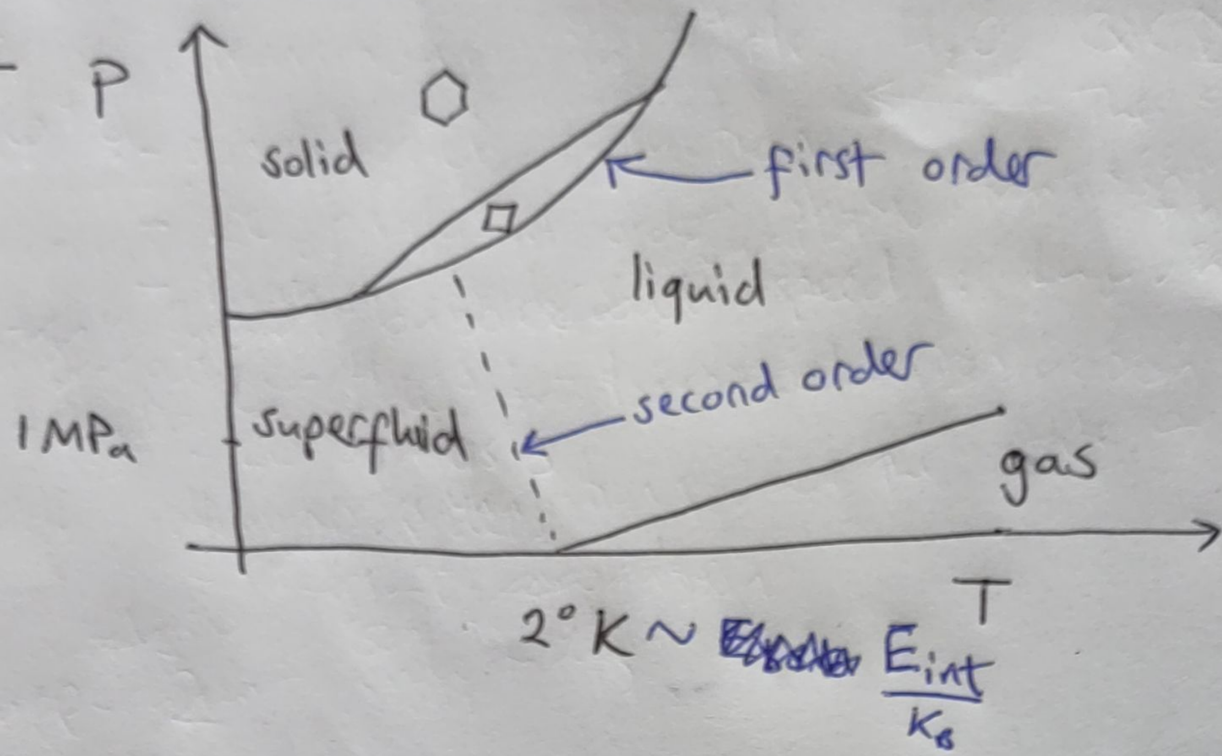
