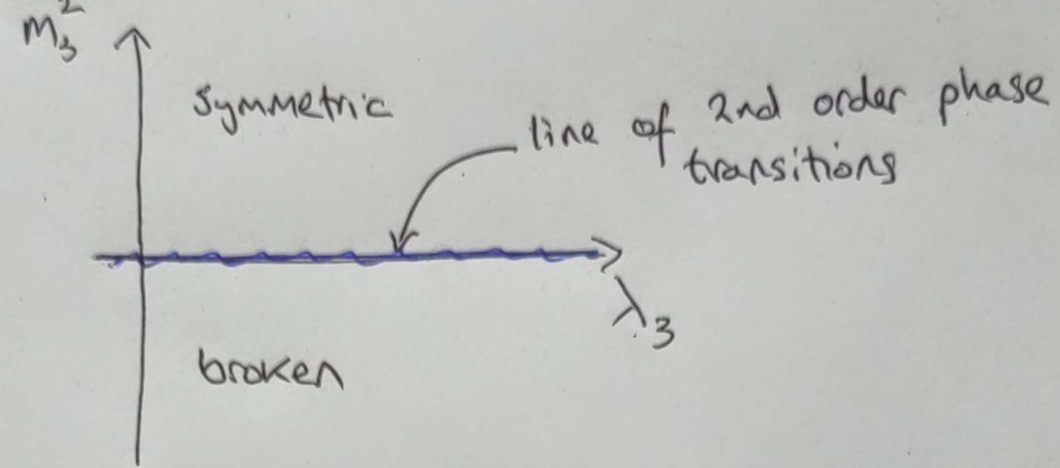
Phase transitions II

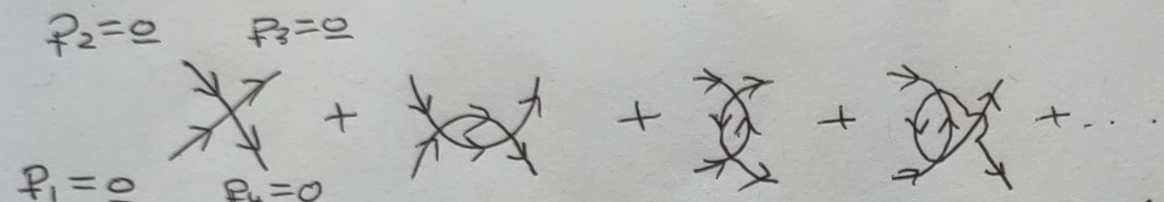
Aim: study our 3d EFT; determine the phase diagram; consider cosmological implications.

Last time:



Our 3d EFT is superrenormalisable
$\Rightarrow$ insensitive to UV but <u>sensitive to IR</u>.

What happens as we go through $m_3^2 = 0$?



N.B. pure scalar for simplicity.

$$= -4\lambda_3 + (4\lambda_3)^2\left(\frac{1}{2}+1+1\right)\int \frac{1}{(q^2+m_3^2)^2} + \ldots$$

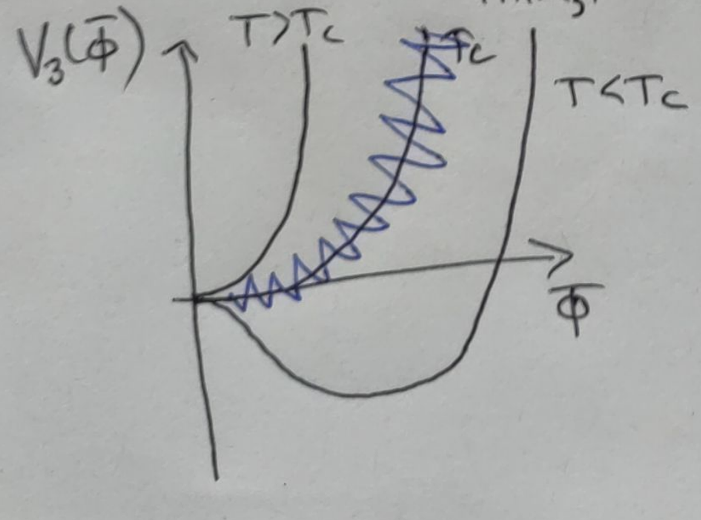
$$= -4\lambda_3\left(1 - \frac{5\lambda_3}{4\pi|m_3|} + \ldots\right)$$

$\Rightarrow \dfrac{\lambda_3}{4\pi|m_3|}$ is the dimensionless loop expansion parameter.

