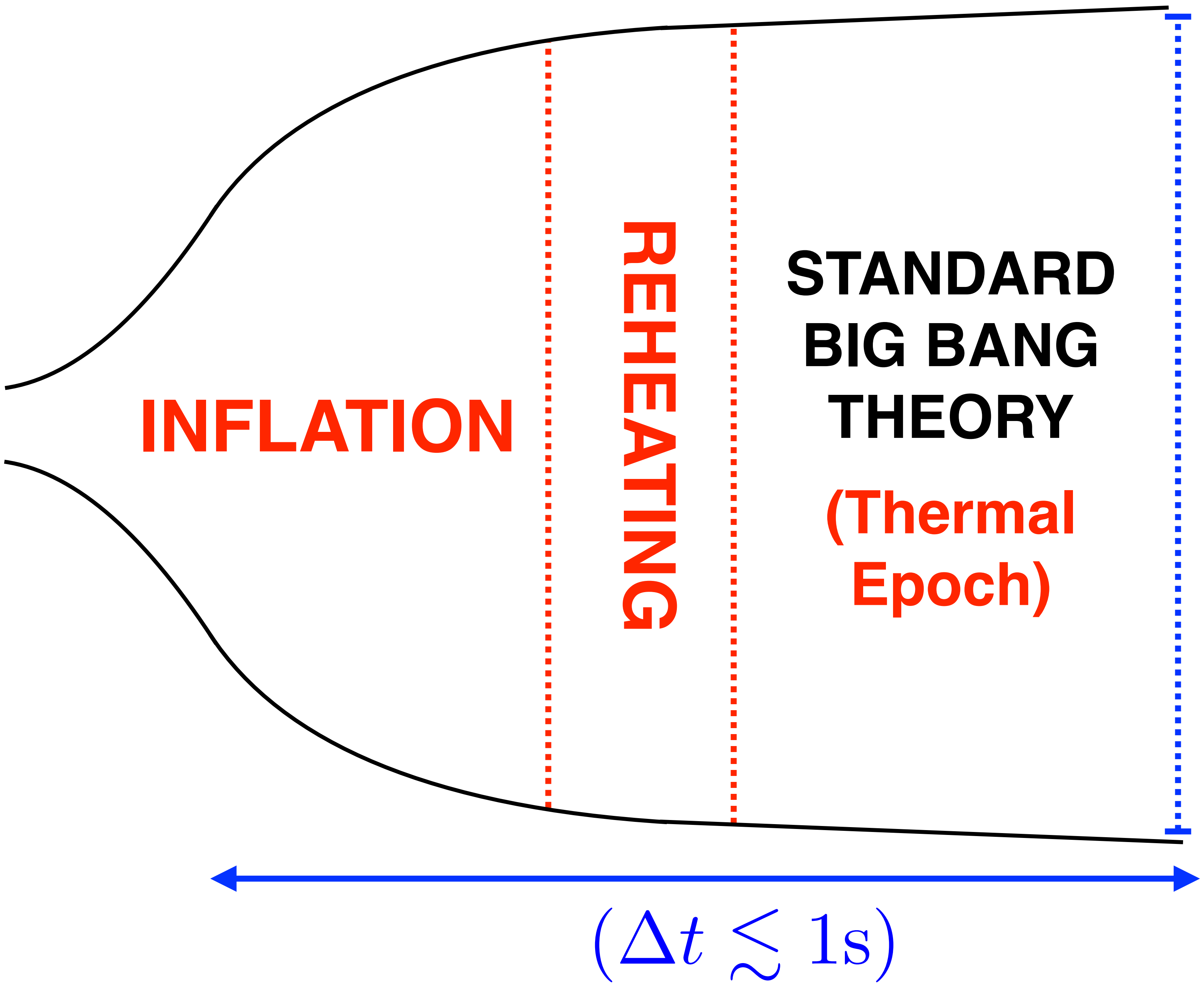
# The Early Universe



INFLATION

REHEATING

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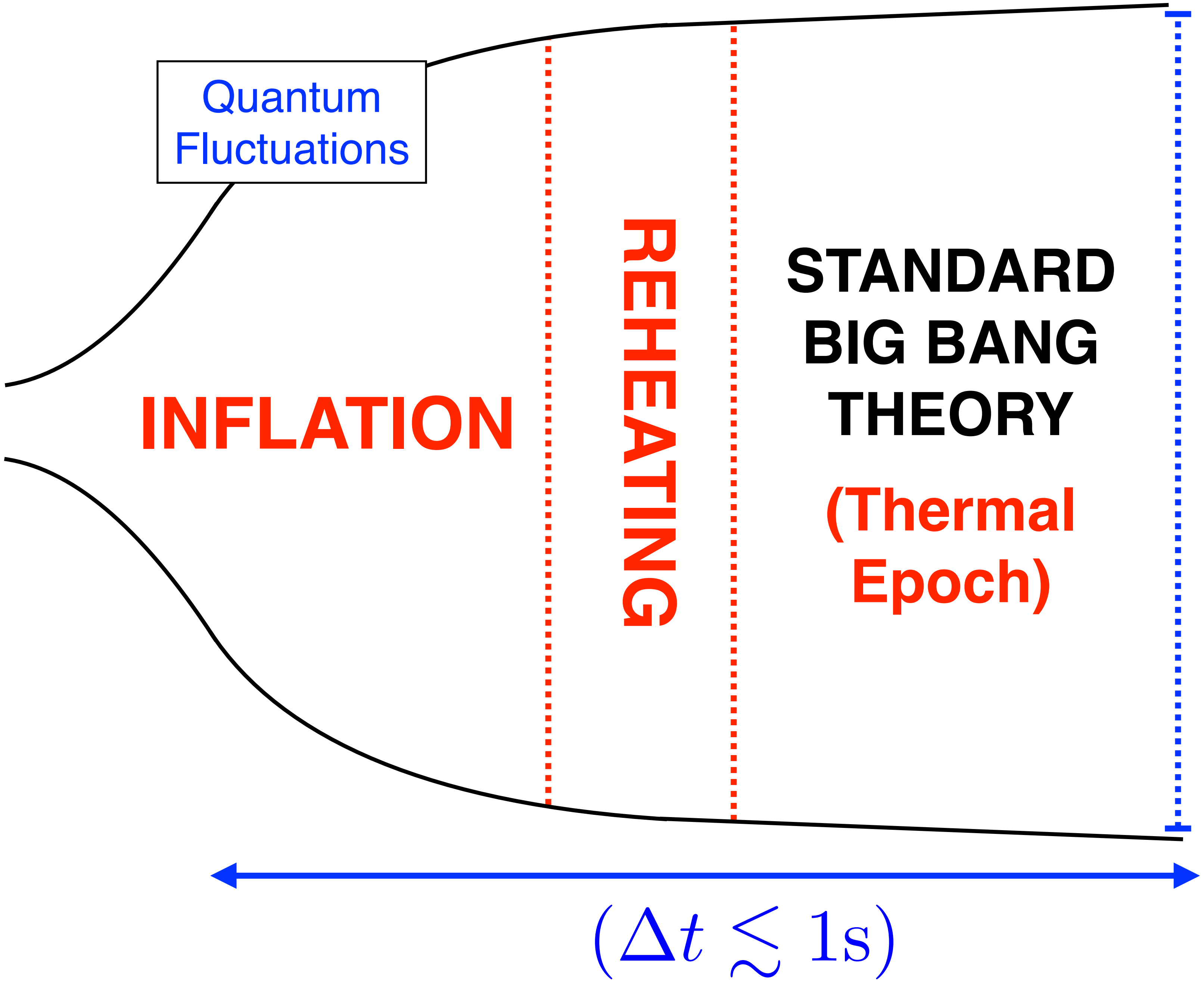
BIG BANG
THEORY

(Thermal
Epoch)

$(\Delta t \lesssim 1\mathrm{s})$

**What phenomena are we interested in ?**

# The Early Universe



Quantum Fluctuations

**INFLATION**

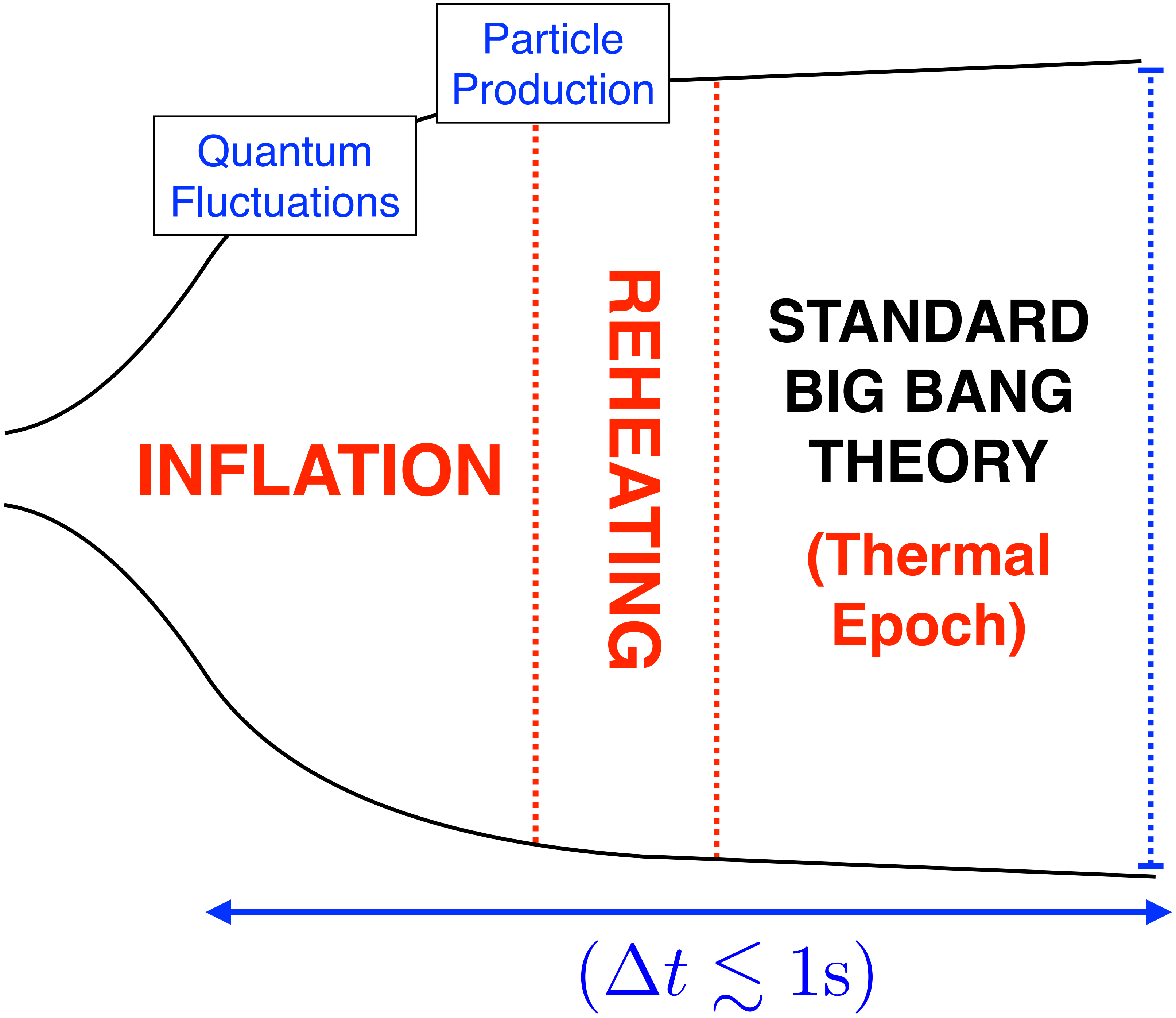
**REHEATING**

**STANDARD BIG BANG THEORY**

**(Thermal Epoch)**

$(\Delta t \lesssim 1\mathrm{s})$

# The Early Universe



Particle Production

Quantum Fluctuations

**INFLATION**

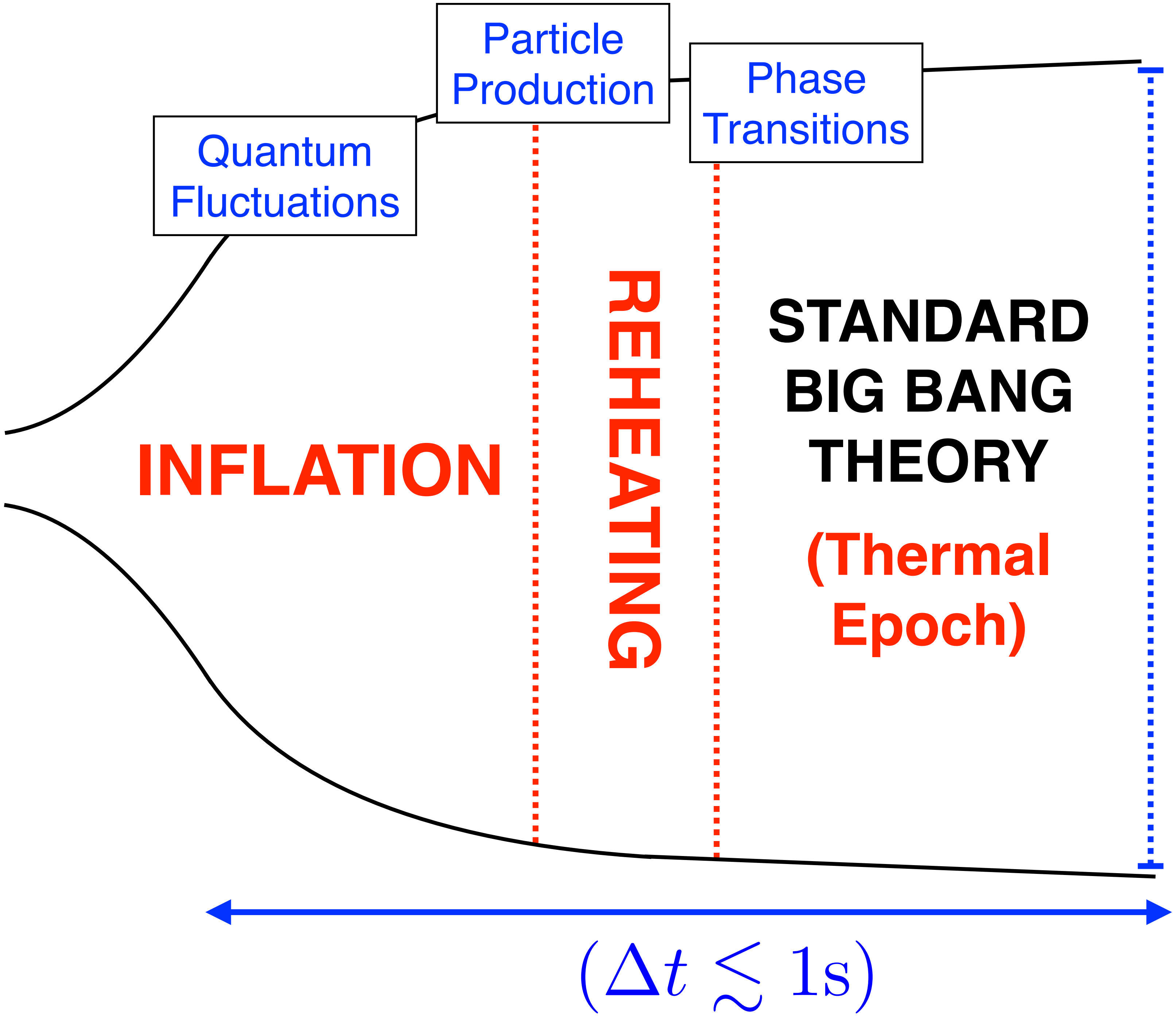
**REHEATING**

**STANDARD BIG BANG THEORY**

**(Thermal Epoch)**

$(\Delta t \lesssim 1\mathrm{s})$

# The Early Universe



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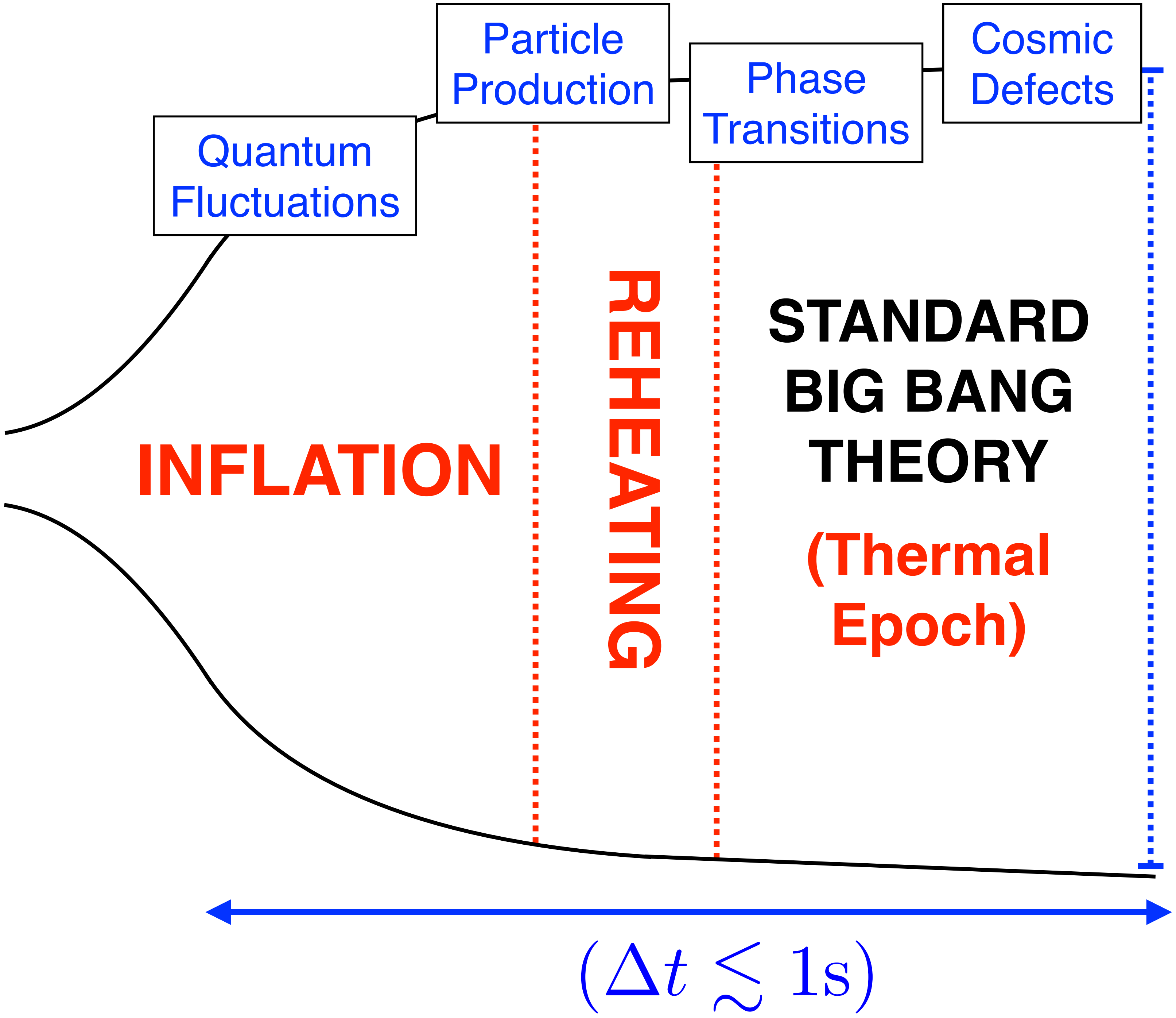
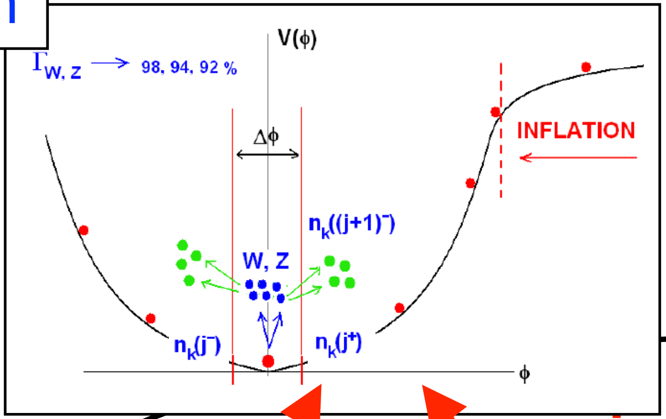
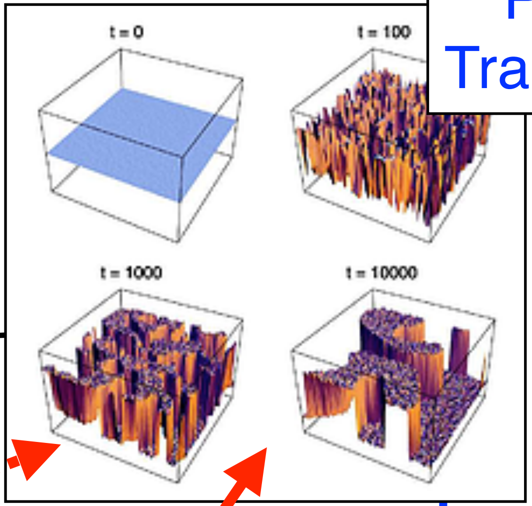


# The Early Universe

Particle Production



Phase Transitions



INFLATION

REHEATING

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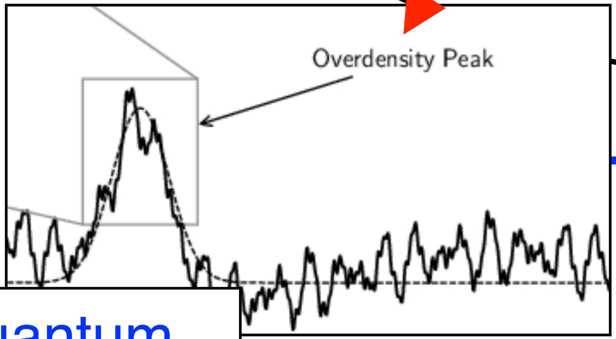
(Thermal Epoch)



Quantum Fluctuations

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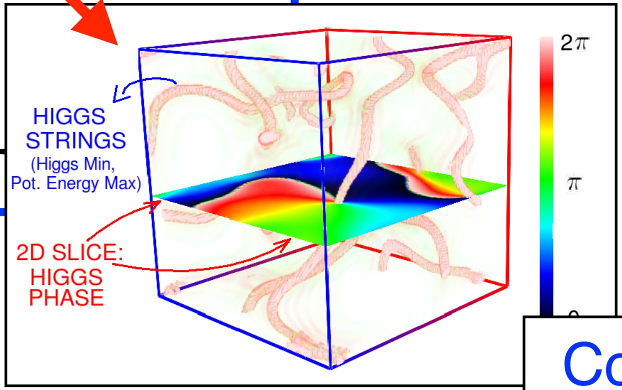
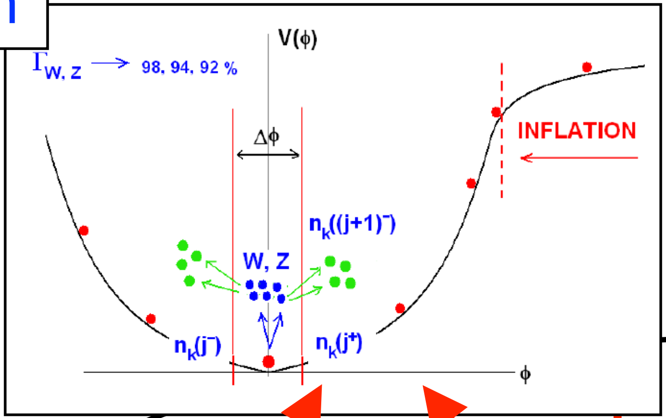


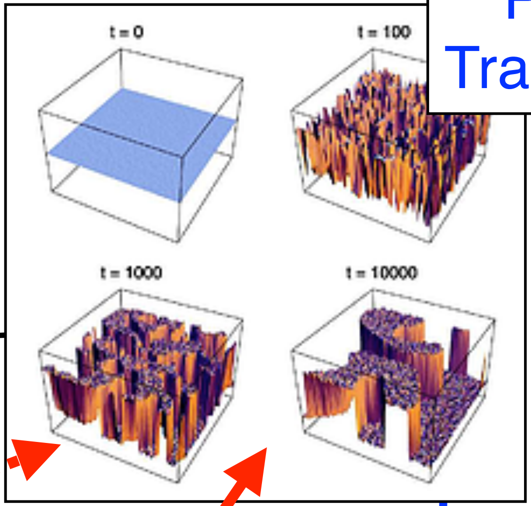
Cosmic Defects

# The Early Universe

Particle Production



Phase Transitions

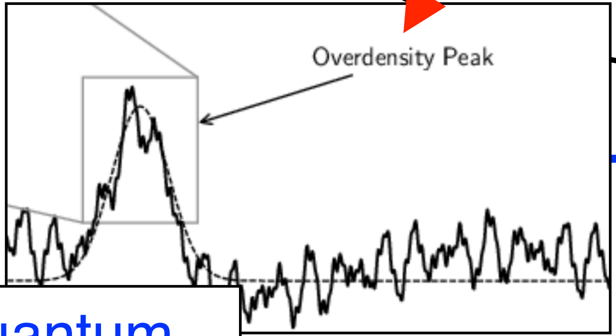


INFLATION

REHEATING

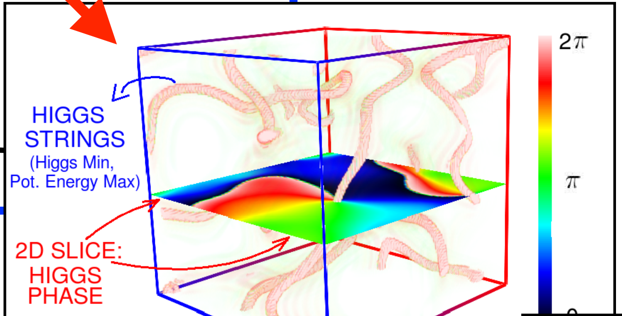
STANDARD BIG BANG THEORY

(Thermal Epoch)

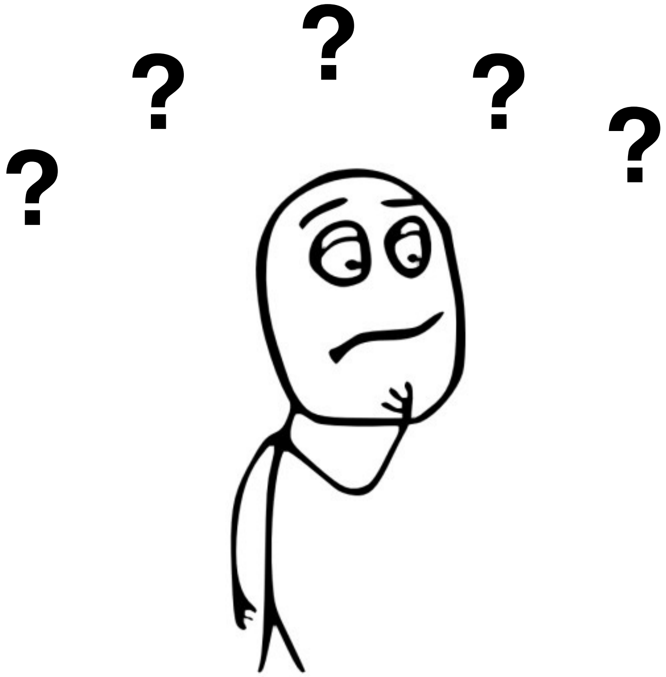


Quantum Fluctuations

$(\Delta t \lesssim 1\text{s})$

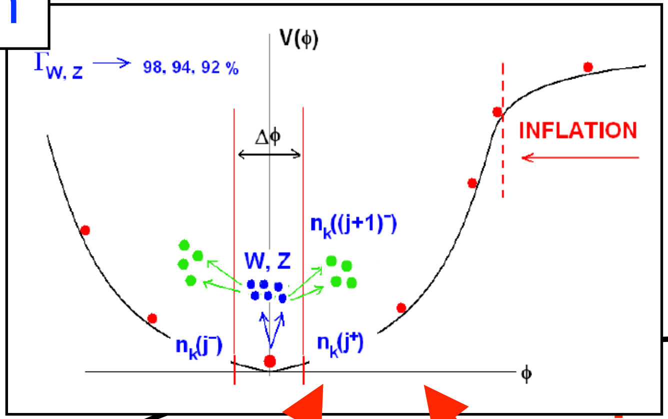


Cosmic Defects

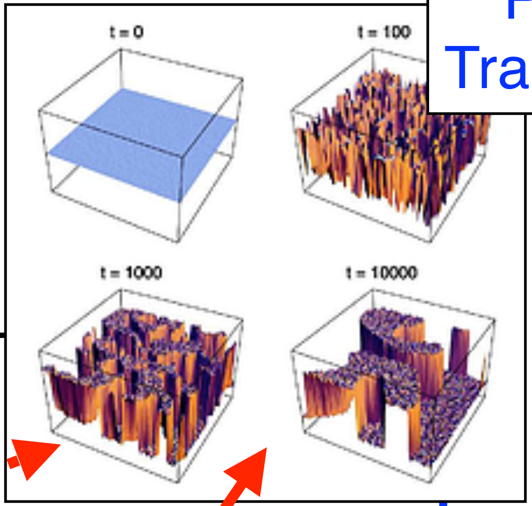


# The Early Universe

Particle Production



Phase Transitions



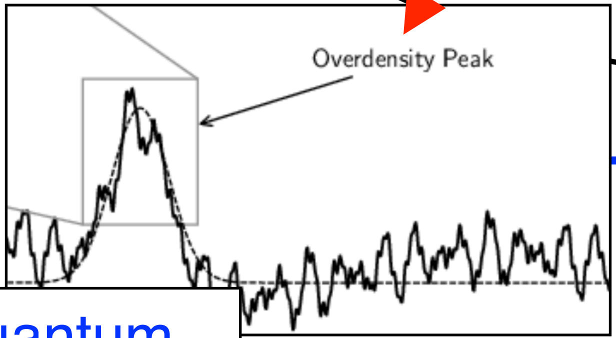
**INFLATION**

**REHEATING**

**STANDARD BIG BANG THEORY**

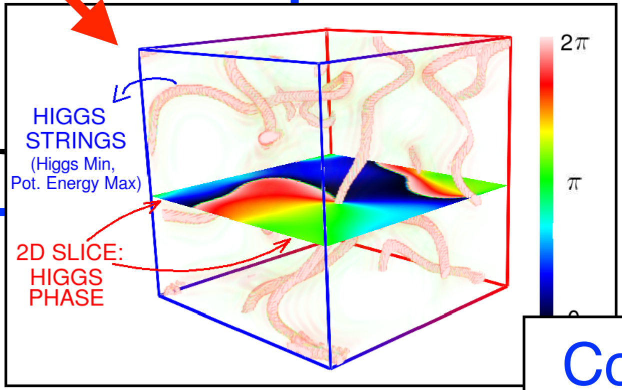
**(Thermal Epoch)**

**Grav. Wave emission !**



Quantum Fluctuations

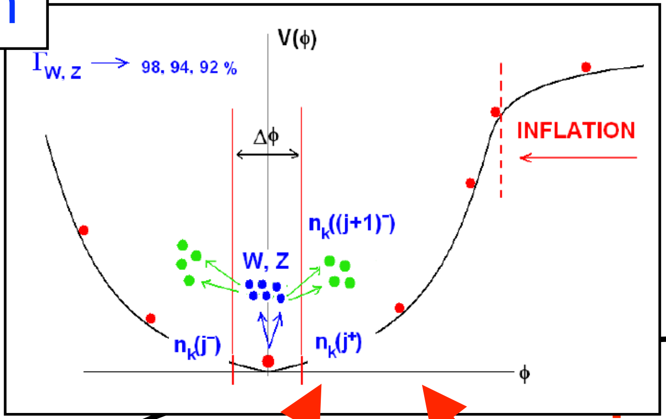
$(\Delta t \lesssim 1\mathrm{s})$



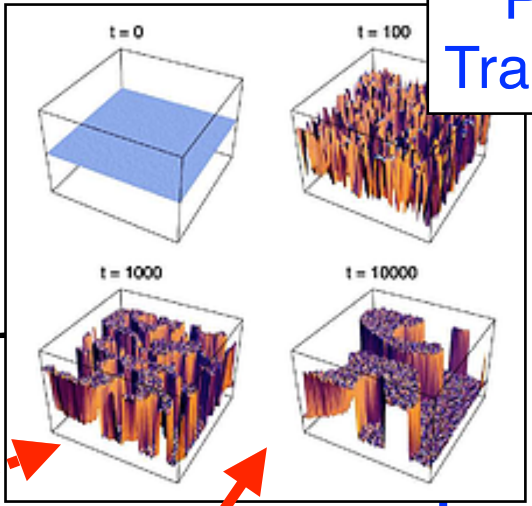
Cosmic Defects

# The Early Universe

Particle Production



Phase Transitions



INFLATION

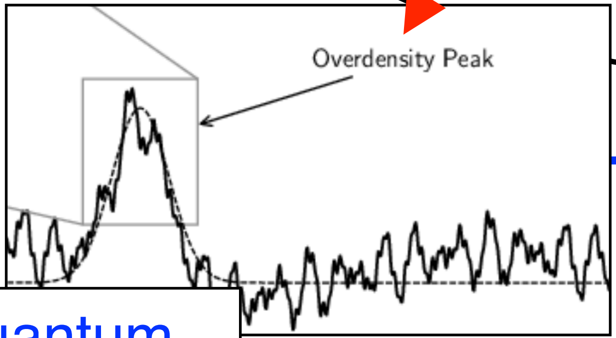
REHEATING

STANDARD BIG BANG THEORY

(Thermal Epoch)

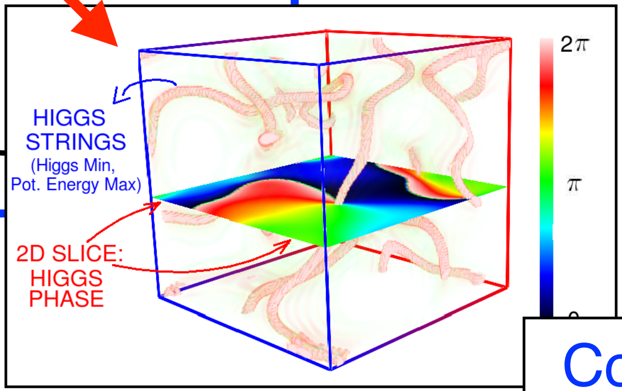
Grav. Wave emission !

( Non-linear Phenomena )



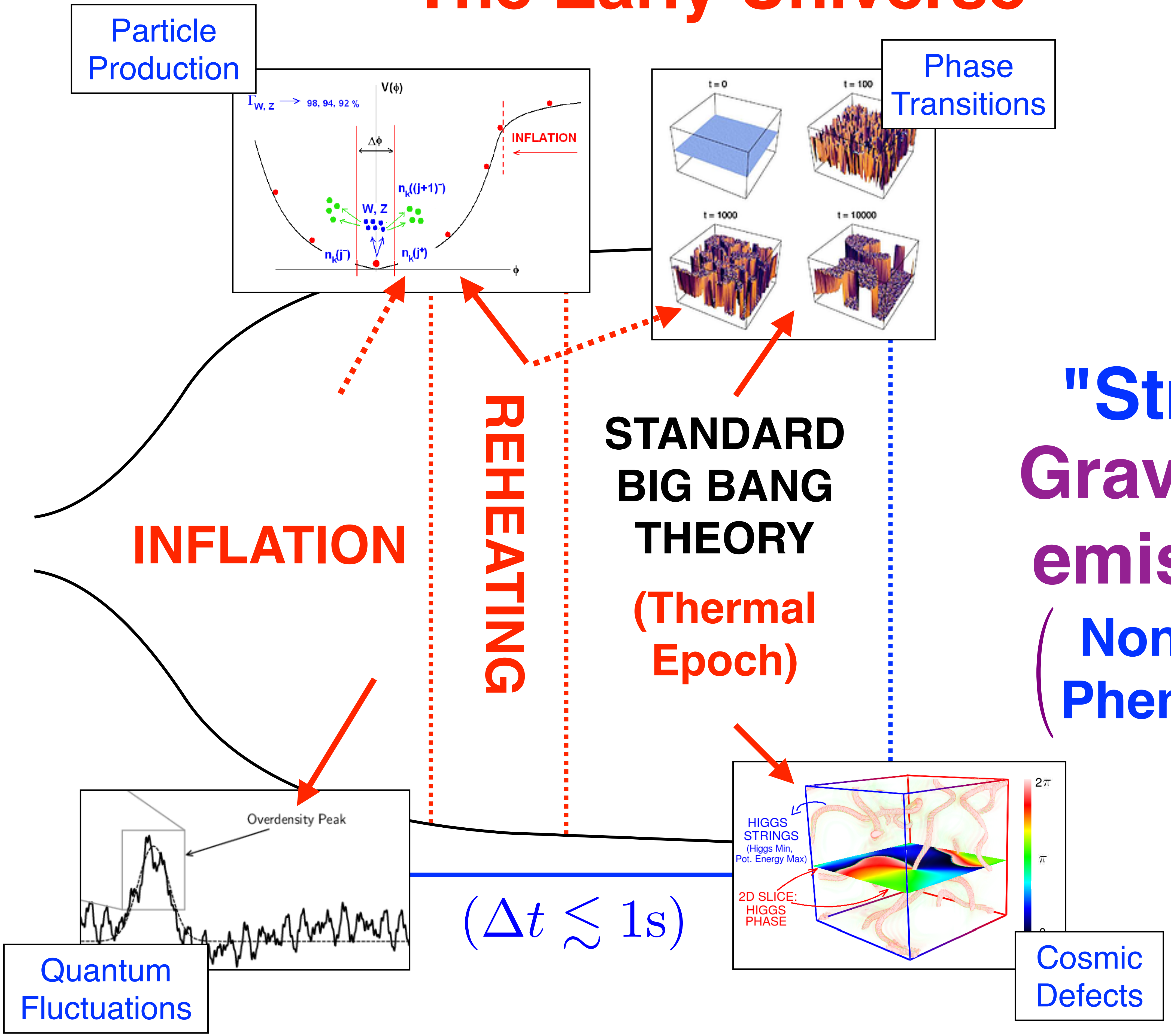
Quantum Fluctuations

$$(\Delta t \lesssim 1\mathrm{s})$$



Cosmic Defects

# The Early Universe



Particle Production

Phase Transitions

INFLATION

REHEATING

STANDARD BIG BANG THEORY

(Thermal Epoch)

Quantum Fluctuations

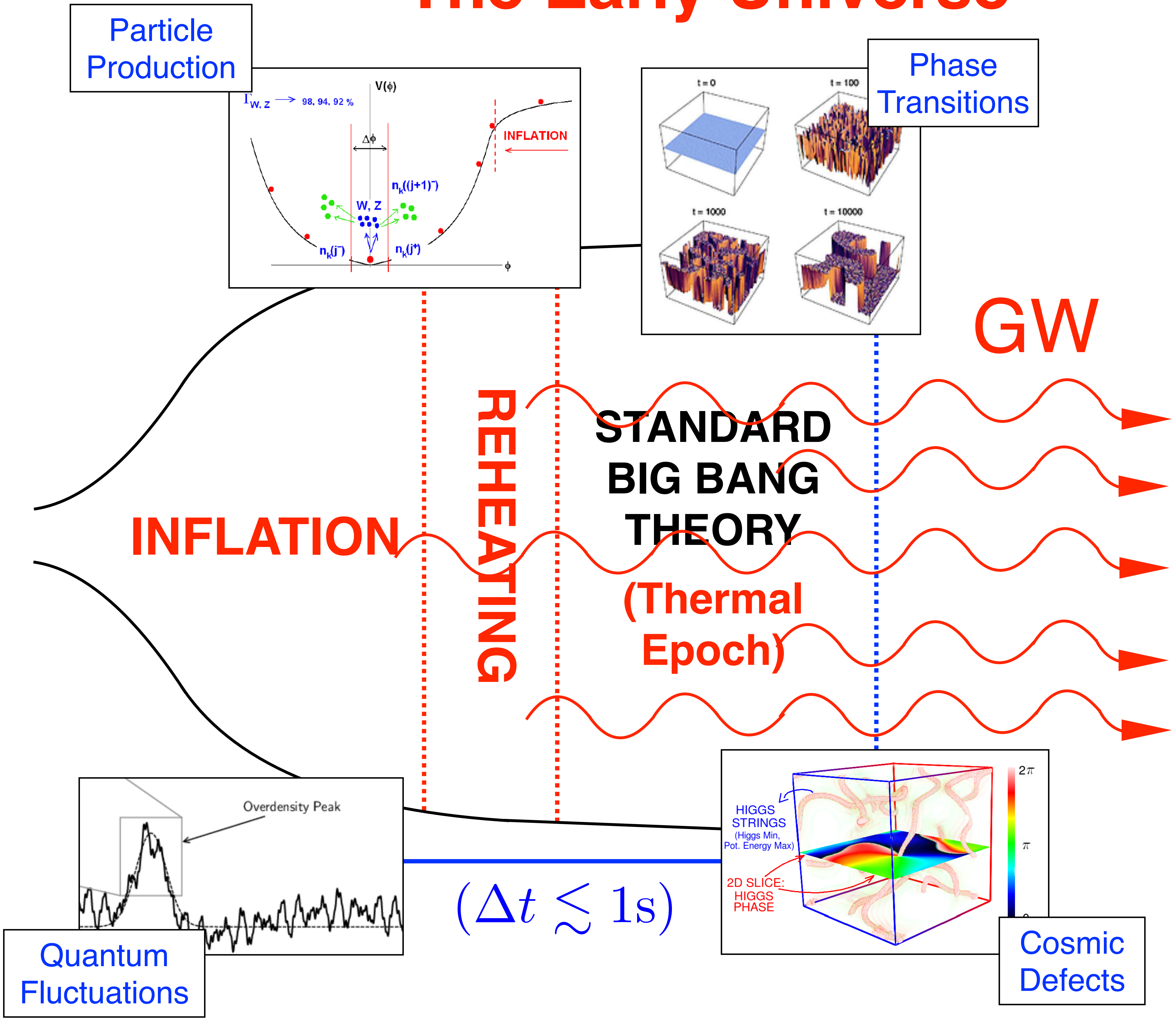
$(\Delta t \lesssim 1\mathrm{s})$

Cosmic Defects

"Strong" Grav. Wave emission !