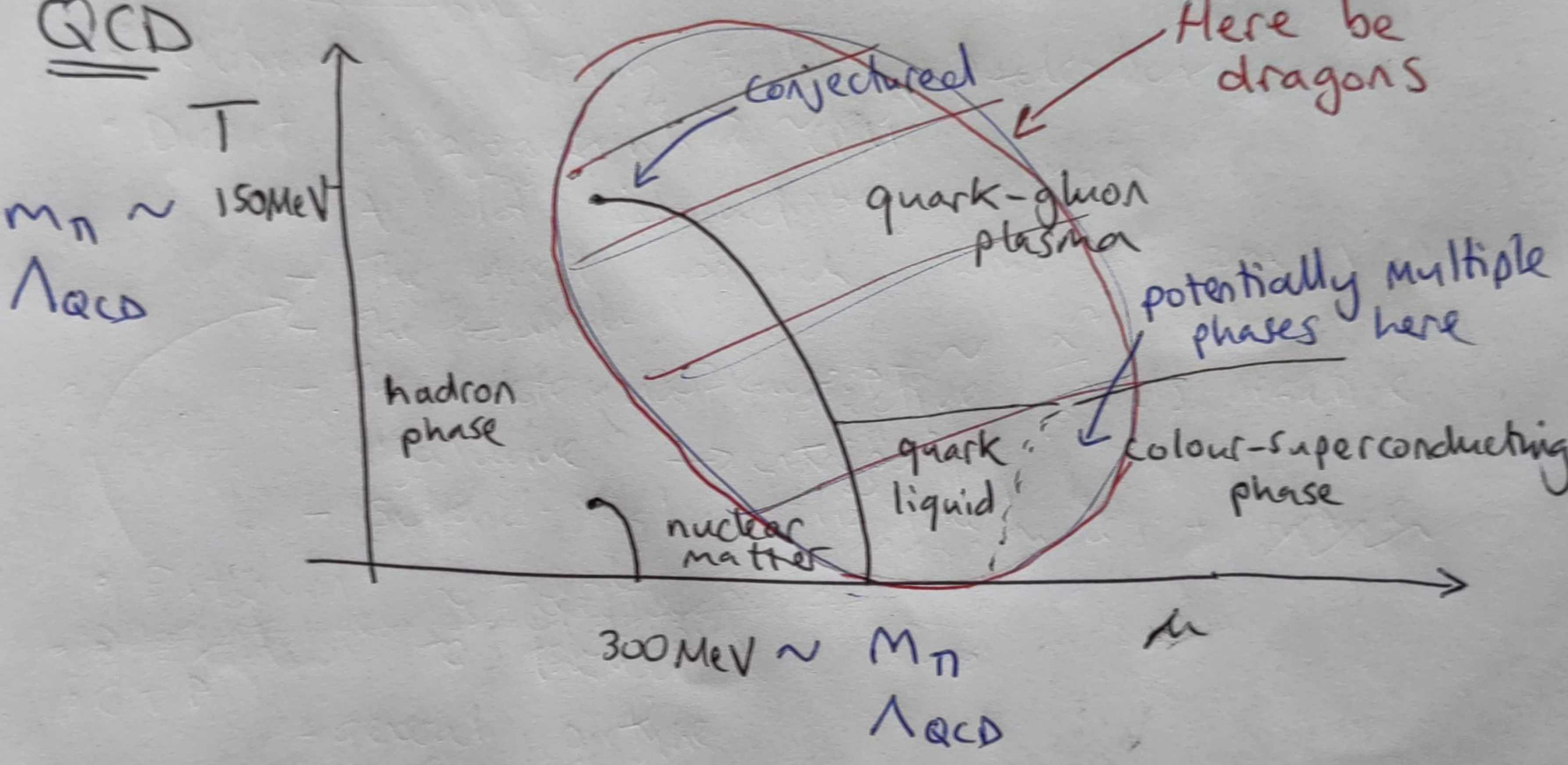
Phase transitions I