Away from $T = T_c$,

$$m_3^2 \simeq -\mu^2 + \left(\frac{\lambda}{3} + \frac{g^2}{4}\right)T^2 \sim \lambda T^2 \sim (\sqrt{\lambda}T)^2$$

$$\Rightarrow \frac{\lambda_3}{4\pi|m_3|} \sim \frac{\lambda T}{4\pi\sqrt{\lambda T^2}} \sim \frac{\sqrt{\lambda}}{4\pi} \sim \frac{\sqrt{g^2}}{4\pi}$$



"hard scale" $\sim \pi T$

"soft scale" $\sim \sqrt{\lambda}\, T$

"ultrasoft scale" $\sim \dfrac{\lambda}{\pi} T$, nonperturbative

To make progress, we will consider $V_{eff}(\bar{\Phi})$ within the EFT.

Expanding, $\phi = \frac{1}{\sqrt{2}}(\bar{\Phi} + \phi_H + i\phi_G)$,

$$\mathcal{L}_{EFT} \supset (D_i\phi)^* D_i\phi \supset \frac{1}{2} g_3^2 \bar{\Phi}^2 A_i A_i \Rightarrow m_V^2 = g_3^2 \bar{\Phi}^2$$

$$m_L^2 = m_D^2 + h_3 \bar{\Phi}^2$$

$$m_H^2 = m_3^2 + 3\lambda_3 \bar{\Phi}^2 , \quad m_G = \underbrace{m_3^2 + \lambda_3 \bar{\Phi}^2}_{\text{zero for } \bar{\Phi}^2 = -\frac{m_3^2}{\lambda_3}} .$$

To compute how these fields back react on the condensate $\bar{\Phi}$, let's return to the definition of V_{eff}.

Writing $\Psi_a = \{A_i, A_0, \phi_H, \phi_G\}$ for the fluctuating fields,

$$e^{-\mathcal{V} V_{eff}(\bar{\Phi})} \equiv \int_{p\neq 0} D\Psi \exp\left[-S_{EFT}[\bar{\Phi}] - \frac{1}{2}\int_{p,q} \Psi_a(p) \frac{\delta^2 S_{EFT}}{\delta\Psi_a(p)\delta\Psi_b(q)}\bigg|_{\Psi=0} \Psi_b(q) - (\text{higher order})\right]$$

$$= e^{-\mathcal{V} V_{EFT}^{tree}(\bar{\Phi})} \det\left(\frac{\delta^2 S_{EFT}}{\delta\Psi_a(p)\delta\Psi_b(q)}\bigg|_{\Psi=0}\right)^{-1/2} (1 + (\text{higher order}))$$

$$\Rightarrow V_{eff}(\bar{\Phi}) = V_{EFT}^{tree}(\bar{\Phi}) + \frac{1}{2}\frac{1}{\mathcal{V}} \operatorname{tr} \log \left(\frac{\delta^2 S_{EFT}}{\delta\Psi_a \delta\Psi_b}\right)\bigg|_{\Psi=0} + \ldots$$

So need eigenvalues of the quadratic operator in the action.

For scalars this is easy,

$$S_{EFT} \supset \int_{q,p} \frac{1}{2} \phi_H(q) \underbrace{(p^2 + m_H^2)}_{\text{eigenvalue}} \phi_H(p)\, \delta(p+q)$$

For the vectors we have,

$$M_{ij} = \underbrace{p^2 \delta_{ij} - p_i p_j}_{\text{from } \frac{1}{4}F_{ij}F_{ij}} + \underbrace{\frac{1}{\xi_3} p_i p_j}_{\text{from } \mathcal{L}_{GF}} + \underbrace{m_V^2 \delta_{ij}}_{\text{from Higgs mechanism}}$$

Writing $\underline{\underline{P}}_T = \delta_{ij} - \frac{p_i p_j}{p^2}$, $\underline{\underline{P}}_L = \frac{p_i p_j}{p^2}$

which satisfy, $\underline{\underline{P}}_T^2 = \underline{\underline{P}}_T$, $\underline{\underline{P}}_L^2 = \underline{\underline{P}}_L$, $\underline{\underline{P}}_T + \underline{\underline{P}}_L = \underline{\underline{\mathbb{1}}}$.