( Non-linear Phenomena )

# The Early Universe



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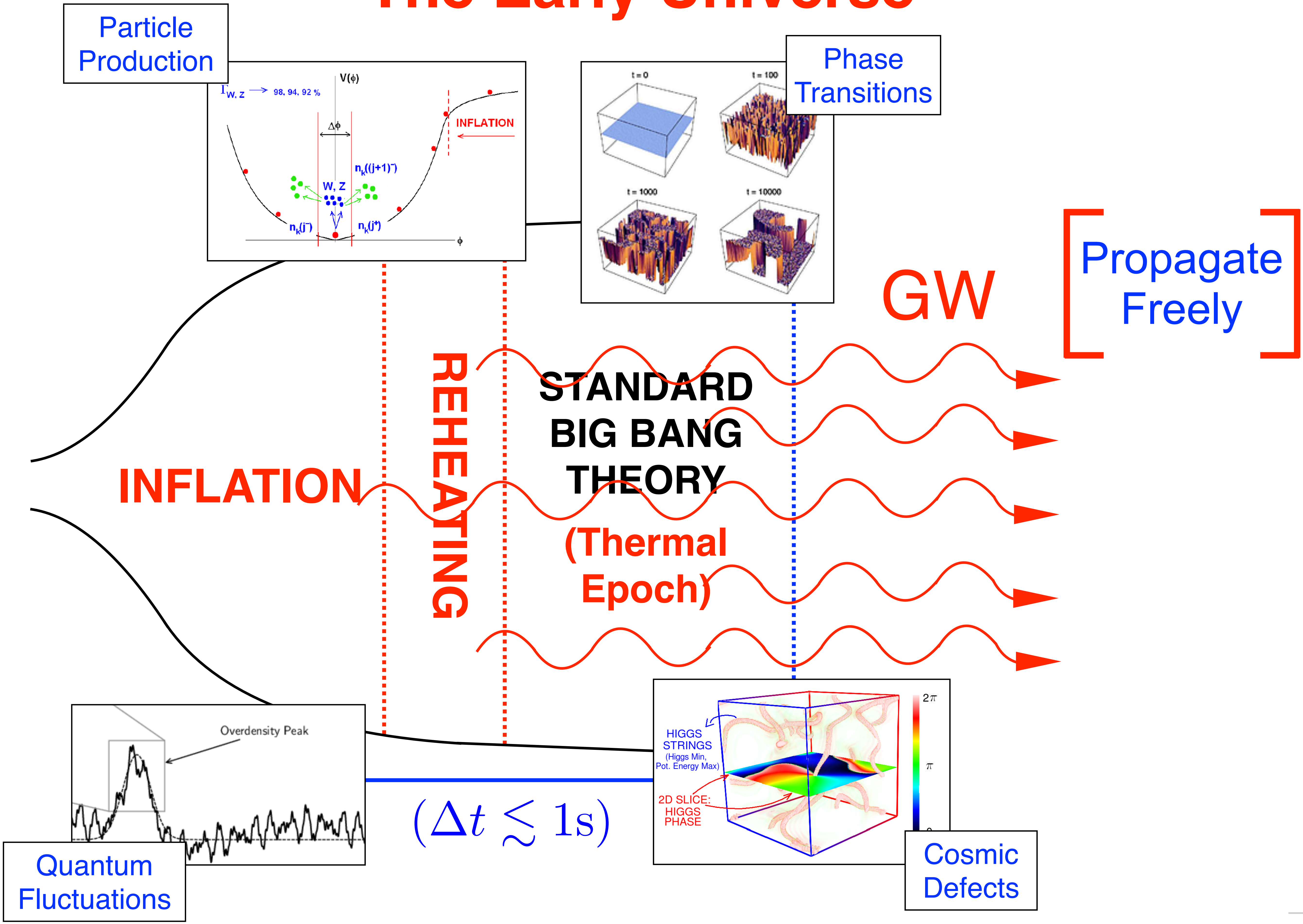


Particle Production

Phase Transitions

GW

Propagate Freely

INFLATION

REHEATING

STANDARD BIG BANG THEORY

(Thermal Epoch)

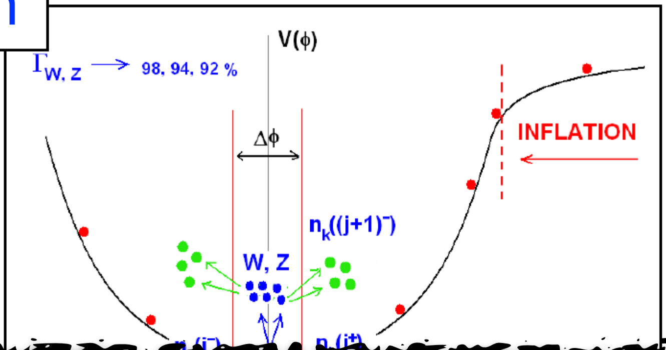
Quantum Fluctuations

$(\Delta t \lesssim 1\mathrm{s})$

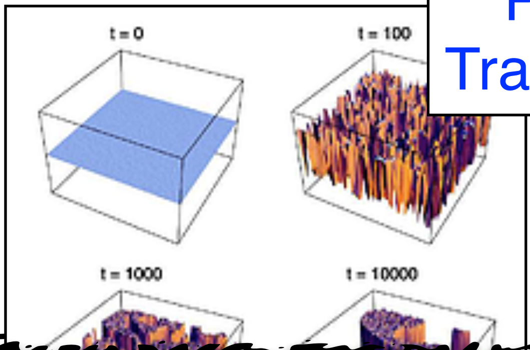
Cosmic Defects

# The Early Universe

Particle Production



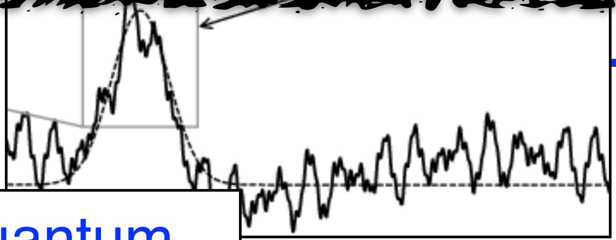
Phase Transitions



**Grav. Wave Backgrounds**

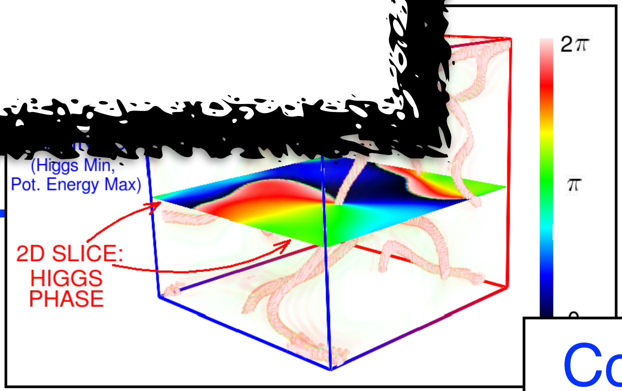
GW

[ Propagate Freely ]



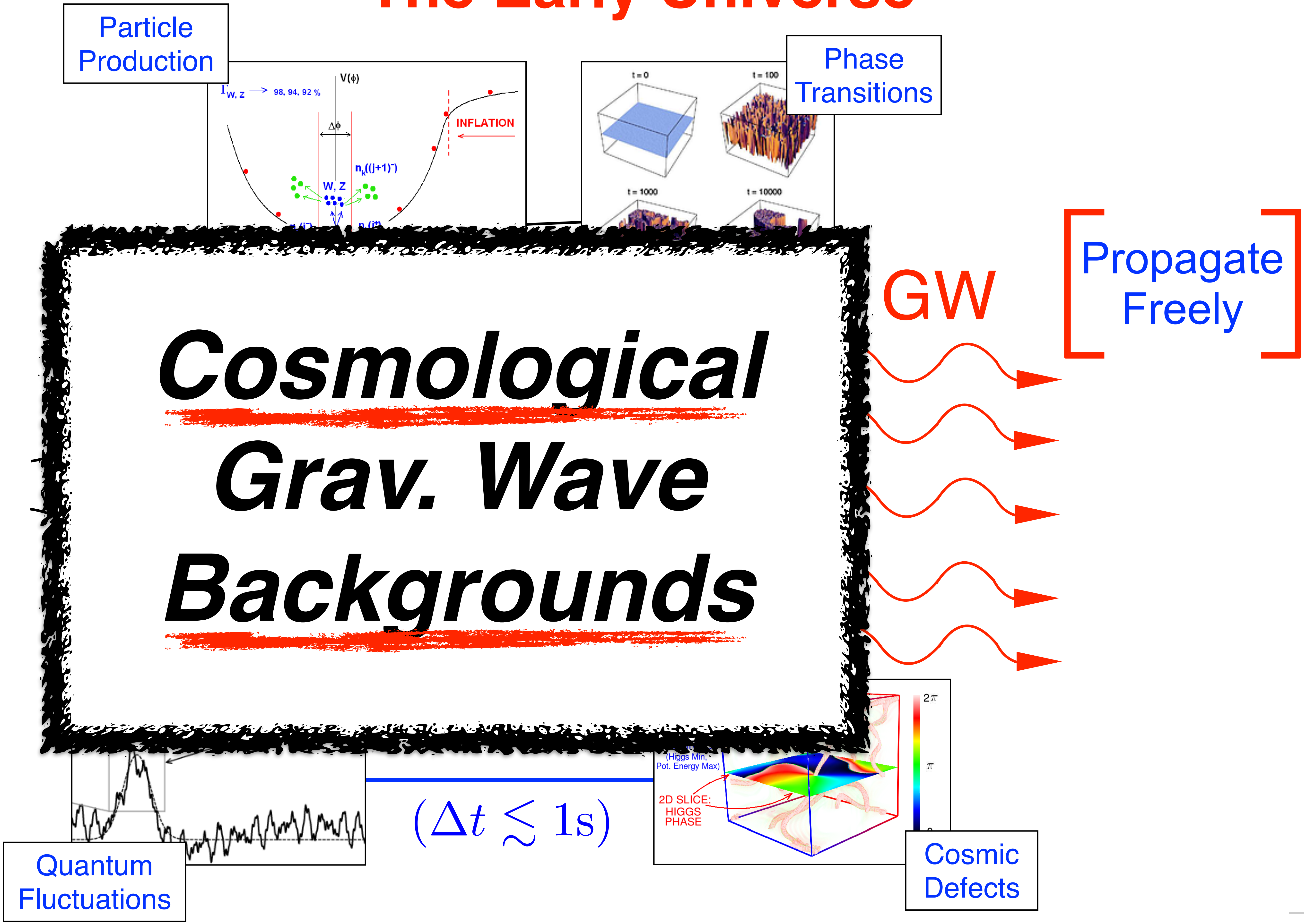
Quantum Fluctuations

$(\Delta t \lesssim 1\mathrm{s})$



Cosmic Defects

# The Early Universe



Particle Production

Phase Transitions

## Cosmological Grav. Wave Backgrounds

GW

[ Propagate Freely ]

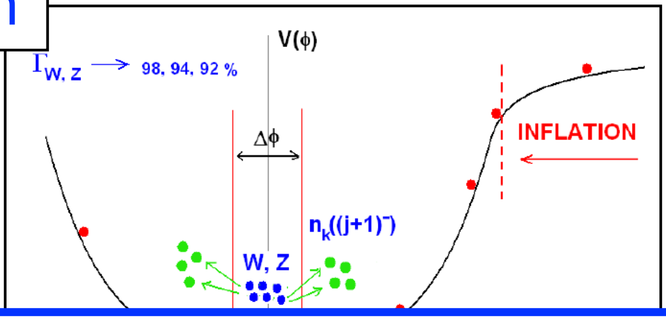
Quantum Fluctuations

Cosmic Defects

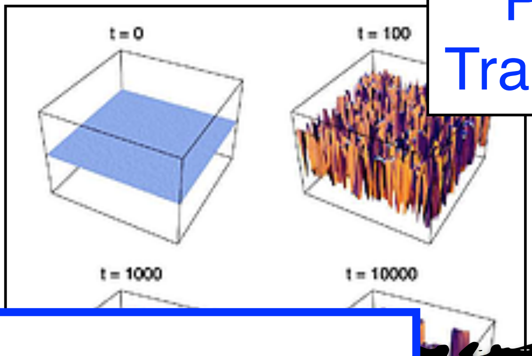
$(\Delta t \lesssim 1\text{s})$

# The Early Universe

Particle Production



Phase Transitions

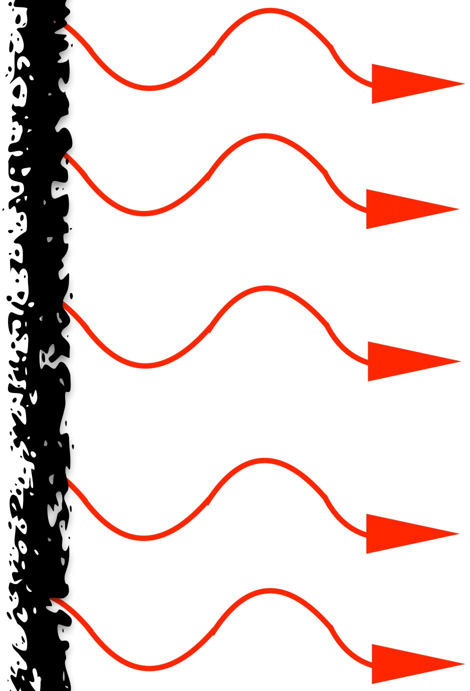


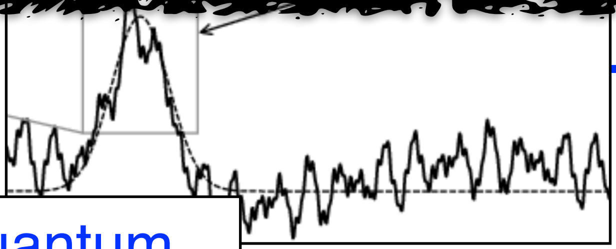
**Unique cosmic messenger**

## ***Cosmological Grav. Wave Backgrounds***

GW

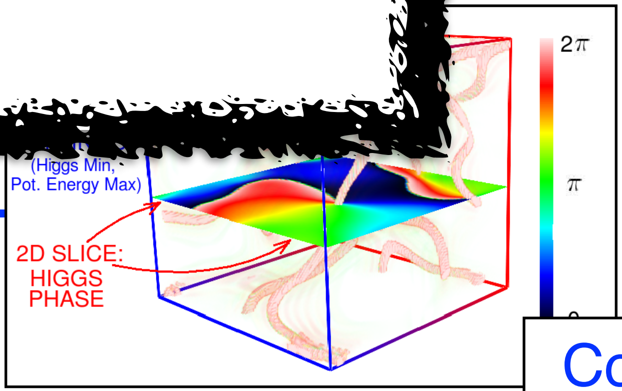
[ Propagate Freely ]





Quantum Fluctuations

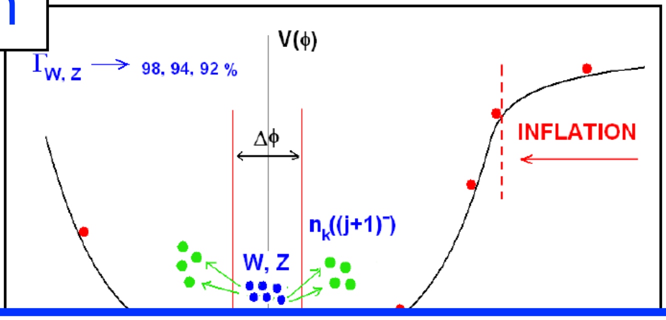
$(\Delta t \lesssim 1\mathrm{s})$



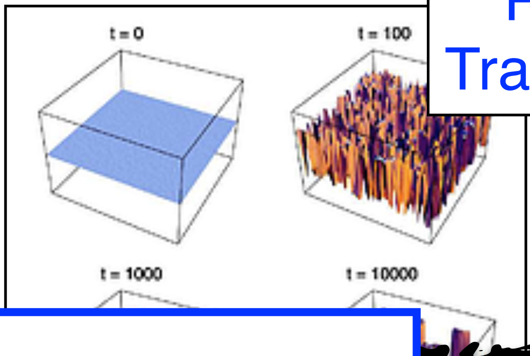
Cosmic Defects

# The Early Universe

Particle Production



Phase Transitions



**Unique cosmic messenger**

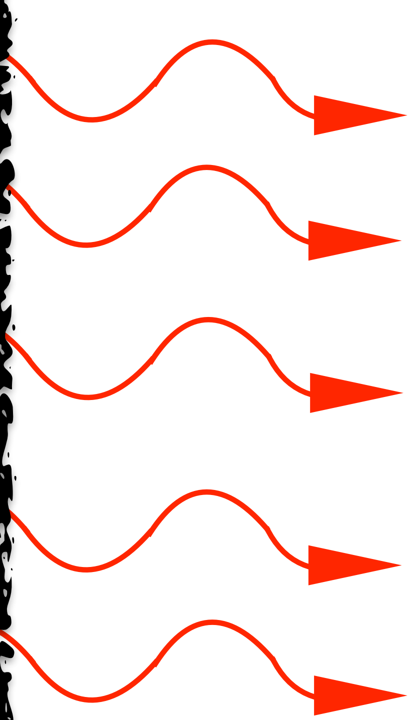
**Cosmological**

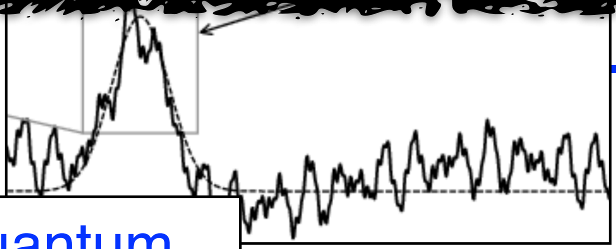
**Grav. Wave**

**Backgrounds**

GW

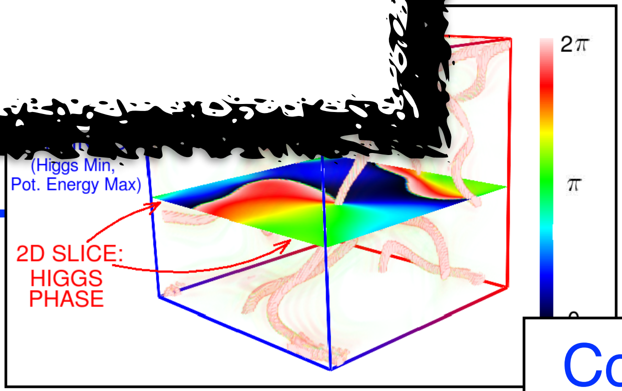
[ Propagate Freely ]





Quantum Fluctuations

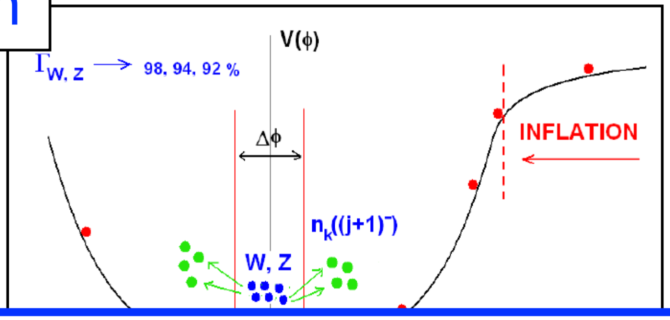
$(\Delta t \lesssim 1\mathrm{s})$



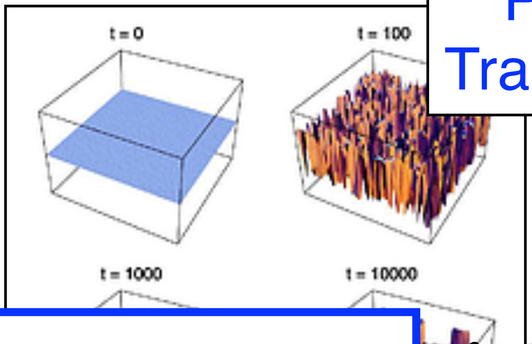
Cosmic Defects

# The Early Universe

Particle Production



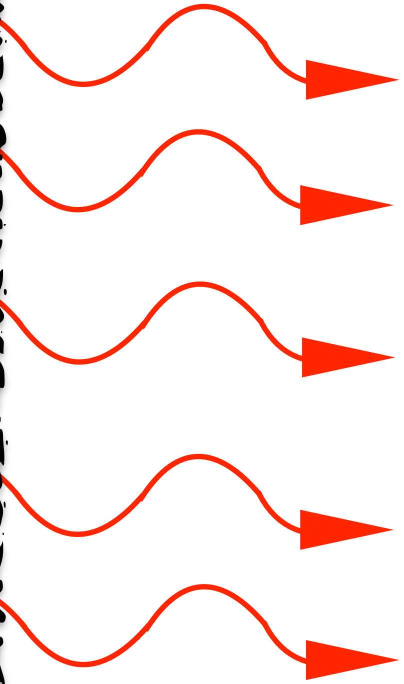
Phase Transitions



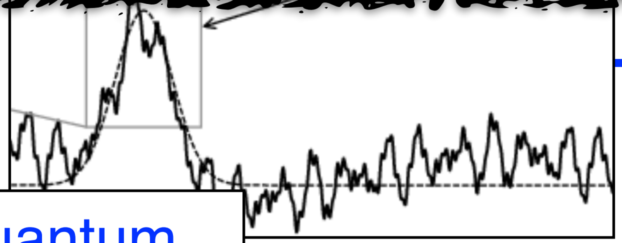
**Unique cosmic messenger**

**Cosmological Grav. Wave Backgrounds**

GW

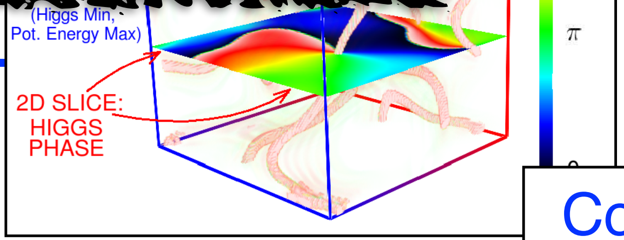


Probe of the early Universe



Quantum Fluctuations

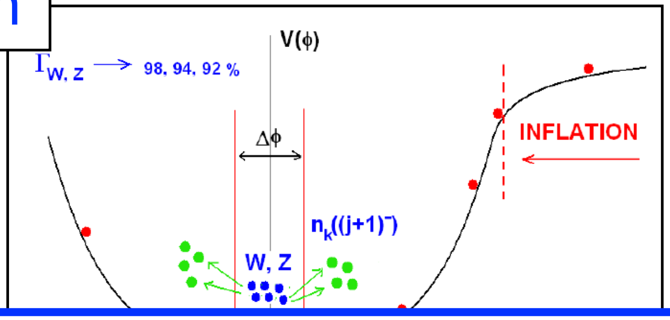
$(\Delta t \lesssim 1\text{s})$



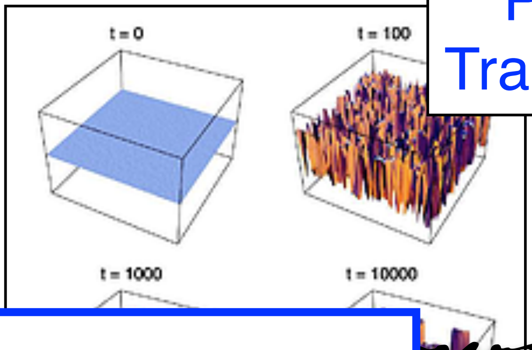
Cosmic Defects

# The Early Universe

Particle Production



Phase Transitions

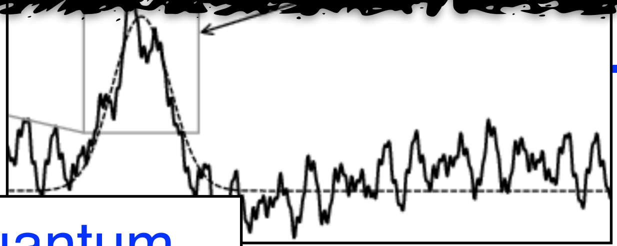


## Unique cosmic messenger

- * Energies above terrestrial means
- * Fundamental Physics
- * Beyond the Standard Model
- * ~ Origin of the Universe

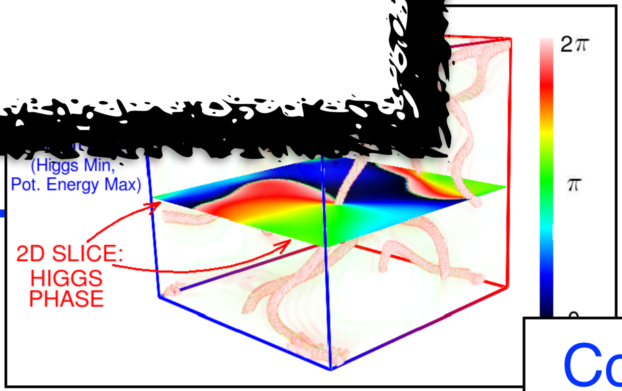
GW

Probe of the early Universe



Quantum Fluctuations

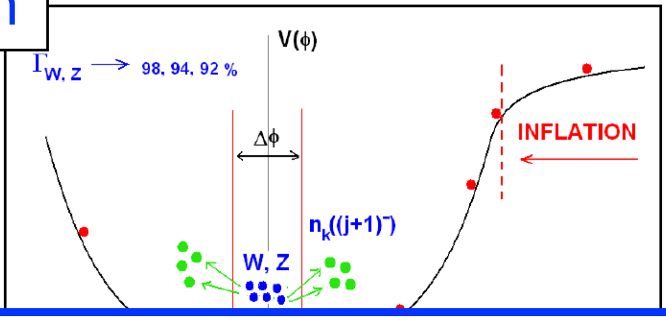
$(\Delta t \lesssim 1\text{s})$



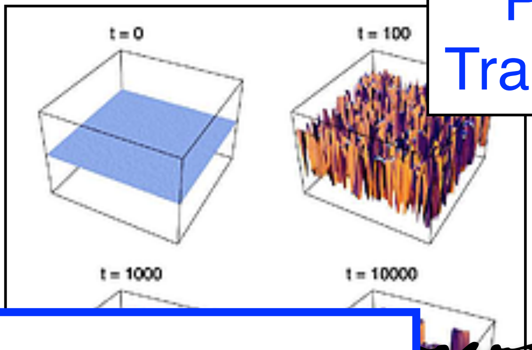
Cosmic Defects

# The Early Universe

Particle Production



Phase Transitions



**Unique cosmic messenger**

* **Energies above terrestrial means**

* **Fundamental Physics**

* **Beyond the Standard Model**

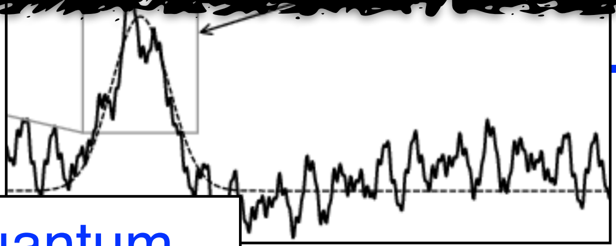
* **~ Origin of the Universe**

**Otherwise inaccessible !**

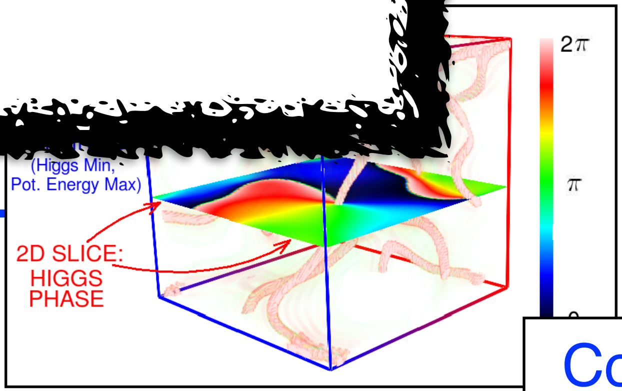
GW

Probe of the early Universe

Quantum Fluctuations



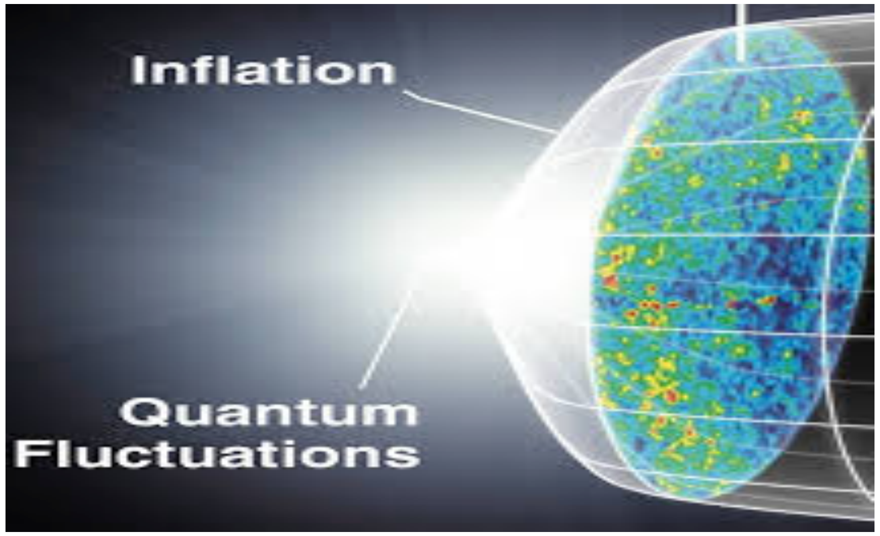
$(\Delta t \lesssim 1\mathrm{s})$



Cosmic Defects

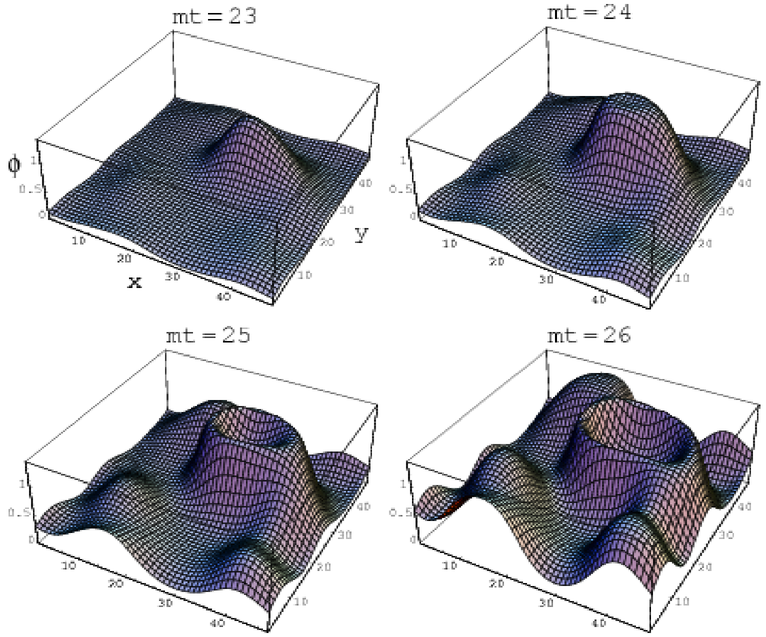
# GW Cosmological BACKGROUNDS

## Inflationary Period



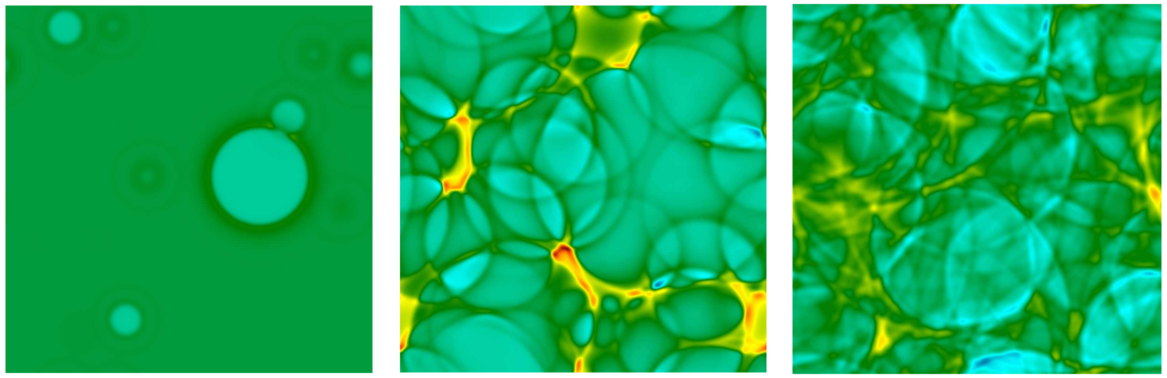
*(Image: Google Search)*

## (p)Reheating



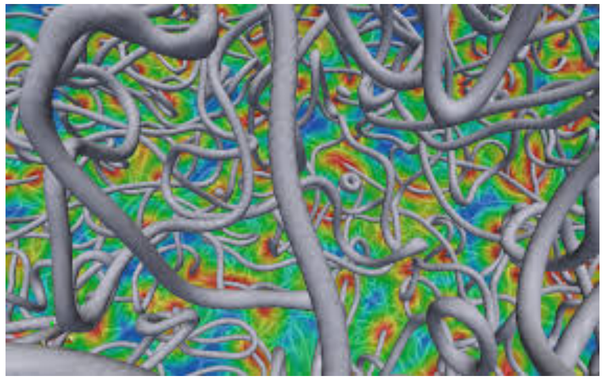
*(Fig. credit: Phys.Rev. D67 103501)*

## Phase Transitions



*(Image: PRL 112 (2014) 041301)*

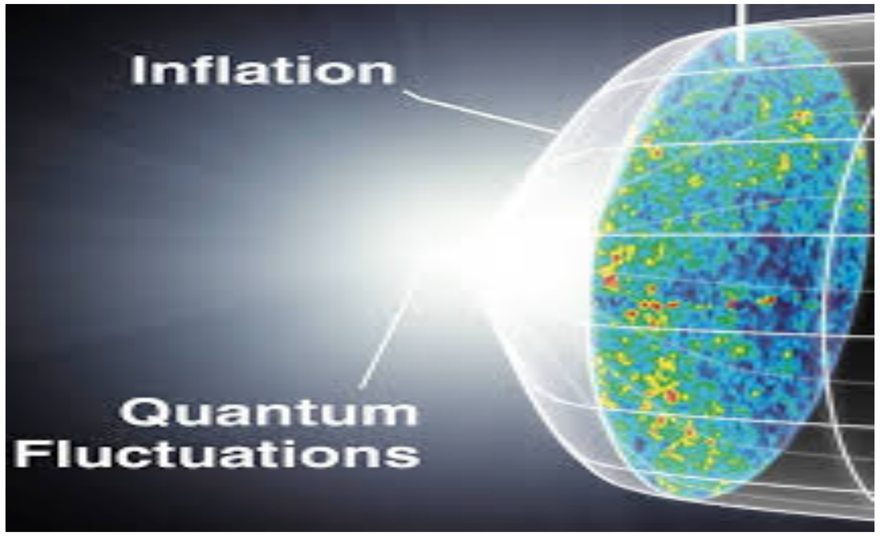
## Cosmic Defects



*(Image: Daverio et al, 2013)*

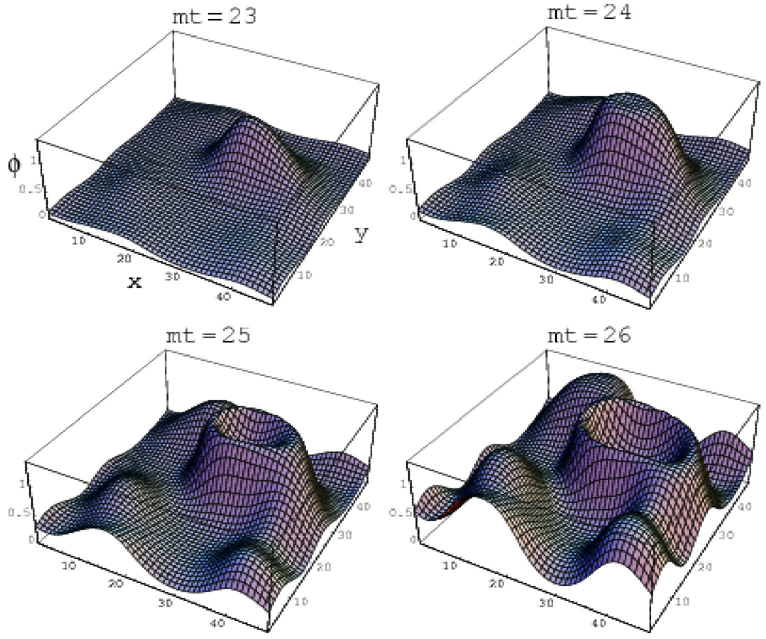
# GW Cosmological BACKGROUNDS

## Inflationary Period



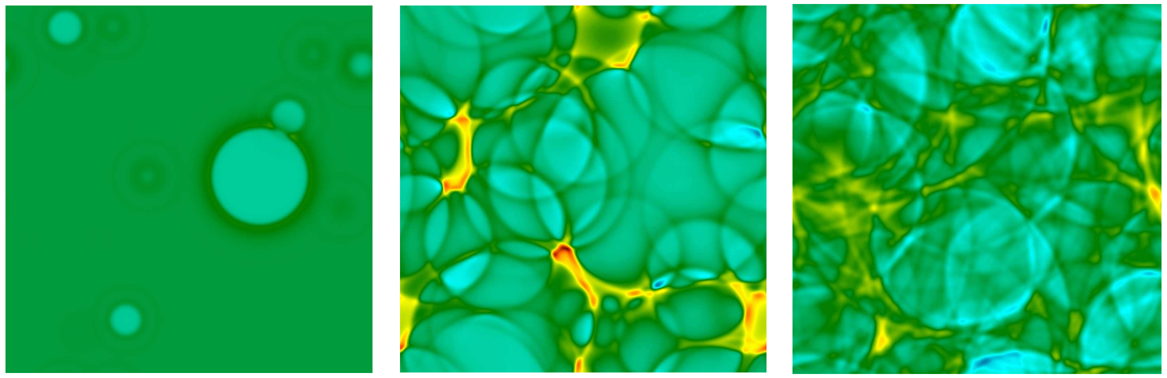
*(Image: Google Search)*

## (p)Reheating



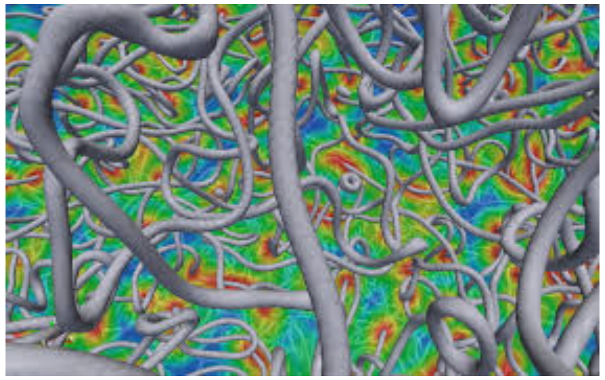
*(Fig. credit: Phys.Rev. D67 103501)*

## Phase Transitions

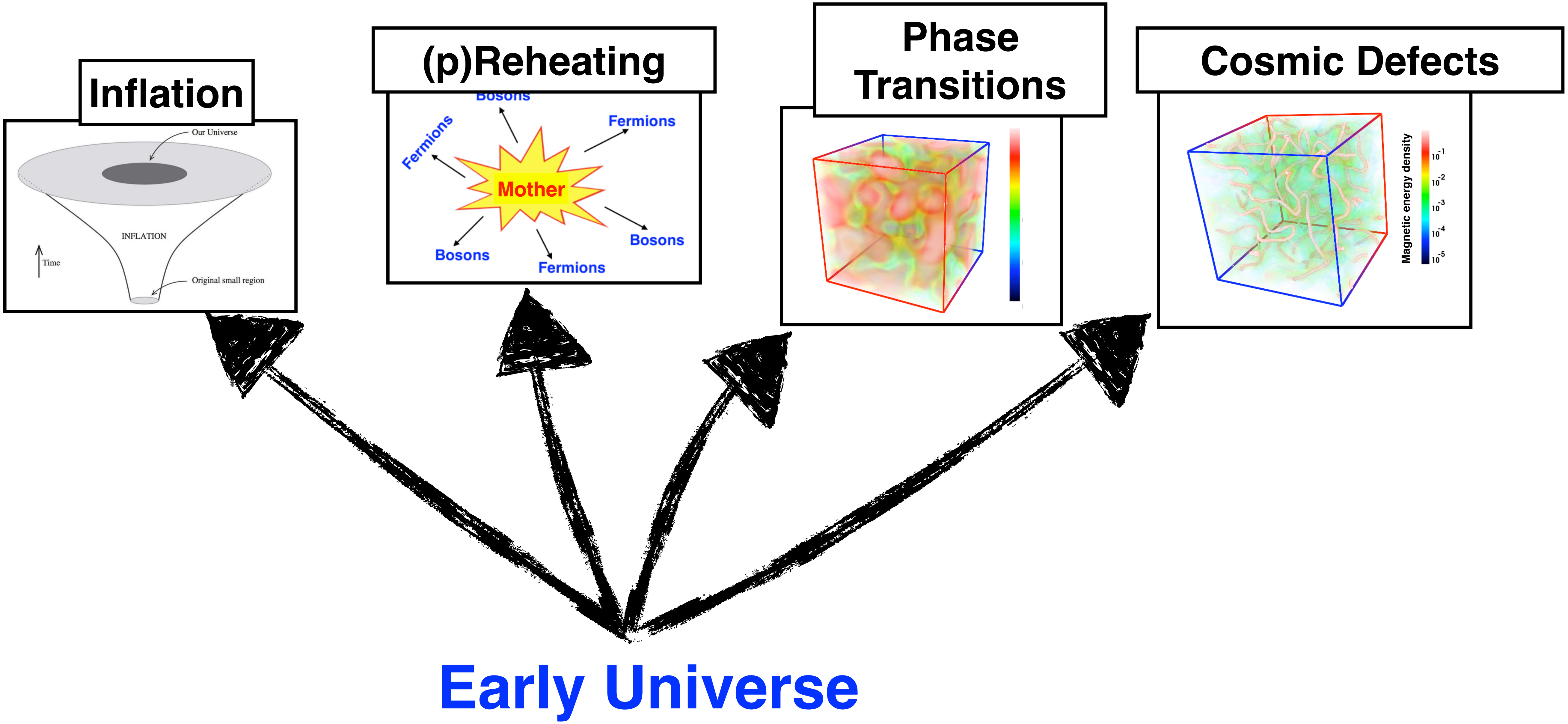


*(Image: PRL 112 (2014) 041301)*

## Cosmic Defects



*(Image: Daverio et al, 2013)*



**Inflation**

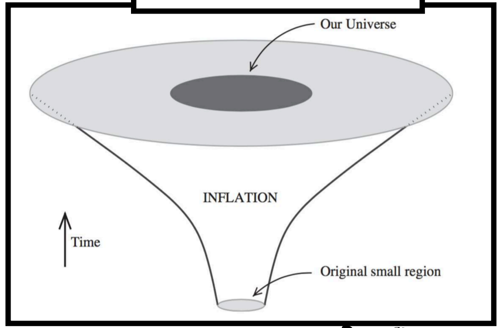
**(p)Reheating**

**Phase Transitions**

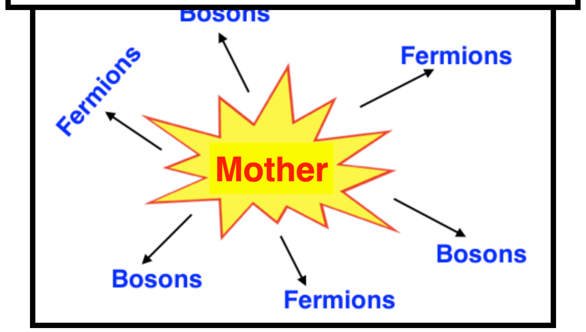
**Cosmic Defects**

**Early Universe**

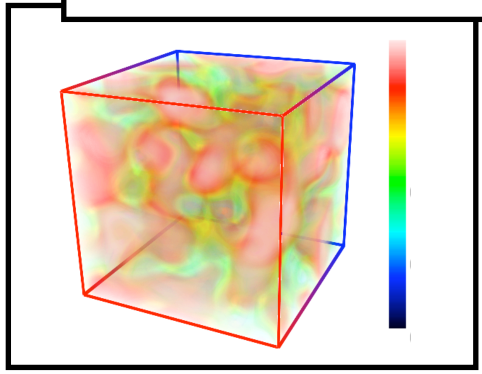
**Inflation**



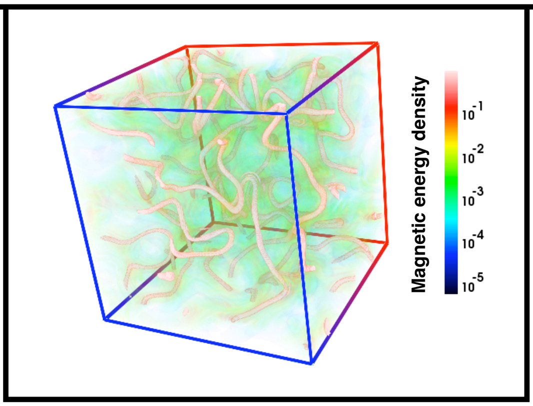
**(p)Reheating**



**Phase Transitions**



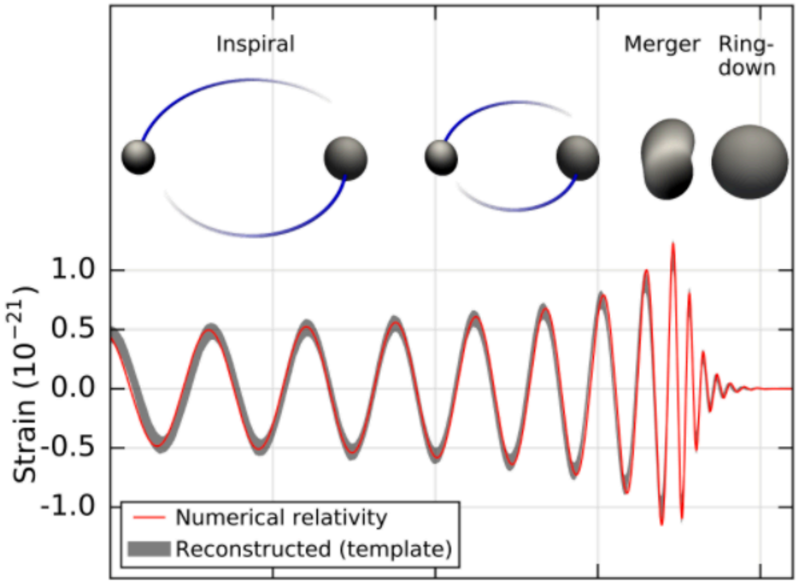
**Cosmic Defects**



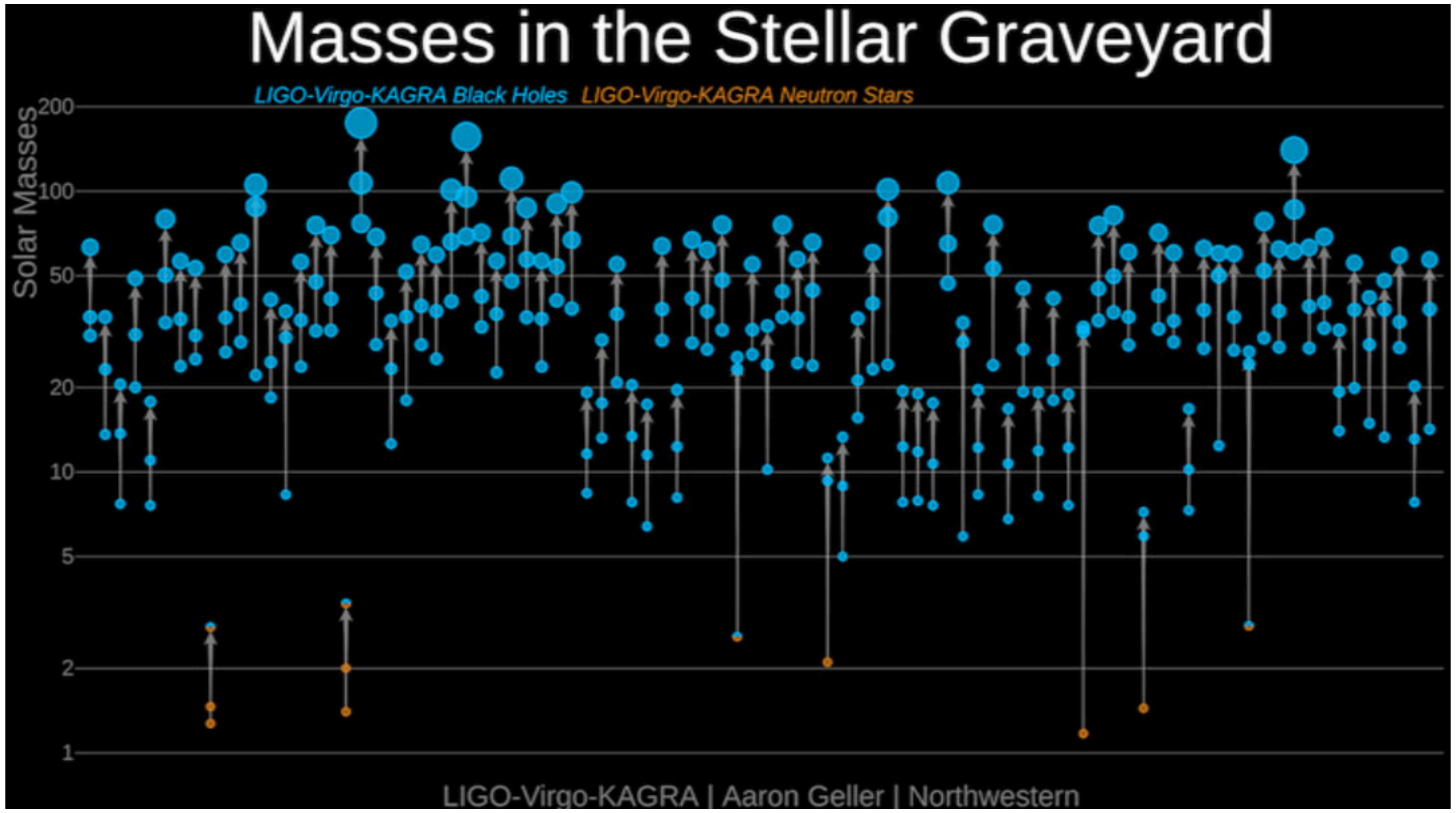
**Early Universe**

**Late Universe ?**

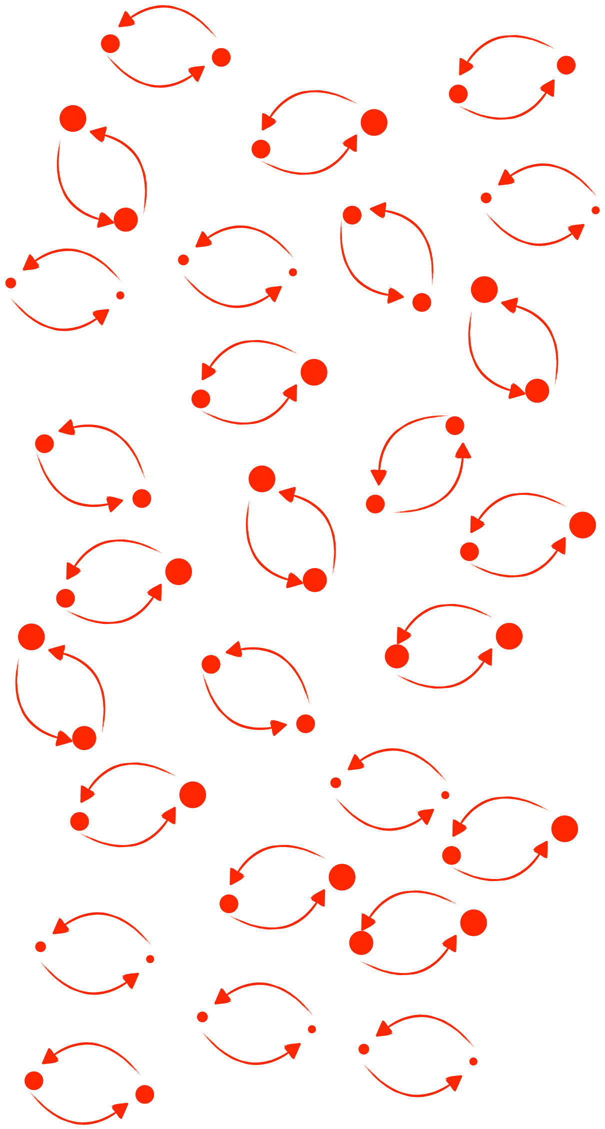
# Late Universe $(0 \leq z \lesssim 10)$