^{4}He



QCD



In cosmological history, after inflation the universe was reheated up to a very high temperature,

$$10\,\text{MeV} \lesssim T_{RH} \lesssim 10^{14}\,\text{GeV}.$$

Then, as the universe expanded, it cooled down to $\sim 2.7°K$ today (out of equilibrium).

How to determine the phase diagram of a given QFT

Order parameters: observables which reveal the phase.

- **Global symmetry breaking**

  e.g. $V = \underbrace{-\frac{\mu^2}{2}\phi^a\phi^a + \frac{\lambda}{4}(\phi^a\phi^a)^2}_{O(N)}$, $a = 1, 2, \ldots, N$

  The state may break this: $\underbrace{\langle \phi^a \rangle = v\delta^a_1}_{O(N-1)}$

- **Higgs mechanism**

  Same story in gauge-fixed perturbation theory, but $\langle\phi\rangle$ is gauge dependent $\Rightarrow \langle\phi\rangle = 0$, Elitzur's Thm

  $\therefore$ gauge symmetries cannot be broken.

  $\langle\phi^\dagger\phi\rangle$ is gauge invariant but never zero.

- **Chiral symmetry breaking**

  In QCD with N_f massless quarks, there is a

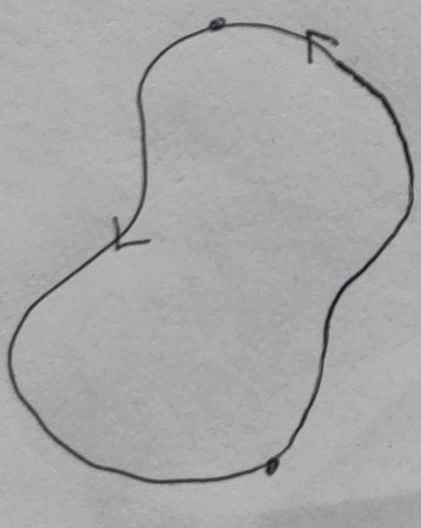
  $$SU(N_f)_L \times SU(N_f)_R$$

  symmetry, which can be spontaneously broken by

  $$\langle \bar{q}^a_R q^a_L + \bar{q}^a_L q_R \rangle \neq 0 \rightarrow SU(N_f)_{L+R}$$

- **Confinement**

  Wilson line order parameter.



$$\langle P e^{ig\oint_C A\cdot dl} \rangle_{\text{gauge}} \sim \begin{cases} e^{-\kappa L[C]}, & \text{nonconfining} \\ e^{-\kappa' A[C]}, & \text{confining} \end{cases}$$

- **Mass spectrum** can be an order parameter.

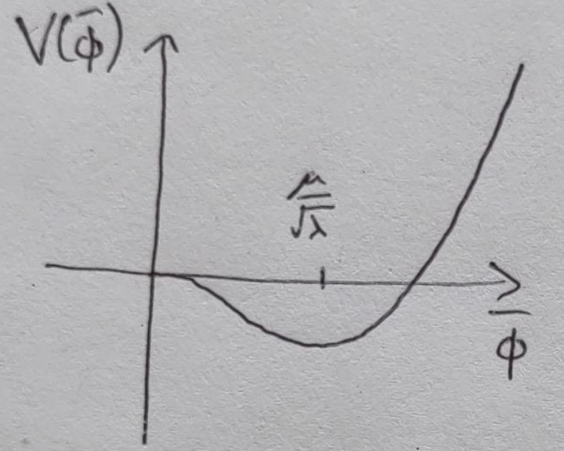
Abelian Higgs model

$$\mathcal{L}_E = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* D_\mu \phi + V(\phi) + \mathcal{L}_{GF}$$

$$V(\phi) = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$$

$$\mathcal{L}_{GF} = \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \underbrace{\partial_\mu \bar{c} \partial_\mu c}_{\text{ghosts decoupled in Abelian theory}} , \quad \xi \to 0_+ \quad \text{Landau gauge}$$

At zero temperature, in perturbation theory, the Higgs mechanism is usually understood in terms of the potential, for $\phi = \frac{1}{\sqrt{2}}(\bar{\Phi} + \delta\phi)$,



At nonzero temperature, we instead consider the effective potential. In finite volume, we can isolate the mode with $P=0$, denoted $\bar{\Phi}$,

$$\langle \bar{\Phi} \rangle = \frac{1}{Z} \int D\phi DA\, e^{-S_E} \cdot \phi$$

$$= \frac{1}{Z} \int d\bar{\Phi}\, \bar{\Phi} \int_{P \neq 0} D\phi DA\, e^{-S_E} \equiv \frac{1}{Z} \int d\bar{\Phi}\, \bar{\Phi}\, e^{-\beta \overset{\text{volume}}{\mathcal{V}} V_{\text{eff}}(\bar{\Phi})}$$

$$= \int d\bar{\Phi}\, \bar{\Phi}\, \rho(\bar{\Phi})$$

$$\Rightarrow \boxed{\rho(\bar{\Phi}) = \frac{1}{Z} e^{-\beta \mathcal{V} V_{\text{eff}}(\bar{\Phi})}} \quad \text{probability density for } \bar{\Phi}$$

For something interesting to happen, $V_{eff}(\bar{\phi})$ must change shape significantly with temperature.

$$V_{eff}(\bar{\phi}) = V_{tree}(\bar{\phi}) + V_{loops}(\bar{\phi})$$

$$\Rightarrow \frac{V_{loops}}{V_{tree}} = \underbrace{(\text{couplings})}_{\text{assumed small}} + \underbrace{(\text{loop } \overset{\text{sum}}{\vee} \text{integrals})}_{f^{n}\text{s of masses and } T} \gtrsim 1$$

perturbative phase transition $\Rightarrow$ (large loop sum-integrals) hierarchies of scale ✱ $\Rightarrow$ $\underbrace{\text{need resummations}}_{\text{EFT}}$