$$\underline{\underline{M}} = p^2 \underline{\underline{P}}_T + \frac{1}{\xi_3} p^2 \underline{\underline{P}}_L + m_V^2 (\underline{\underline{P}}_T + \underline{\underline{P}}_L)$$

$$= (p^2 + m_V^2)\, \underline{\underline{P}}_T + \left(\frac{1}{\xi_3} p^2 + m_V^2\right) \underline{\underline{P}}_L$$

The projection operators have eigenvalues 1 and 0. $\underline{\underline{P}}_T$ has 2 nonzero (unity) eigenvalues and $\underline{\underline{P}}_L$ has 1.

Putting it all together,

$$V_{EFT}^{1\text{-loop}}(\bar{\Phi}) = \frac{1}{2}\int_q \log(q^2 + m_H^2) + \frac{1}{2}\int_q \log(q^2 + m_G^2) + \frac{1}{2}\int_q \log(q^2 + m_L^2)$$

$$+ \frac{1}{2}\cdot 2 \int_q \log(q^2 + m_V^2) + \frac{1}{2}\int_q \log\Big(\underbrace{\cancel{\tfrac{1}{\xi_3} p^2} + m_V^2}_{\text{limit } \xi_3 \to 0 \text{ taken at the end.}}\Big)$$

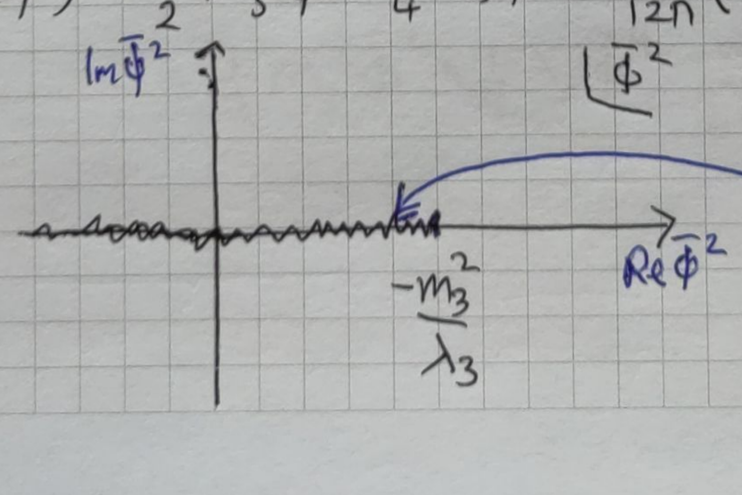
field independent

$$= -\frac{1}{12\pi}\left(m_H^3 + m_G^3 + m_L^3 + 2 m_V^3 \right), \quad m \equiv \sqrt{m^2}$$

+ const.

Pure scalar theory,

$$V_{eff}(\bar{\Phi}) = \frac{1}{2} m_3^2 \bar{\Phi}^2 + \frac{1}{4}\lambda_3 \bar{\Phi}^4 - \frac{1}{12\pi}\Big(\underbrace{(m_3^2 + 3\lambda_3\bar{\Phi}^2)^{3/2} + (m_3^2 + \lambda_3 \bar{\Phi}^2)^{3/2}}_{\text{what happens for } m_3^2 < 0?}\Big)$$



branch cut, V_{eff} not defined here. Meaning of V_{eff}?

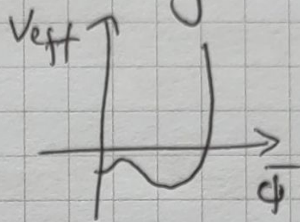
$V^{1\text{-loop}}_{EFT}(\bar{\Phi})$ gives a small correction deep in the symmetric or broken phases.

In general,

$$\frac{V^{1\text{-loop}}_{EFT}}{V^{tree}_{EFT}} \sim \frac{\lambda_3}{\pi m_3}.$$

We can get interesting behaviour, including multiple minima, but only when,

$$\frac{\lambda_3}{\pi m_3} \gtrsim 1 \quad \longleftrightarrow$$



where we can't trust perturbation theory.