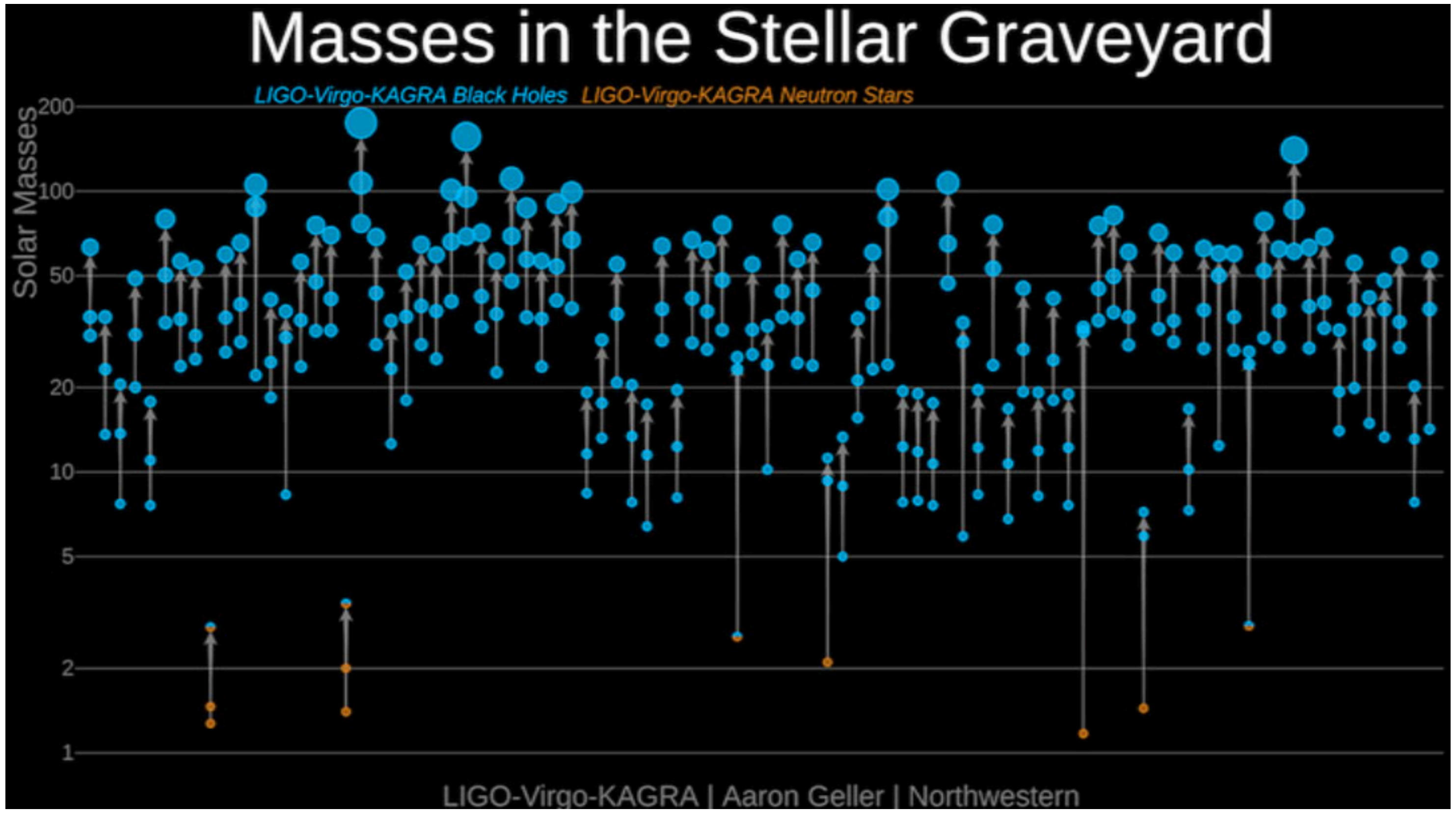


**LIGO/VIRGO**
**2015-now**

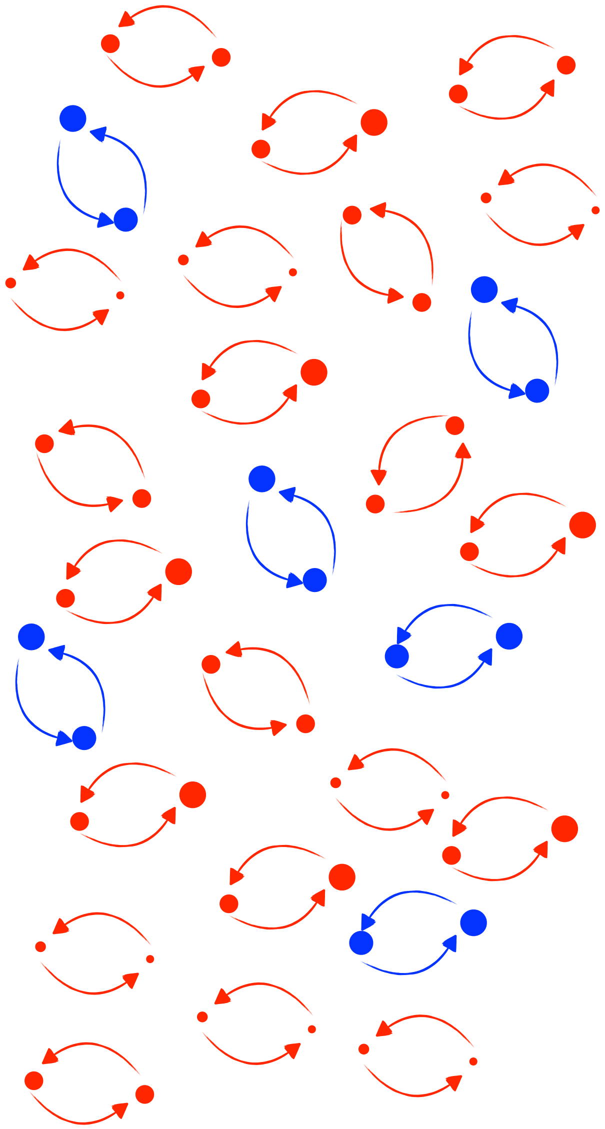


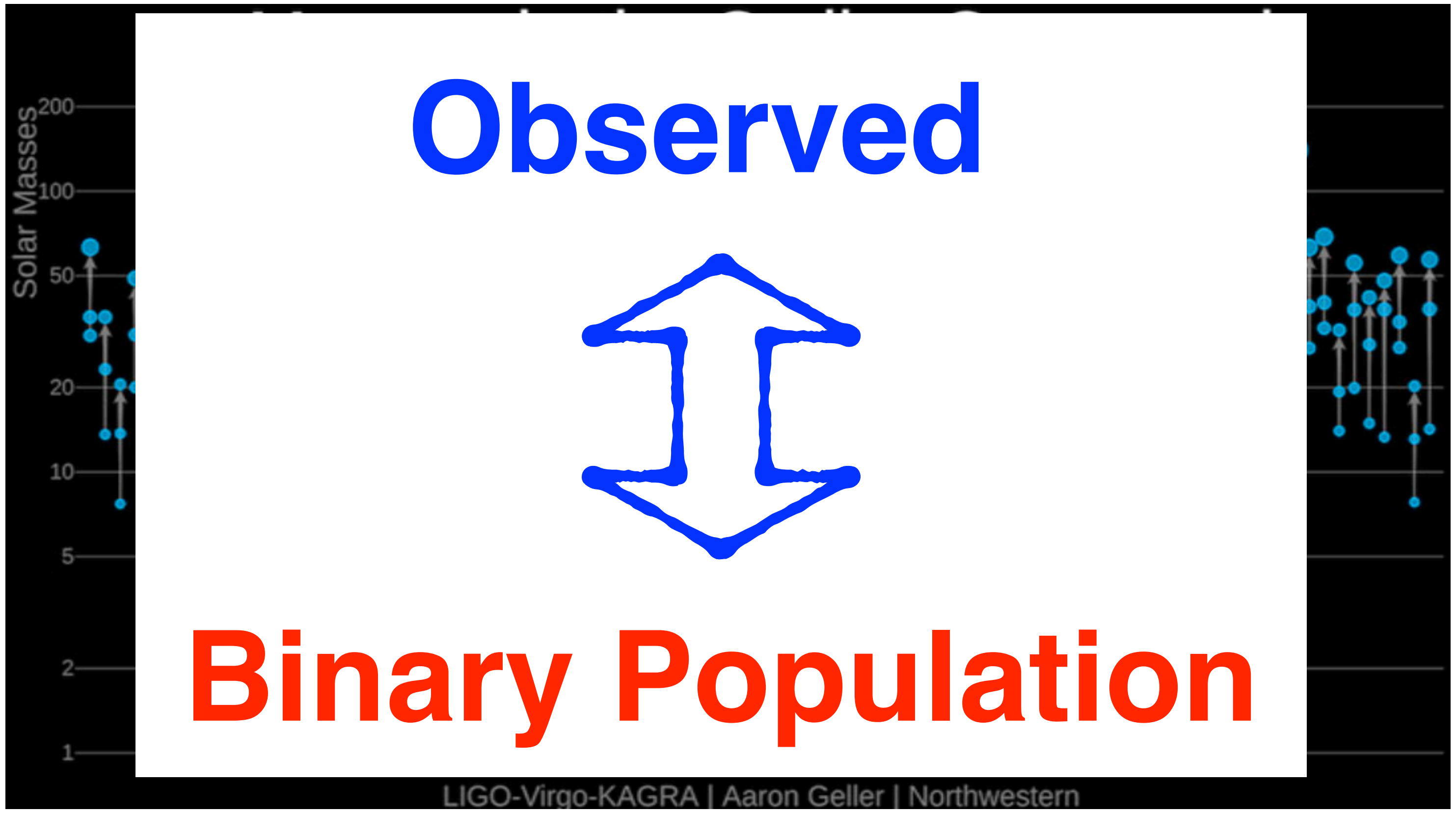
# Late Universe $(0 \leq z \lesssim 10)$





# Late Universe $(0 \le z \lesssim 10)$





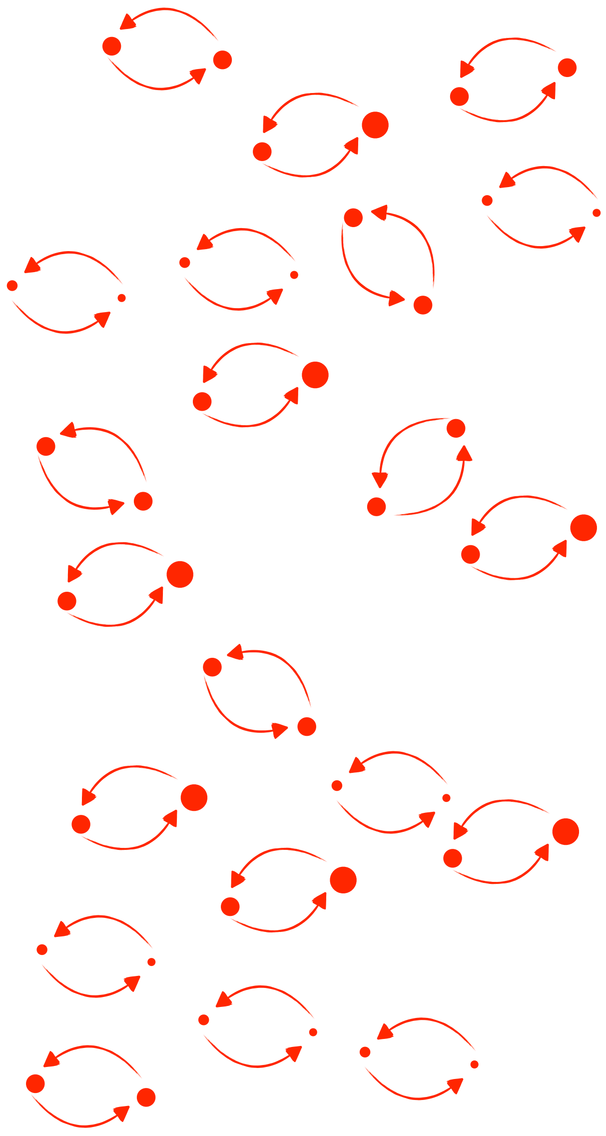
**Observed**

⇕

**Binary Population**

LIGO-Virgo-KAGRA | Aaron Geller | Northwestern

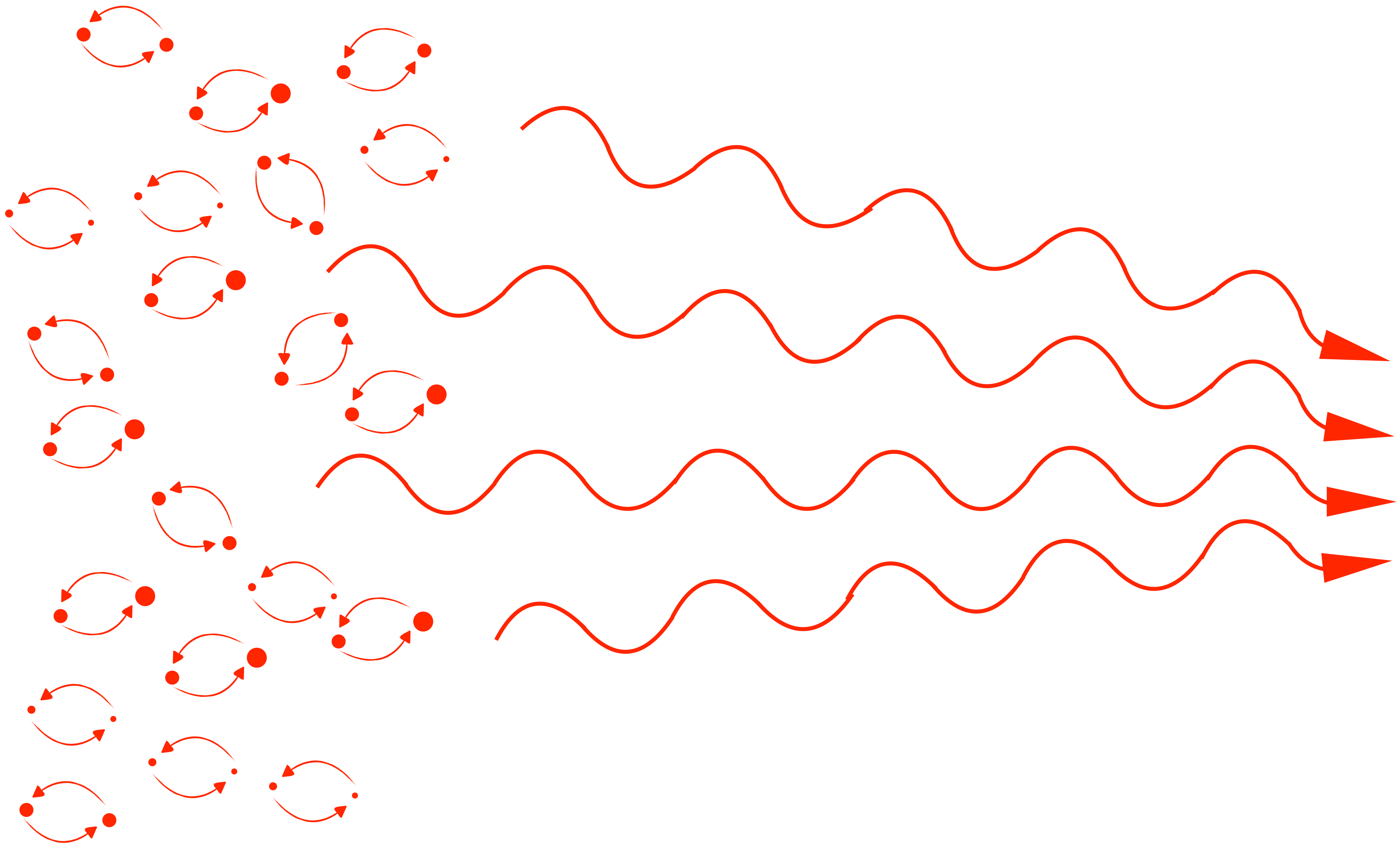
# Late Universe $(0 \leq z \lesssim 10)$



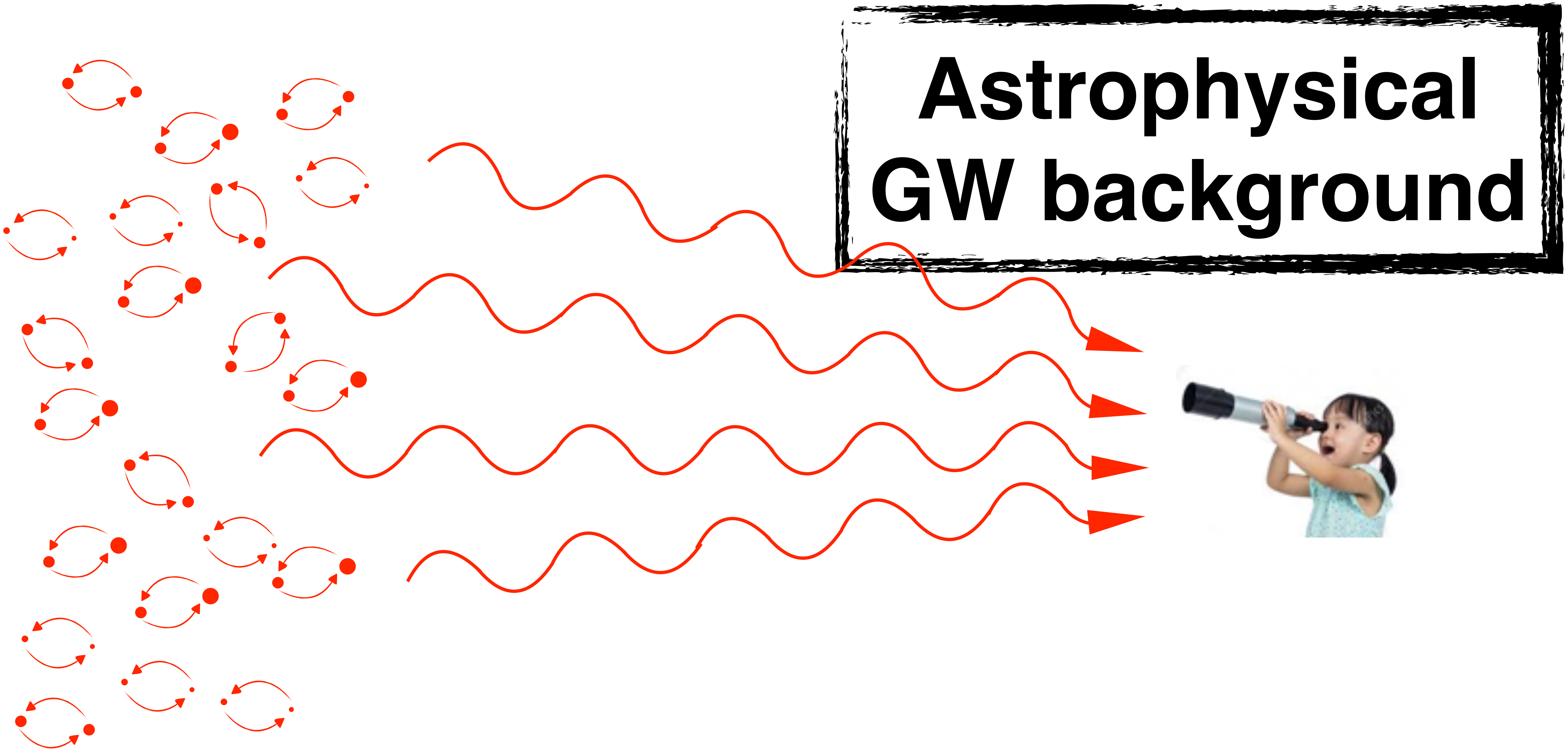
**Mostly unresolved binaries !**

LIGO-Virgo-KAGRA | Aaron Geller | Northwestern

## Late Universe $(0 \leq z \lesssim 10)$

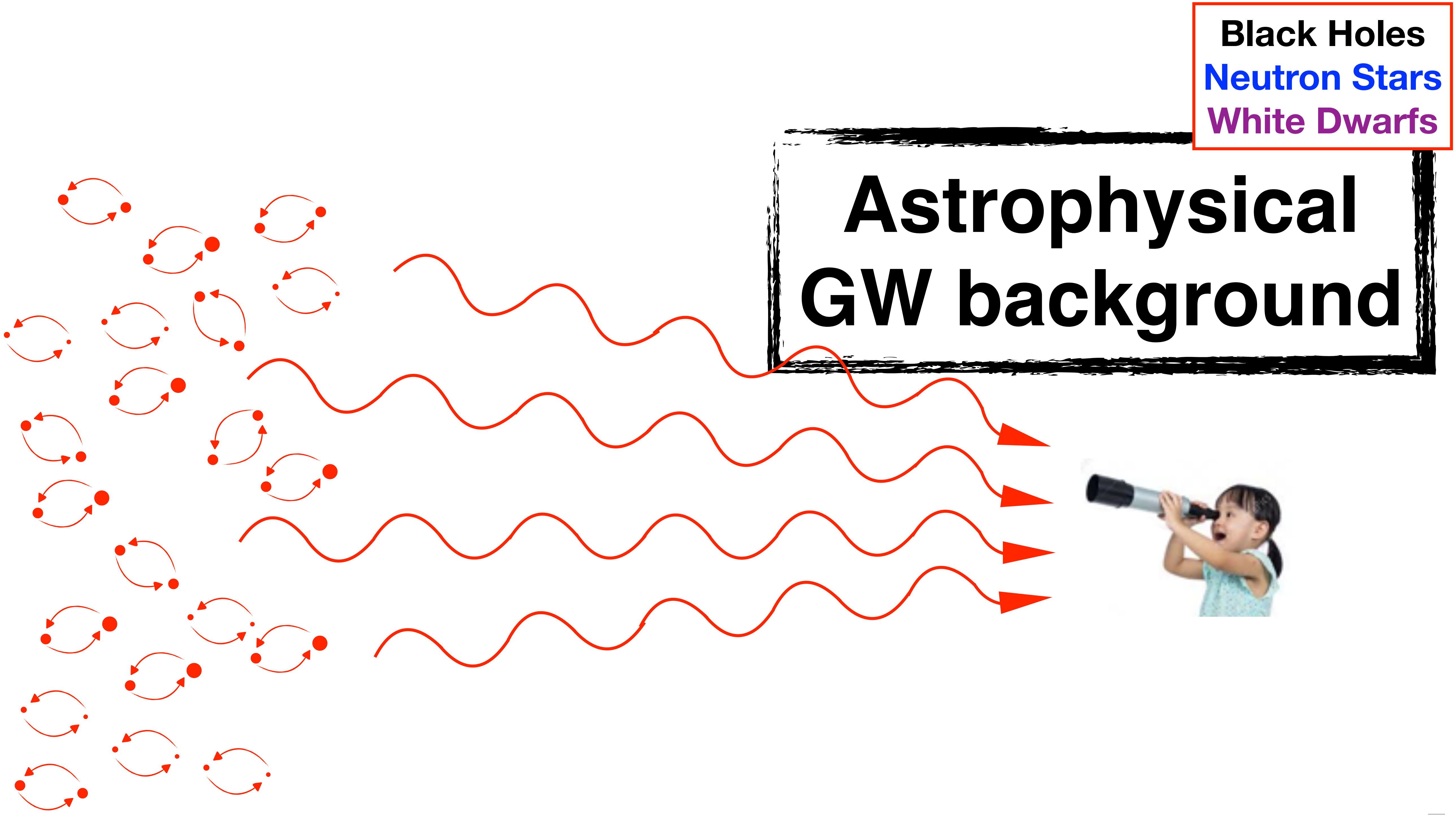


## Late Universe $(0 \leq z \lesssim 10)$



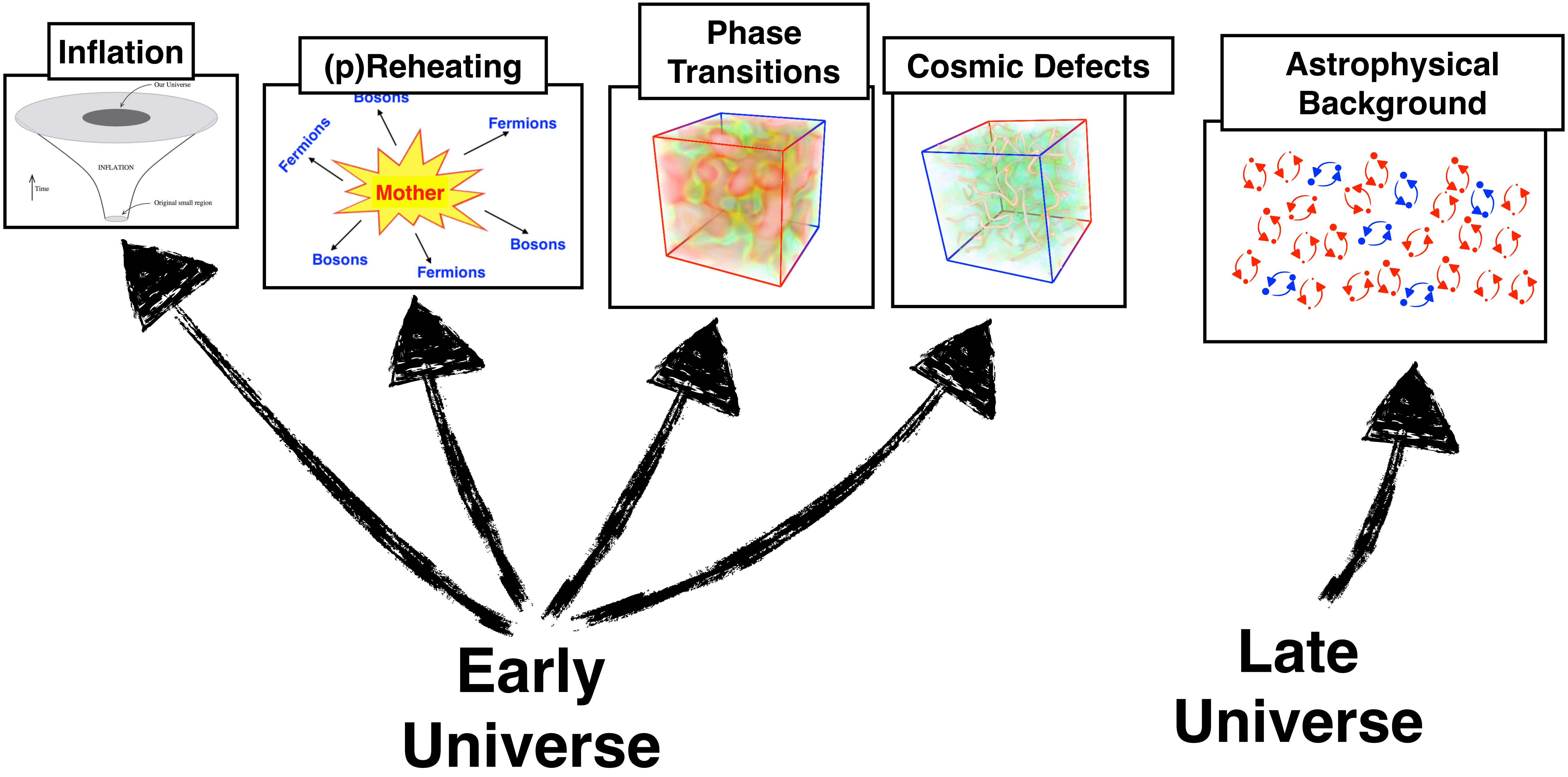
**Astrophysical GW background**

# Late Universe $(0 \leq z \lesssim 10)$



Black Holes
Neutron Stars
White Dwarfs

**Astrophysical GW background**



**Inflation**

**(p)Reheating**

**Phase Transitions**

**Cosmic Defects**

**Astrophysical Background**

**Early Universe**

**Late Universe**

# Gravitational Wave Backgrounds

# Summary & Perspective

# Perspective

**Gravitational Wave Backgrounds**

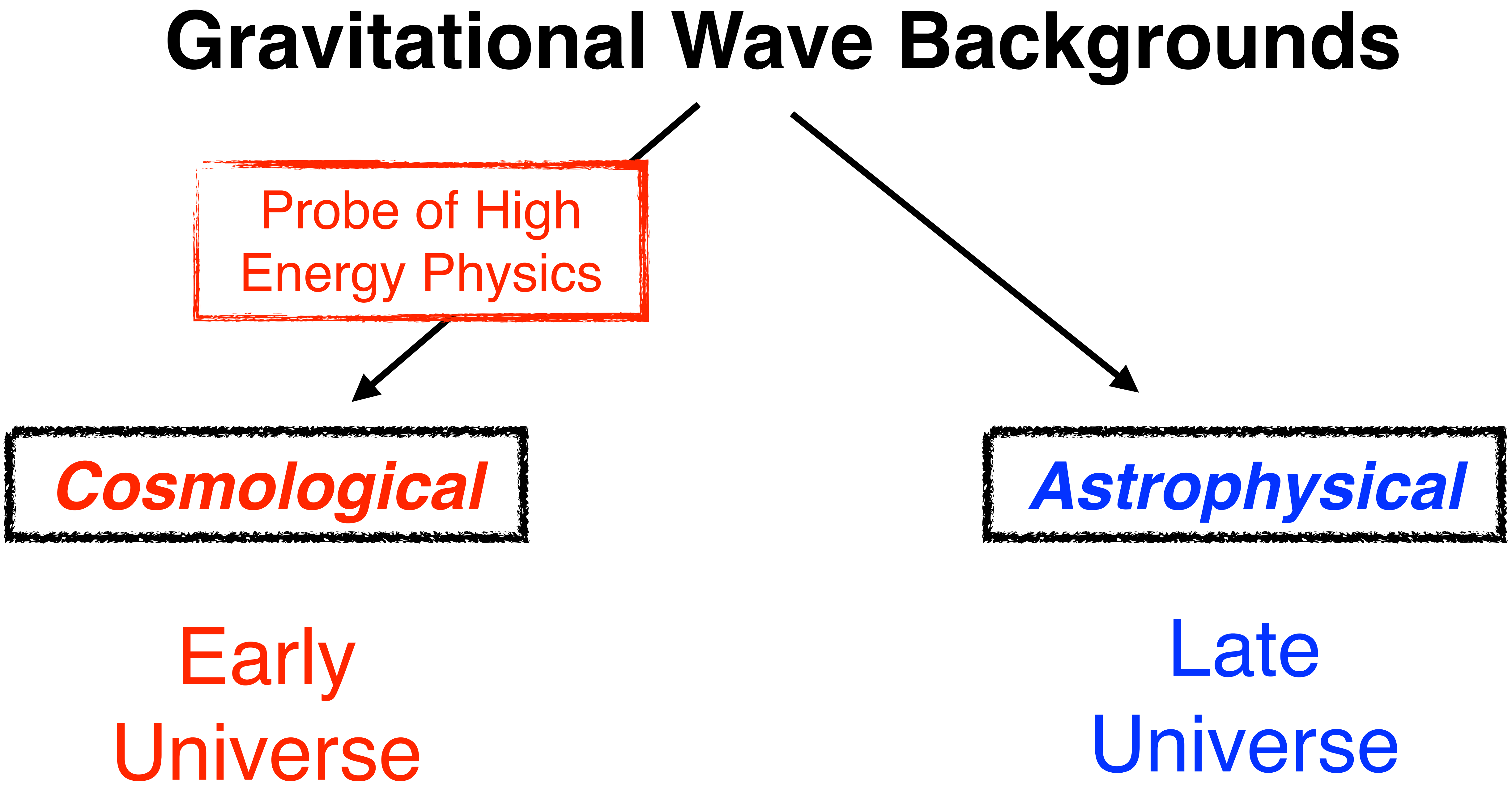
***Cosmological***

Early
Universe

***Astrophysical***

Late
Universe

# Perspective



**Gravitational Wave Backgrounds**

Probe of High
Energy Physics

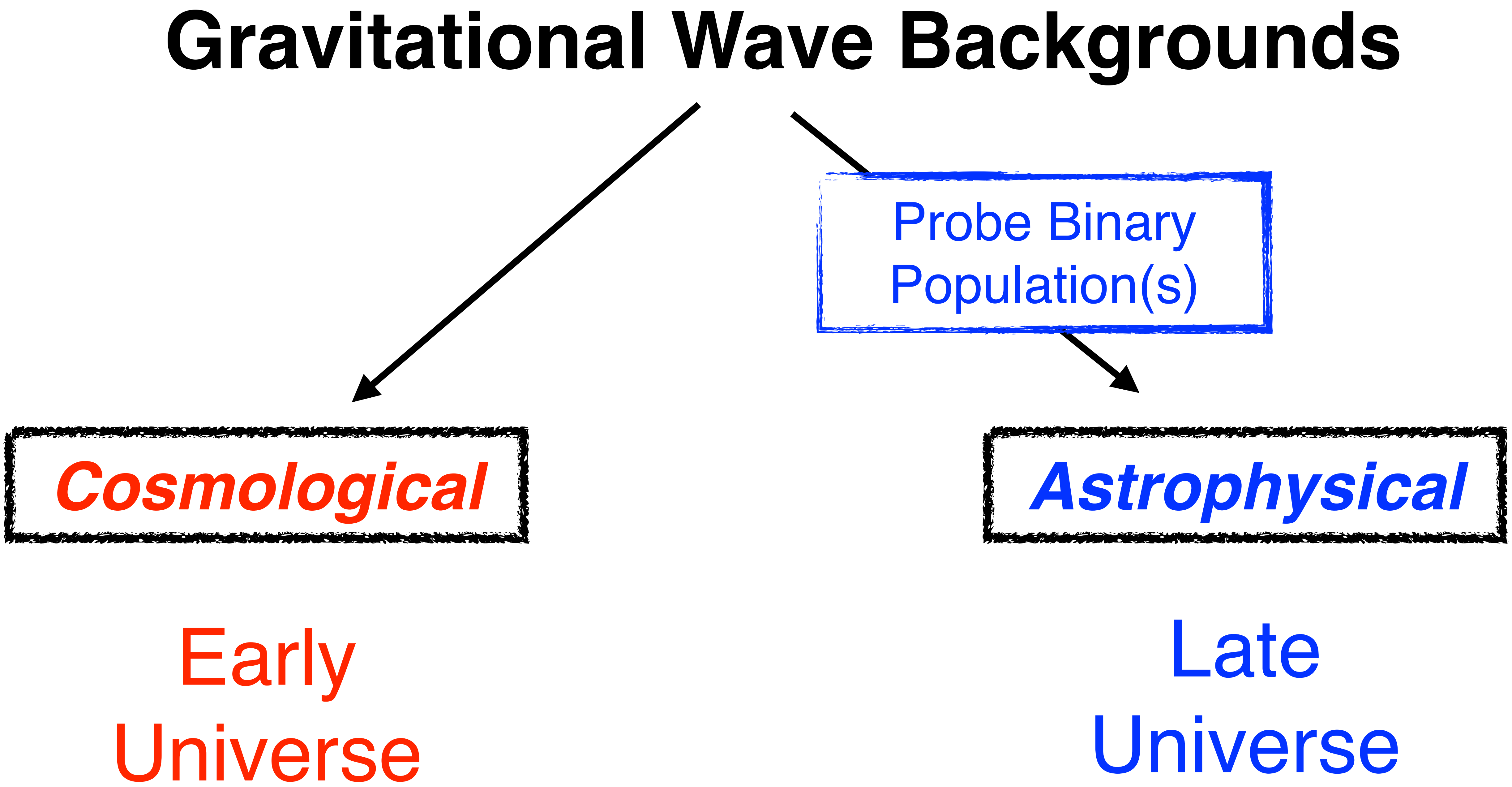
***Cosmological***

***Astrophysical***

Early
Universe

Late
Universe

# Perspective



**Gravitational Wave Backgrounds**

Probe Binary Population(s)

***Cosmological***

***Astrophysical***

Early Universe

Late Universe

# Perspective

**Gravitational Wave Backgrounds**

**HOLY GRAIL**

***Cosmological***

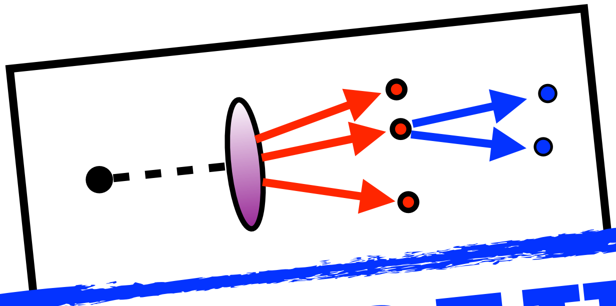
***Astrophysical***

Early Universe

Late Universe

# Perspective

**Gravitational Wave Backgrounds**



**HOLY GRAIL**

***Cosmological***

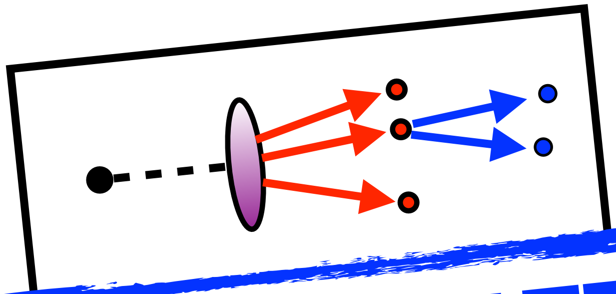
***Astrophysical***

Early
Universe

Late
Universe

# Perspective

**Gravitational Wave Backgrounds**



**HOLY GRAIL**

***Cosmological***

Early
Universe

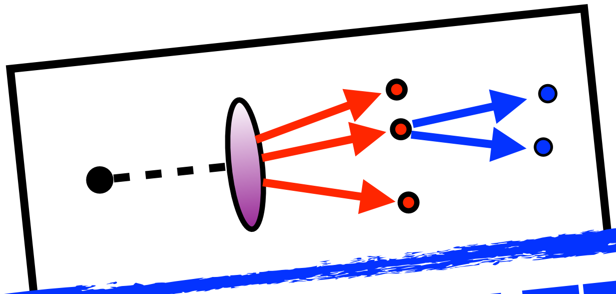
**GUARANTEED**

***Astrophysical***

Late
Universe

# Perspective

**Gravitational Wave Backgrounds**



**HOLY GRAIL**

***Cosmological***

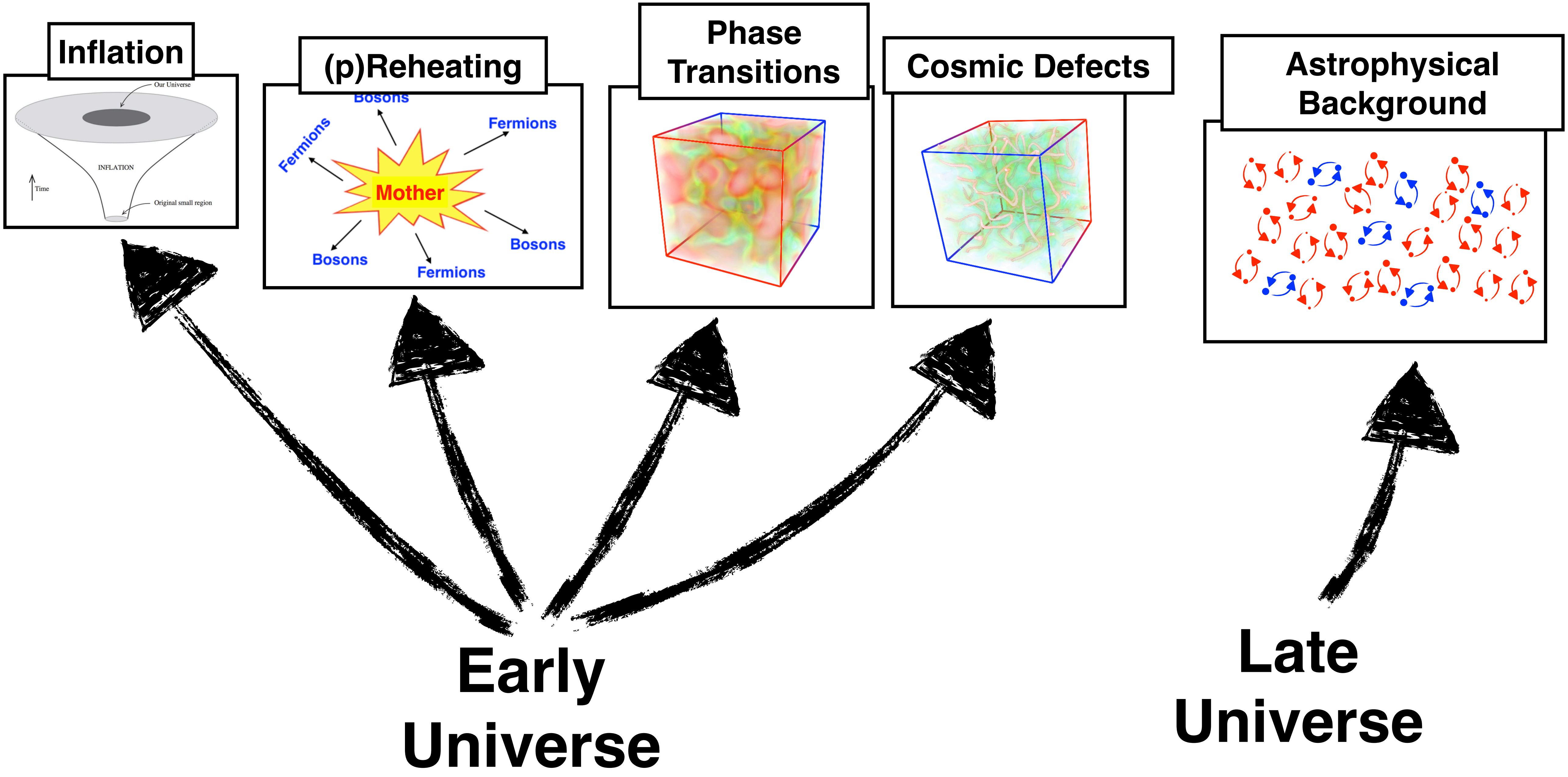
Early Universe

**GUARANTEED**

***Astrophysical***

Late Universe

**Foreground**



**Inflation**

**(p)Reheating**

**Phase Transitions**

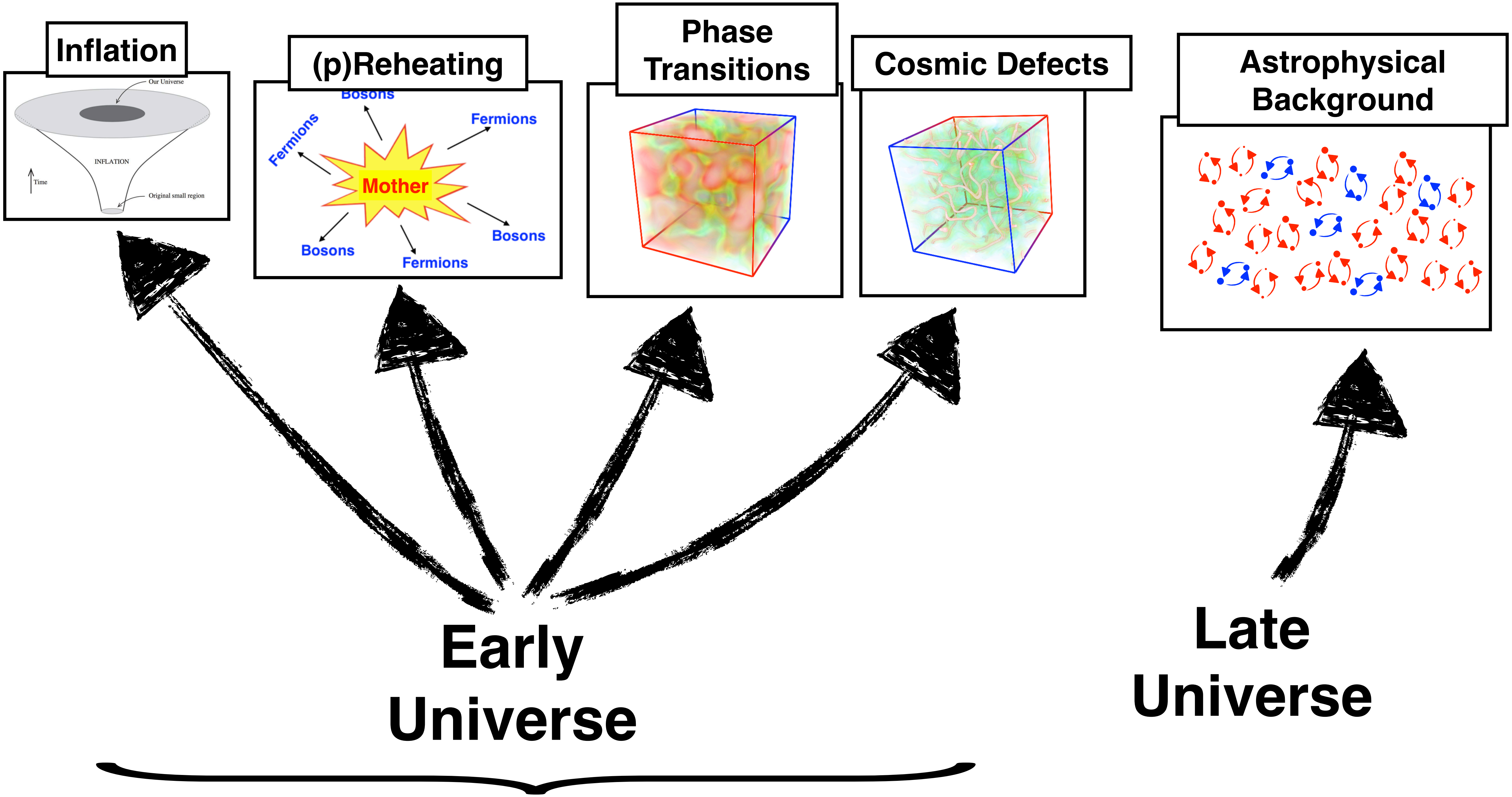
**Cosmic Defects**

**Astrophysical Background**

**Early Universe**

**Late Universe**

# Gravitational Wave Backgrounds



Inflation

(p)Reheating

Phase Transitions

Cosmic Defects

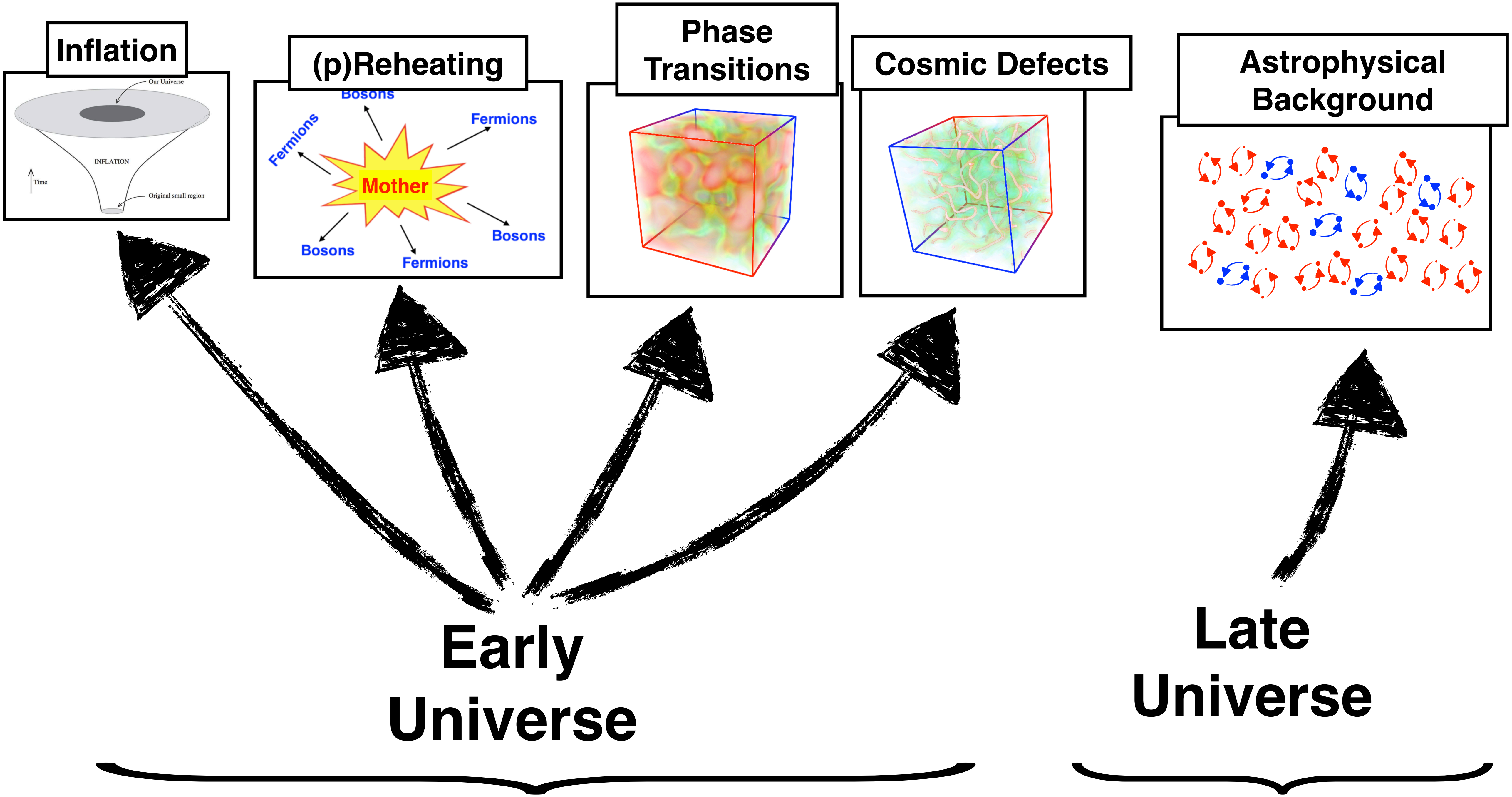
Astrophysical Background

Early Universe

Late Universe

Core of the lectures !