High temperature dimensional reduction

$$\psi(\tau, \underline{x}) = \sum_n \psi_n(\underline{x})\, e^{i\omega_n \tau}$$

$$\omega_n = n\pi T, \quad n = \begin{cases} \text{even}, & \text{boson} \\ \text{odd}, & \text{fermion} \end{cases}$$

If $\pi T \gg m$, we have a hierarchy of scales. ✱ There is an EFT for energies $\ll \pi T$, consisting of the purely spatial $n=0$ bosonic modes.

purely spatial $\Rightarrow A_\mu = \begin{pmatrix} A_0 \\ A_i \end{pmatrix} \rightarrow$ A_0 scalar in 3d

$A_i + \partial_i \Omega$ vector in 3d

The most general Lagrangian satisfying the symmetries,

$$\mathcal{L}_{3d} = \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \partial_i A_0 \partial_i A_0 + (D_i \phi)^* D_i \phi + V_3(\phi, A_0) + \mathcal{L}_{GF,3d}$$

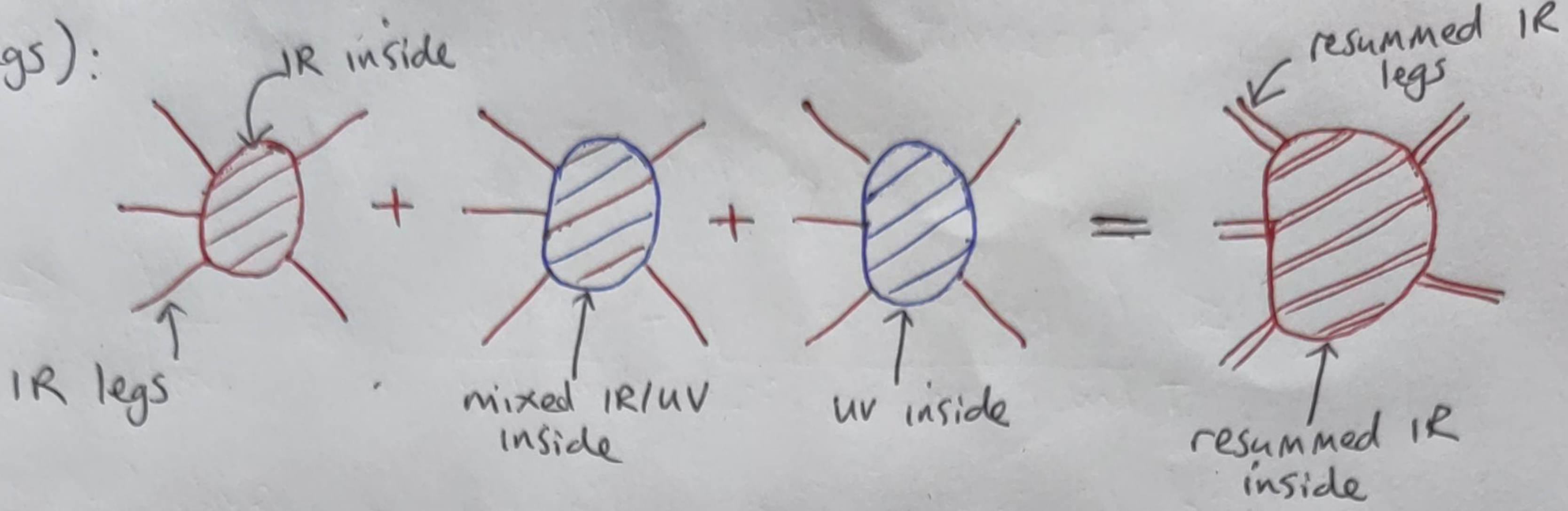
$$V_3(\phi, A_0) = m_3^2 \phi^* \phi + \lambda_3 (\phi^* \phi)^2 + \underbrace{\frac{1}{2} m_D^2 A_0^2 + \frac{1}{2} h_3 A_0^2 \phi^* \phi + \frac{1}{4} \lambda_A A_0^4}_{\text{new terms allowed}}.$$

$$\mathcal{L}_{GF,3} = \frac{1}{2\xi_3} (\partial_i A_i)^2 + \underbrace{\frac{1}{2} \partial_i \bar{c} \partial_i c}_{\text{ghosts decoupled again}},$$

$$\xi_3 \rightarrow 0_+$$

Next we must match the EFT parameters, once done the EFT will capture ALL IR observables.