<u>Why do things go wrong?</u>

$$m_3^2 = -\mu^2 + \left(\frac{\lambda}{3} + \frac{g^2}{4}\right)T^2 \ll \lambda T^2$$

$\Rightarrow$ a new scale hierarchy!

$$m_G, m_H \ll m_L, m_V$$

This is a good thing, because we know how to proceed $\longrightarrow$ integrate out the heavy fields.

$$m_H^2 \ll m_V^2 \Leftarrow \lambda_3 \bar{\Phi}^2 \ll g_3^2 \bar{\Phi}^2$$

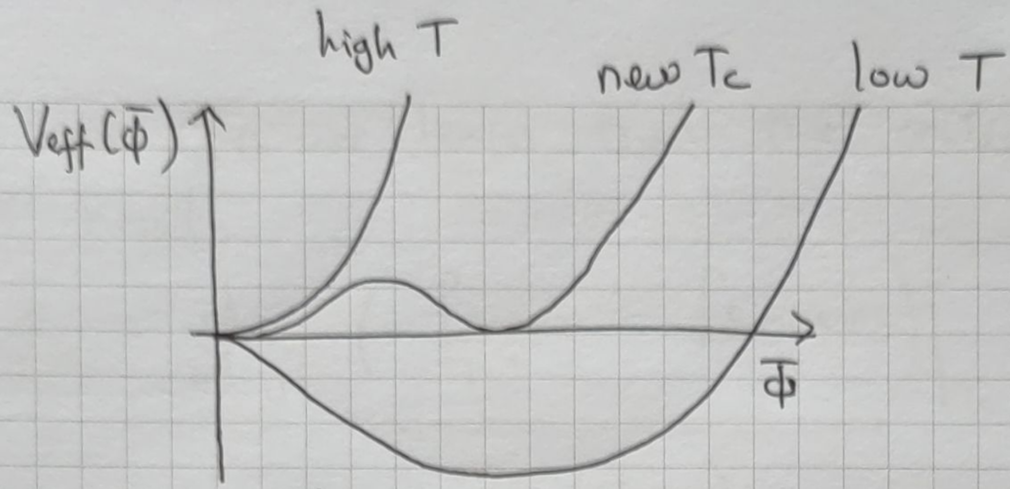
$$\Rightarrow \lambda \ll g^2.$$

[Dropping A_0 for simplicity.]

$$V_{eff}(\bar{\Phi}) = V^{tree}_{\substack{New\\EFT}}(\bar{\Phi}) = \frac{1}{2} m_3^2 \bar{\Phi}^2 + \frac{1}{4}\lambda_3 \bar{\Phi}^4 - \frac{1}{12\pi} \cdot 2 \cdot m_V^3$$

$$= \frac{1}{2} m_3^2 \bar{\Phi}^2 + \frac{1}{4}\lambda_3 \bar{\Phi}^4 - \frac{g_3^3}{6\pi}|\bar{\Phi}|^3$$

⑤



Solving $V_{eff}'(\bar\Phi_{min}) = 0$, $V_{eff}(\bar\Phi_{min}) = V_{eff}(0)$

$$\Rightarrow \quad \bar\Phi_c = \frac{g_3^3}{3\pi\lambda_3}, \quad m_{3,c}^2 = \frac{2g_3^6}{(6\pi)^2\lambda_3}.$$

This is a first-order phase transition.

What about perturbation theory?

$$\text{scalar loop} \sim \frac{\lambda_3}{\pi m_H} \sim \frac{\lambda_3 \pi\sqrt{\lambda_3}}{\pi g_3^3} \sim \left(\frac{\lambda_3}{g_3^2}\right)^{3/2}$$