As these backgrounds probe Fundamental Physics (HEP)



**Inflation**

**(p)Reheating**

**Phase Transitions**

**Cosmic Defects**

**Astrophysical Background**

**Early Universe**

**Late Universe**

**Core of the lectures !**

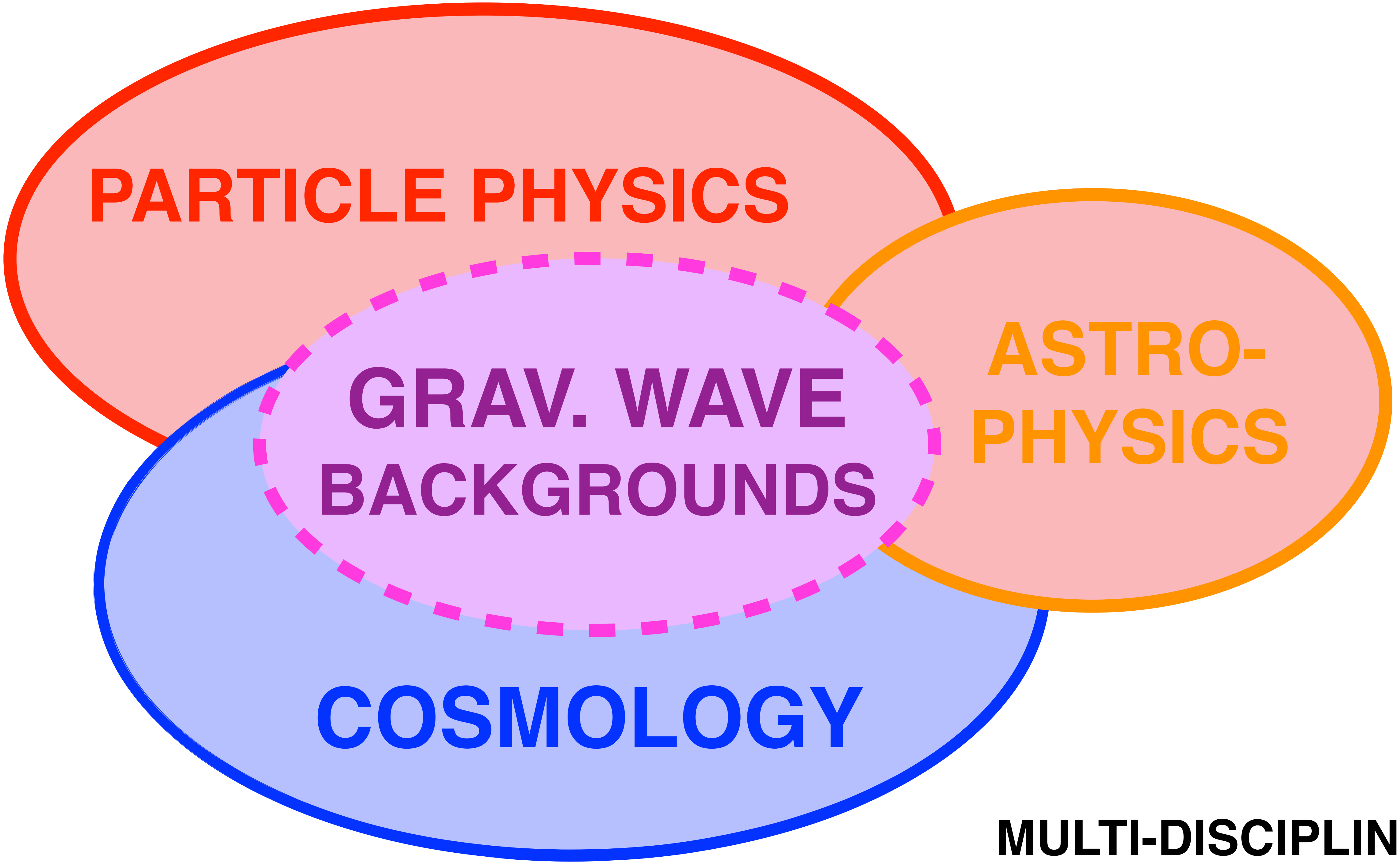
**As these backgrounds probe Fundamental Physics (HEP)**

**Peripheral**

**Backgrounds act (for us) as a foreground !**

# Gravitational Wave Backgrounds

BROAD



PARTICLE PHYSICS

ASTRO-PHYSICS

GRAV. WAVE BACKGROUNDS

COSMOLOGY

MULTI-DISCIPLINARY

# Gravitational Wave Backgrounds

## OUTLINE

**Early Universe Sources**

1) GWs from Inflation
2) GWs from Preheating
3) GWs from Phase Transitions
4) GWs from Cosmic Defects

# Gravitational Wave Backgrounds

## OUTLINE

0) Grav. Waves (GWs)

Early Universe Sources

1) GWs from Inflation
2) GWs from Preheating
3) GWs from Phase Transitions
4) GWs from Cosmic Defects

# Gravitational Wave Backgrounds

## OUTLINE

1) Grav. Waves (GWs)

**Early Universe Sources**

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

# Gravitational Wave Backgrounds

## OUTLINE

**Grav. Th.**

1) Grav. Waves (GWs)

**Early Universe Sources**

2) GWs from Inflation
3) GWs from Preheating
4) GWs from Phase Transitions
5) GWs from Cosmic Defects

# Gravitational Wave Backgrounds

## OUTLINE

**Grav. Th.**

- 1) Grav. Waves (GWs)

**Early Universe Sources**

- 2) GWs from Inflation
- 3) GWs from Preheating
- 4) GWs from Phase Transitions
- 5) GWs from Cosmic Defects

**Late Universe & Experiments**

- 6) Astrophysical Background(s)
- 7) Observational Constraints/Prospects

← (Briefly)

# Gravitational Wave Backgrounds

1st Topic
(Formal Th.)

## OUTLINE

1) Grav. Waves (GWs)

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

# Gravitational Wave Backgrounds

## OUTLINE

1) Grav. Waves (GWs)

Main Topics (Pheno / Th.)

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

# Gravitational Wave Backgrounds

## OUTLINE

1) Grav. Waves (GWs)

Main Topics (Pheno / Th.)

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

Oliver Gould Lectures

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

# Gravitational Wave Backgrounds

## OUTLINE

1) Grav. Waves (GWs)

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

Bonus

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

(Briefly)

# Gravitational Wave Backgrounds

## OUTLINE

1) Grav. Waves (GWs)

2) GWs from Inflation

3) GWs from Preheating

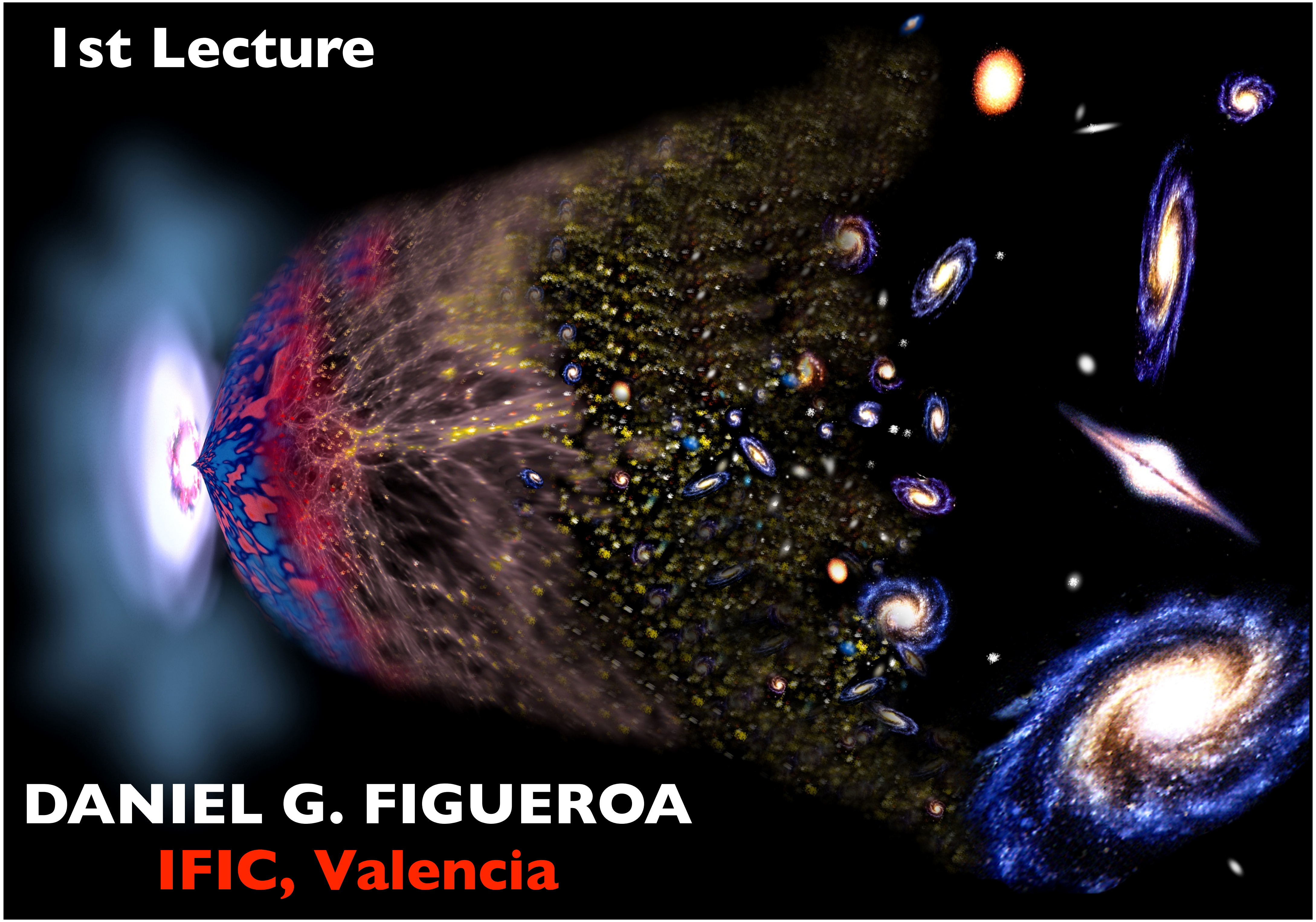
4) GWs from Phase Transitions

5) GWs from Cosmic Defects

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

# GRAVITATIONAL WAVE
# — BACKGROUNDS —



1st Lecture

**DANIEL G. FIGUEROA**

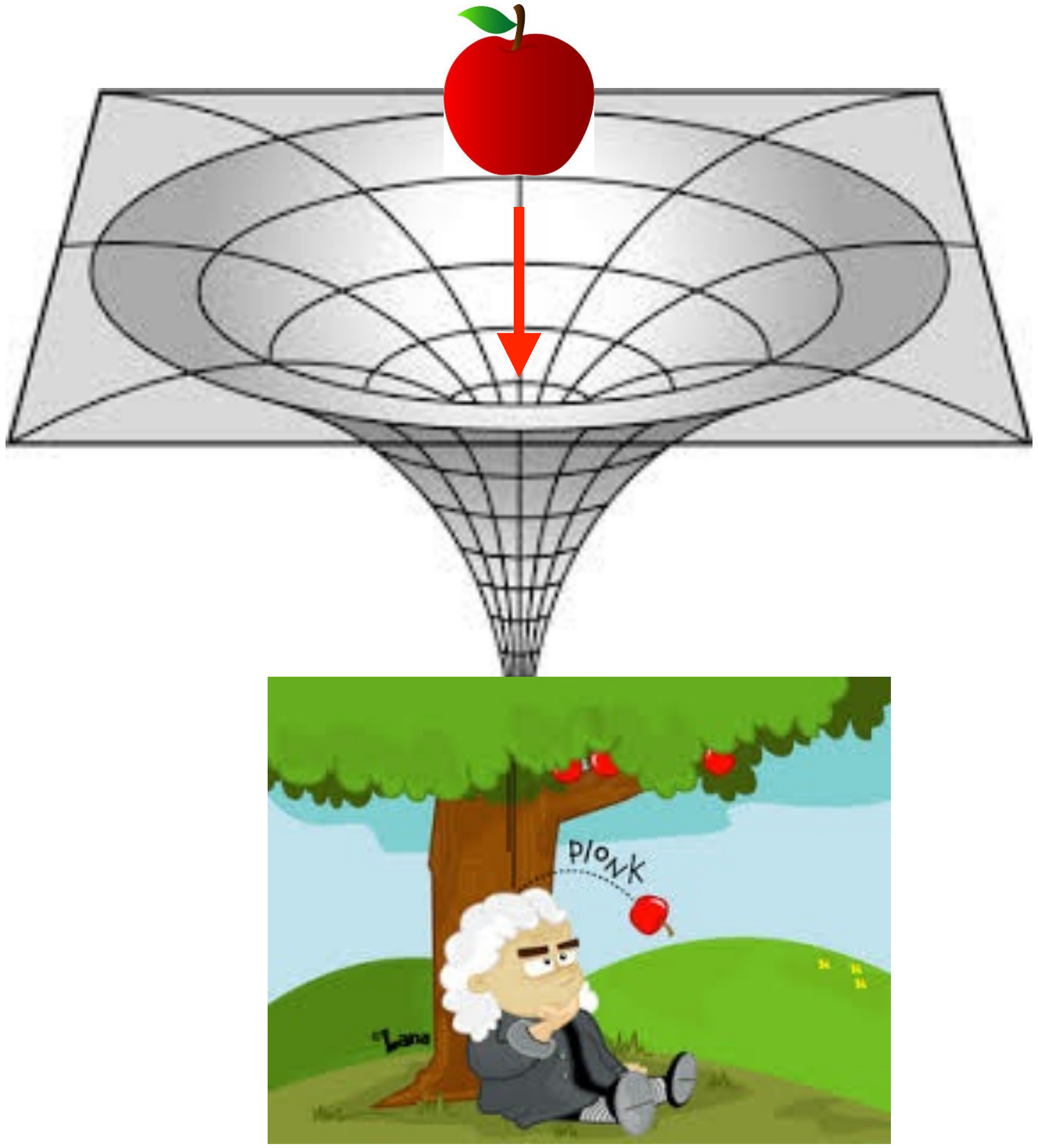
**IFIC, Valencia**

MITP Summer School - CrossLinks of Early Universe Cosmology, 15 July - August 2, 2024

# Primer on Gravitational Waves

# Gravitational Framework

**General Relativity (GR)**



$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

$$\left[ m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$\mathrm{DIFF}: \quad x^\mu \to x'^\mu(x)$$

symmetry

# Gravitational Framework

**General Relativity (GR)**

$$\underset{\text{geometry}}{G_{\mu\nu}} = \frac{1}{m_p^2}\underset{\text{matter}}{T_{\mu\nu}}$$

$$G_{\mu\nu} \equiv \underset{\substack{\big\downarrow \\ \text{2nd order, non-Linear}}}{\mathcal{D}}[\overset{\substack{\text{metric} \\ \big\uparrow}}{g_{\alpha\beta}}] \quad = \quad \underset{\text{source}}{m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, \ldots)}$$

# Gravitational Framework

**General Relativity (GR)**

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# How do we define GWs ?

# Gravitational Framework

**General Relativity (GR)**

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[\overset{\uparrow}{g}_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, ...)$$

source

$\downarrow$

**expand in perturbations**

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

# How do we define GWs ?

# Gravitational Framework

**General Relativity (GR)**

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

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↓

**expand in perturbations**

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

**Some part sources GWs**

## How do we define GWs ?

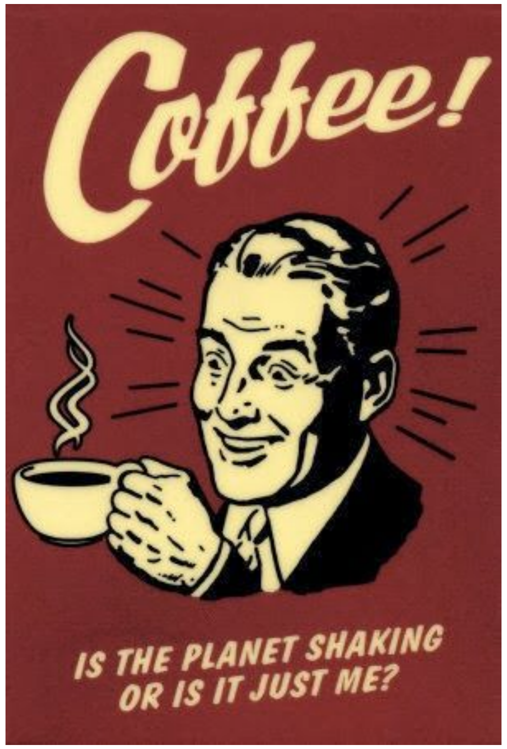
$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

# Perturbative Approach…

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

# Perturbative Approach...

I hope you took a
good load of coffee
('cause you are gonna need it)



# Definition of GWs

## 1st Approach

# Gravitational Wave Definition

**1st approach to GWs**

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

# Gravitational Wave Definition

LINEARIZED GRAVITY

$$g_{\mu\nu} = \overset{\overset{\text{Minkowski}}{\uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

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**1st approach to GWs**

Minkowski

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$$\mathrm{DIFF}: \quad x^\mu \to x'^\mu(x)$$

symmetry **?**

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$$\text{DIFF :} \quad x^\mu \xcancel{\rightarrow} x'^\mu(x)$$

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x)$$

$$\boxed{(\ |\partial_\mu \xi_\nu(x)| \lesssim |h_{\mu\nu}|\ )}$$

**residual symm.**

$$\boxed{h_{\mu\nu}(x) \rightarrow h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)}}$$

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**Notation:**

$$\begin{cases} \partial_{(\mu}\xi_{\nu)} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \partial_{[\mu}\xi_{\nu]} \equiv \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \end{cases}$$

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**Let's expand Einstein Equations !**

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$$( \ |h_{\mu\nu}| \ll 1 \ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

$$( \bar{h} = - h )$$

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(After some algebra ….)

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**Einstein tensor expanded**

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$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

**Lorentz gauge**

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**Technical Note:** If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

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$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \ \partial_{(\mu}\xi_{\nu)}$$

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(solution always!)

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(10 - 4 = 6 d.o.f. )

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**Is that all ?**

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**Is that all ? Not really …**

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$$( \ |h_{\mu\nu}| \ll 1 \ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

**(further residual gauge)**

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with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

**(further residual gauge)**

$$(\partial^\mu \bar{h}_{\mu\nu} = 0 \quad \rightarrow \quad \partial'^\mu \bar{h}'_{\mu\nu} = 0)$$

**(Lorentz preserving)**

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→

**IF** $T_{\mu\nu} = 0$

**Outside Source**

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**(further residual gauge)**

**IF** $T_{\mu\nu} = 0$

**Outside Source**

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

(**transverse-traceless gauge**)

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**1st approach to GWs**

$$g_{\mu\nu} = \overset{\substack{\text{Minkowski}\\ \uparrow}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

**fixed frame**

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

**(further residual gauge)**

$$\Longrightarrow$$

**IF** $T_{\mu\nu} = 0$

**Outside Source**

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

(**transverse-traceless gauge**)

(6 - 4 = 2 d.o.f. )

# Gravitational Wave Definition

**1st approach to GWs**

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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**IF** $T_{\mu\nu} \neq 0$

**Inside Source !**

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

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**6 - 4 = 2 d.o.f. ?**

**?**

(**transverse-traceless gauge**)

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**Cannot make** $h_{*0} = 0$

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

**(6 - 4 = 2 d.o.f. )**

**Yet there are still only 2 radiative *dof* !**

# Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f. )

**1st approach to GWs**

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

---

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(TT gauge: 6 - 4 = 2 d.o.f. )

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**Wave Eq. → Gravitational Waves !**

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**can GW be 'gauged away' ?**

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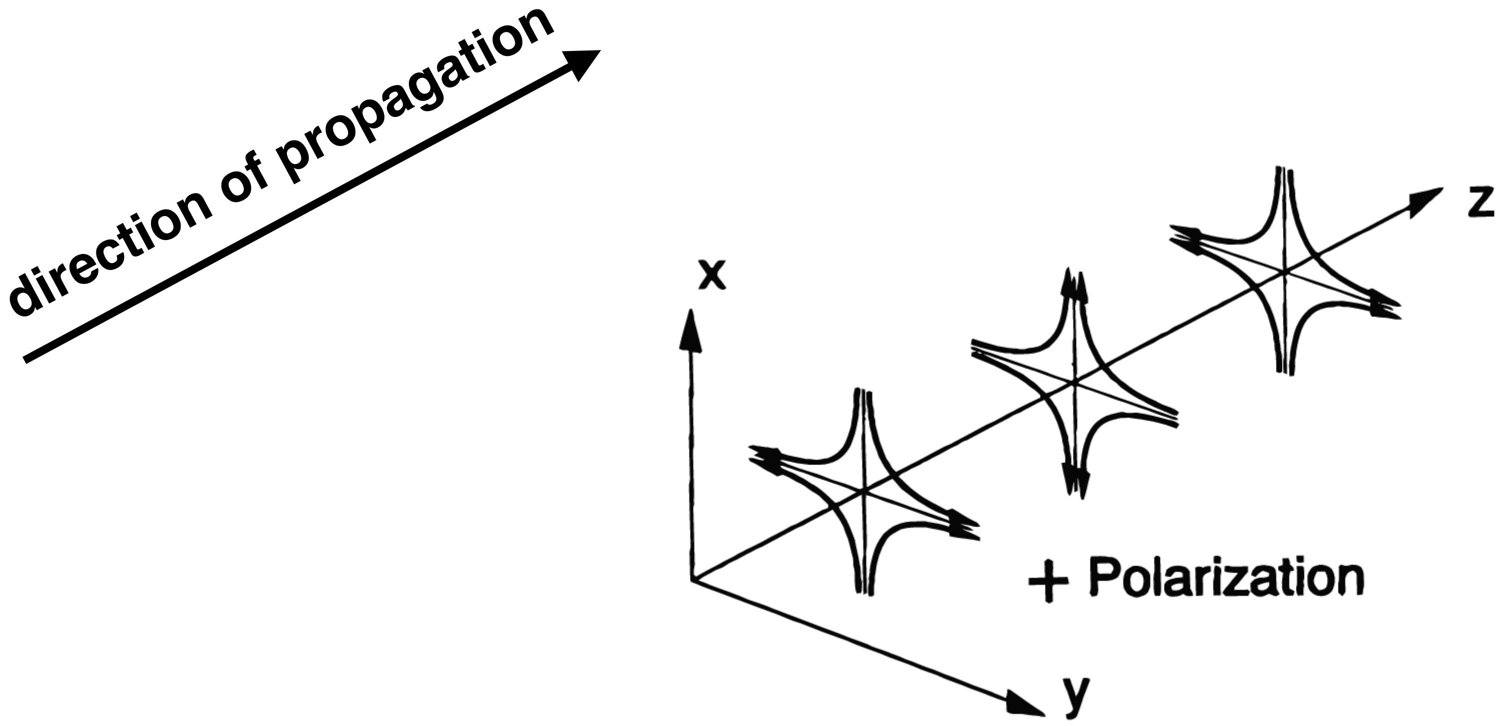

**Transverse**
**(& Traceless)**

# Gravitational Wave Definition

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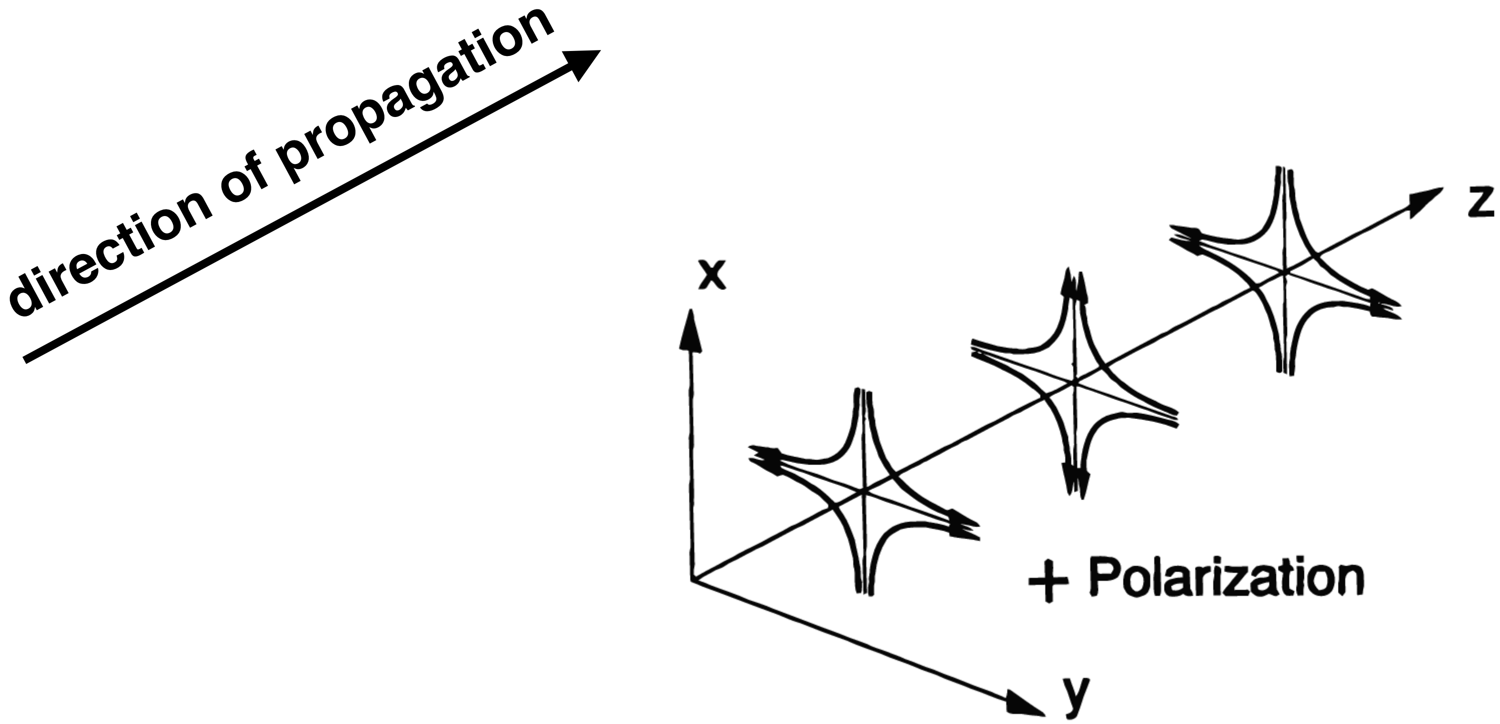

**2 *dof* =**
**2 polarizations**

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**2 *dof* = 2 polarizations**

$$h_{ab}(t, \mathbf{x}) = \int_{-\infty}^{\infty} df \int d\hat{n}\, h_{ab}(f, \hat{n}) e^{-2\pi i f (t - \hat{n}\mathbf{x})}$$

(plane wave)

transverse plane

# Gravitational Wave Definition

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transverse plane

(plane wave)

$$h_{ab}(f,\hat{n}) = \sum_{A=+,\times} h_A(f,\hat{n})\epsilon^{(A)}_{ab}(\hat{n}) \;=\; \begin{pmatrix} h_\times & h_x & 0 \\ h_x & -h_\times & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Transverse-
Traceless
(2 dof)

# Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f. )

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(plane wave)

$$h_{ab}(f, \hat{n}) = \sum_{A=+,\times} h_A(f, \hat{n}) \epsilon^{(A)}_{ab}(\hat{n}) \;=\; h_+ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\epsilon^{(+)}_{ab}} + h_\times \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\epsilon^{(\times)}_{ab}}$$

Transverse-Traceless (2 dof)

# Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f. )

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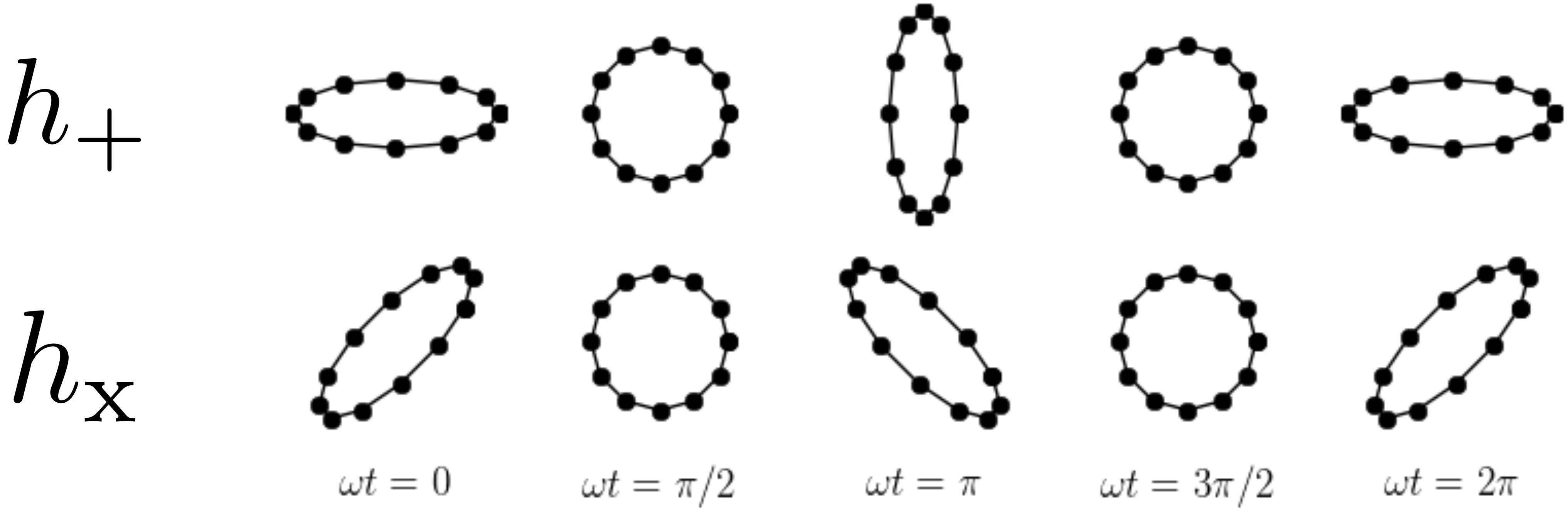

https://www.youtube.com/watch?v=U_hLM1WPDqM

https://www.youtube.com/watch?v=EtL9UyRx_Us

# Definition of GWs

## 2nd approach

# Gravitational Wave Definition

**2nd approach to GWs**

(gauge invariant def.)

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\overset{\uparrow}{\eta_{\mu\nu}}} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

# Gravitational Wave Definition

**2nd approach to GWs**

(gauge invariant def.)

Minkowski

$$g_{\mu\nu} = \underset{\uparrow}{\eta_{\mu\nu}} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

(svt decomposition)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

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s: scalar
v: vector
t: tensor

# Gravitational Wave Definition

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16 degrees of freedom

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16 degrees of freedom

In order NOT to over-count degrees of freedom $\left\{ \right.$

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16 degrees of freedom

In order NOT to over-count degrees of freedom

$$\left\{ \left. \begin{array}{ll} \partial_i S_i = 0 \ (1 \text{ constraint}), & \partial_i F_i = 0 \ (1 \text{ constraint}), \\ \partial_i h_{ij} = 0 \ (3 \text{ constraints}), & h_{ii} = 0 \ (1 \text{ constraint}) \end{array} \right\} \right.$$

Metric perturbations

# Gravitational Wave Definition

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16 degrees of freedom

In order NOT to over-count degrees of freedom

$$\left\{\begin{array}{l} \left.\begin{array}{ll} \partial_i S_i = 0 \text{ (1 constraint)}, & \partial_i F_i = 0 \text{ (1 constraint)}, \\ \partial_i h_{ij} = 0 \text{ (3 constraints)}, & h_{ii} = 0 \text{ (1 constraint)} \end{array}\right\} \\ \\ \left.\begin{array}{ll} \partial_i u_i = 0 \text{ (1 constraint)}, & \partial_i v_i = 0 \text{ (1 constraint)}, \\ \partial_i \Pi_{ij} = 0 \text{ (3 constraints)}, & \Pi_{ii} = 0 \text{ (1 constraint)}, \end{array}\right\} \end{array}\right.$$

Metric perturbations

Energy/Momentum tensor

# Gravitational Wave Definition

(svt metric perturbations)

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6 constraints for metric perturbations

6 constraints for E/p tensor components

# Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

10

~~16~~ degrees of freedom

(svt E/p-tensor components)

$$T_{00} = \rho,$$

$$T_{0i} = T_{i0} = \partial_i u + u_i,$$

$$T_{ij} = T_{ji} = p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij}.$$

10

~~16~~ degrees of freedom

In order NOT to over-count degrees of freedom

$$\left\{\begin{array}{l} \left.\begin{array}{ll} \partial_i S_i = 0 \text{ (1 constraint)}, & \partial_i F_i = 0 \text{ (1 constraint)}, \\ \partial_i h_{ij} = 0 \text{ (3 constraints)}, & h_{ii} = 0 \text{ (1 constraint)} \end{array}\right\} \\ \\ \left.\begin{array}{ll} \partial_i u_i = 0 \text{ (1 constraint)}, & \partial_i v_i = 0 \text{ (1 constraint)}, \\ \partial_i \Pi_{ij} = 0 \text{ (3 constraints)}, & \Pi_{ii} = 0 \text{ (1 constraint)}, \end{array}\right\} \end{array}\right.$$

6 constraints for metric perturbations

6 constraints for E/p tensor components

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(svt metric perturbations)

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10 degrees of freedom

(svt E/p-tensor components)

$$T_{00} = \rho,$$

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10 degrees of freedom

In order NOT to over-count degrees of freedom

$$\left\{\begin{array}{l}
\left.\begin{array}{ll}
\partial_i S_i = 0 \ (1 \text{ constraint}), & \partial_i F_i = 0 \ (1 \text{ constraint}), \\
\partial_i h_{ij} = 0 \ (3 \text{ constraints}), & h_{ii} = 0 \ (1 \text{ constraint})
\end{array}\right\} \\
\\
\left.\begin{array}{ll}
\partial_i u_i = 0 \ (1 \text{ constraint}), & \partial_i v_i = 0 \ (1 \text{ constraint}), \\
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\end{array}\right\}
\end{array}\right.$$

6 constraints for metric perturbations

6 constraints for E/p tensor components

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10 degrees of freedom

Physical Constraints

$$\partial^\mu T_{\mu\nu} = 0$$

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10 degrees of freedom

Physical Constraints

$$\partial^\mu T_{\mu\nu} = 0, \quad \Rightarrow \quad \left\{ \begin{array}{l} \nabla^2 u = \dot{\rho} \;\; (1 \text{ constraint}), \\ \\ \nabla^2 \sigma = \frac{3}{2}(\dot{u} - p) \;\; (1 \text{ constraint}), \\ \\ \nabla^2 v_i = \dot{u}_i \;\; (2 \text{ constraints}). \end{array} \right\}$$

4 constraints (due to E/p conservation)

# Gravitational Wave Definition

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$$\delta g_{00} = -2\phi,$$

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of freedom

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Physical
Constraints

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6 degrees of freedom

Physical Constraints

$$\partial^{\mu} G_{\mu\nu} = 0 \quad \Rightarrow \quad [\ldots]$$

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6 degrees of freedom

Physical Symmetry

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$$

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6 degrees of freedom

Physical Symmetry

( 4 *d.o.f.* spurious )

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$$

$$\xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i)$$

$$\text{with } \partial_i d_i = 0,$$

# Gravitational Wave Definition

(svt metric perturbations)

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6 degrees
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Physical
Symmetry
( 4 *d.o.f.*
spurious )

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⇨

**Gauge Invariant !**

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E},$$

$$\Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E,$$

$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i,$$

with $\partial_i\Sigma_i = 0$

# Gravitational Wave Definition

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**6** degrees of freedom

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6 degrees of freedom

Physical
Symmetry
( 4 *d.o.f.*
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$$\left\{ \begin{array}{ll} \phi \longrightarrow \phi - \dot{d}_0, & B \longrightarrow B - d_0 - \dot{d}, \\ \psi \longrightarrow \psi + \frac{1}{3}\nabla^2 d, & E \longrightarrow E - 2d, \\ S_i \longrightarrow S_i - \dot{d}_i, & F_i \longrightarrow F_i - 2d_i, \\ \boxed{h_{ij} \longrightarrow h_{ij}.} \quad (\mathbf{2}) & \end{array} \right\} \Rightarrow$$

**Gauge Invariant !**

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad (\mathbf{1})$$

$$\Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E, \qquad (\mathbf{1})$$

$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i, \qquad (\mathbf{2})$$

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# Gravitational Wave Definition

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$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad (1)$$

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$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i, \qquad (\partial_i \Sigma_i = 0) \qquad (2)$$

$$h_{ij} \equiv h_{ij}\ , \qquad (h_{ii} = \partial_i h_{ij} = 0) \qquad (2)$$

**6 gauge invariant**
**degrees of freedom**

# Gravitational Wave Definition

**Gauge Invariant !**

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad \text{(1)}$$

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$$h_{ij} \equiv h_{ij}\,, \qquad (h_{ii} = \partial_i h_{ij} = 0) \qquad \text{(2)}$$

**6 gauge invariant degrees of freedom**

**Gauge Invariant Einstein Tensor**

$$G_{00} = -\nabla^2 \Theta,$$

$$G_{0i} = -\frac{1}{2}\nabla^2 \Sigma_i - \partial_i \dot{\Theta},$$

$$G_{ij} = -\frac{1}{2}\Box h_{ij} - \partial_{(i}\dot{\Sigma}_{j)} - \frac{1}{2}\partial_i \partial_j \left(2\Phi + \Theta\right) + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi + \Theta\right) - \ddot{\Theta}\right].$$

# Gravitational Wave Definition

**Gauge Invariant !**

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad (1)$$

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**6 gauge invariant**
**degrees of freedom**

**Gauge Invariant (perturbed) Einstein Eqs.**

$$\nabla^2 \Theta = -\frac{1}{m_p^2}\rho, \qquad \nabla^2 \Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right)$$

$$\nabla^2 \Sigma_i = -\frac{2}{m_p^2}u_i, \qquad \Box h_{ij} = -\frac{2}{m_p^2}\Pi_{ij}.$$

# Gravitational Wave Definition

**Gauge Invariant !**

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad (1)$$

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**6 gauge invariant degrees of freedom**

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# Gravitational Wave Definition

**6 gauge invariant *d.o.f.***

Gauge Invariant
(perturbed)
Einstein Eqs.

$$\nabla^2\Theta = -\frac{1}{m_p^2}\rho, \text{(1)} \qquad \nabla^2\Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right) \text{ (1)}$$

$$\nabla^2\Sigma_i = -\frac{2}{m_p^2}u_i, \text{(2)} \qquad \Box h_{ij} = -\frac{2}{m_p^2}\Pi_{ij}. \quad \text{(2)}$$

# Gravitational Wave Definition

**6 gauge invariant *d.o.f.***

**Gauge Invariant (perturbed) Einstein Eqs.**

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$$\boxed{h_{ij}\,,\quad (h_{ii} = \partial_i h_{ij} = 0)}$$

**transverse**
**& traceless**
**(tensor *dof*)**

# Gravitational Wave Definition

**6 gauge invariant *d.o.f.***

**Gauge Invariant (perturbed) Einstein Eqs.**

$$\nabla^2\Theta = -\frac{1}{m_p^2}\rho, (1) \qquad \nabla^2\Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right)\ (1)$$

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$$\boxed{h_{ij}\,,\quad (h_{ii} = \partial_i h_{ij} = 0)}$$

**transverse**
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**(tensor *dof*)**

**Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !**

# Gravitational Wave Definition

**6 gauge invariant *d.o.f.***

**Gauge Invariant (perturbed) Einstein Eqs.**

$$\nabla^2\Theta = -\frac{1}{m_p^2}\rho, \text{(1)} \qquad \nabla^2\Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right) \text{ (1)}$$

$$\nabla^2\Sigma_i = -\frac{2}{m_p^2}u_i, \text{(2)} \qquad \boxed{\Box h_{ij} = -\frac{2}{m_p^2}\Pi_{ij}.} \quad \text{(2)}$$

$$\boxed{h_{ij}\,, \quad (h_{ii} = \partial_i h_{ij} = 0)}$$

**transverse & traceless (tensor *dof*)**

**Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !**

**Gravitational Waves (GWs) are TT *d.o.f.* metric perturbations, independently of system of reference**

# Definition of GWs

## 3rd approach

# Gravitational Wave Definition

**3rd approach to GWs**

(for a FLRW space-time)

$$g_{\mu\nu}(x) = \underbrace{\bar{g}_{\mu\nu}(x)}_{\text{(FLRW)}} + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$$

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where $h_{0\mu} = 0\,, \quad h_{ii} = 0\,, \quad \partial_i h_{ij} = 0$ **Transverse-Traceless (TT) *d.o.f.***

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Shall I use the blackboard !?