We match IR observables (correlators with IR legs):



- The <u>Appelquist–Carazzone theorem</u> states that this is possible, diagram by diagram, order by order.
- We should match 1-light-particle-irreducible correlation functions, to ensure factorisation.
- To simplify matching, we should project out the IR parts of loop integrals with the method of regions. In dimensional regularisation, this means Taylor expand loop integrands in IR quantities:

e.g.
$$\sum\!\!\!\!\!\!\!\int_Q \frac{1}{(\underbrace{Q^2}_{UV}+\underbrace{m^2}_{IR})^2}\Bigg|_{UV} = \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^4}\left(1 - 2\frac{m^2}{Q^2} + O\left(\frac{m^4}{Q^4}\right)\right)$$
$$\simeq \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^4}$$

At leading order, we need only to care about diagrams enhanced by positive powers of πT,
$$\sum\!\!\!\!\!\!\!\int_Q \frac{1}{(Q^2)^a} \propto (\pi T)^{4-2a}, \quad a=1 \text{ enhanced.}$$

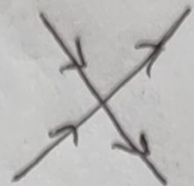
Let's match the Higgs mass at leading order,

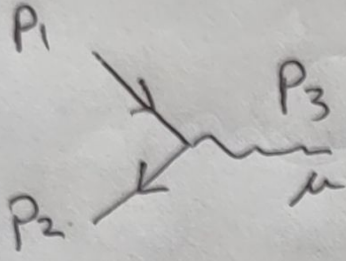
$$-\Gamma^{\phi\phi}_{EFT}(\underline{0})\Big|_{UV} = +M_3^2 +$$


None of these diagrams have any UV component

Feynman rules in unbroken phase (Euclidean):

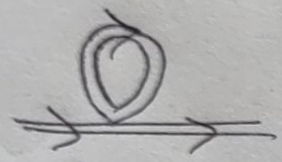
$$\frac{\delta^4(-S_E)}{\delta\phi(p_1)\delta\phi(p_2)\delta A_\mu(p_3)\delta A_\nu(p_4)} = -2g^2\delta_{\mu\nu}\,\delta(p_1+p_2+p_3+p_4)$$

$\Rightarrow$  $= -2g^2\delta_{\mu\nu}$,  $= -4\lambda$

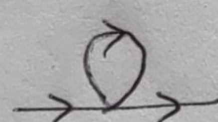


$= -g(p_1+p_2)_\mu$,

$$\overbracket{\phi\phi^*} = \frac{1}{p^2+\underbrace{m^2}_{-\mu^2}} , \quad \overbracket{A_\mu A_\nu} = \frac{1}{p^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)$$

 $\Big|_{UV} = -4\lambda\int_q \frac{1}{q^2+m^2}\Big|_{UV} = -4\lambda\int_q \frac{1}{q^2}\left(1-\frac{m^2}{q^2}+O\left(\frac{m^4}{q^4}\right)\right)$

$$= 0 \Rightarrow \Gamma^{\phi\phi}_{EFT}(0) = m_3^2 .$$

$-\Gamma^{\phi\phi}_{(0)}\Big|_{UV} = \mu^2 +$  $+$  $+$  $\Big|_{UV}$

$$= \mu^2 - 4\lambda \SumInt_Q \frac{1}{Q^2} - \frac{1}{2}\cdot 2g^2 \SumInt_Q \frac{\delta_{\mu\nu}}{Q^2}\left(\delta_{\mu\nu}-\frac{Q_\mu Q_\nu}{Q^2}\right)$$

$$+ g^2 \SumInt_Q \frac{Q_\mu Q_\nu}{Q^2}\left(\delta_{\mu\nu} - \underbrace{\frac{Q_\mu Q_\nu}{Q^2}}_{0}\right)$$