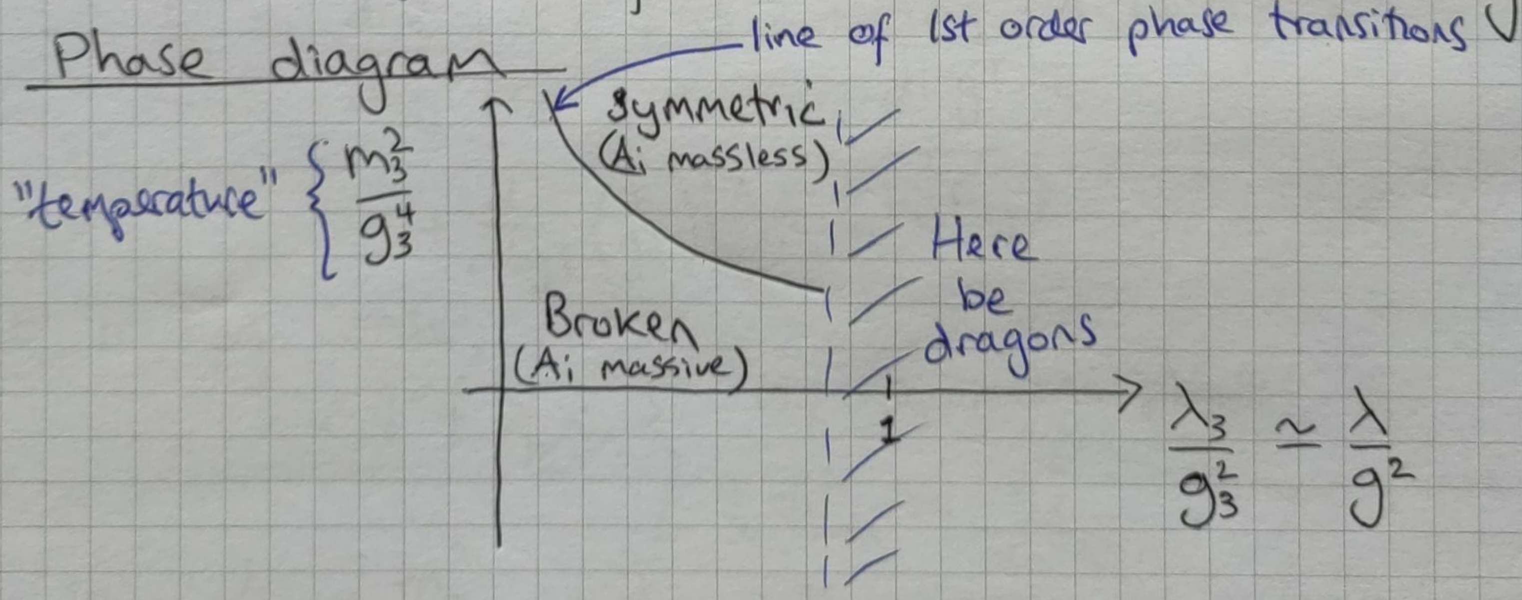
$$\text{vector loop} \sim \frac{g_3^2}{\pi m_V} \sim \frac{g_3^2 \cdot \pi\lambda_3}{\pi g_3^4} \sim \left(\frac{\lambda_3}{g_3^2}\right)$$

So, perturbative corrections are controlled by

$$\frac{\lambda_3}{g_3^2} \sim \frac{m_H^2}{m_V^2}$$

and are small if there is a scale hierarchy.

Phase diagram



This story actually unfolded over decades of study.

What else can we learn than the order?

Latent heat:

$$\frac{L}{T_c^4} = \frac{d}{d\log T}\Bigg(\underbrace{\frac{f_{broken}}{T^4} - \frac{f_{symmetric}}{T^4}}_{\frac{V_{eff}(\bar\Phi_{min})}{T^3}.}\Bigg)\Bigg|_{T_c}$$

$$= \frac{d}{d\log T}\left(\frac{V_{eff}(\bar\Phi_{min}, m_3^2, \lambda_3, g_3^2)}{T^3}\right)\Bigg|_{T_c}$$

$$= \cancel{\frac{\partial V_{eff}(\bar\Phi_{min})}{\partial\bar\Phi_{min}}}^{\to 0}\frac{d\bar\Phi_{min}}{dT}\Bigg|_{T_c} + \underbrace{\frac{\partial V_{eff}(\bar\Phi_{min})}{\partial m_3^2}}_{\text{pure 3d}}\underbrace{\frac{dm_3^2}{d\log T}}_{\text{4d}-\text{3d}}\Bigg|_{T_c} + \ldots$$

What is $\frac{\partial V_{eff}(\bar\Phi_{min})}{\partial m_3^2}$?