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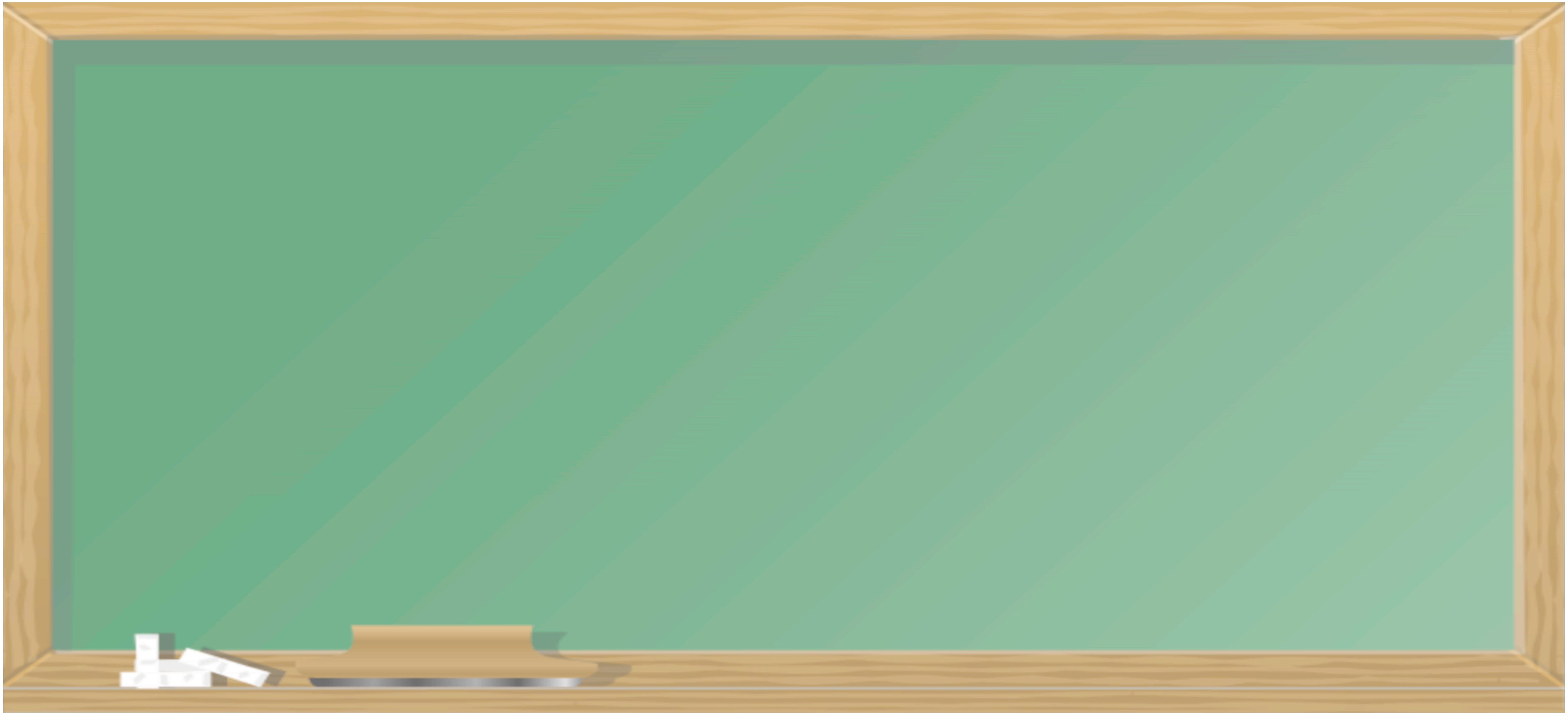

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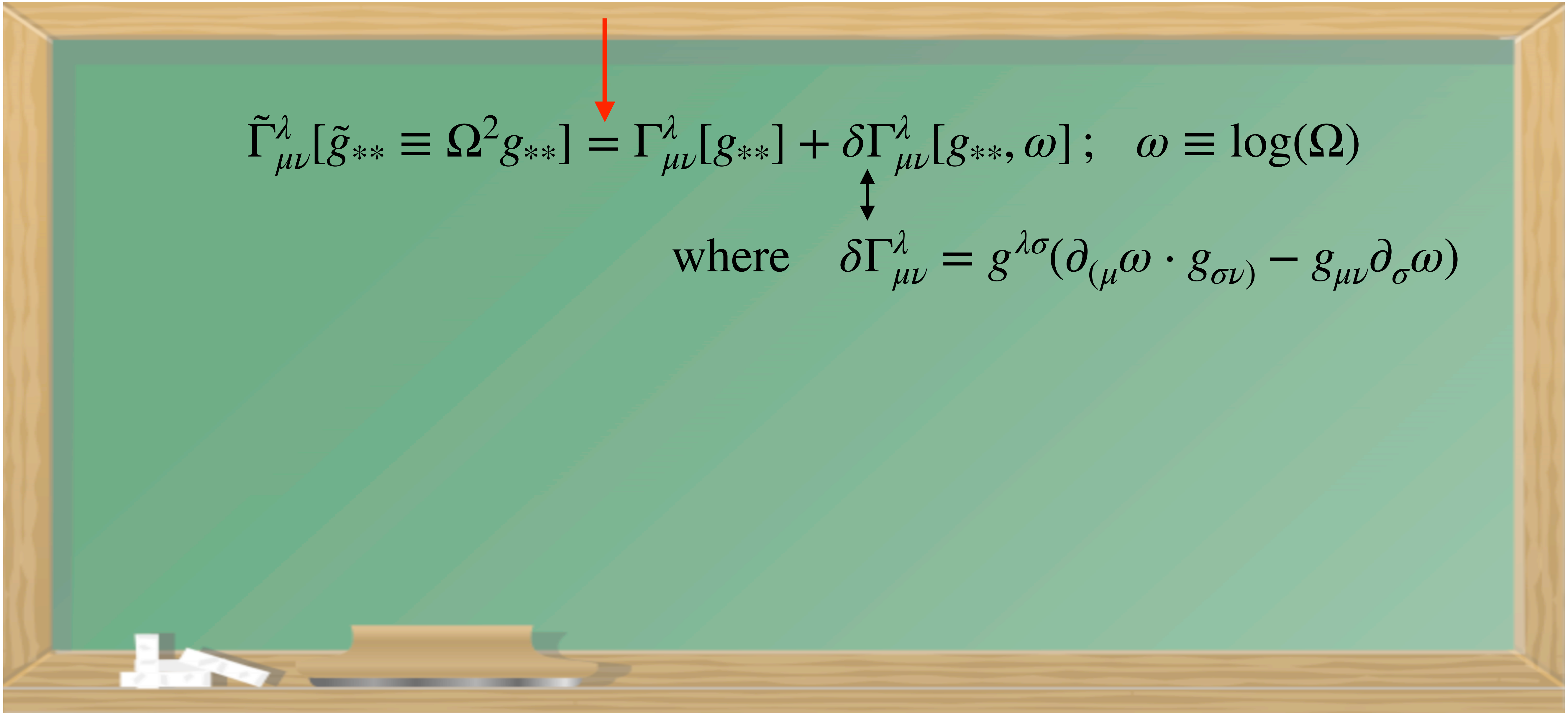

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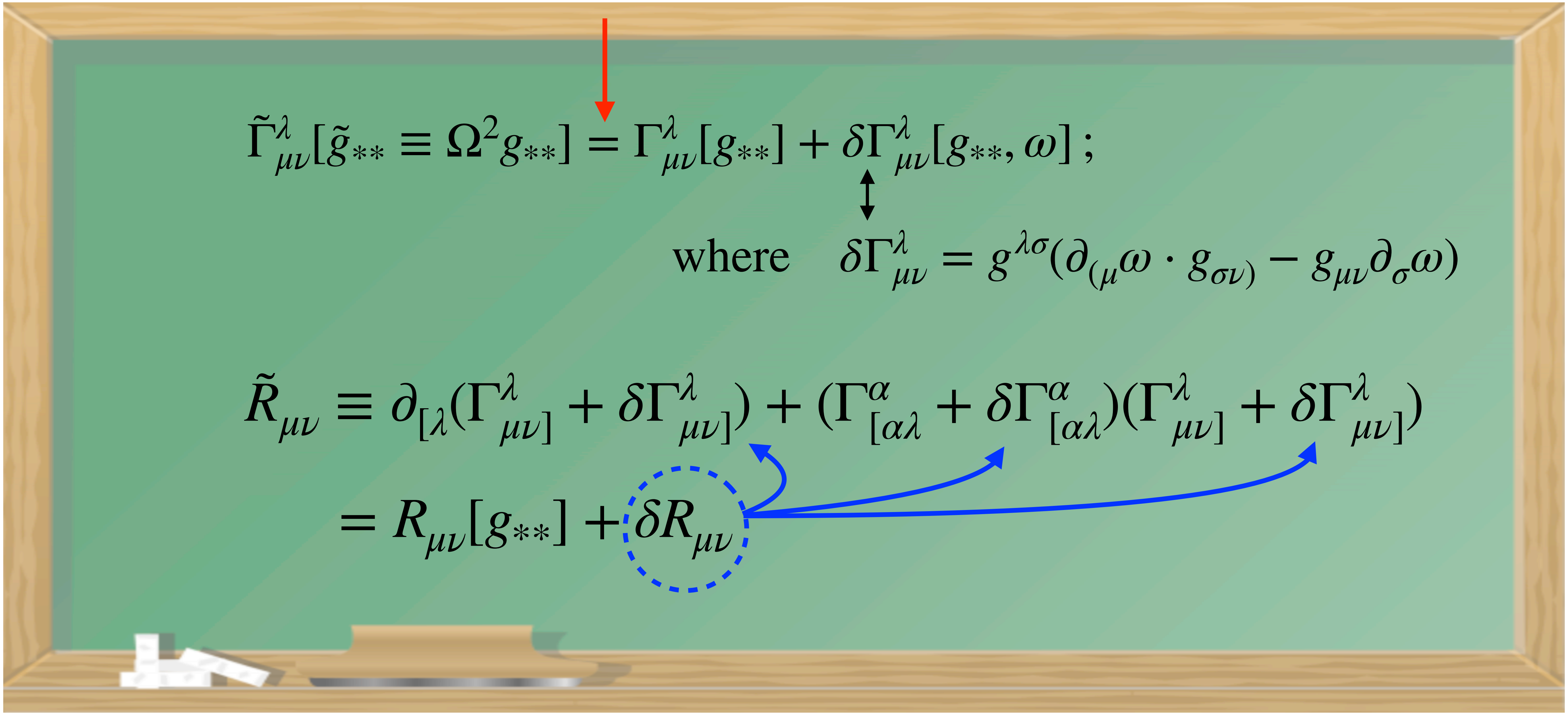

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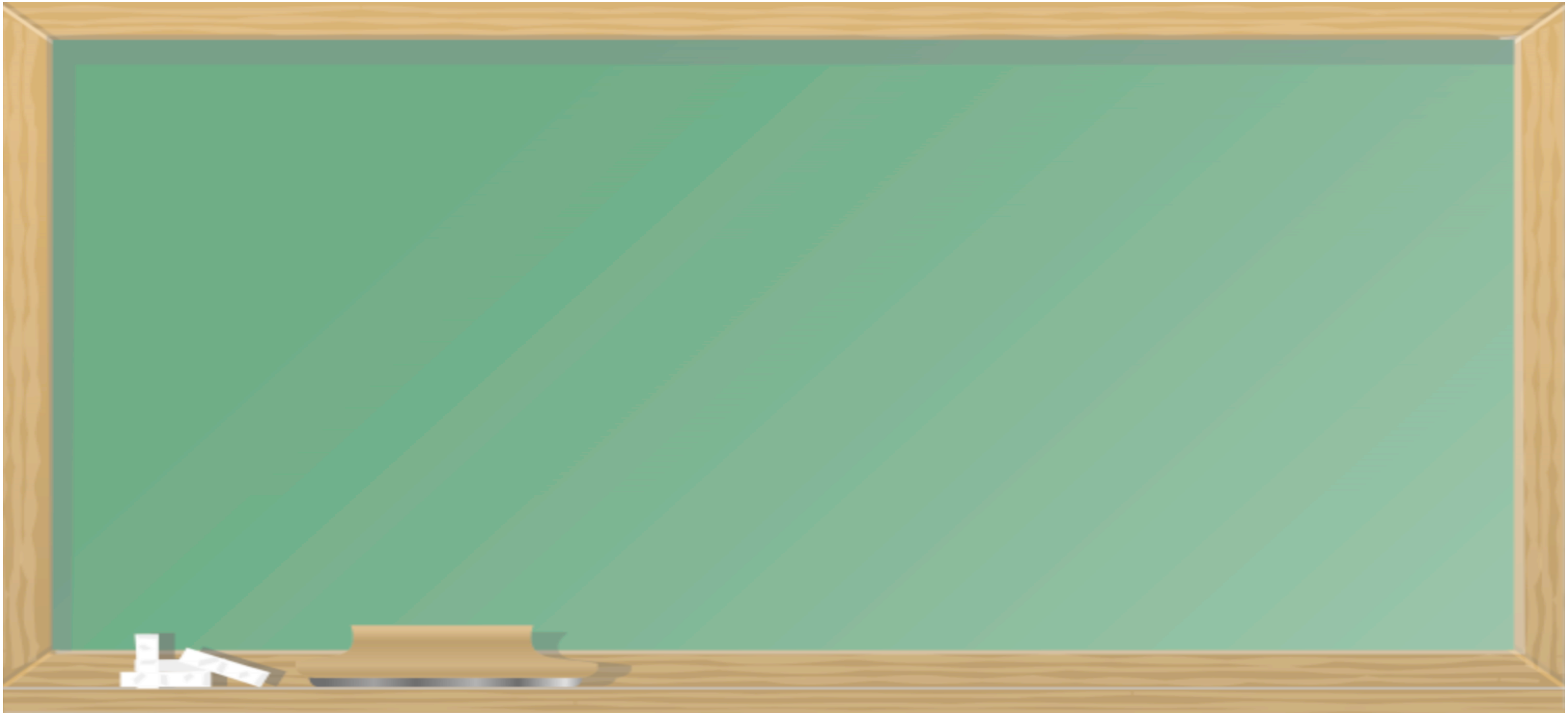

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**How does it look** $\delta R_{\mu\nu}$ **?**

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$$\left(\ \tilde{g}_{\mu\nu} = \overbrace{\Omega^2(x)}^{a^2(t)}\, \overbrace{g_{\mu\nu}(x)}^{(\eta_{\mu\nu} + h_{\mu\nu})}\ \right)$$

$$\delta R_{\mu\nu} = \partial_{[\lambda}\delta\Gamma^{\lambda}_{\mu\nu]} + \delta\Gamma^{\lambda}_{[\lambda\sigma}\Gamma^{\sigma}_{\mu\nu]} + \Gamma^{\lambda}_{[\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu]} + \delta\Gamma^{\lambda}_{[\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu]}$$

where $\quad \delta\Gamma^{\lambda}_{\mu\nu} = \omega_{(\mu}\delta^{\lambda}{}_{\nu)} - g_{\mu\nu}\omega^{\lambda}\,; \quad \omega_{\mu} \equiv \omega_{,\mu}\,; \quad \omega^{\mu} \equiv \omega^{,\mu}$

$$\delta R_{\mu\nu}[g_{**}, \omega] \equiv A\omega_{\mu}\omega_{\nu} + B\omega_{\mu;\nu} + Cg_{\mu\nu}\omega_{\alpha}\omega^{\alpha} + Dg_{\mu\nu}(\omega^{\alpha})_{;\alpha}$$

*(A, B, C, D constants)*

**It can only take this form !**

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu}[\tilde{g}_{**}] \equiv R_{\mu\nu}[g_{**}] + \delta R_{\mu\nu}$

$$[\ \omega \equiv \log(\Omega)\ ]$$

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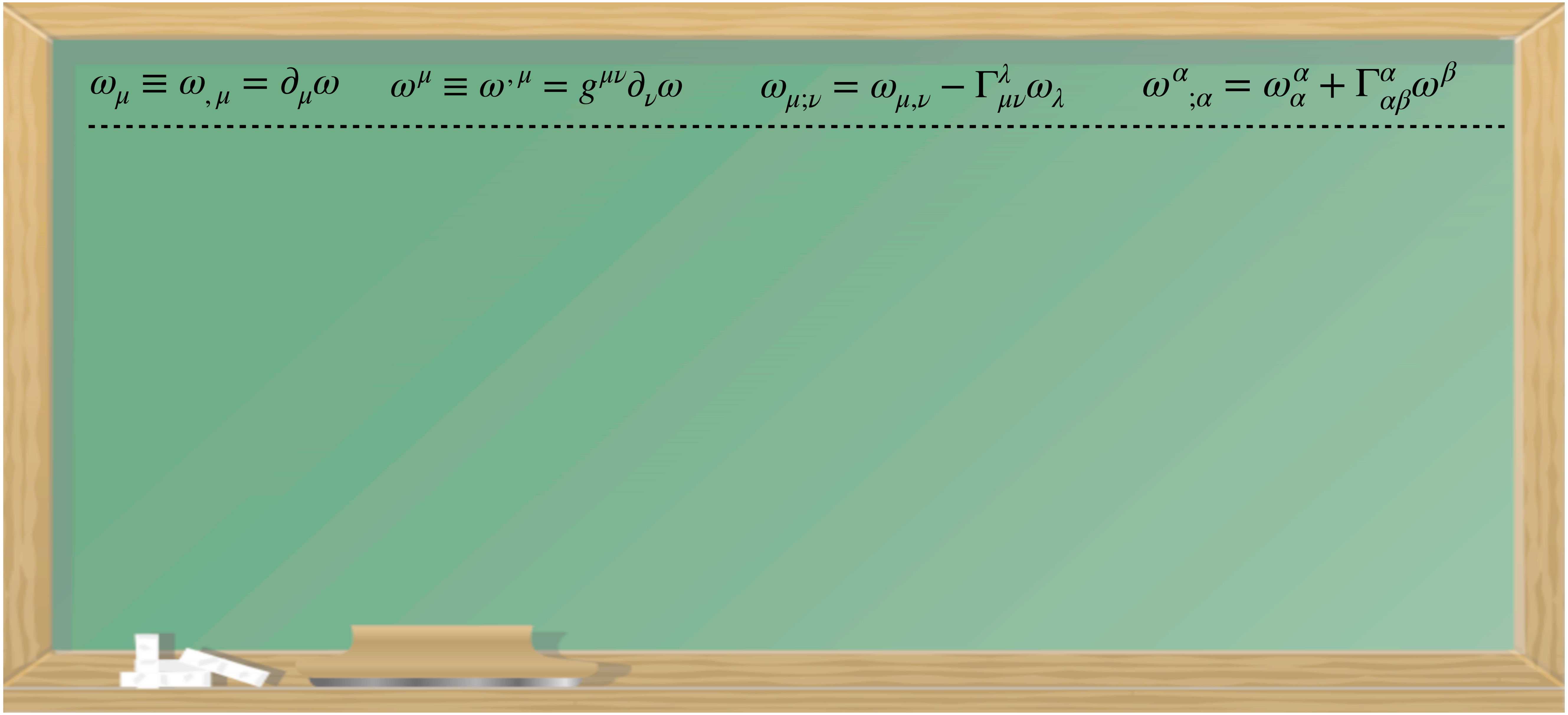
**After some Calculation…** $A = +2\,, B = -2\,, C = -2\,, D = -1$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu}[\tilde{g}_{**}] \equiv R_{\mu\nu}[g_{**}] + \delta R_{\mu\nu}$ ; $[g_{**} = \eta_{**} + h_{**}]$

$$\left[ \delta R_{\mu\nu} = 2\omega_\mu\omega_\mu - 2\omega_{\mu;\nu} - 2g_{\mu\nu}\omega_\alpha\omega^\alpha - g_{\mu\nu}(\omega^\alpha)_{;\alpha} \right] \quad ; \quad \omega \equiv \log a(t)$$



$\omega_\mu \equiv \omega_{,\mu} = \partial_\mu\omega$ $\quad$ $\omega^\mu \equiv \omega^{,\mu} = g^{\mu\nu}\partial_\nu\omega$ $\quad$ $\omega_{\mu;\nu} = \omega_{\mu,\nu} - \Gamma^\lambda_{\mu\nu}\omega_\lambda$ $\quad$ $\omega^\alpha{}_{;\alpha} = \omega^\alpha_\alpha + \Gamma^\alpha_{\alpha\beta}\omega^\beta$

# Gravitational Wave Definition

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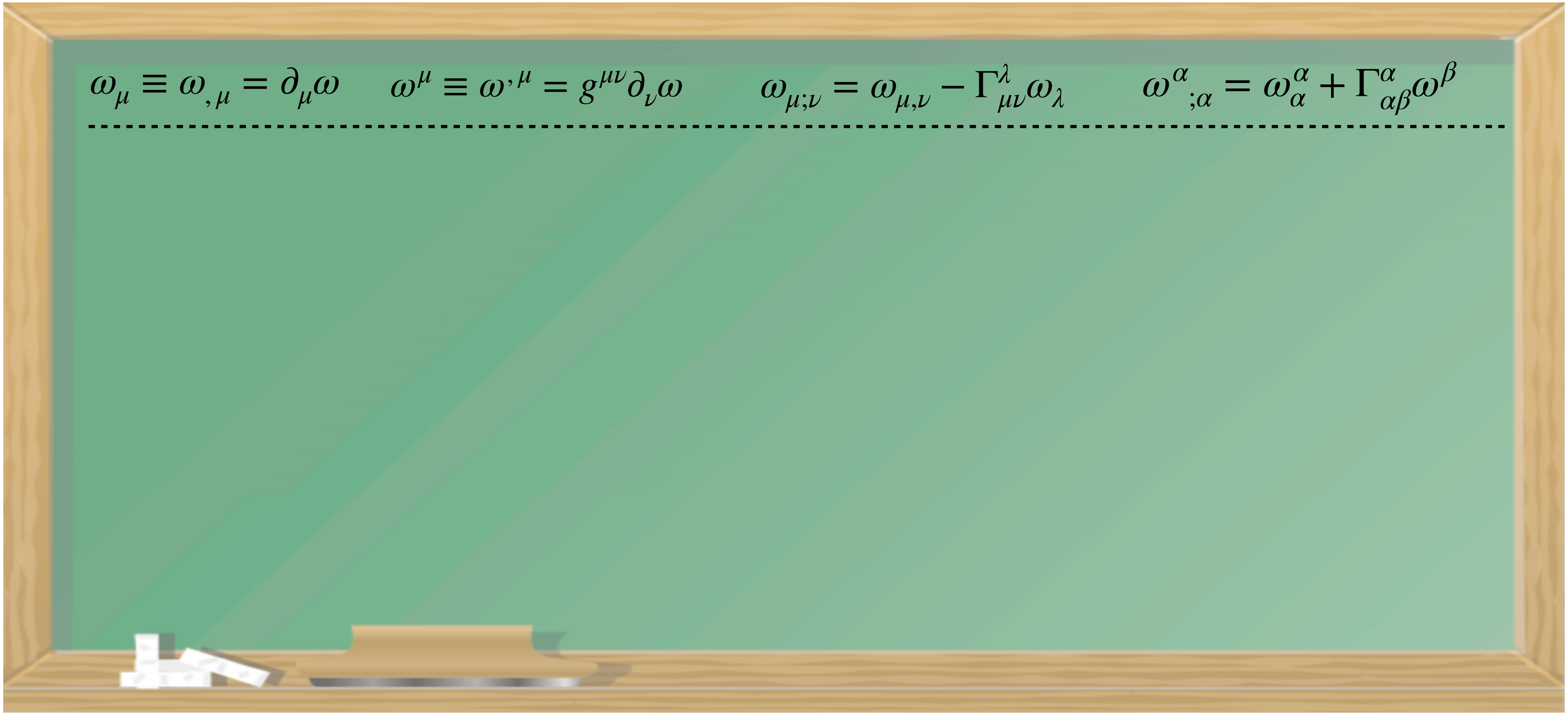

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$$2\omega_\mu\omega_\nu = 2\mathscr{H}^2\delta_{\mu0}\delta_{\nu0}\,, \qquad \mathscr{H} \equiv a'/a$$

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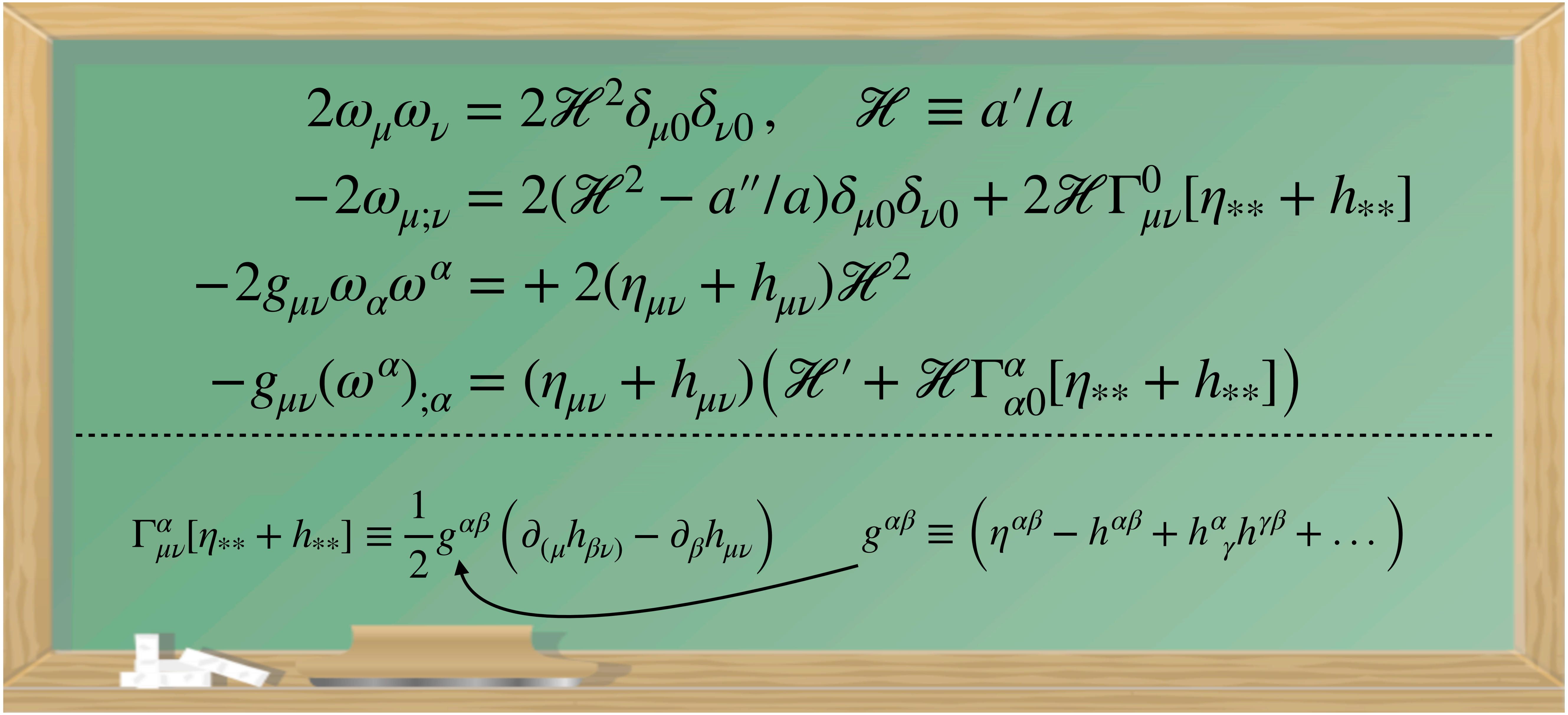

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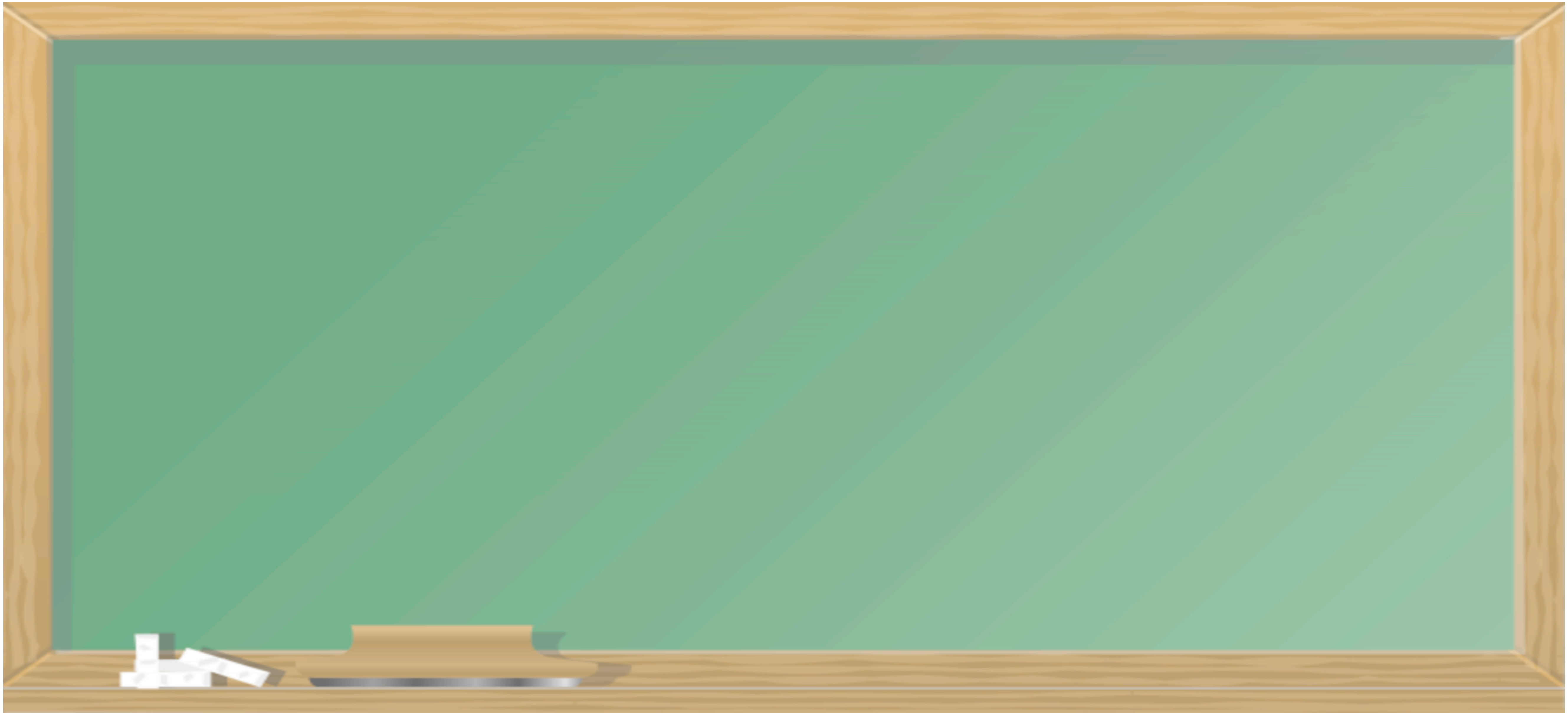

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$$\left[\ \mathscr{D}_{\mu\nu}\omega = 2\omega_{\mu}\omega_{\mu} - 2\omega_{\mu;\nu} - 2g_{\mu\nu}\omega_{\alpha}\omega^{\alpha} - g_{\mu\nu}(\omega^{\alpha})_{;\alpha}\ \right] \quad ; \quad \omega \equiv \log a(t)$$

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$$-2\omega_{\mu;\nu} = 2(\mathscr{H}^2 - a''/a)\delta_{\mu0}\delta_{\nu0} + 2\mathscr{H}(\overset{(1)}{\Gamma}{}^{0}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{0}_{\mu\nu})$$

$$-2g_{\mu\nu}\omega_{\alpha}\omega^{\alpha} = +2(\eta_{\mu\nu} + h_{\mu\nu})\mathscr{H}^2$$

$$-g_{\mu\nu}(\omega^{\alpha})_{;\alpha} = (\eta_{\mu\nu} + h_{\mu\nu})\Big(\mathscr{H}' + \mathscr{H}(\overset{(1)}{\Gamma}{}^{\alpha}_{\alpha0} + \overset{(2)}{\Gamma}{}^{\alpha}_{\alpha0})\Big)$$

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$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \ldots \quad \begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv R_{\mu\nu}[\eta_{**} + h_{**}] + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + \underline{(\mathscr{D}_{\mu\nu}\omega)^{(1)}} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

$$\Big[\, \mathscr{D}_{\mu\nu}\omega = 2\omega_\mu\omega_\mu - 2\omega_{\mu;\nu} - 2g_{\mu\nu}\omega_\alpha\omega^\alpha - g_{\mu\nu}(\omega^\alpha)_{;\alpha} \,\Big] \quad ; \quad \omega \equiv \log a(t)$$

$$2\omega_\mu\omega_\nu = 2\mathscr{H}^2\delta_{\mu 0}\delta_{\nu 0}\,, \qquad \mathscr{H} \equiv a'/a$$

$$-2\omega_{\mu;\nu} = 2(\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + \boxed{2\mathscr{H}(\overset{(1)}{\Gamma}{}^0_{\mu\nu}} + \overset{(2)}{\Gamma}{}^0_{\mu\nu})$$

$$-2g_{\mu\nu}\omega_\alpha\omega^\alpha = +\,2(\eta_{\mu\nu} \boxed{+\,h_{\mu\nu})\mathscr{H}^2}$$

$$-g_{\mu\nu}(\omega^\alpha)_{;\alpha} = \boxed{(\eta_{\mu\nu}} + \boxed{h_{\mu\nu})\Big(\mathscr{H}'} + \boxed{\mathscr{H}(\overset{(1)}{\Gamma}{}^\alpha_{\alpha 0}} + \overset{(2)}{\Gamma}{}^\alpha_{\alpha 0})\Big)$$

$$\Gamma^\alpha_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} + \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} + \ldots \quad \begin{cases} \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \end{cases}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv R_{\mu\nu}[\eta_{**}+h_{**}] + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + \underline{(\mathscr{D}_{\mu\nu}\omega)^{(2)}}$$

$$\left[\ \mathscr{D}_{\mu\nu}\omega = 2\omega_\mu\omega_\mu - 2\omega_{\mu;\nu} - 2g_{\mu\nu}\omega_\alpha\omega^\alpha - g_{\mu\nu}(\omega^\alpha)_{;\alpha}\ \right] \quad ; \quad \omega \equiv \log a(t)$$

$$2\omega_\mu\omega_\nu = 2\mathscr{H}^2\delta_{\mu0}\delta_{\nu0}\,, \qquad \mathscr{H} \equiv a'/a$$

$$-2\omega_{\mu;\nu} = 2(\mathscr{H}^2 - a''/a)\delta_{\mu0}\delta_{\nu0} + \enclose{circle}{2\mathscr{H}}(\overset{(1)}{\Gamma}{}^0_{\mu\nu} + \enclose{circle}{\overset{(2)}{\Gamma}{}^0_{\mu\nu})}$$

$$-2g_{\mu\nu}\omega_\alpha\omega^\alpha = +2(\eta_{\mu\nu}+h_{\mu\nu})\mathscr{H}^2$$

$$-g_{\mu\nu}(\omega^\alpha)_{;\alpha} = \enclose{circle}{(\eta_{\mu\nu}} + \boxed{h_{\mu\nu})}\Big(\mathscr{H}' + \mathscr{H}(\boxed{\overset{(1)}{\Gamma}{}^\alpha_{\alpha0}} + \enclose{circle}{\overset{(2)}{\Gamma}{}^\alpha_{\alpha0})}\Big)$$

$$\Gamma^\alpha_{\mu\nu}[\eta_{**}+h_{**}] \equiv \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} + \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} + \ldots \qquad \begin{cases} \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \\[2ex] \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \end{cases}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv R_{\mu\nu}[\eta_{**} + h_{**}] + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

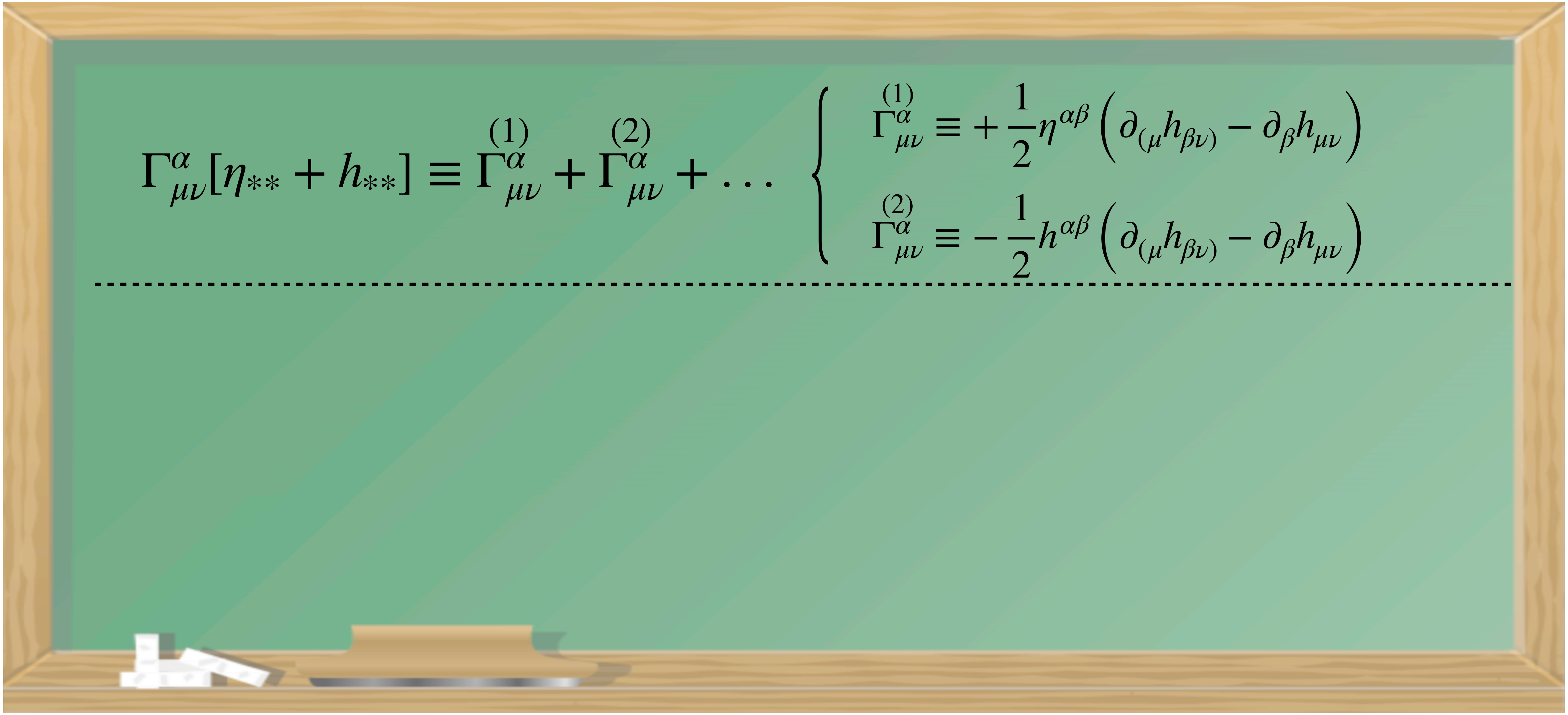
$$\left[\ \mathscr{D}_{\mu\nu}\omega = 2\omega_\mu\omega_\mu - 2\omega_{\mu;\nu} - 2g_{\mu\nu}\omega_\alpha\omega^\alpha - g_{\mu\nu}(\omega^\alpha)_{;\alpha}\ \right] \quad ; \quad \omega \equiv \log a(t)$$

$$\Gamma^\alpha_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} + \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} + \dots \quad \left\{ \begin{array}{l} \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} \equiv +\frac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} \equiv -\frac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \end{array} \right.$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

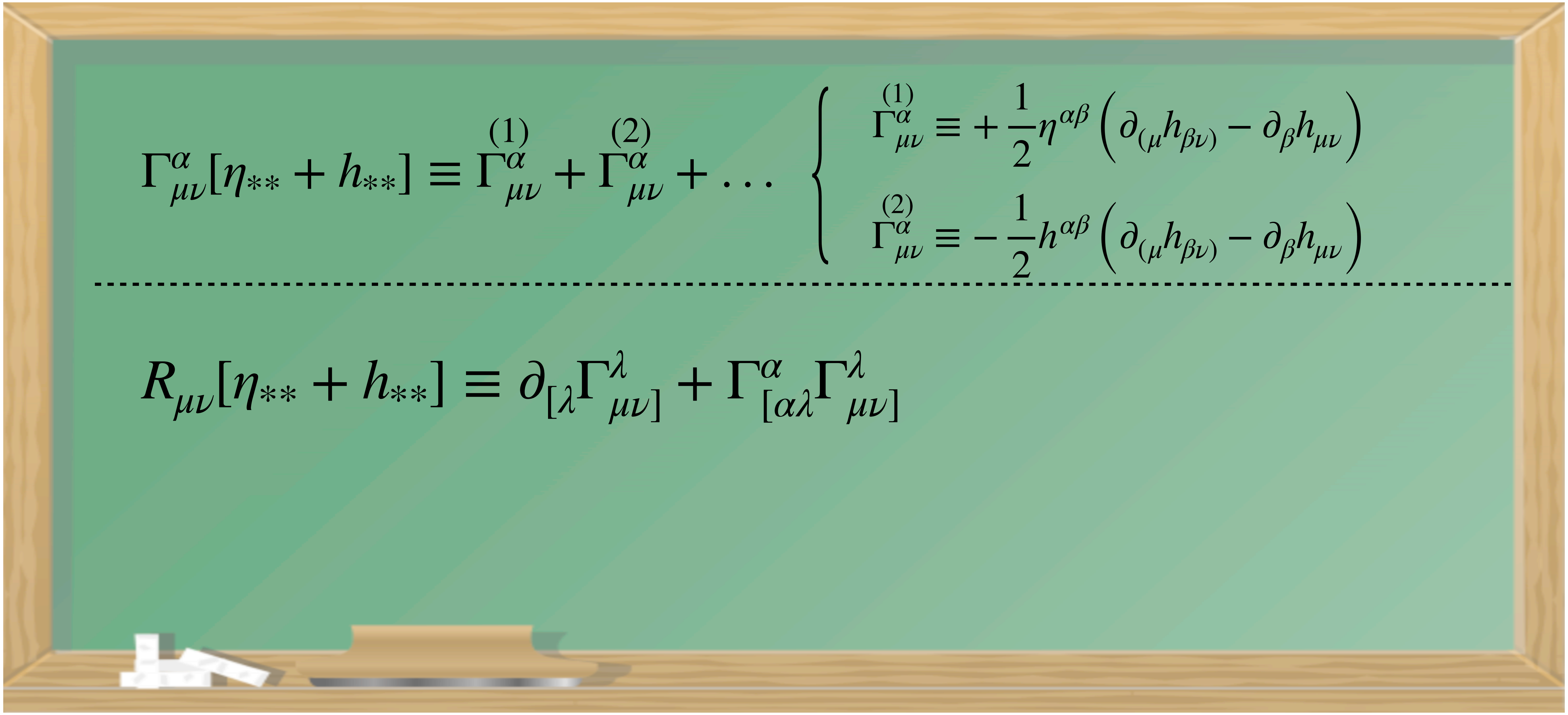


$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \dots \quad \left\{ \begin{array}{l} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{array} \right.$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$



$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \ldots \quad \begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \partial_{[\lambda}\Gamma^{\lambda}_{\mu\nu]} + \Gamma^{\alpha}_{[\alpha\lambda}\Gamma^{\lambda}_{\mu\nu]}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \dots \quad \begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

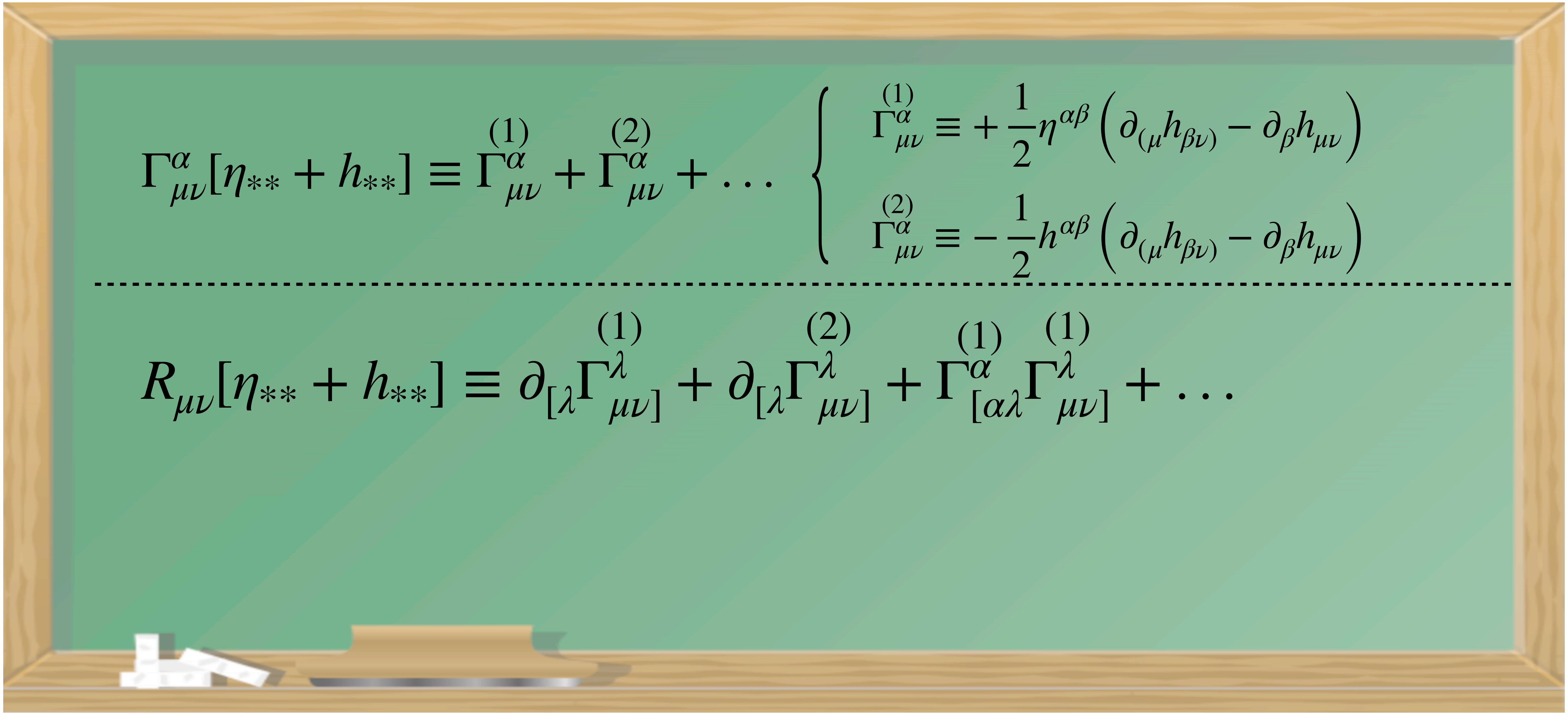
$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \partial_{[\lambda}\Gamma^{\lambda}_{\mu\nu]} + \Gamma^{\alpha}_{[\alpha\lambda}\Gamma^{\lambda}_{\mu\nu]}$$

$$\partial_{[\lambda}(\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \dots) + (\overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda} + \overset{(2)}{\Gamma}{}^{\alpha}_{[\alpha\lambda} + \dots)(\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \dots)$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$



$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \dots \quad \begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\frac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\frac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \partial_{[\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \dots$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \dots \quad \begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \underbrace{\partial_{[\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(1)}{\delta R}_{\mu\nu}} + \underbrace{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(2)}{\delta R}_{\mu\nu}} + \dots$$

# Gravitational Wave Definition

**Then:**

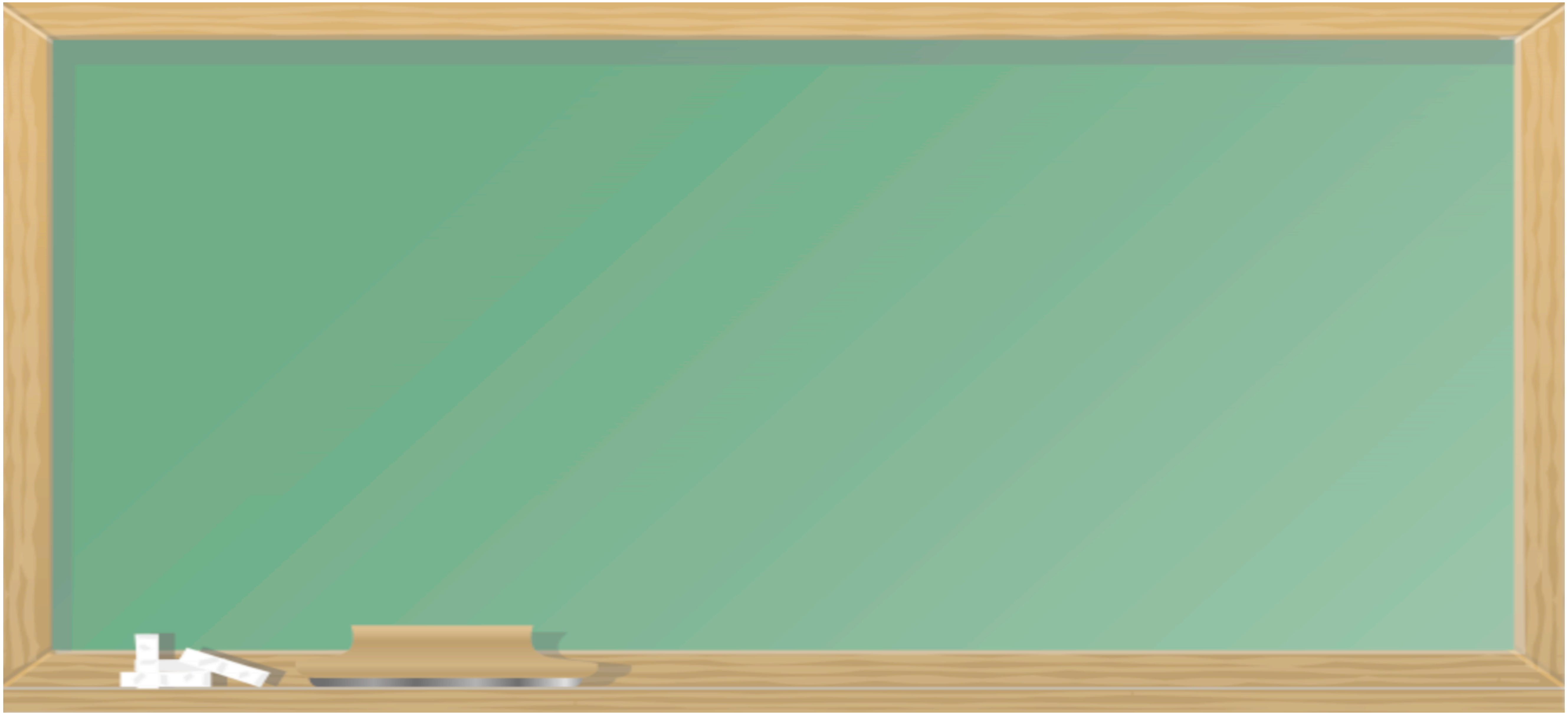
$$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{0 + \overset{(1)}{\delta R}_{\mu\nu} + \overset{(2)}{\delta R}_{\mu\nu}} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \dots \quad \left\{ \begin{array}{l} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{array} \right.$$