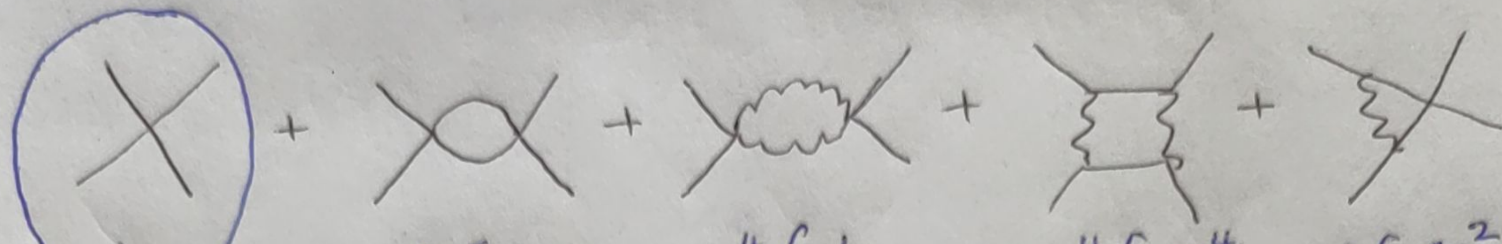
dropped subleading m^2 etc.

$$= \mu^2 - 4\lambda\frac{T^2}{12} - g^2\cdot 3\cdot\frac{T^2}{12}$$

$$+\Gamma^{\phi\phi}_{(0)}\Big|_{UV} = +\Gamma^{\phi\phi}_{EFT}(0)\Big|_{UV} \Rightarrow m_3^2 = \underbrace{-\mu^2 + \left(\frac{\lambda}{3}+\frac{g^2}{4}\right)T^2}$$

−ve at low T
+ve at high T

N.B. corrections to couplings are not enhanced.



$$\lambda \qquad \lambda^2 \sum\!\!\!\!\!\!\int \frac{1}{Q^4} \qquad g^4 \sum\!\!\!\!\!\!\int \frac{1}{Q^4} \qquad g^4 \sum\!\!\!\!\!\!\int \frac{Q^4}{Q^8} \qquad g^4 \sum\!\!\!\!\!\!\int \frac{Q^2}{Q^6}$$

so, at leading order we can stop at tree-level.

Mass dimensions

Canonical normalisation in 3d and 4d

$$\int_0^\beta d\tau \int d^3x \, \frac{1}{2} \partial_\mu \phi^{(0)}_{4d} \partial_\mu \phi^{(0)}_{4d} = \int d^3x \, \frac{1}{2} \partial_i \phi_{3d} \partial_i \phi_{3d}$$

$$\Rightarrow \phi_{3d} = \frac{\phi_{4d}}{\sqrt{T}} \quad \Rightarrow \quad \lambda_3 \simeq \lambda T \text{ etc.}$$

What have we achieved?

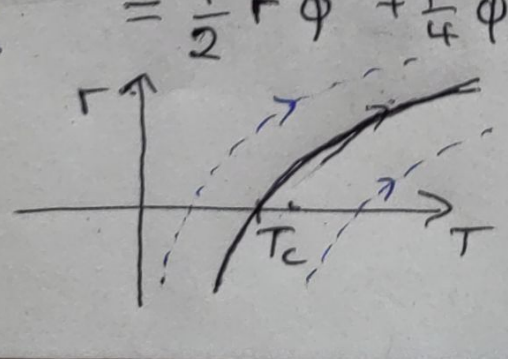
$$V_{EFT}(\bar{\phi}, \bar{A}_0) = \frac{1}{2} m_3^2 \bar{\phi}^2 + \frac{1}{4} \lambda_3 \bar{\phi}^4 + \frac{1}{2} m_D^2 A_0^2 + \frac{1}{4} h_3 \bar{A}_0^2 \bar{\phi}^2 + \frac{1}{4} \lambda_A \bar{A}_0^4,$$