$$\frac{\partial}{\partial m_3^2}\left(-\frac{1}{V}\log\int D\phi DA\, e^{-S_{EFT}}\right) = +\frac{1}{VZ}\int D\phi DA\,\frac{\partial S_{EFT}}{\partial m_3^2}e^{-S_{EFT}}$$

$$= \frac{1}{Z}\int D\phi DA\,\frac{1}{V}\left(\int d^3x\,\phi^*\phi\right)e^{-S_{EFT}}$$

$$= \langle\overline{\phi^*\phi}\rangle$$

volume average

TFT average

So,

$$\frac{L}{T_c^4} = \frac{dm_3^2}{d\log T}\Bigg|_{T_c}\Delta\langle\overline{\phi^*\phi}\rangle + \frac{d\lambda_3}{d\log T}\Bigg|_{T_c}\Delta\langle\overline{(\phi^*\phi)^2}\rangle$$

$$+ \frac{d}{d\log T}\left(\frac{1}{g_3^2}\right)\Bigg|_{T_c}\Delta\langle\overline{\tfrac{1}{4}F_{ij}F_{ij}}\rangle$$

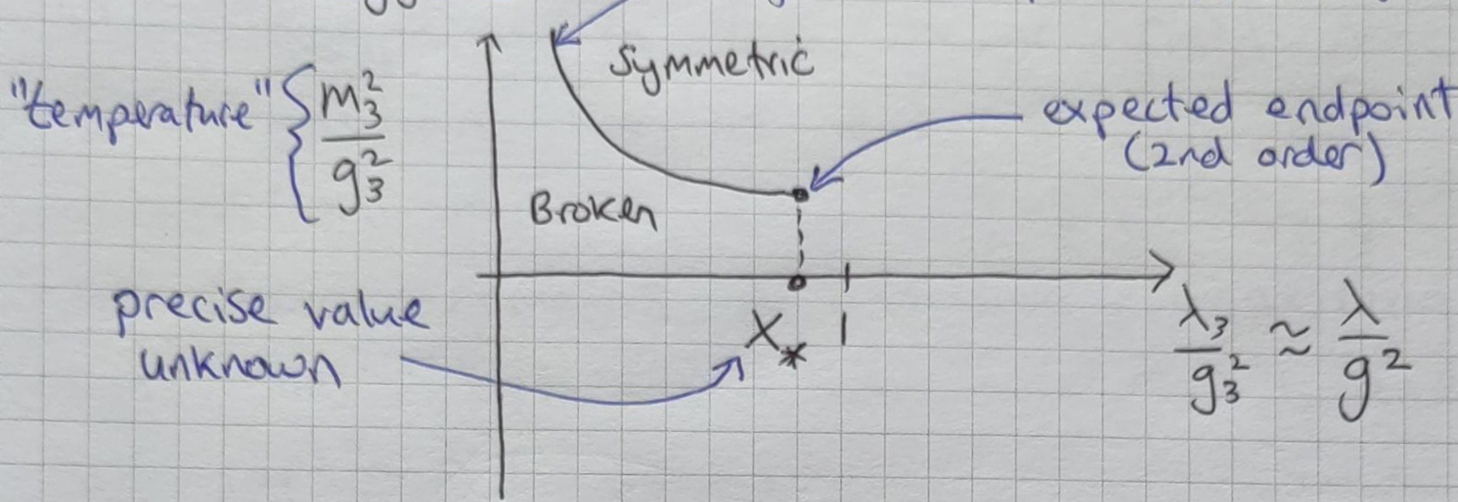
At leading order,

$$\langle\overline{\phi^*\phi}\rangle\Big|_{T_c} = \begin{cases} 0, & \text{symmetric phase} \\ \frac{1}{2}\bar\Phi_{min}^2 = \frac{g_3^6}{18\pi^2\lambda_3^2}, & \text{broken phase}\end{cases}$$

so, $\frac{L}{T_c^4} \propto \frac{m_V^2}{m_H^2}$ and gets stronger for a larger hierarchy.

Conversely, the transition gets weaker as we approach strong coupling, at $\frac{\lambda}{g^2} \sim 1$.

This <u>suggests</u>



So, if $\frac{\lambda}{g^2} < x_* \Rightarrow$ transition is 1st order

$\frac{\lambda}{g^2} = x_* \Rightarrow$ 2nd order

$\frac{\lambda}{g^2} > x_* \Rightarrow$ crossover $\left(\frac{d^n f}{dT^n} \text{ all finite}\right)$

For the electroweak theory, it is very similar

$$x_* \sim 0.11$$

$$x_{SM} \sim 0.3 > x_* \Rightarrow \text{crossover.}$$

This was studied using lattice Monte-Carlo simulations of the 3d EFT for the electroweak sector.