$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \underbrace{\partial_{[\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(1)}{\delta R}_{\mu\nu}} + \underbrace{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(2)}{\delta R}_{\mu\nu}} + \dots$$

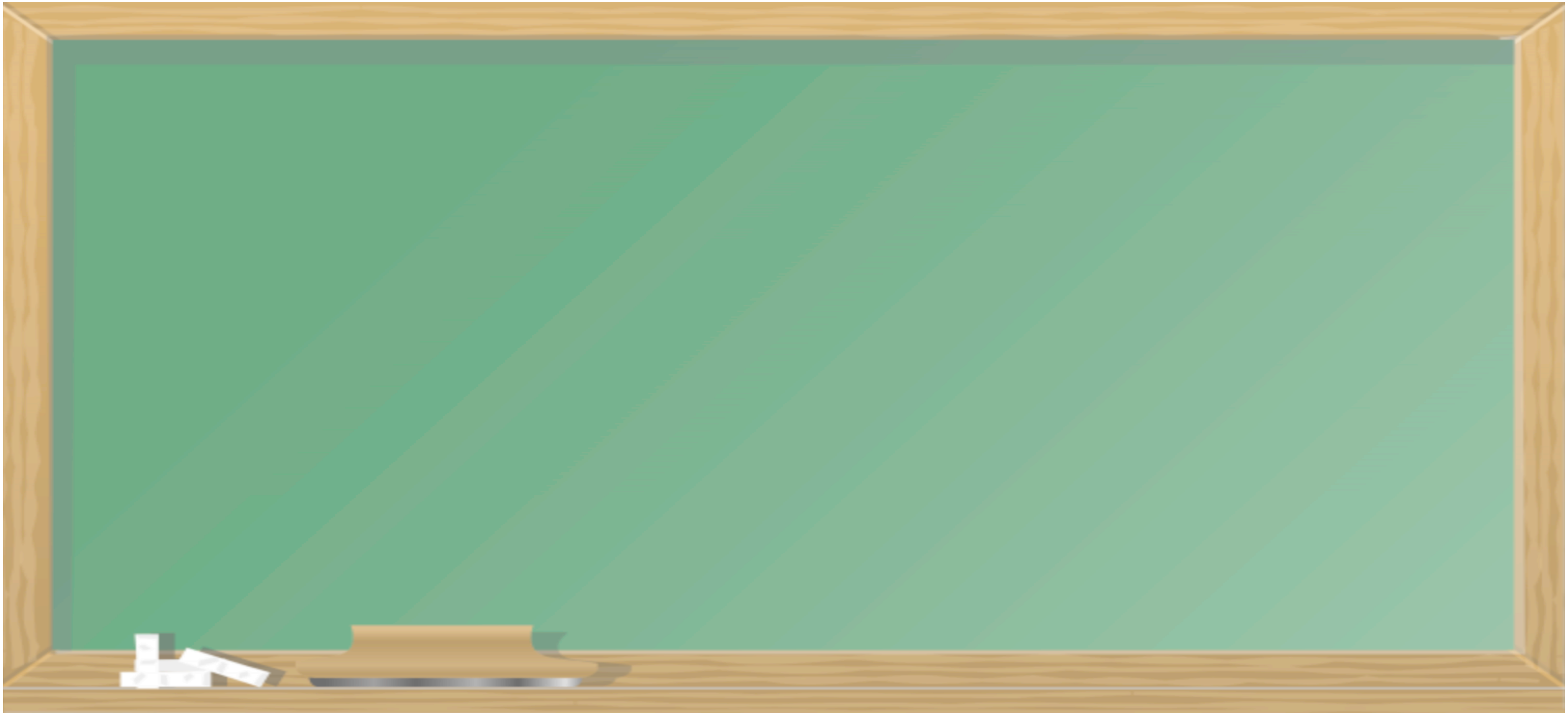
# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(1)}{\delta R}_{\mu\nu} + \overset{(2)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$



# Gravitational Wave Definition

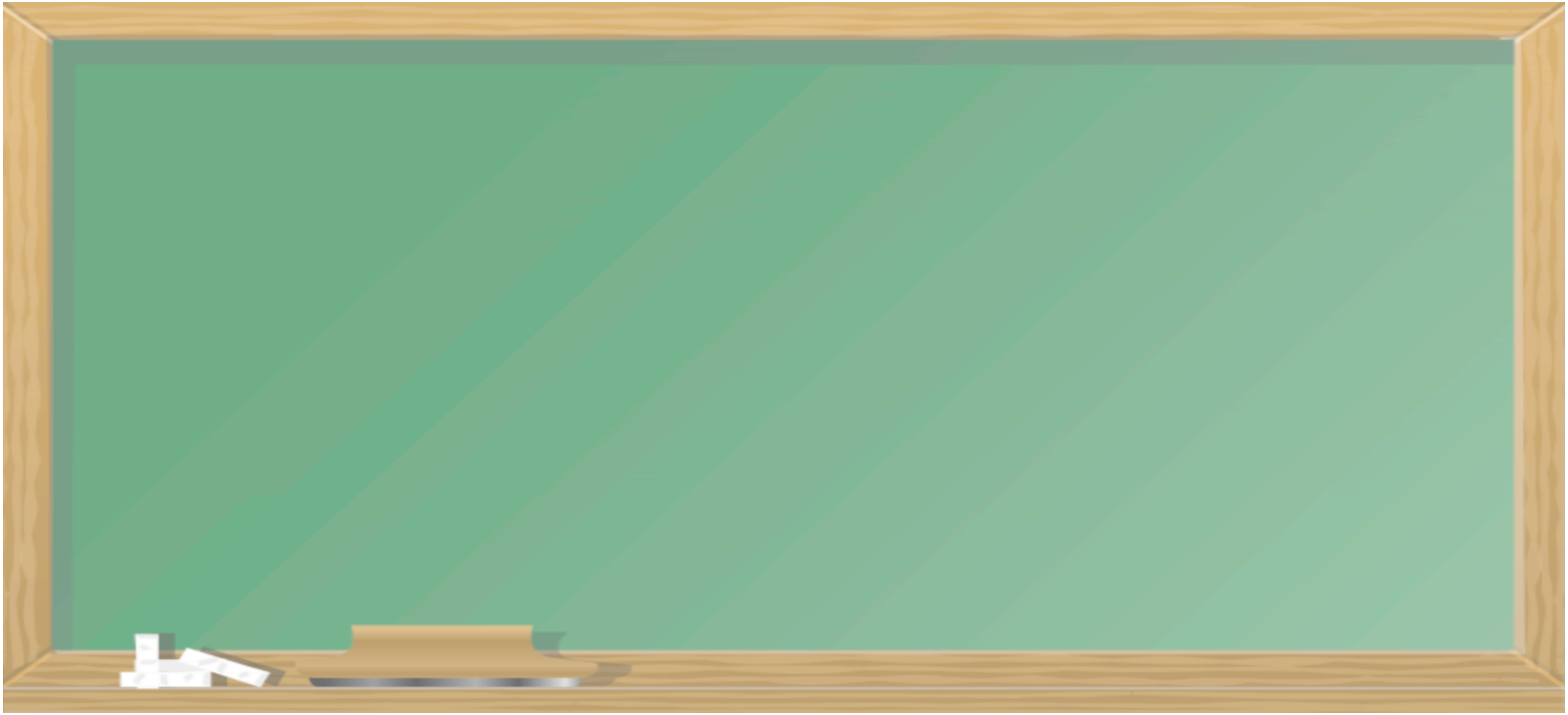
**Then:** $$\tilde{R}_{\mu\nu} \equiv (\mathscr{D}_{\mu\nu}\omega)^{(0)} + \left(\delta \overset{(1)}{R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(1)}\right) + \left(\delta \overset{(2)}{R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}\right)$$



# Gravitational Wave Definition

**Then:**

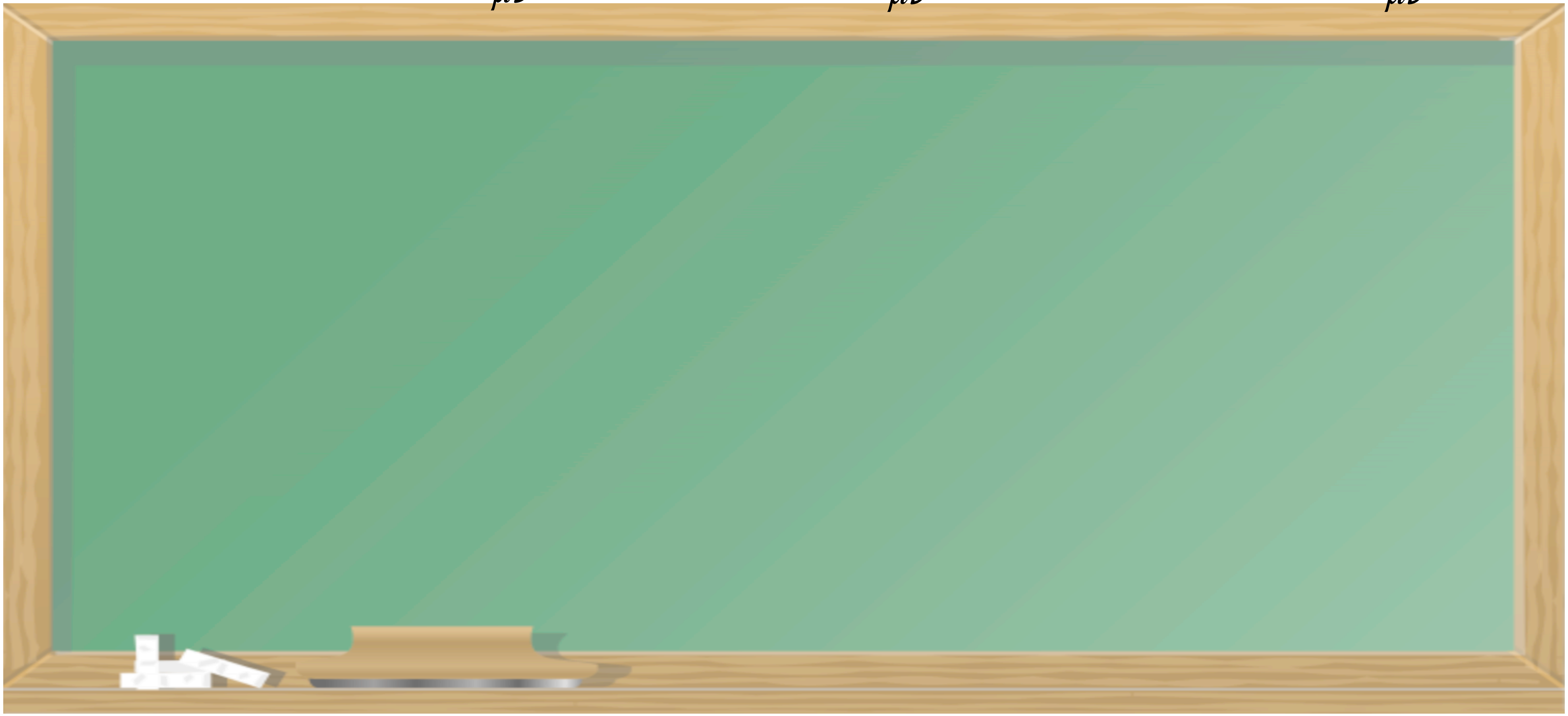
$$\tilde{R}_{\mu\nu} \equiv \underbrace{(\mathscr{D}_{\mu\nu}\omega)^{(0)}}_{\mathcal{O}(h^0_{**})} + \underbrace{\left(\overset{(1)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(1)}\right)}_{\mathcal{O}(h_{**})} + \underbrace{\left(\overset{(2)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}\right)}_{\mathcal{O}(h^2_{**})}$$



# Gravitational Wave Definition

**Then:**

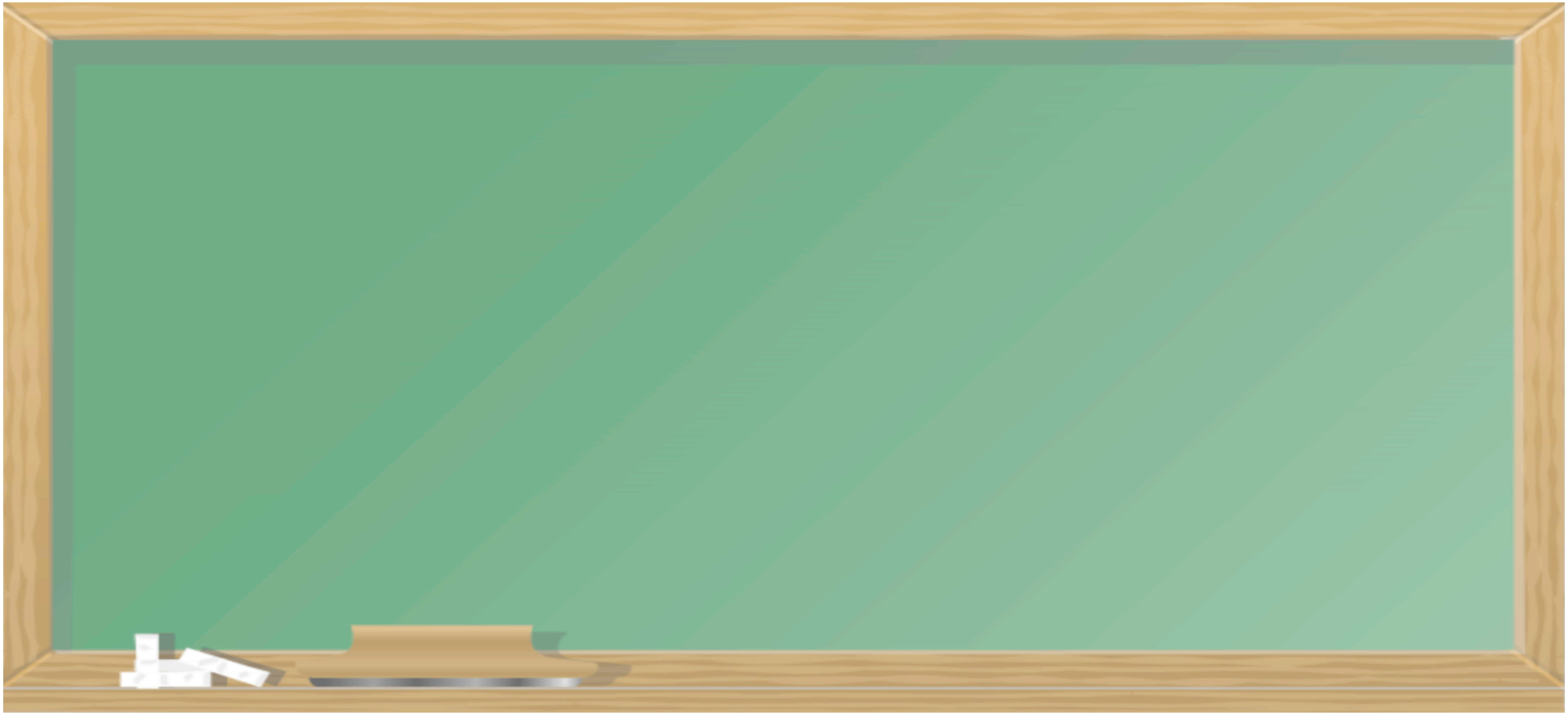
$$\tilde{R}_{\mu\nu} \equiv \underbrace{(\mathscr{D}_{\mu\nu}\omega)^{(0)}}_{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underbrace{\left(\delta\overset{(1)}{R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(1)}\right)}_{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underbrace{\left(\delta\overset{(2)}{R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}\right)}_{\overset{(2)}{\tilde{R}}_{\mu\nu}}$$



# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

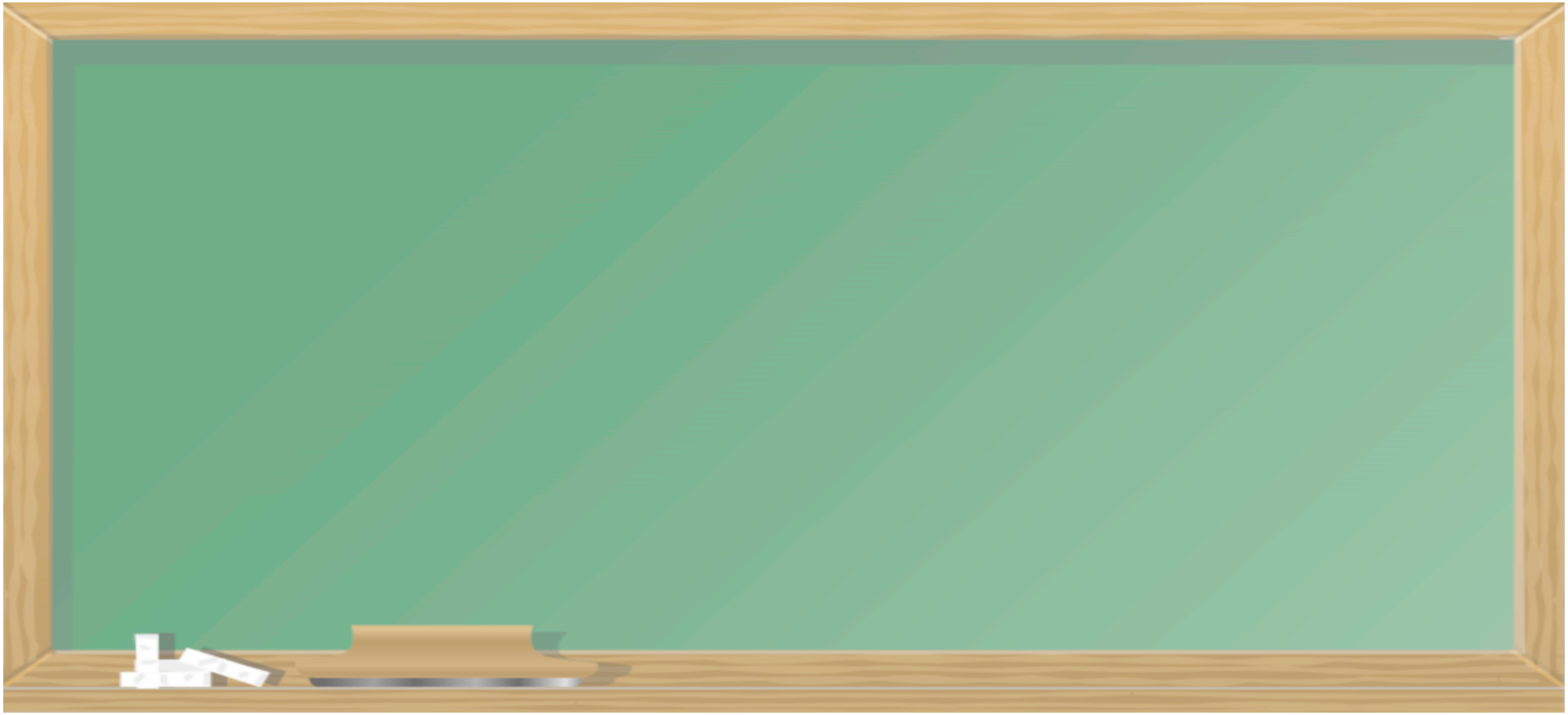
**Up to here, valid for all perturbations (s,v,t)**



# Gravitational Wave Definition

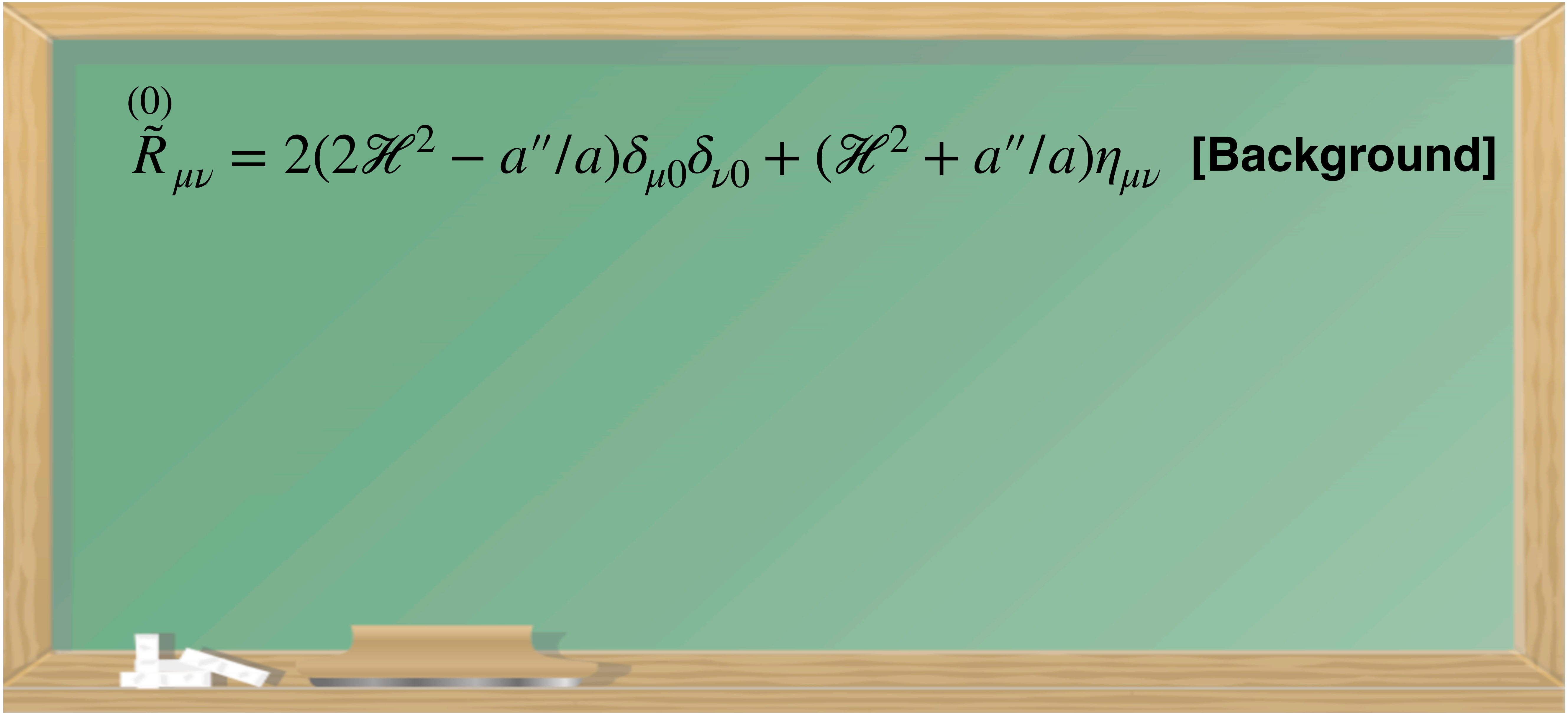
**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

But let's keep now only TT-part of perturbations …



# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$



$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[ specialised now to TT parts … ]**

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu0}\delta_{\nu0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left( -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H}h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[ specialised now to TT parts … ]**

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left( -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

$$\overset{(2)}{\tilde{R}}_{\mu\nu} = -\frac{1}{2}\mathscr{H}\eta_{\mu\nu}h^{\alpha\beta}h'_{\alpha\beta} + \underbrace{\overset{(2)}{\delta R}_{\mu\nu}}_{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**Let's forget for the moment of second order parts …**

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

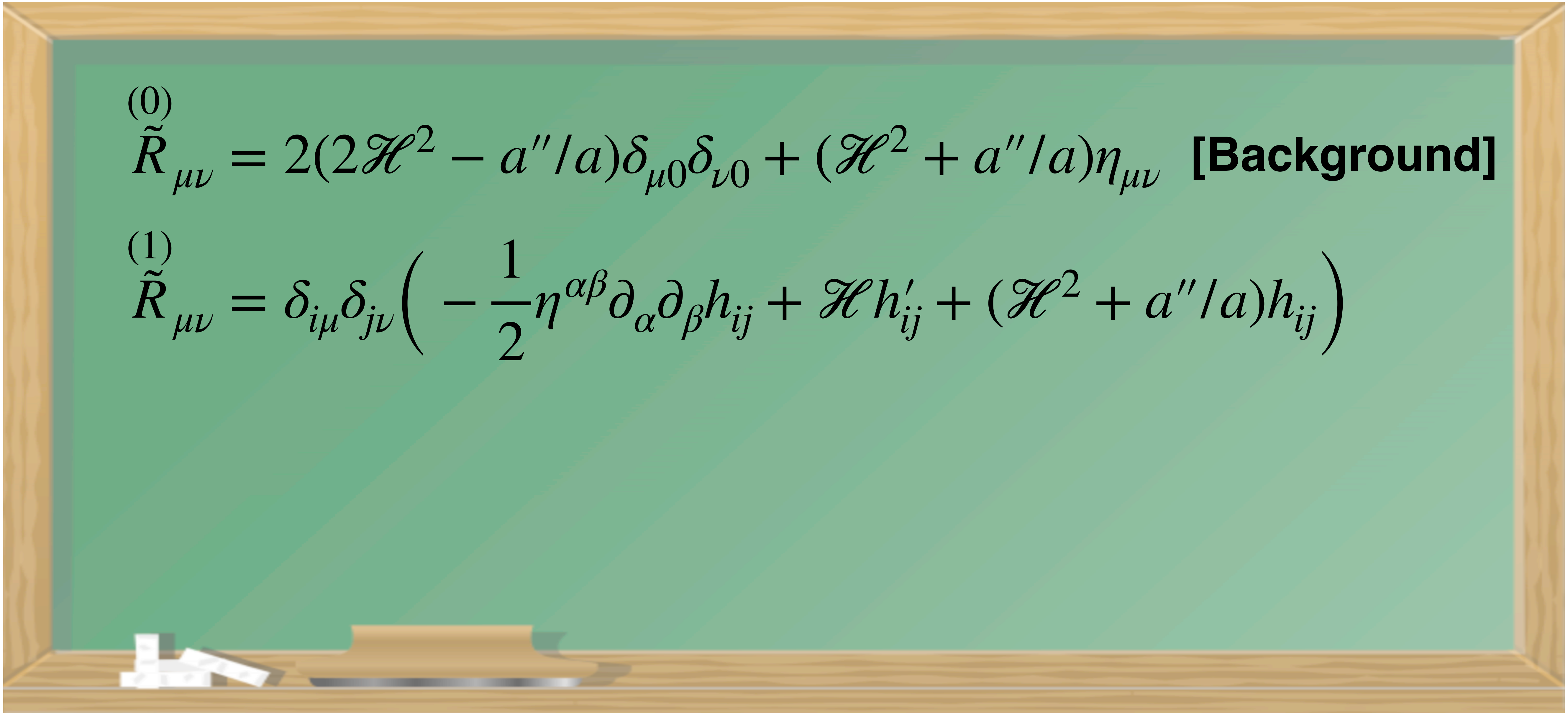
$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\Big( -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \Big)$$

$$\overset{(2)}{\tilde{R}}_{\mu\nu} = -\frac{1}{2}\mathscr{H}\eta_{\mu\nu}h^{\alpha\beta}h'_{\alpha\beta} \underbrace{+ \overset{(2)}{\delta R}_{\mu\nu}}_{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}$

Let's forget for the moment of second order parts …



$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij}\right)$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}$

**Let's focus on the Einstein Equations**

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij}\right)$$

# Gravitational Wave Definition

**Then:** $$\tilde R_{\mu\nu} \equiv \overset{(0)}{\tilde R}_{\mu\nu} + \overset{(1)}{\tilde R}_{\mu\nu} \;;\qquad m_p^2 \tilde R_{\mu\nu} = S_{\mu\nu}$$

$$[\ S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\tilde g_{\mu\nu}\tilde g^{\alpha\beta}T_{\alpha\beta}\ ]$$

$$\overset{(0)}{\tilde R}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu0}\delta_{\nu0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde R}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left( -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H}h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \quad ; \quad m_p^2\tilde{R}_{\mu\nu} = S_{\mu\nu}$$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$

**[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H}h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij}\right)$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$ ; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left( -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \quad ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left( -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \quad ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$S_{\mu\nu} = \underbrace{(\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)\tilde{g}_{\mu\nu}}_{\textbf{Perfect fluid}} + \underbrace{\Pi_{ij}}_{\textbf{Anisotropic Stress}} \quad ; \quad u_\mu \equiv (a,0,0,0)$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \Rightarrow \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

Perturbation

$$S_{\mu\nu} = \underbrace{(\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)\tilde{g}_{\mu\nu}}_{\textbf{Perfect fluid}} + \underbrace{\Pi_{ij}}_{\textbf{Anisotropic Stress}} \quad ; \quad u_\mu \equiv (a,0,0,0)$$

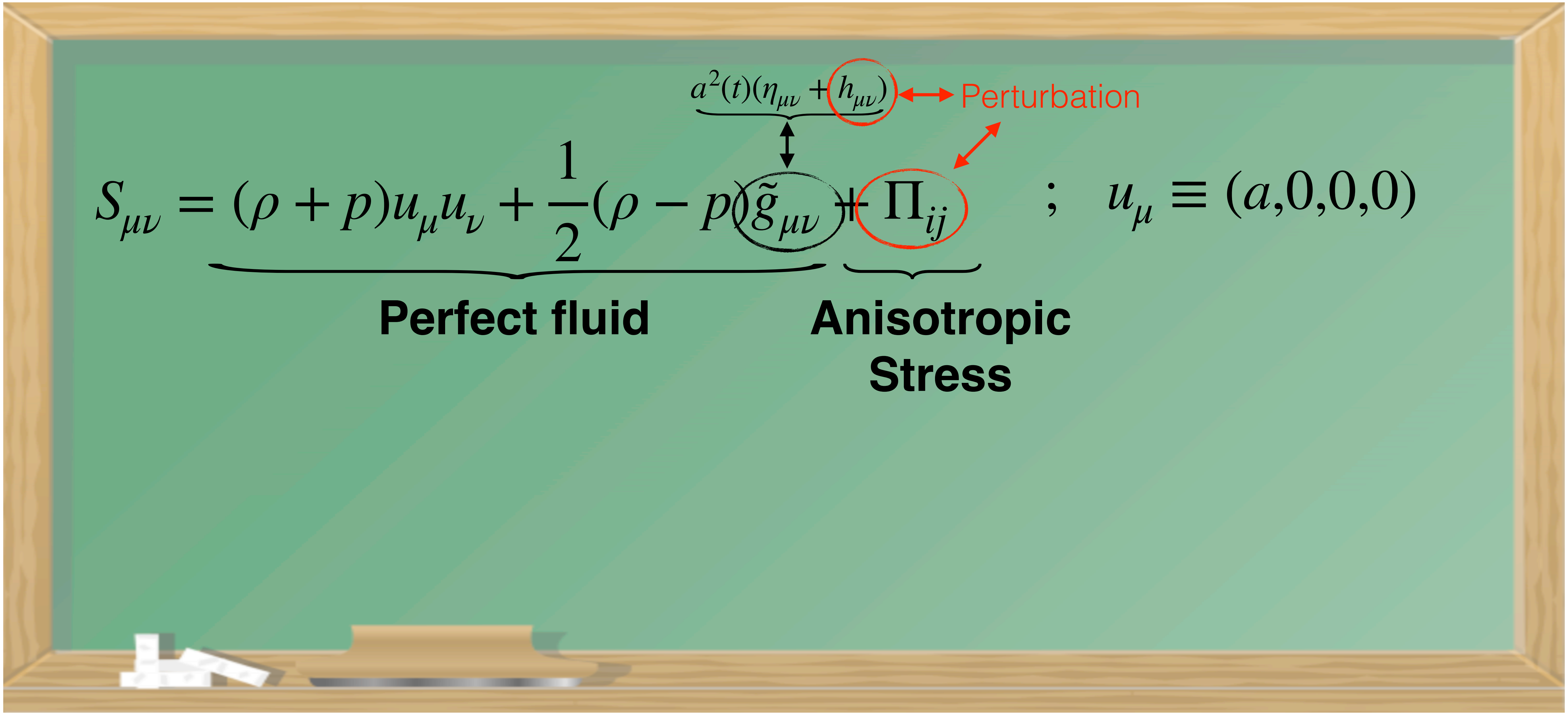
# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \quad ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$S_{\mu\nu} = \underbrace{(\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)\overset{\overbrace{a^2(t)(\eta_{\mu\nu} + h_{\mu\nu})}}{\overset{\updownarrow}{\tilde{g}_{\mu\nu}}}}_{\textbf{Perfect fluid}} + \underbrace{\overset{\text{Perturbation}}{\Pi_{ij}}}_{\textbf{Anisotropic Stress}} \quad ; \quad u_\mu \equiv (a,0,0,0)$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$ ; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$



$$S_{\mu\nu} = \underbrace{(\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)\overset{\underbrace{a^2(t)(\eta_{\mu\nu} + \boxed{h_{\mu\nu}})}}{\boxed{\tilde{g}_{\mu\nu}}}}_{\textbf{Perfect fluid}} + \underbrace{\boxed{\Pi_{ij}}}_{\textbf{Anisotropic Stress}} \quad ; \quad u_\mu \equiv (a,0,0,0)$$

Perturbation

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \quad ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$\underbrace{a^2(t)(\eta_{\mu\nu} + h_{\mu\nu})}$$

↔ Perturbation

$$S_{\mu\nu} = (\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)\tilde{g}_{\mu\nu} + \Pi_{ij} \quad ; \quad u_\mu \equiv (a, 0, 0, 0)$$

$$= \underbrace{(\rho + p)a^2\delta_{\mu 0}\delta_{\mu 0} + \frac{1}{2}(\rho - p)a^2\eta_{\mu\nu}} + \underbrace{\frac{1}{2}(\rho - p)a^2 h_{\mu\nu} + \Pi_{ij}}$$

$$= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \Rightarrow \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$\overbrace{a^2(t)(\eta_{\mu\nu} + h_{\mu\nu})} \leftrightarrow \text{Perturbation}$$

$$S_{\mu\nu} = (\rho + p) u_\mu u_\nu + \frac{1}{2}(\rho - p) \tilde{g}_{\mu\nu} + \Pi_{ij} \quad ; \quad u_\mu \equiv (a,0,0,0)$$

$$= \underbrace{(\rho + p) a^2 \delta_{\mu 0} \delta_{\mu 0} + \frac{1}{2}(\rho - p) a^2 \eta_{\mu\nu}}_{\overset{(0)}{S}_{\mu\nu}} + \underbrace{\frac{1}{2}(\rho - p) a^2 h_{\mu\nu} + \Pi_{ij}^{(\mathrm{T})}}_{\overset{(1)}{S}_{\mu\nu}}$$

$$[\ \Pi_{ij} = \Pi_{ij}^{(\mathrm{S})} + \Pi_{ij}^{(\mathrm{V})} + \Pi_{ij}^{(\mathrm{T})}\ ]$$

# Gravitational Wave Definition

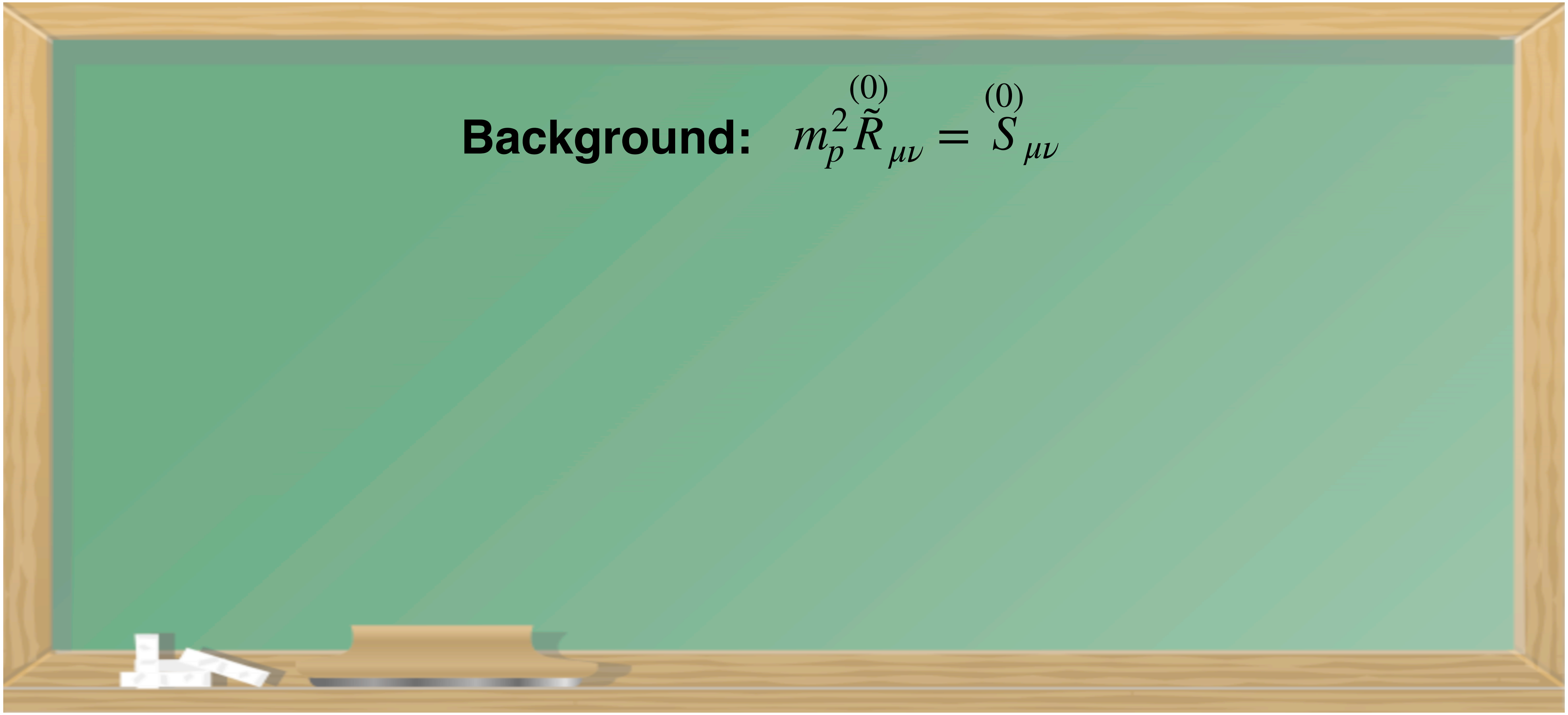
**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$m_p^2\left( \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} \right) = \underbrace{(\rho + p)a^2\delta_{\mu 0}\delta_{\mu 0} + \frac{1}{2}(\rho - p)a^2\eta_{\mu\nu}}_{\overset{(0)}{S}_{\mu\nu}} + \underbrace{\frac{1}{2}(\rho - p)a^2 h_{\mu\nu} + \Pi_{ij}^{(T)}}_{\overset{(1)}{S}_{\mu\nu}}$$

# Gravitational Wave Definition

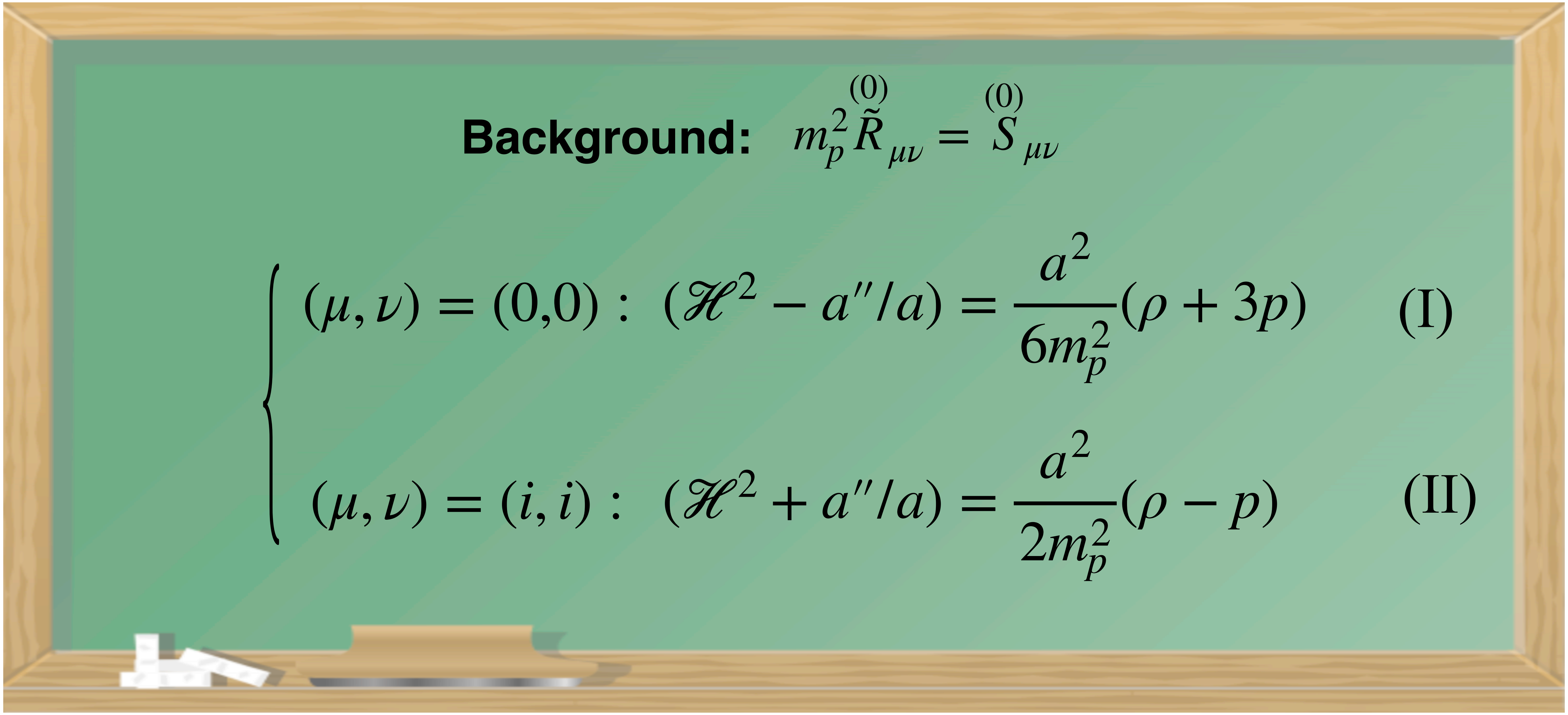
**Then:** $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \Rightarrow \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$



**Background:** $m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu}$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$ ; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$



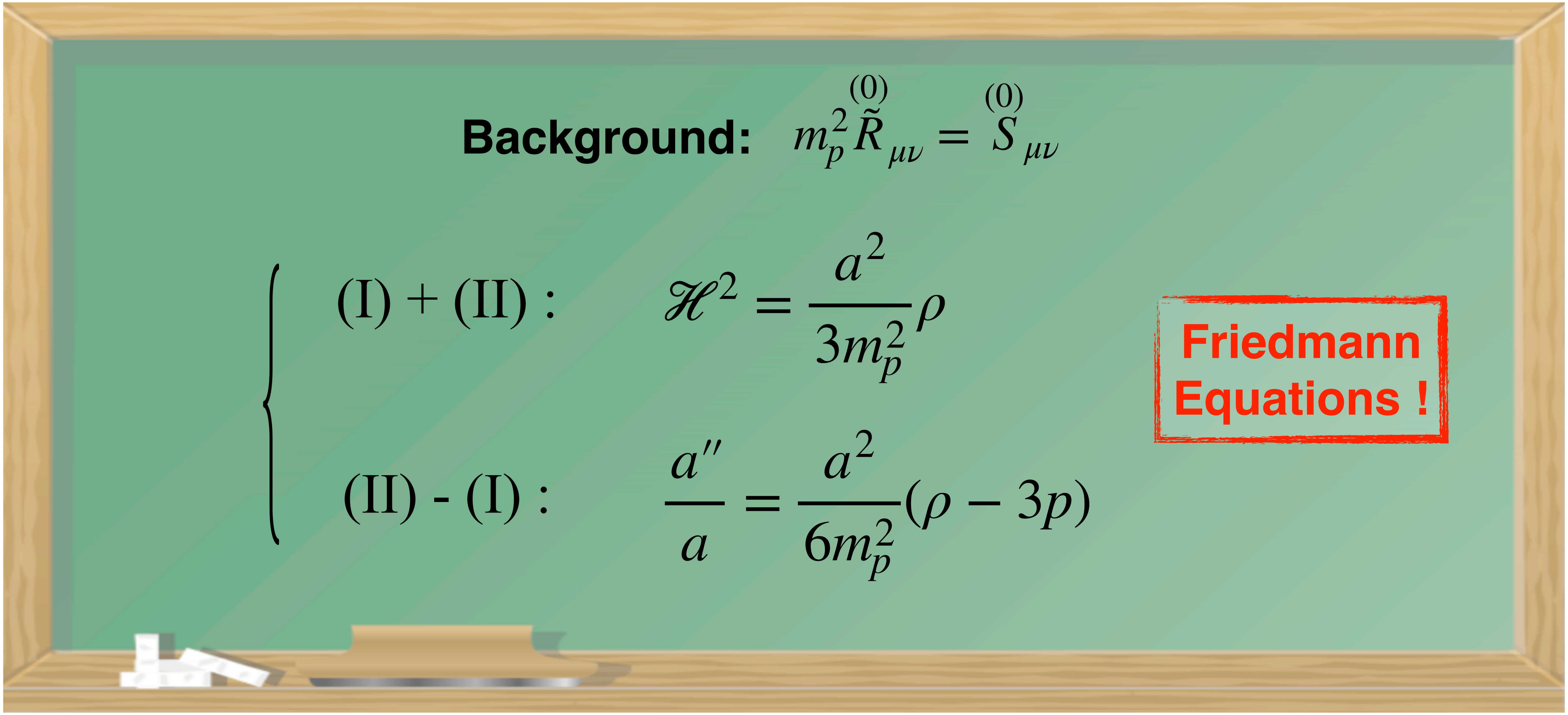
**Background:** $m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu}$

$$\begin{cases} (\mu,\nu) = (0,0): \ (\mathscr{H}^2 - a''/a) = \dfrac{a^2}{6m_p^2}(\rho + 3p) & \text{(I)} \\ (\mu,\nu) = (i,i): \ (\mathscr{H}^2 + a''/a) = \dfrac{a^2}{2m_p^2}(\rho - p) & \text{(II)} \end{cases}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \;;\qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \;\Longrightarrow\; \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$



**Background:** $m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu}$

$$\begin{cases} \text{(I) + (II) :} \qquad \mathscr{H}^2 = \dfrac{a^2}{3m_p^2}\rho \\ \\ \text{(II) - (I) :} \qquad \dfrac{a''}{a} = \dfrac{a^2}{6m_p^2}(\rho - 3p) \end{cases}$$

$$\boxed{\textbf{Friedmann Equations !}}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \Rightarrow \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

**First Order:** $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}\ ; \qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

**First Order:** $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij} + 2(\mathscr{H}^2 + a''/a)h_{ij} = \frac{2}{m_p^2}\Pi^{(\mathrm{T})}_{ij} + \frac{a^2(\rho - p)}{m_p^2} h_{ij}$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$ ; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

**First Order:** $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\underbrace{h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij}}_{\text{wave operator}} + \underbrace{2(\mathscr{H}^2 + a''/a)h_{ij}}_{\text{mass term?}} = \frac{2}{m_p^2}\Pi^{(\mathrm{T})}_{ij} + \frac{a^2(\rho - p)}{m_p^2} h_{ij}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \Longrightarrow \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

**First Order:** $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\underbrace{h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij}}_{\text{wave operator}} + \underbrace{\cancel{2(\mathscr{H}^2 + a''/a)h_{ij}}}_{\text{mass term?}} = \frac{2}{m_p^2}\Pi_{ij}^{(\mathrm{T})} + \cancel{\frac{a^2(\rho - p)}{m_p^2}} h_{ij}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

**First Order:** $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\boxed{h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij} = \frac{2}{m_p^2} \Pi^{(\mathrm{T})}_{ij}}$$

**Grav. Wave Eq. of motion**

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \;;\qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

**First Order:** $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\boxed{h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij} = \frac{2}{m_p^2}\Pi_{ij}^{(\mathrm{T})}}$$

**Grav. Wave Eq. of motion**

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

**First Order:** $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\boxed{h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij} = \frac{2}{m_p^2} \Pi^{(\mathrm{T})}_{ij}}$$

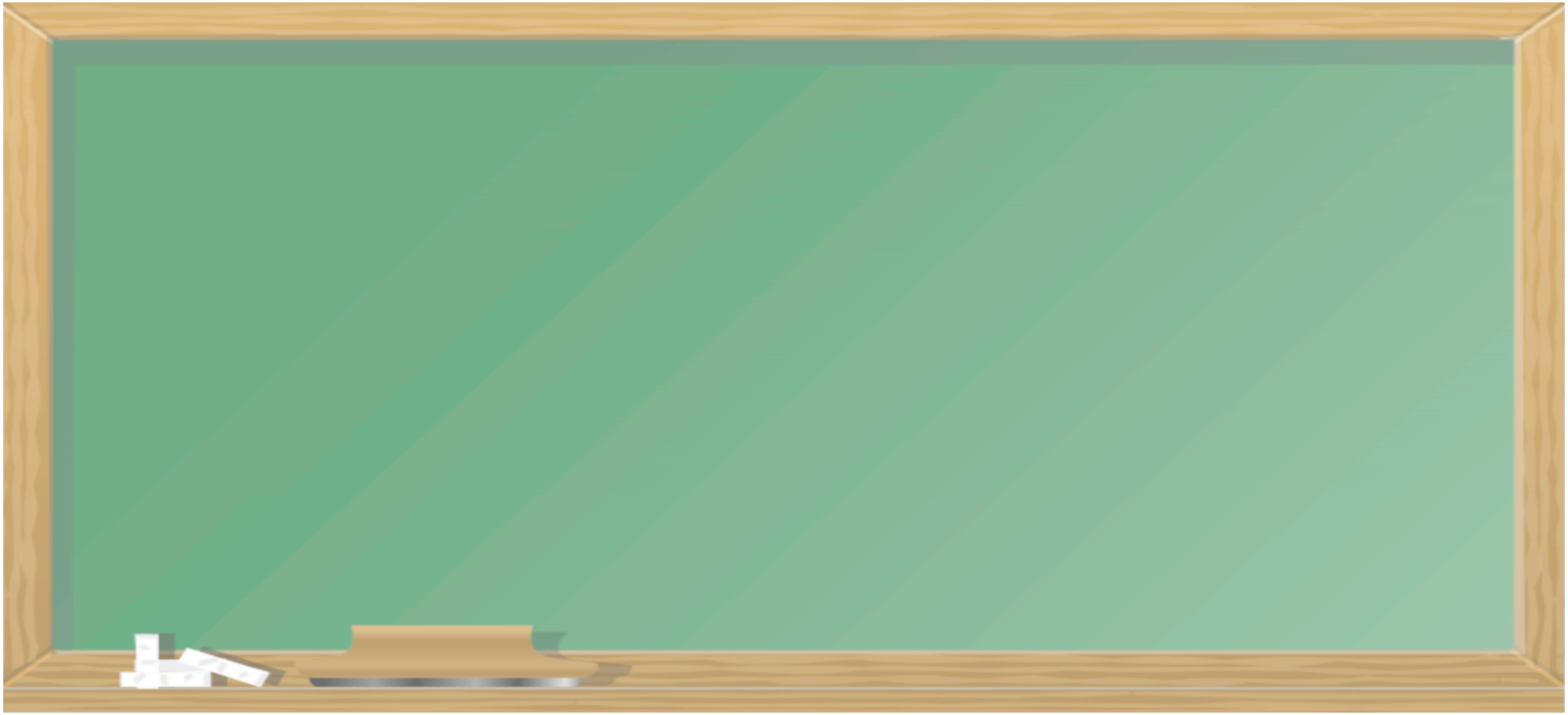
**Grav. Wave Eq. of motion**

**Friction**

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]**



# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** **?**

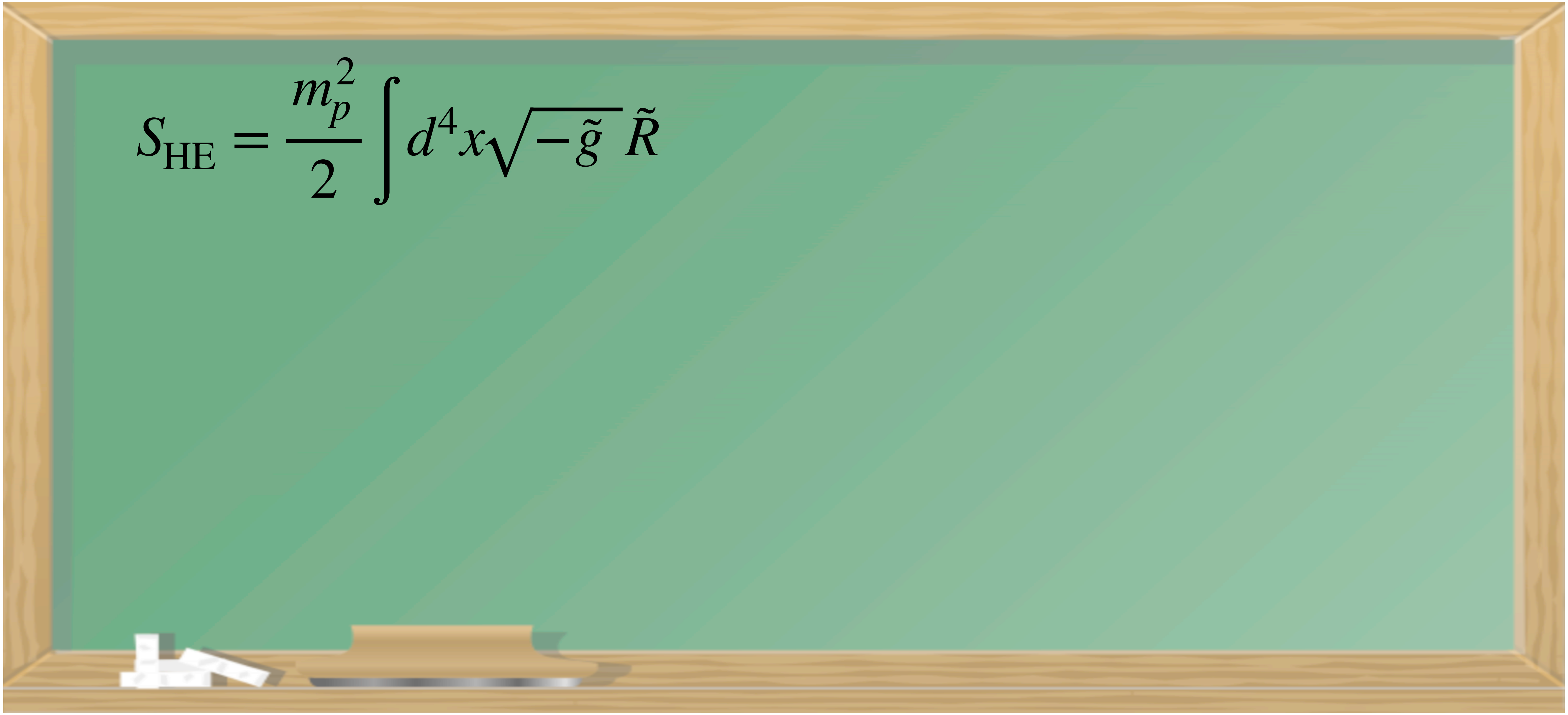
$$\overset{(2)}{\tilde{R}}_{\mu\nu} = -\frac{1}{2}\mathscr{H}\eta_{\mu\nu}h^{\alpha\beta}h'_{\alpha\beta} + \overset{(2)}{\delta R}_{\mu\nu} \quad ; \quad \overset{(2)}{\delta R}_{\mu\nu} \equiv \partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}$$

$$\begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\frac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\frac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underset{\text{[Friedmann Equations]}}{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underset{\text{[GW Eq. motion]}}{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underset{\text{?}}{\overset{(2)}{\tilde{R}}_{\mu\nu}}$$



$$S_{\mathrm{HE}} = \frac{m_p^2}{2} \int d^4x \sqrt{-\tilde{g}}\, \tilde{R}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** **?**

$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\sqrt{-\tilde{g}}\;\tilde{R}$$

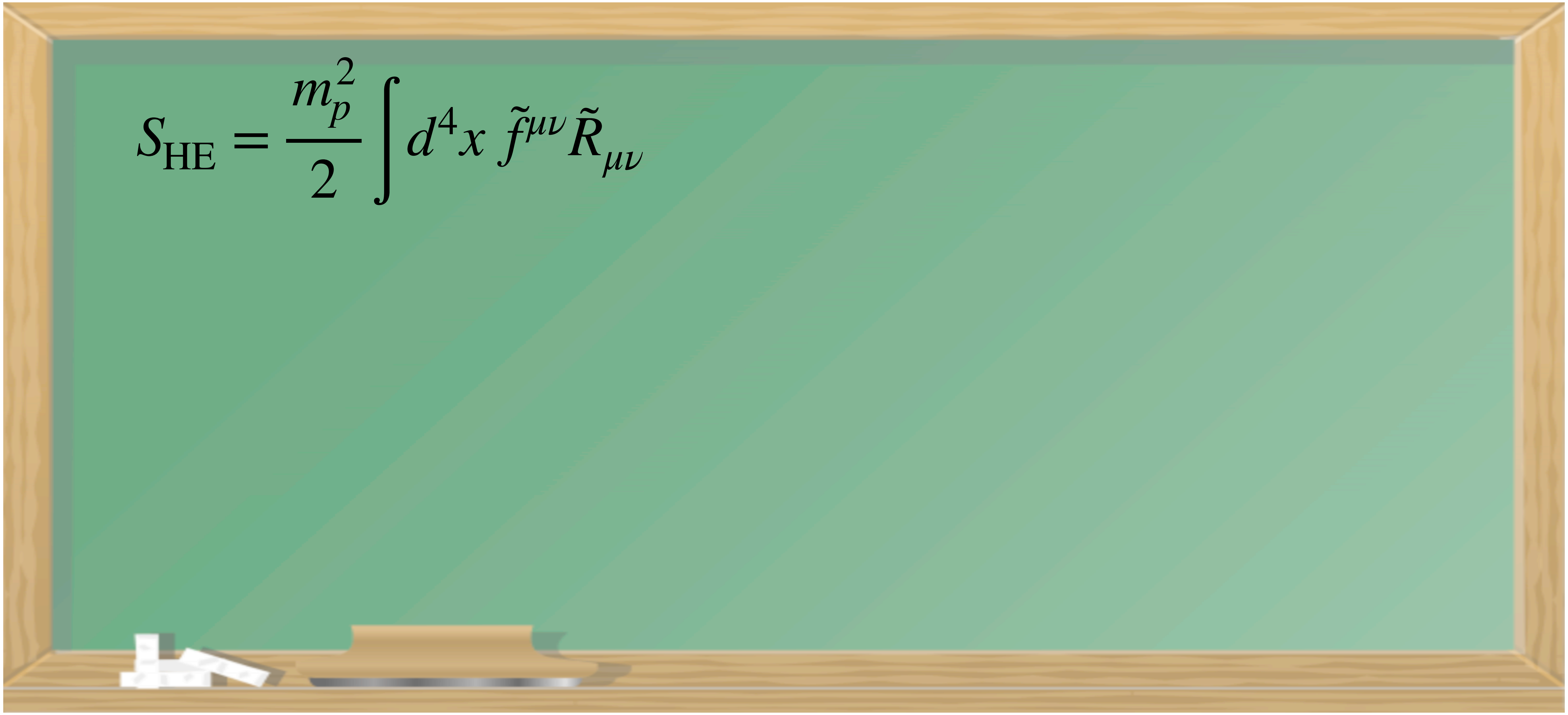
$$= \frac{m_p^2}{2}\int d^4x\sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$= \frac{m_p^2}{2}\int d^4x\;\tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underset{\text{[Friedmann Equations]}}{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underset{\text{[GW Eq. motion]}}{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underset{\textbf{?}}{\overset{(2)}{\tilde{R}}_{\mu\nu}}$$



$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\, \tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

**[Recall: specialised to TT parts only !]**

$$S_{\rm HE} = \frac{m_p^2}{2} \int d^4x \, \tilde{f}^{\mu\nu} \tilde{R}_{\mu\nu}$$

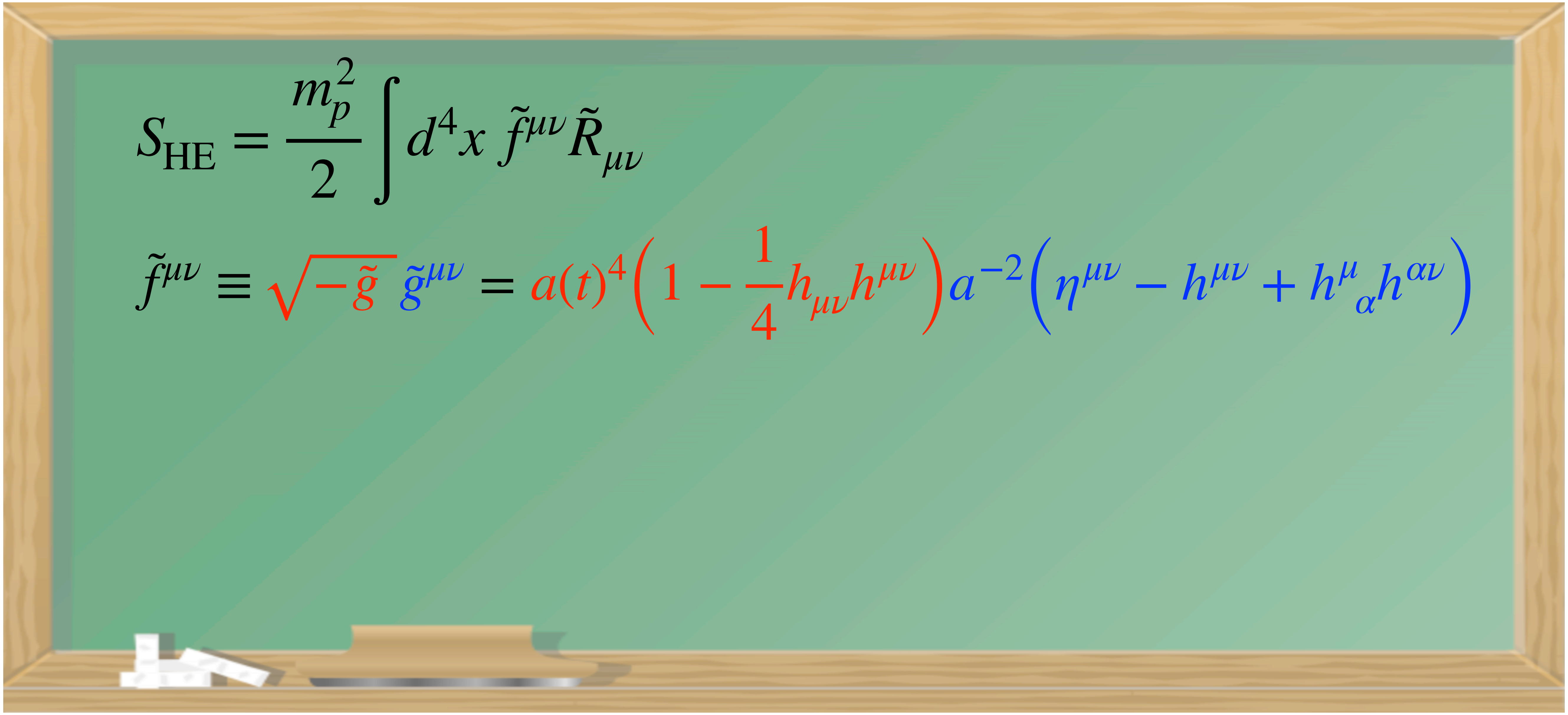
# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**



$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\, \tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$\tilde{f}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\; \tilde{g}^{\mu\nu} = a(t)^4\left(1 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)a^{-2}\left(\eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\alpha}h^{\alpha\nu}\right)$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\, \tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$\tilde{f}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu} = a(t)^4\left(1-\frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)a^{-2}\left(\eta^{\mu\nu}-h^{\mu\nu}+h^{\mu}{}_{\alpha}h^{\alpha\nu}\right)$$

$$= \underbrace{a(t)^2\eta^{\mu\nu}}_{\overset{(0)}{\tilde{f}^{\mu\nu}}} \underbrace{- a(t)^2 h^{\mu\nu}}_{+\overset{(1)}{\tilde{f}^{\mu\nu}}} + \underbrace{h^{\mu}{}_{\alpha}h^{\alpha\nu} - \frac{1}{4}\eta^{\mu\nu}h_{\alpha\beta}h^{\alpha\beta}}_{+\overset{(2)}{\tilde{f}^{\mu\nu}}}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$S_{\text{HE}} = \frac{m_p^2}{2}\int d^4x\, \tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$= \frac{m_p^2}{2}\int d^4x \left(\overset{(0)}{\tilde{f}^{\mu\nu}} + \overset{(1)}{\tilde{f}^{\mu\nu}} + \overset{(2)}{\tilde{f}^{\mu\nu}}\right)\left(\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}\right)$$

$$= \overset{(0)}{S}_{\text{HE}} + \overset{(1)}{S}_{\text{HE}} + \overset{(2)}{S}_{\text{HE}}$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$S_{\text{HE}} = \overset{(0)}{S}_{\text{HE}} + \overset{(1)}{S}_{\text{HE}} + \overset{(2)}{S}_{\text{HE}}$$

$$\overset{(0)}{S}_{\text{HE}} \equiv \frac{m_p^2}{2} \int d^4x \, \overset{(0)}{\tilde{f}^{\mu\nu}} \overset{(0)}{\tilde{R}}_{\mu\nu}$$

$$\overset{(1)}{S}_{\text{HE}} \equiv \frac{m_p^2}{2} \int d^4x \left( \overset{(0)}{\tilde{f}^{\mu\nu}} \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{f}^{\mu\nu}} \overset{(0)}{\tilde{R}}_{\mu\nu} \right)$$

$$\overset{(2)}{S}_{\text{HE}} \equiv \frac{m_p^2}{2} \int d^4x \left( \overset{(0)}{\tilde{f}^{\mu\nu}} \overset{(2)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{f}^{\mu\nu}} \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{f}^{\mu\nu}} \overset{(0)}{\tilde{R}}_{\mu\nu} \right)$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$S_{\rm HE} = \overset{(0)}{S}_{\rm HE} + \overset{(1)}{S}_{\rm HE} + \overset{(2)}{S}_{\rm HE}$$

$$\overset{(0)}{S}_{\rm HE} = 3m_p^2 \int d^4x\ a(t)a''(t)$$

$$\overset{(1)}{S}_{\rm HE} = 0$$

$$\overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H}\, h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$\overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H}\, h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** **?**

$$\overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H} h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

**Consistency check: Find Eq.'s of motion of** $h_{ij}$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \underset{\text{[Friedmann Equations]}}{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underset{\text{[GW Eq. motion]}}{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underset{\textbf{?}}{\overset{(2)}{\tilde{R}}_{\mu\nu}}$$

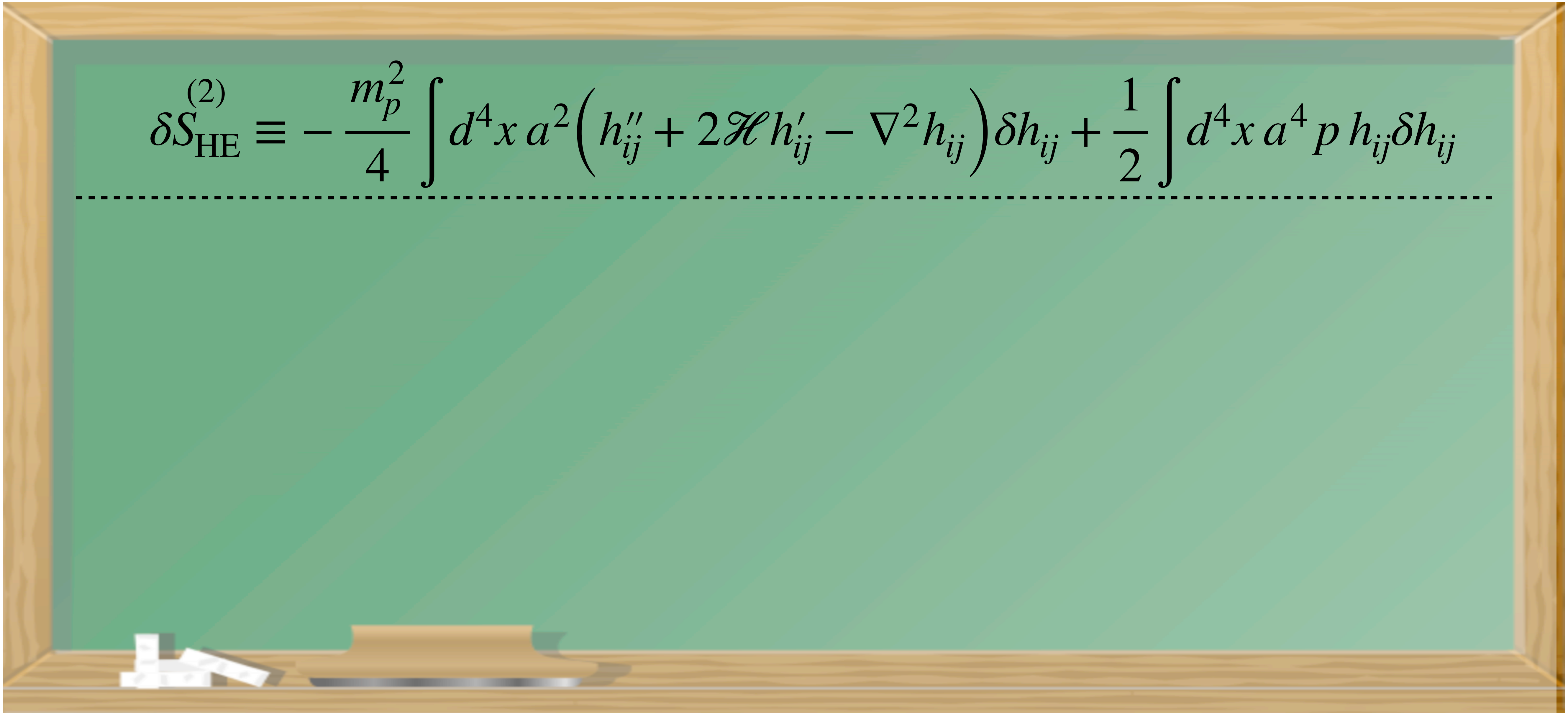
$$\overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H} h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

$$\delta\overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\Big(\underbrace{h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}}_{\text{wave operator}} + \underbrace{2(\mathscr{H}' + a''/a)}_{-\frac{2a^2p}{m_p^2}}h_{ij}\Big)\delta h_{ij}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** **?**



$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H}h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[Friedmann Equations]** **[GW Eq. motion]** **?**

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$S_{\rm m} \equiv \int d^4x \sqrt{-\tilde{g}}\, \mathscr{L}_{\rm m}$$

(**matter sector**)

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** **?**

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

---

$$S_{\rm m} \equiv \int d^4x \sqrt{-\tilde{g}}\, \mathscr{L}_{\rm m} = \overset{(0)}{S}_{\rm m} + \overset{(1)}{S}_{\rm m} + \overset{(2)}{S}_{\rm m} + \ldots$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \underset{\text{[Friedmann Equations]}}{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underset{\text{[GW Eq. motion]}}{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underset{\textbf{?}}{\overset{(2)}{\tilde{R}}_{\mu\nu}}$$

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$S_{\rm m} \equiv \int d^4x\sqrt{-\tilde{g}}\,\mathscr{L}_{\rm m} = \overset{(0)}{S}_{\rm m} + \overset{(1)}{S}_{\rm m} + \overset{(2)}{S}_{\rm m} + \ldots$$

$$\overset{(2)}{S}_{\rm m} \equiv -\frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\,T_{\mu\nu}\,\delta\tilde{g}^{\mu\nu} - \frac{1}{4}\int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\;T_{\mu\nu})}{\delta\tilde{g}^{\alpha\beta}}\right)\delta\tilde{g}^{\mu\nu}\delta\tilde{g}^{\alpha\beta}$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[Friedmann Equations]** **[GW Eq. motion]** **?**

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$S_{\rm m} \equiv \int d^4x\sqrt{-\tilde{g}}\;\mathscr{L}_{\rm m} = \overset{(0)}{S}_{\rm m} + \overset{(1)}{S}_{\rm m} + \overset{(2)}{S}_{\rm m} + \dots$$

$$\overset{(2)}{S}_{\rm m} \equiv -\underbrace{\frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\;T_{\mu\nu}\,\delta\tilde{g}^{\mu\nu}}_{\frac{1}{2}\int d^4x\, a^2(t)\,\Pi^{(\rm T)}_{ij}\,h_{ij}} \underbrace{-\frac{1}{4}\int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\;T_{\mu\nu})}{\delta\tilde{g}^{\alpha\beta}}\right)\delta\tilde{g}^{\mu\nu}\delta\tilde{g}^{\alpha\beta}}_{-\frac{1}{4}\int d^4x\, a^4(t)\, p\, h_{ij}h_{ij}}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$\overset{(2)}{\delta S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$\overset{(2)}{S}_{\mathrm{m}} \equiv \frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(\mathrm{T})}_{ij}\, h_{ij} - \frac{1}{4}\int d^4x\, a^4(t)\, p\, h_{ij}h_{ij}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$\overset{(2)}{\delta S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(\mathrm{T})}_{ij}\, \delta h_{ij} - \frac{1}{2}\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations]

[GW Eq. motion]

?

$$\overset{(2)}{\delta S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H}h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\rm m} = \frac{1}{2}\int d^4x\, a^2(t)\,\Pi^{(\rm T)}_{ij}\,\delta h_{ij} - \frac{1}{2}\cancel{\int d^4x\, a^4(t)\, p\, h_{ij}\,\delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\rm m} + \overset{(2)}{\delta S}_{\rm HE} = 0$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$\overset{(2)}{\delta S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H}h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(\mathrm{T})}_{ij}\, \delta h_{ij} - \frac{1}{2}\cancel{\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} + \overset{(2)}{\delta S}_{\mathrm{HE}} = 0 = \int d^4x\, a^2\left[-\frac{m_p^2}{4}\left(h''_{ij} + 2\mathscr{H}h'_{ij} - \nabla^2 h_{ij}\right) + \frac{1}{2}\Pi^{(\mathrm{T})}_{ij}\right]\delta h_{ij}$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**

$$\overset{(2)}{\delta S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

---

$$\overset{(2)}{\delta S}_{\mathrm{m}} = \frac{1}{2}\int d^4x\, a^2(t)\,\Pi^{(\mathrm{T})}_{ij}\,\delta h_{ij} - \frac{1}{2}\cancel{\int d^4x\, a^4(t)\, p\, h_{ij}\,\delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} + \overset{(2)}{\delta S}_{\mathrm{HE}} = -\frac{m_p^2}{4}\int d^4x\, a^2\Big(\underbrace{h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}}_{\text{wave operator}} - \underbrace{\frac{2}{m_p^2}\Pi^{(\mathrm{T})}_{ij}}_{\text{Source}}\Big)\delta h_{ij} = 0$$

# Gravitational Wave Definition

**Then:**

$$\tilde{R}_{\mu\nu} \equiv \underset{\text{[Friedmann Equations]}}{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underset{\text{[GW Eq. motion]}}{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underset{\text{?}}{\overset{(2)}{\tilde{R}}_{\mu\nu}}$$

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

$$\delta \overset{(2)}{S}_{\rm m} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(T)}_{ij}\, \delta h_{ij} - \frac{1}{2}\cancel{\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}}$$

$$\delta \overset{(2)}{S}_{\rm m} + \delta \overset{(2)}{S}_{\rm HE} = -\frac{m_p^2}{4}\int d^4x\, a^2 \underbrace{\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij} - \frac{2}{m_p^2}\Pi^{(T)}_{ij}\right)}_{\text{Correct Eq. of motion !}}\delta h_{ij} = 0$$

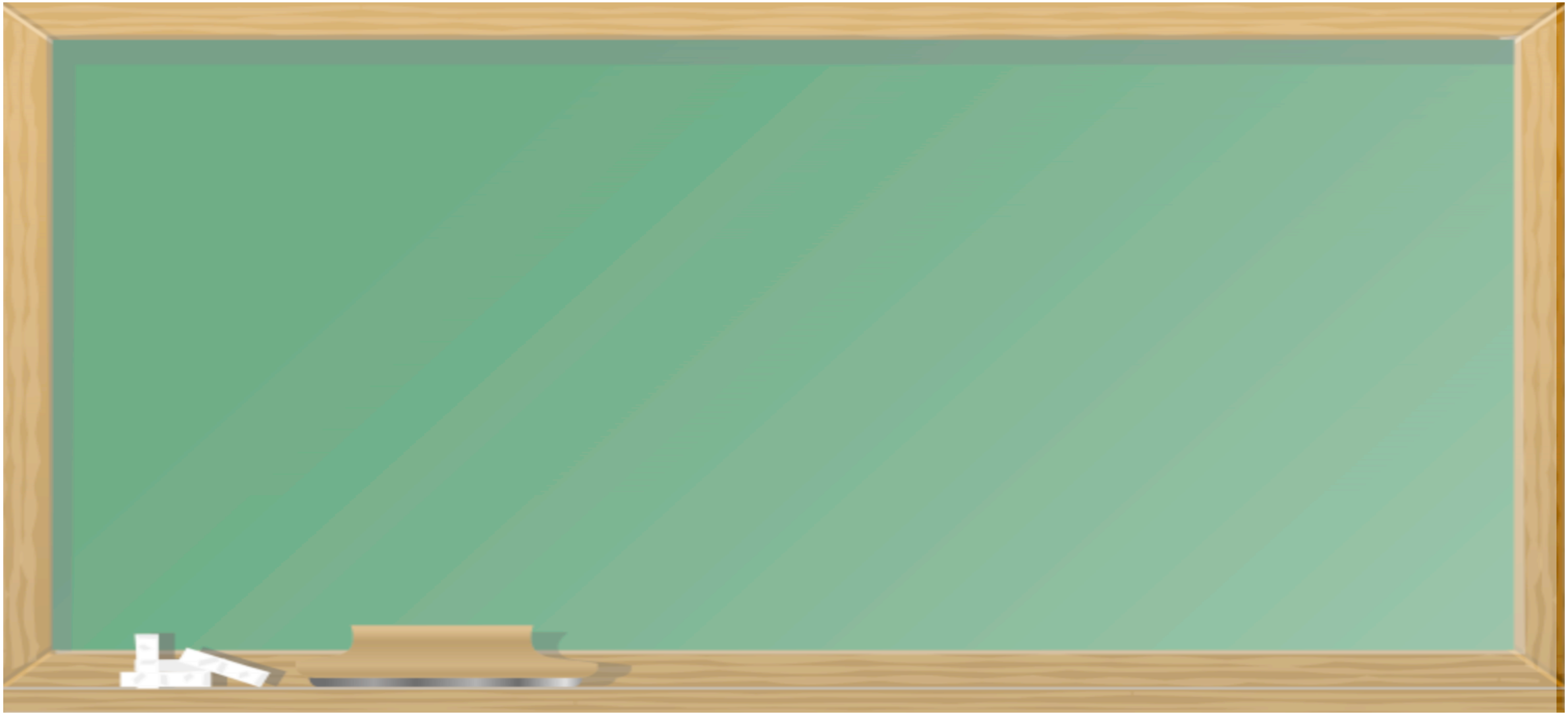
# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

**?**



# Gravitational Wave Definition

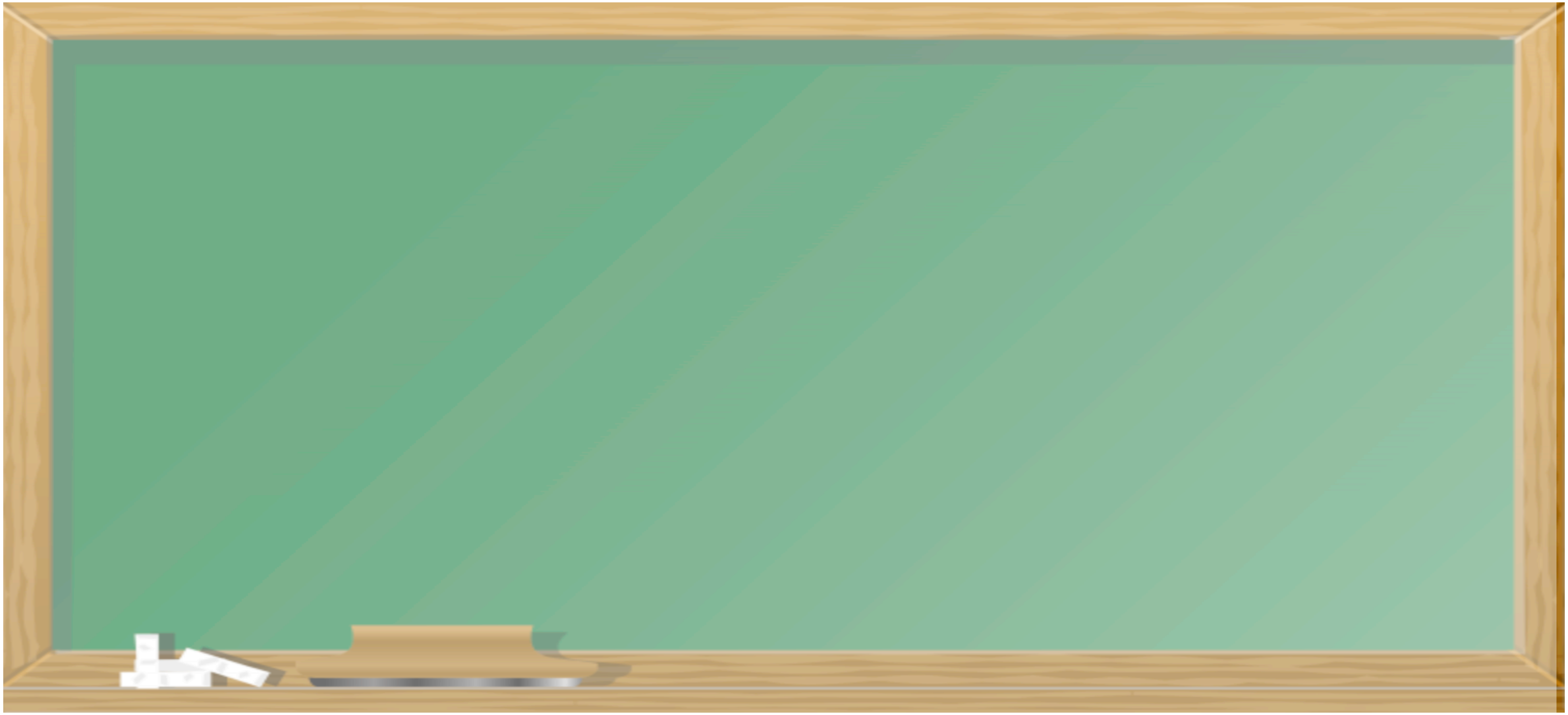
**Then:**

$$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

$\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**



# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

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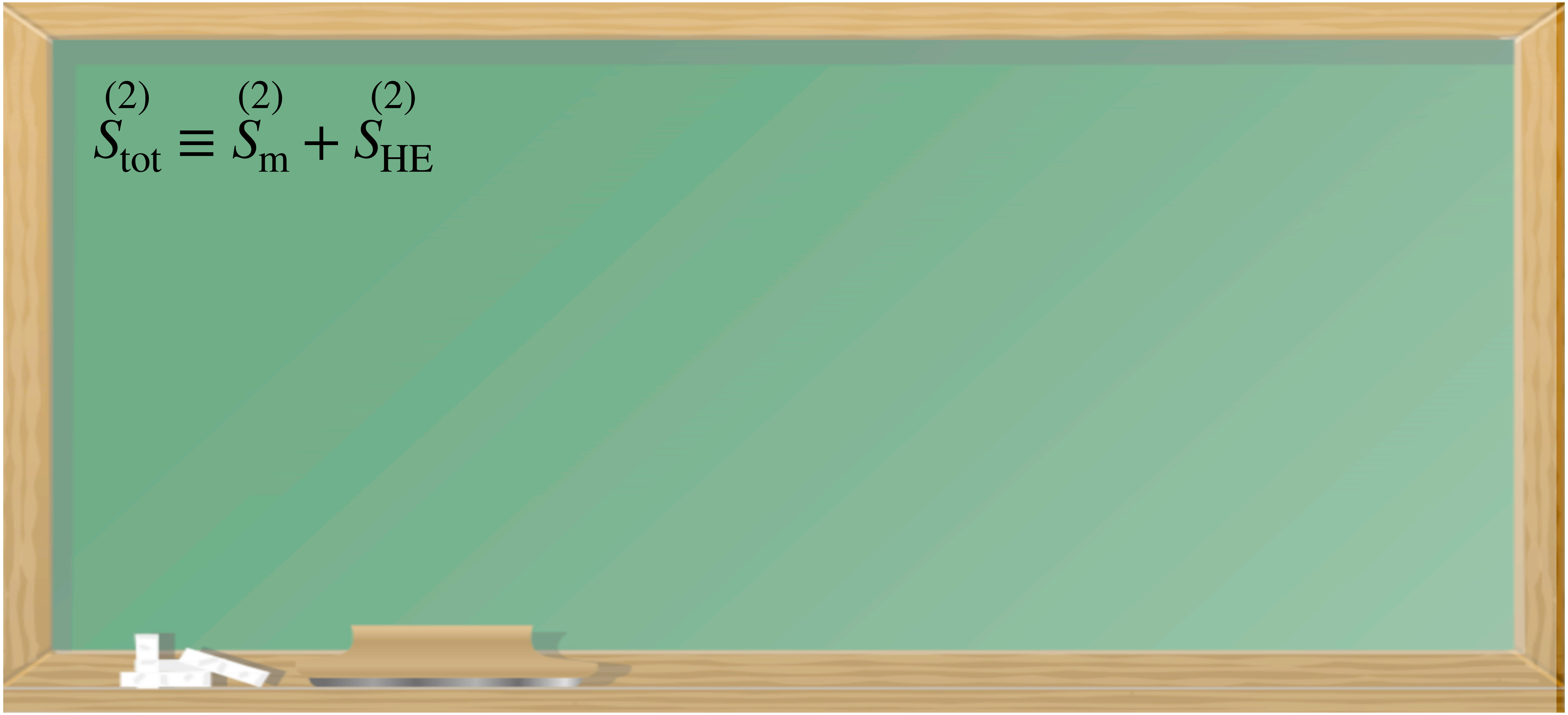
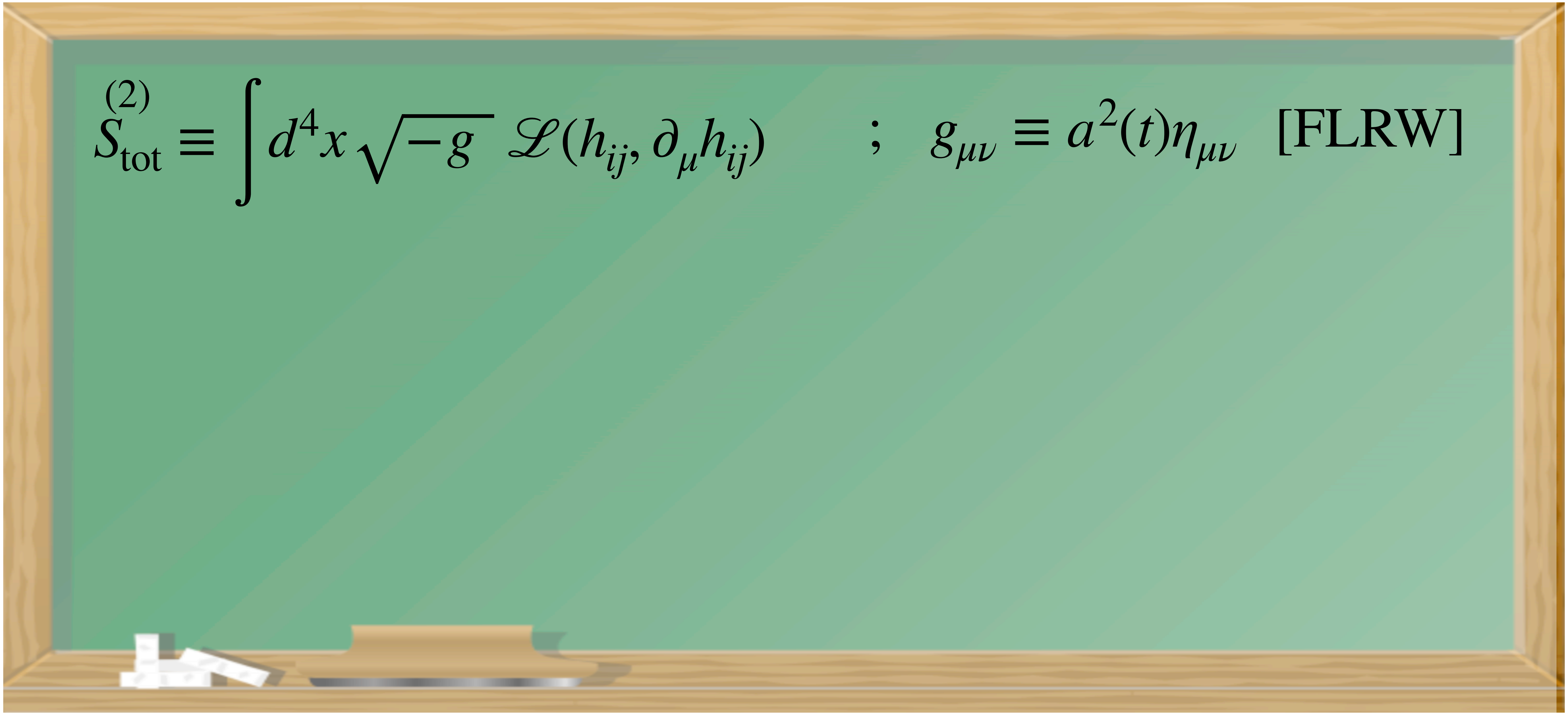


$$\overset{(2)}{S}_{\rm tot} \equiv \overset{(2)}{S}_{\rm m} + \overset{(2)}{S}_{\rm HE}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**



$$\overset{(2)}{S}_{\rm tot} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

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**Noether's Theorem:** $$T_{\mu\nu} \equiv -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L}$$

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

**[Friedmann Equations]** **[GW Eq. motion]** $\mathcal{O}(h^2_{**}) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\rm tot} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

**Noether's Theorem:** $T_{\mu\nu} \equiv -\dfrac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} + \partial_\lambda f^{\lambda\mu\nu}$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** $\mathcal{O}(h^2_{**}) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\ \mathcal{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

**Noether's Theorem:** $$T_{\mu\nu} \equiv \left\langle -\frac{\partial \mathcal{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathcal{L} + \partial_\lambda f^{\lambda\mu\nu} \right\rangle$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

$\mathcal{O}(h^2_{**}) \longrightarrow$ **GW's Energy-momentum ?**

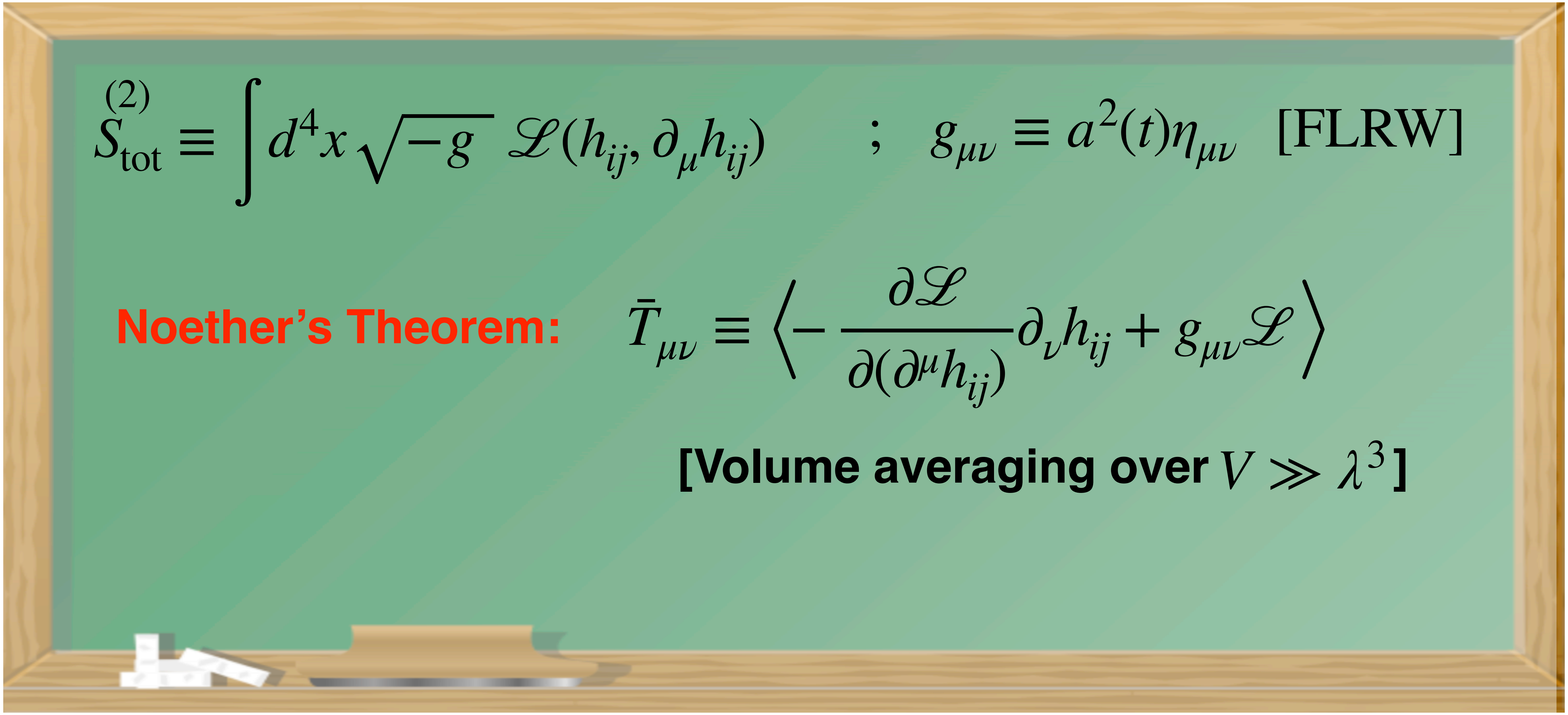
$$\overset{(2)}{S}_{\rm tot} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

**Noether's Theorem:** $$T_{\mu\nu} \equiv \left\langle -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} \;+\; \cancel{\partial_\lambda f^{\lambda\mu\nu}} \right\rangle$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**



$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

**Noether's Theorem:** $$\bar{T}_{\mu\nu} \equiv \left\langle -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} \right\rangle$$

**[Volume averaging over $V \gg \lambda^3$]**

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]**

**[GW Eq. motion]**

$\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

**Noether's Theorem:** $$\bar{T}_{\mu\nu} \equiv \left\langle -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} \right\rangle$$

$$\rho_{\text{GW}} = a^{-2}\bar{T}_{00} \equiv \left\langle \frac{m_p^2}{4}\left( \frac{1}{2a^2}(h'_{ij})^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + 4\mathscr{H} h_{ij}h'_{ij} \right) - \frac{1}{2a^2}\Pi^{(\text{T})}_{ij} h_{ij} \right\rangle$$

# Gravitational Wave Definition

**Then:** $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

**[Friedmann Equations]** **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

**Noether's Theorem:** $$\bar{T}_{\mu\nu} \equiv \left\langle -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} \right\rangle$$

$$\rho_{\text{GW}} = a^{-2}\bar{T}_{00} \equiv \left\langle \frac{m_p^2}{4}\left( \frac{1}{2a^2}(h_{ij}')^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + 4\mathscr{H} h_{ij}h_{ij}' \right) - \frac{1}{2a^2}\Pi_{ij}^{(\text{T})}h_{ij} \right\rangle$$

**Kinetic** **Gradient** **Cross** **Interaction**

# Gravitational Wave Definition

**Then:** $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

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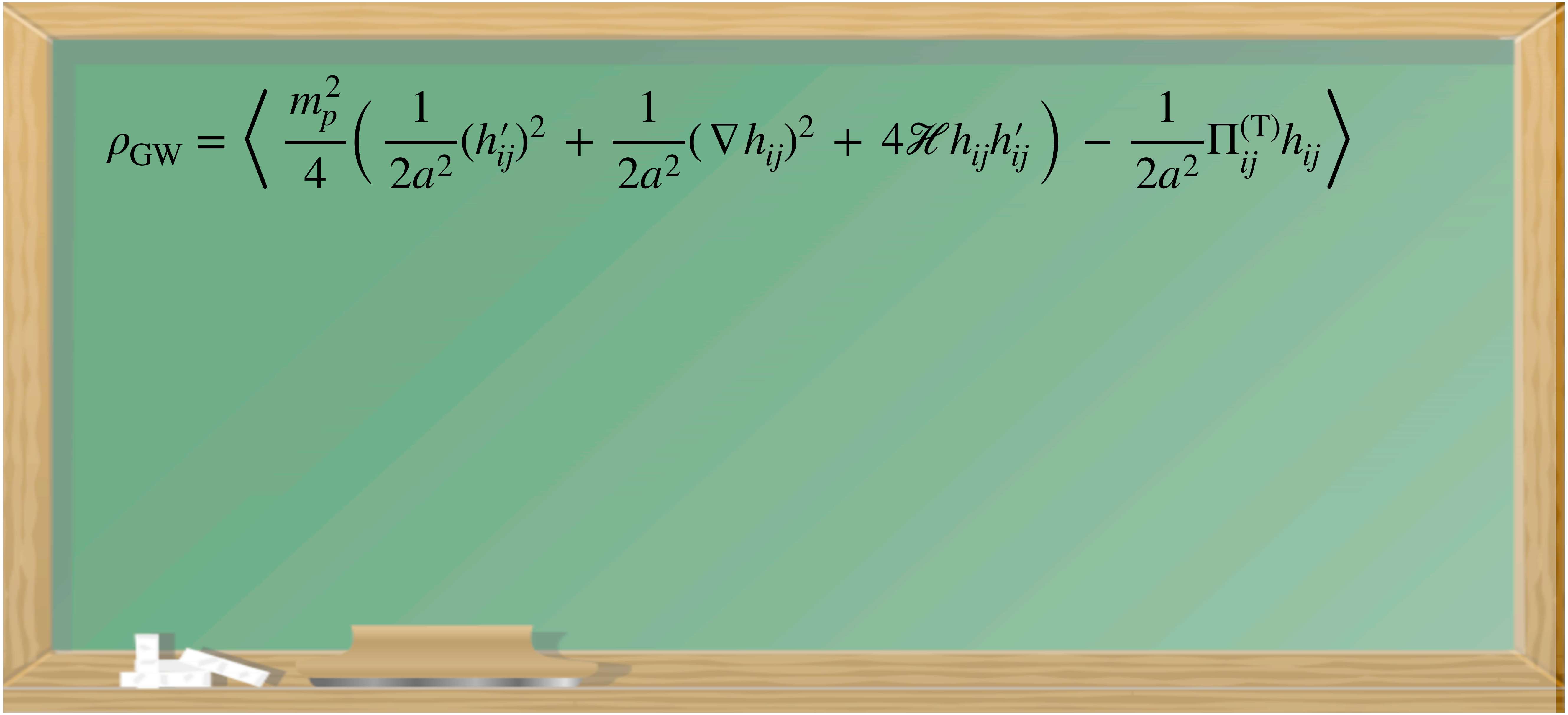

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$$\sim k^2h^2 \gg \sim \mathscr{H}\omega h$$

**Sub-horizon :**

$(k \gg \mathscr{H})$

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(after emission)

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$$\boxed{\rho_{\rm GW} = \frac{m_p^2}{4a^2}\left\langle (h_{ij}')^2 \right\rangle}$$

**Energy density carried by Grav. Waves**

**Sub-horizon** $(k \gg \mathscr{H})$ **&** **Free fields** (after emission)

**[Volume averaging over** $V \gg \lambda^3$ **]**

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**&**

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**[GW Eq. motion]**

**GW energy-momentum over background !** → **How gravity gravitates !**

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(after emission)

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**[GW Eq. motion]**

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$$\boxed{\rho_{\mathrm{GW}} = \frac{1}{32\pi G}\left\langle \dot{h}_{ij}\dot{h}_{ij} \right\rangle_{V\gg\lambda^3}}$$

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**Sub-horizon** $(k \gg \mathscr{H})$ **&** **Free fields** (after emission)

$$\rho_{\rm GW} = \int d\log f \left( \frac{\partial \rho_{\rm GW}}{d\log f} \right)$$

→ **Energy density Spectrum of Gravitational Waves**

# Gravitational Wave Backgrounds

## OUTLINE

✓ 1) Grav. Waves (GWs)

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

6) Astrophysical Background(s)

7) Observational Constraints/Prospects

# Gravitational Wave Backgrounds

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✓ 1) Grav. Waves (GWs)

**Early Universe Sources**

- 2) GWs from Inflation
- 3) GWs from Preheating
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**Core Topics**

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# Gravitational Wave Backgrounds

## OUTLINE

1) Grav. Waves (GWs)

**Early Universe Sources**

2) GWs f

3) G

4) G ns

5) GW mic Defects

**Core Topics**

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# The Gravity of the Situation !

# GW Propagation/Creation in Cosmology

**FLRW:** $ds^2 = a^2(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j), \qquad \mathrm{TT}: \begin{cases} h_{ii} = 0 \\ h_{ij},_j = 0 \end{cases}$

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**Creation of GWs in curved space-time**

**Source: Anisotropic Stress**

Eom: $\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi^{\mathrm{TT}}_{ij},} \qquad \boxed{\Pi_{ij} = T_{ij} - \langle T_{ij} \rangle_{\mathrm{FRW}}}$

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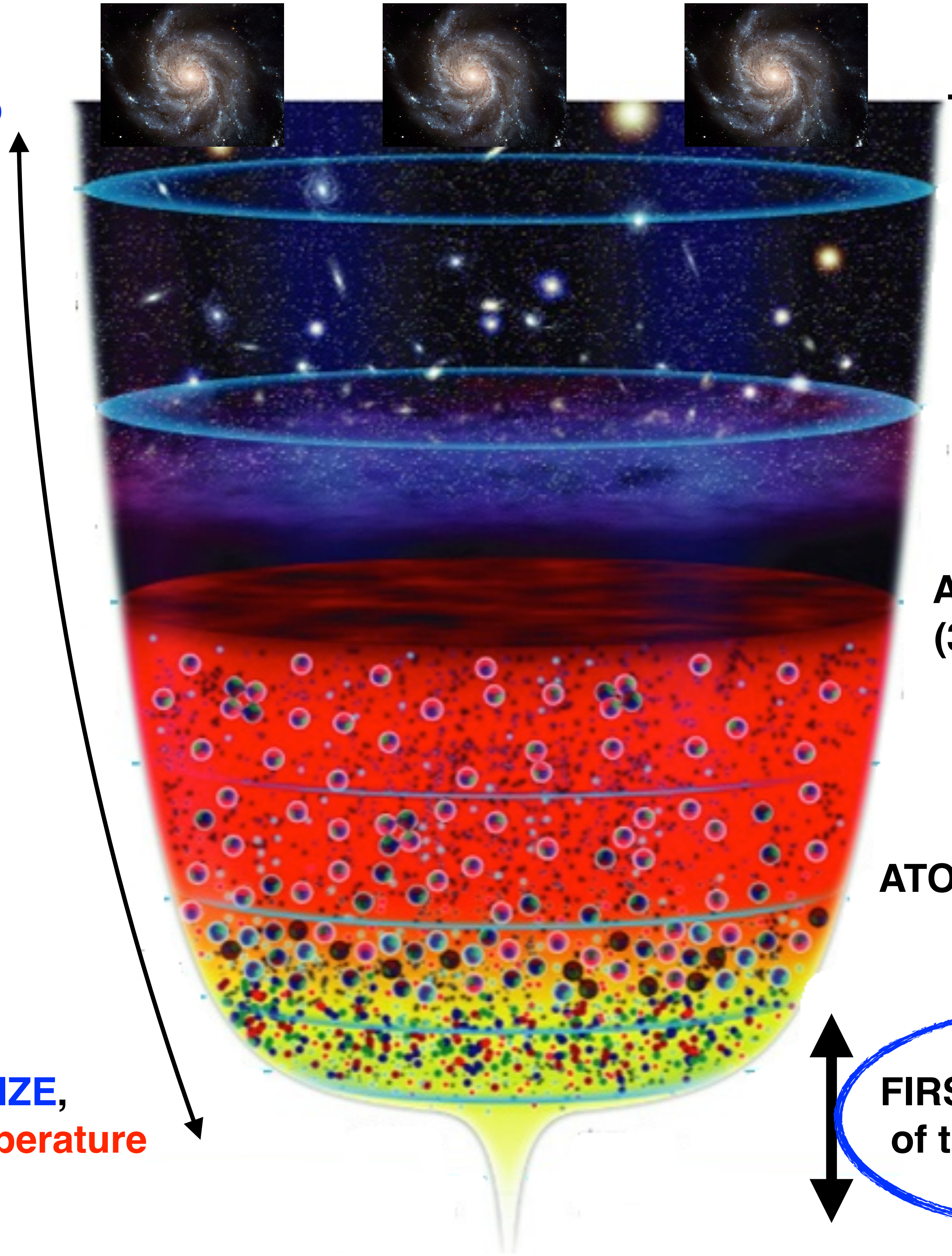
GW Source(s): ( SCALARS , VECTOR , FERMIONS )

$$\Pi^{TT}_{ij} \propto \{\partial_i\chi^a\partial_j\chi^a\}^{TT}, \ \{E_iE_j + B_iB_j\}^{TT}, \ \{\bar{\psi}\gamma_i D_j\psi\}^{TT}$$

# Cosmic History



**BiGGER size,**
**SMALLER Temp**

**SMALLER SIZE,**
**LARGER Temperature**

**TODAY [Galaxies, Clusters, …]**
**(13.700 Million years)**

**FIRST GALAXIES**
**(500 Millions years)**

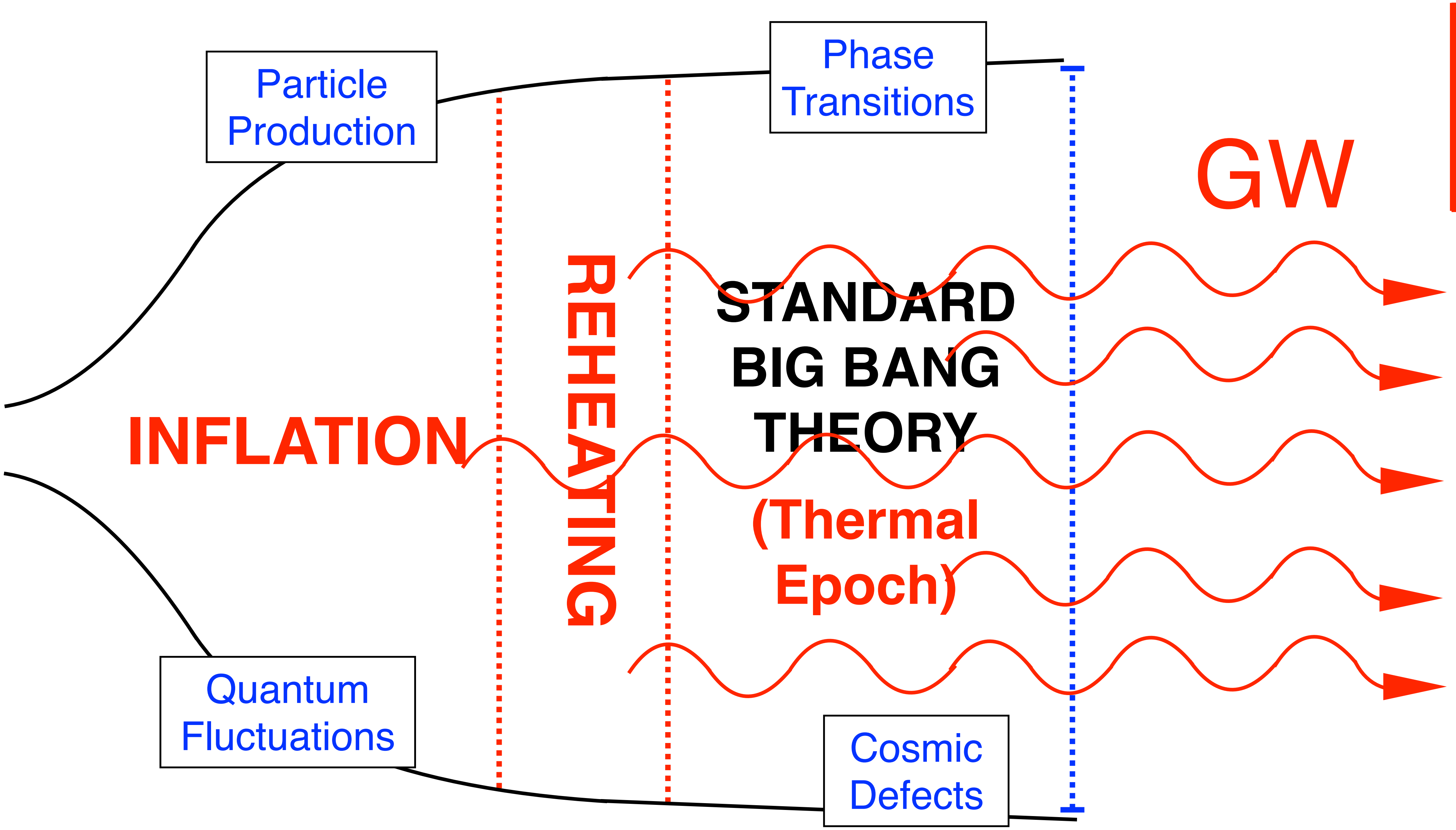
**ATOMS CREATION**
**(300.000-400.000 years)**

**ATOMIC NUCLEI CREATION**
**(3 minutes !)**

**FIRST SECOND**
**of the UNIVERSE !**

$\Pi_{ij}^{TT}$

# The Early Universe



Particle Production

Phase Transitions

INFLATION

REHEATING

STANDARD BIG BANG THEORY

(Thermal Epoch)

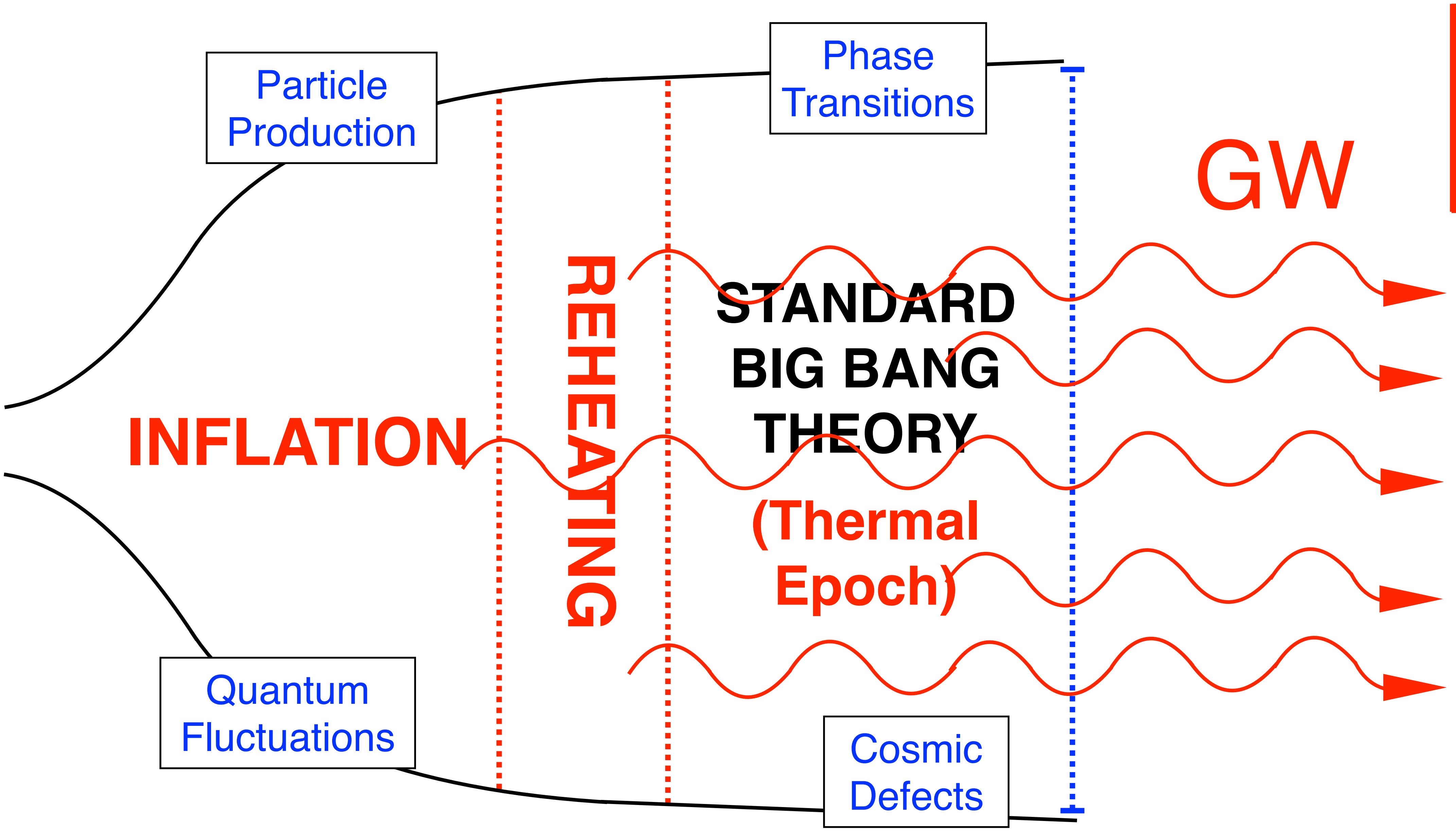
Quantum Fluctuations

Cosmic Defects

GW

[ Probe of the early Universe ]

# The Early Universe



Particle Production

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Quantum Fluctuations

REHEATING

Phase Transitions

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(Thermal Epoch)

Cosmic Defects

GW

Probe of the early Universe

$$\Pi_{ij}^{\mathrm{TT}}[\phi, A_\mu, \psi, ...] \xrightarrow{\textbf{GWs}} \frac{d\rho_{\mathrm{GW}}}{d\log f}$$

# Definition of GWs

## 4th approach

# Gravitational Wave Definition

**4th approach to GWs**

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

(separation not well defined)

# Gravitational Wave Definition

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See e.g. Maggiore's 1st Book on GWs

More subtle problem! Solution: Separation of scales !

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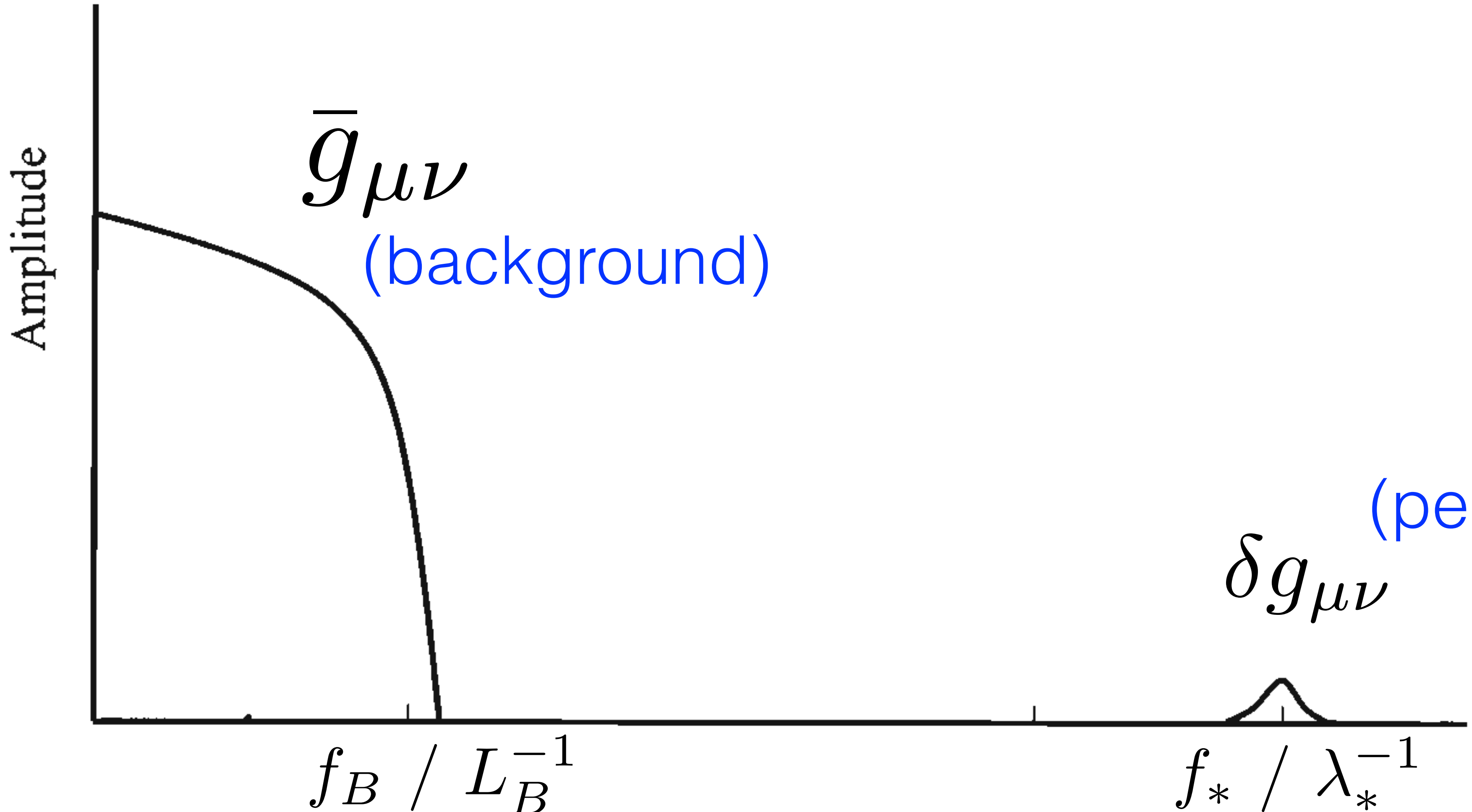

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$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

---

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---

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$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \Rightarrow R_{\mu\nu} = \underbrace{\bar{R}_{\mu\nu}}_{\text{(background)}} + \underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}(\delta g)} + \underbrace{R^{(2)}_{\mu\nu}}_{\mathcal{O}(\delta g^2)}$$

---

Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{Low}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{Low}}$

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$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \quad ➡ \quad R_{\mu\nu} \;=\; \underset{\text{(background)}}{\bar{R}_{\mu\nu}} \;+\; \underset{\mathcal{O}(\delta g)}{R^{(1)}_{\mu\nu}} \;+\; \underset{\mathcal{O}(\delta g^2)}{R^{(2)}_{\mu\nu}}$$

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High Freq. / Short Scale: $R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$

# Gravitational Wave Definition

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$\mathcal{O}(\delta g^2)$

# Gravitational Wave Definition

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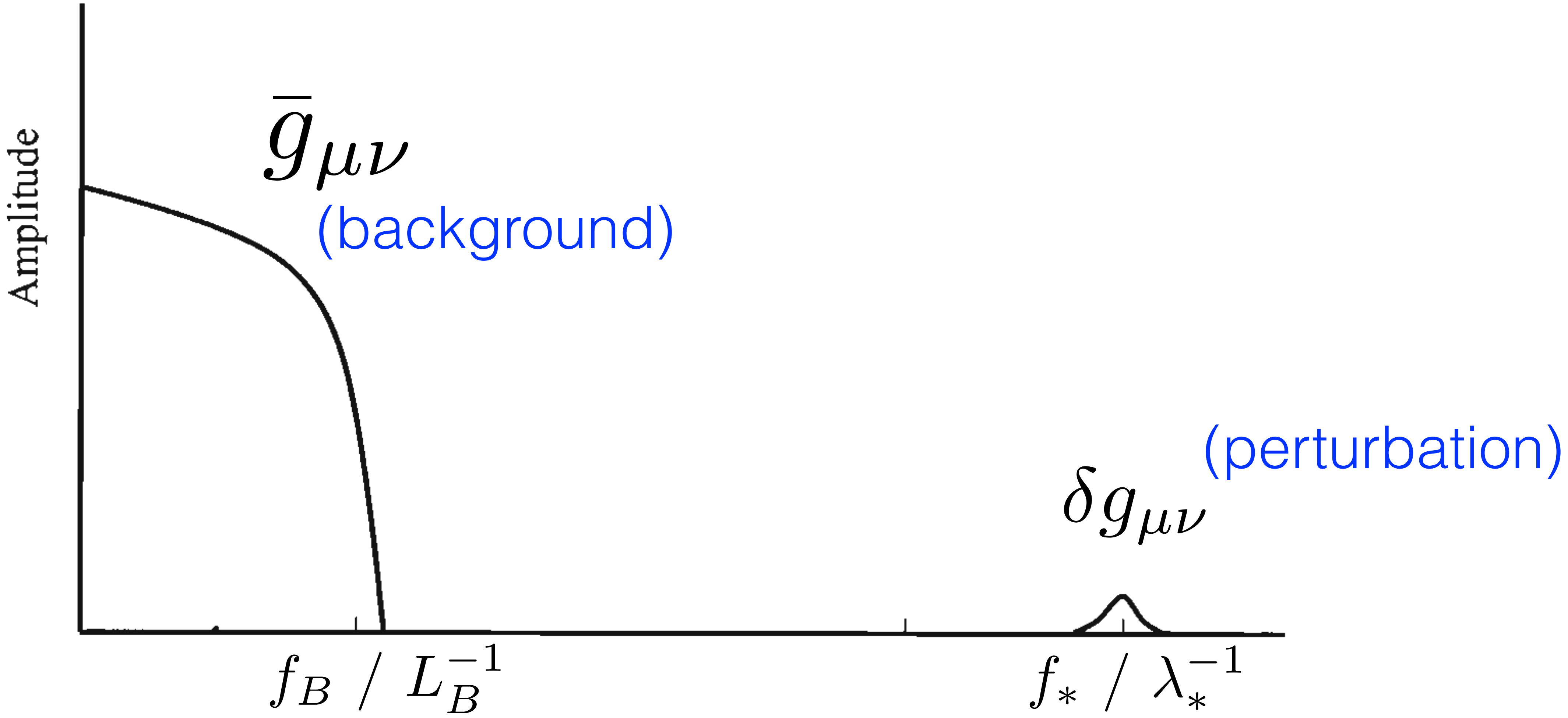

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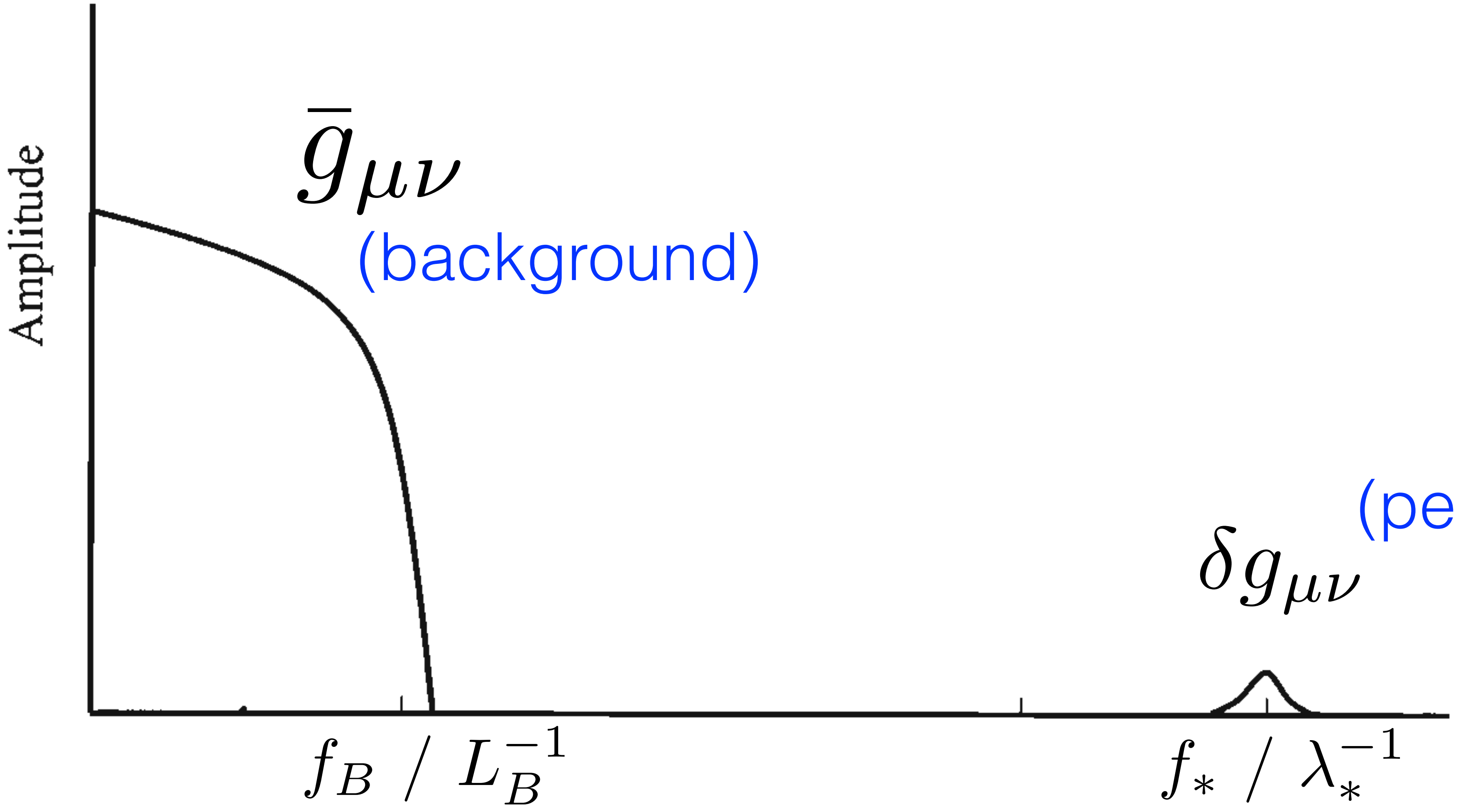
**(space/time average)**

---



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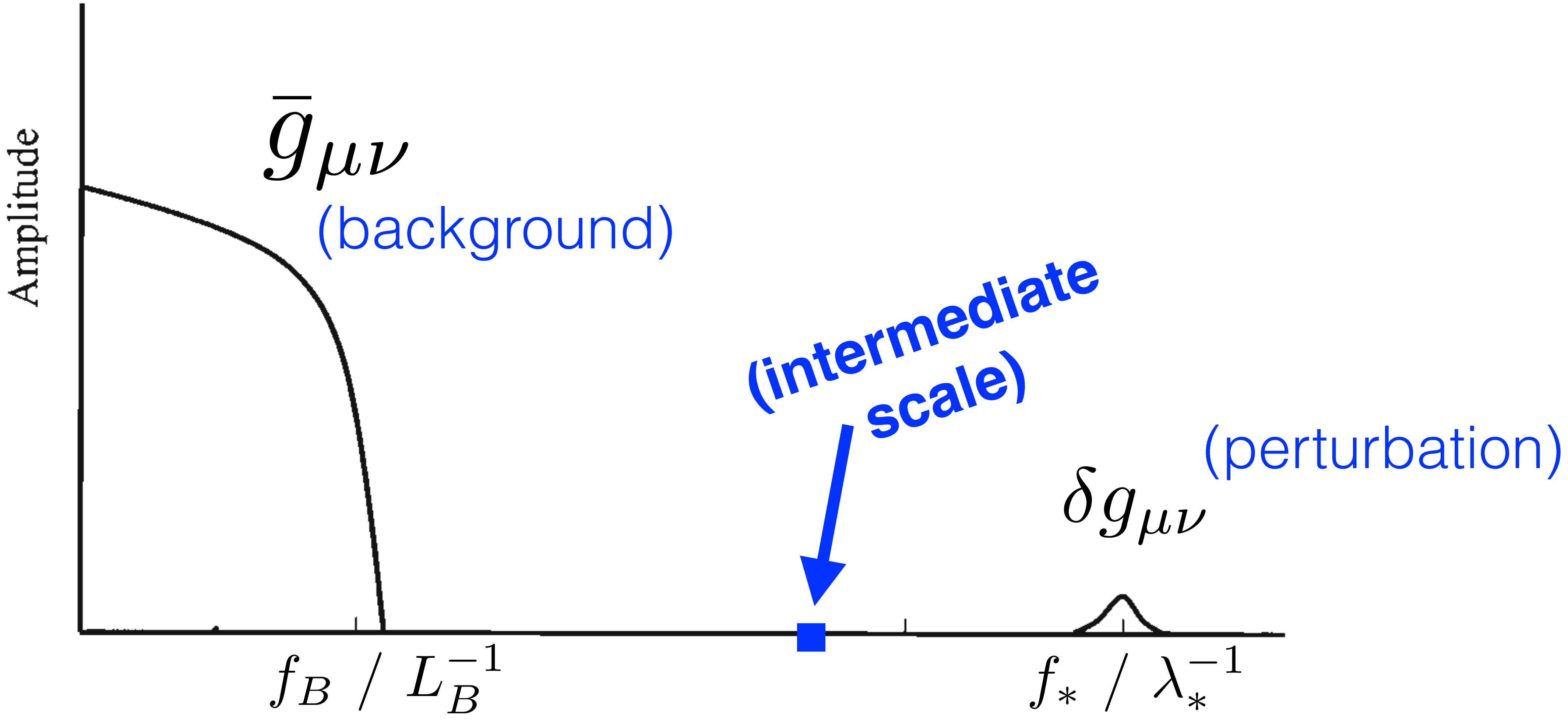

# Gravitational Wave Definition

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**(space/time average)**

$$t_{\mu\nu} = -\frac{1}{m_p^2}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^{(2)} \right\rangle$$

$$\left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle \equiv \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} T$$

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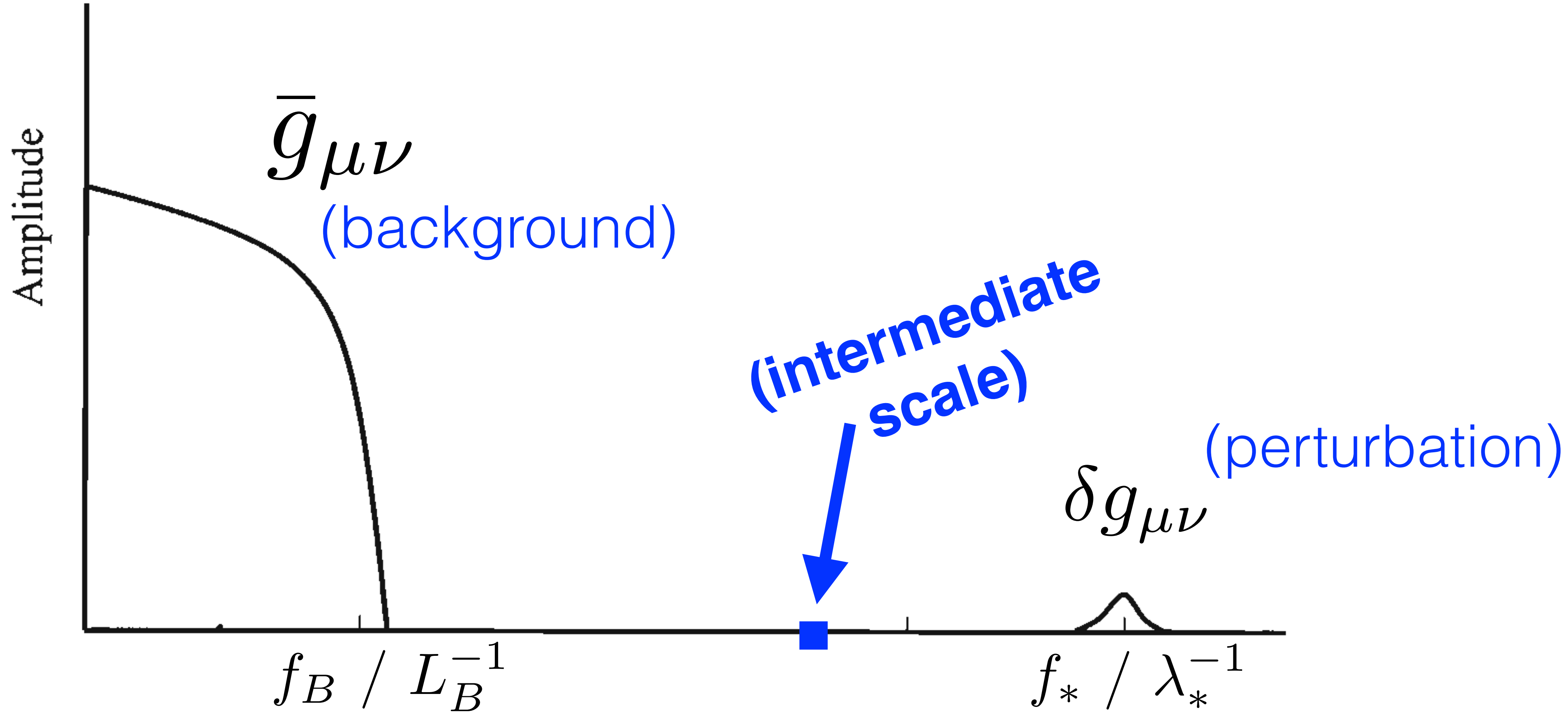

# Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

$$t_{\mu\nu} = -\frac{1}{m_p^2}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle \qquad \left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle \equiv \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}T$$

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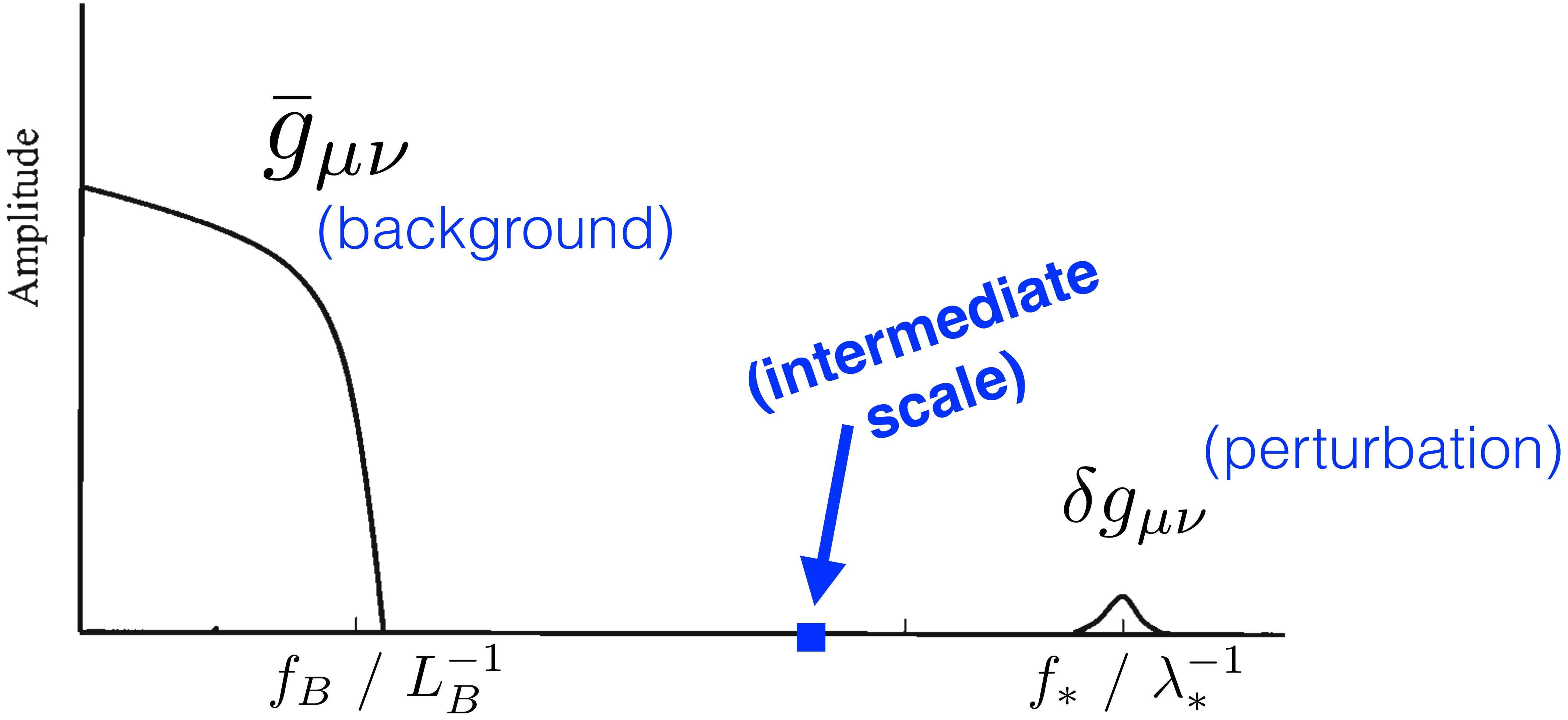

# Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

$$t_{\mu\nu} = -\frac{1}{m_p^2}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle \qquad \left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle \equiv \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}T$$

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$$\langle R^{(2)}_{\mu\nu} \rangle = -\frac{1}{4}\,\langle \partial_\mu \delta g_{\alpha\beta}\,\partial_\nu \delta g^{\alpha\beta} \rangle \quad \Longrightarrow \quad t_{\mu\nu} \equiv \frac{m_p^2}{4}\langle\, \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\,\rangle$$

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GW energy-momentum tensor

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$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\,\langle \partial_\mu \delta g^{\rm TT}_{ij}\,\partial_\nu \delta g^{\rm TT}_{ij} \rangle} \xrightarrow[(\delta g_{ij} \equiv h_{ij})]{} \boxed{\rho_{\rm GW} \equiv \frac{m_p^2}{4}\,\langle \dot{h}^{\rm TT}_{ij}\,\dot{h}^{\rm TT}_{ij} \rangle}$$

GW energy-momentum tensor

**GW energy density**

# Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{High}}$$

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$$\left.\begin{aligned} R^{(1)}_{\mu\nu} &= \bar{g}^{\alpha\beta}\left(D_\alpha D_{(\mu}\delta g_{\nu)\beta} - D_\mu D_\nu \delta g_{\alpha\beta} - D_\alpha D_\beta \delta g_{\mu\nu}\right) \\ D_\mu \overline{\delta g}_{\mu\nu} &= 0 \quad \left(\, \overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta}\,\right) \end{aligned}\right]$$

Lorentz
gauge

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vacuum

$$\longrightarrow \boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = 0}$$

Propagation of GWs
in curved space-time

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Lorentz gauge

vacuum

$$\Longrightarrow \quad \boxed{D_\alpha D^\alpha \delta g^{\rm TT}_{ij} = 0}$$

Propagation of GWs
in curved space-time

$\left(\ D_i \delta g^{\rm TT}_{ij} = \bar{g}^{ij}\delta g^{\rm TT}_{ij} = 0\ \right)$

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Lorentz gauge

→

→

matter

$$\boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = \Pi_{\mu\nu}}$$

Creation of GWs
in curved space-time

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Lorentz gauge

$$\Longrightarrow$$

matter

$$\Longrightarrow \quad \boxed{D_\alpha D^\alpha \overline{\delta g}^{\text{TT}}_{\mu\nu} = \Pi^{\text{TT}}_{\mu\nu}}$$

Creation of GWs
in curved space-time

**TT dof = truly radiative !**
**[no gauge choice]**

# Definition of GWs

* 1st approach: Lin Grav in Minkowski ✓

* 2nd approach: SVT decomp. ✓

* 3rd approach: FLRW background ✓

* 4rd approach: General backgrounds ✓