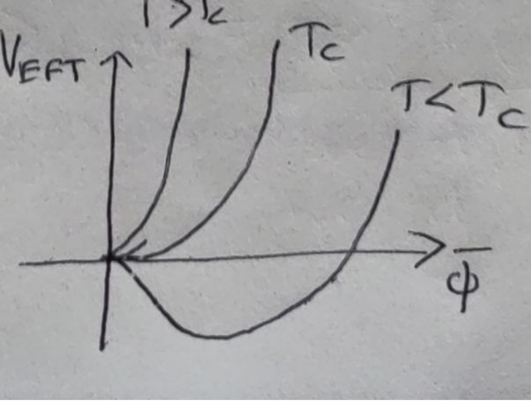
where here $\phi = \frac{1}{\sqrt{2}}(\bar{\Phi})$, $A_0 = \bar{A}_0$. We find that $m_D^2 > 0$, so not a transition there, but m_3^2 changes sign. We also have that $\lambda_3 > 0$, so we can scale this out:

$$\tilde{V}_{EFT}(\tilde{\phi}, 0) = \frac{V_{EFT}(\bar{\phi}, 0)}{\lambda_3^3} = \frac{1}{2} \overbrace{\left(\frac{m_3^2}{\lambda_3^2}\right)}^{r} \overbrace{\left(\frac{\bar{\phi}^2}{\lambda^3}\right)}^{\tilde{\phi}^2} + \frac{1}{4} \overbrace{\left(\frac{\bar{\phi}^4}{\lambda_3^2}\right)}^{\tilde{\phi}^4}$$

Universal form independent of 4d couplings

$$= \frac{1}{2} r \tilde{\phi}^2 + \frac{1}{4} \tilde{\phi}^4.$$

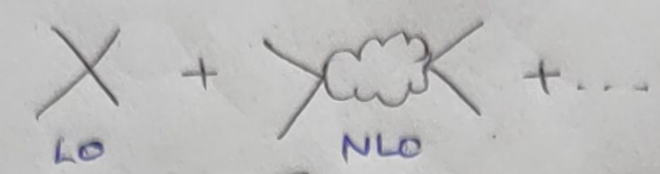


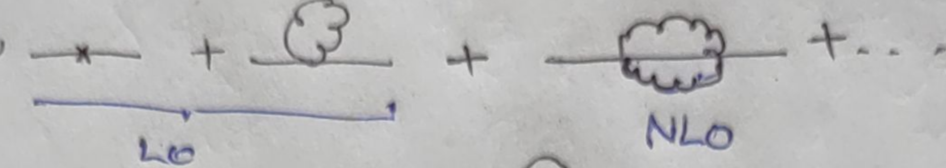


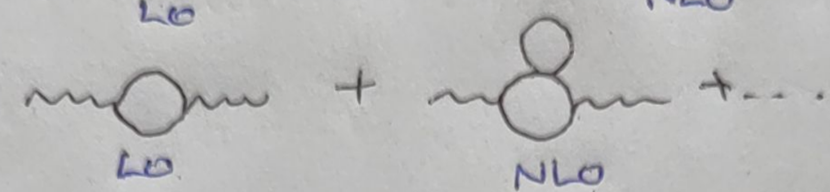
$\Rightarrow$ 2nd order, discontinuity in $\frac{d^2 f}{dT^2}$.

~~Rea~~ Higher order corrections to matching

We have worked at leading order. At NLO:

$\lambda_3 \sim$ 

$m_3^2 \sim$ 

$m_D^2 \sim$ 

$\vdots$

Modifies e.g. m_3^2 as

$$m_3^2 = -\mu^2 + \left(\frac{\lambda}{3} + \frac{g^2}{4}\right)T^2 + \underbrace{\# \frac{g^4 T^2}{(4\pi)^2} \log\left(\frac{\Lambda}{4\pi}\right)}_{\text{2-loops.}} + \dots$$

Note that for masses, because of the enhancement by the scale hierarchy, 2-loop diagrams are equivalent to one-loop diagrams at $T=0$.

At NLO, and using LO (one-loop) β functions one finds that:

$$\Lambda \frac{d\lambda_3}{d\Lambda} = \Lambda \frac{d g_3^2}{d\Lambda} = \Lambda \frac{d h_3}{d\Lambda} = 0 \quad \Big\} \text{ effective couplings are independent of RG scale.}$$

$$\Lambda \frac{d m_3^2}{d\Lambda} \neq 0 \,, \quad \Lambda \frac{d m_D^2}{d\Lambda} \neq 0 \quad \Big\} \text{ effective masses run.}$$

All this is important for accuracy, but higher order corrections to matching don't change the picture, or the order of the transition, just $r(T)$: