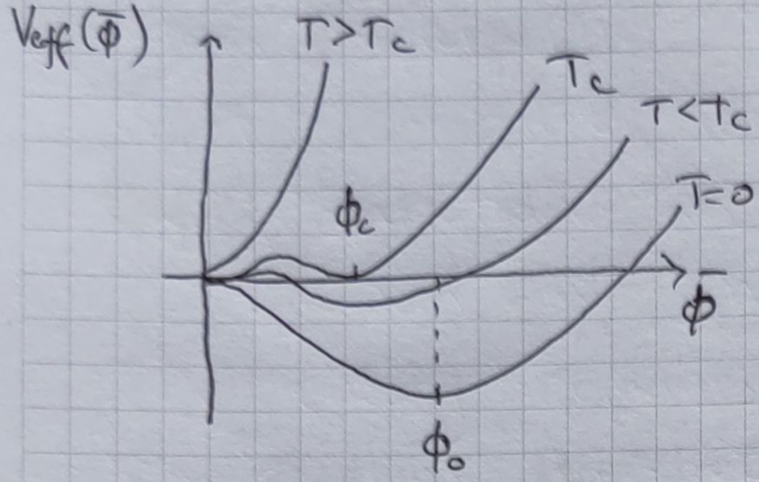
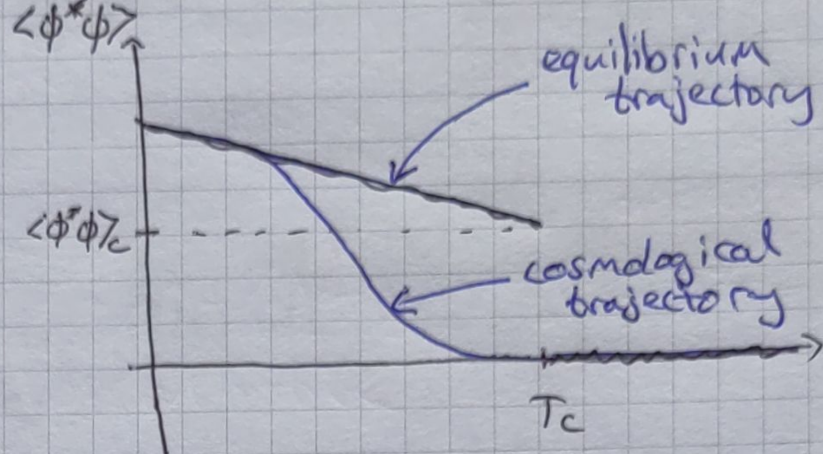
Phase transitions III

Cosmological evolution of a 1st order phase transition





Transition mechanism:

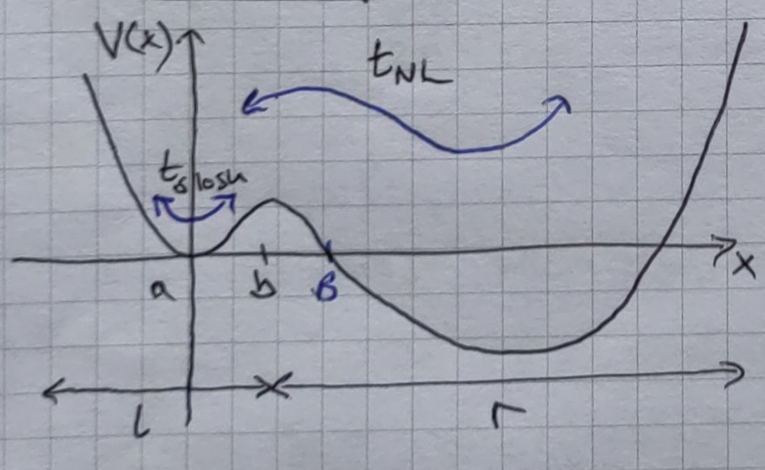
- Transition for volume L^3 suppressed by $e^{-V_{eff}(\bar{\Phi}_{barrier})L^3}$
  $\Rightarrow$ must be local
- Thermal fluctuation over barrier
- Quantum tunnelling through barrier

For cosmology, the transition rate must be faster than Hubble

$$\underbrace{\Gamma}_{\substack{\text{transition rate}\\ \text{per unit}\\ \text{volume}}} \gtrsim H^4 \Rightarrow \gtrsim \text{once per Hubble patch}$$

Need to understand these rates, Γ.

Particle escape rate



Start at 'a' in metastable well. How long until $x>b$?

Assume, $\rho(x>b,p)=0$
$\rho(x=b,p<0)=0$

$$\underbrace{t_{slosh} \ll \overbrace{\Gamma^{-1} \ll t_{NL}}}_{\rho(x<b,p)\,\propto\, e^{-\beta H}}$$

Classical, thermal

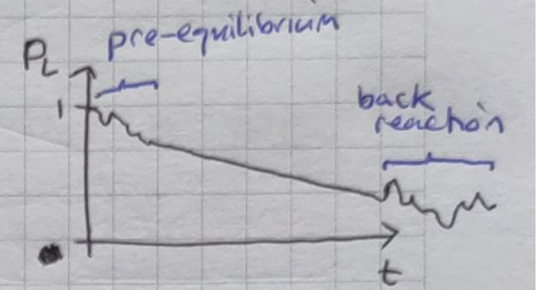
$$H=\frac{p^2}{2m}+V(x)$$

Liouville's equation for probability density:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\underbrace{\left(\frac{p}{m}\rho\right)}_{J_x = \dot{x}\rho} + \frac{\partial}{\partial p}\underbrace{\left(-V'(x)\rho\right)}_{J_p = \dot{p}\rho} = \frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 .$$

Define rate as:

$$\Gamma \equiv \lim_{\substack{t/t_{slosh} \to \infty \\ t/t_{NL} \to 0}} \frac{1}{P_L(t)} \frac{d P_r(t)}{dt}$$



And as $P_r = 1 - P_L$,

$$\Rightarrow \dot{P}_L = -\Gamma P_L \Rightarrow P_L(t) = e^{-\Gamma t} .$$

$$\frac{dP_r}{dt} = \frac{d}{dt}\int_r dx\,dp\,\rho(x,p) = -\int_r dx\,dp\,\nabla\cdot J$$

$$= \int_{x=b} J\cdot\hat{x} = \int dp \cdot \frac{p}{m} \mathcal{N} e^{-\beta H(b,p)} \,\theta(p)$$

$$= \left[-T\mathcal{N} e^{-\beta\left(\frac{p^2}{2m} + V(b)\right)}\right]_{p=0}^{p=\infty} = T\mathcal{N} e^{-\beta V(b)}$$

$$P_L = \int dx\,dp\,\mathcal{N} e^{-\beta H} \simeq \int dx\,dp\,\mathcal{N} e^{-\beta\left(\frac{p^2}{2m} + \frac{1}{2} m\omega_a^2 x^2\right)}$$

$$= \sqrt{2\pi m T}\sqrt{\frac{2\pi T}{m\omega_a^2}}\,\mathcal{N} = \frac{2\pi T\mathcal{N}}{\omega_a}$$

$$\Rightarrow \Gamma = \underbrace{\frac{\omega_a}{2\pi}}_{\text{frequency of 'attempts'}} \underbrace{e^{-\beta V(b)}}_{\text{Boltzmann suppression}}$$

This assumes Hamiltonian evolution. For Langevin evolution (with noise and damping) the result is similar, but the 'frequency of attempts' is reduced by the damping, γ; Kramers (1940) Physica 7 284.

$$\Gamma = \left(\frac{\sqrt{\omega_b^2 + \gamma^2/4} - \gamma/2}{\omega_b}\right) \cdot \frac{\omega_a}{2\pi} e^{-\beta V(b)}$$

Now let's try the quantum case $(T=0)$.

$$P_R(t) = \int_R dx_f \, |\underbrace{\langle x_f, t | a, 0 \rangle}_{D_F(a,0;x_f,t)}|^2$$

initial state

final state

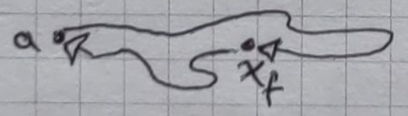
To transfer easily to field theory, we develop a path integral approach,

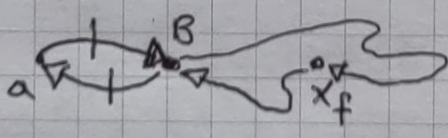
$$D_F(a,0;x_f,t) = \int_{\substack{x(0)=a \\ x(t)=x_f}} Dx \, e^{iS[x]}$$

When $t \ll t_{NL}$ we don't expect the evolution for $x > B$ to affect the rate, so let's factor it out

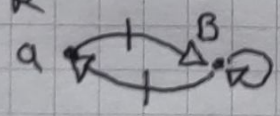
$$D_F(a,0;\underbrace{x_f}_{x_f \in R},t) = \int_0^t dt_0 \underbrace{\int_{\substack{x(0)=a \\ x(t_0)=B}} Dx \, \delta(\underbrace{t_B[x]}_{\text{returns first time } x \text{ crosses } B} - t_0) e^{iS[x]}}_{\bar{D}_F(a,0;B,t_0)} \underbrace{\int_{\substack{x(t_0)=B \\ x(t)=x_f}} Dx \, e^{iS[x]}}_{D_F(B,t_0;x_f,t)}$$

Rather that writing everything out longhand, I'm going to use: $\longrightarrow\!\triangleright \equiv D_F$, $\,\longrightarrow\!\!\!\!\!\!+\;\triangleright \equiv \bar{D}_F$

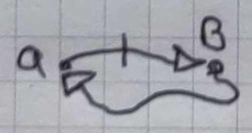
$P_R(t) =$ 

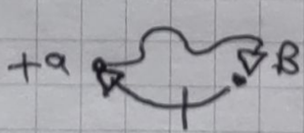
$= \int_R dx_f$ 

(integrals over intermediate times implied.)

$\simeq$ 

$t \ll t_{NL} \Rightarrow \int_R dx_f \langle B, t_0 | x_f, t \rangle \langle x_f, t | B, t_0' \rangle$
$\simeq \langle B, t_0 | B, t_0' \rangle$

$\simeq$ 

$+$ 

Andreassen, Frost and Schwartz 1604.06070

$$\Gamma \equiv \lim_{t\to\infty} \frac{\bar{D}_F(a,0;B,t)\,D^*_F(a,0;B,t) + c.c.}{\int dx\, |D_F(a,0;x,t)|^2}$$

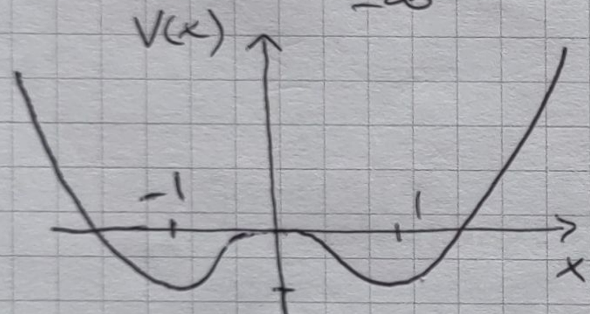
This can be simplified by Wick rotation ($\tau \to it$), noting that:

- D_F and D_F^* rotate with opposite phases
- The $\delta(t_B[x]-t_0)$ factors pick up 'i'.
- The two terms in the numerator can be combined into one going from $-\tau$ to τ; $x(\pm\tau)=a$.

$$\boxed{\Gamma = \lim_{\tau\to\infty} 2\,\mathrm{Im}\left(\frac{\int \mathcal{D}x\, e^{-S_E[x]}\,\delta(\tau_B[x])}{\int \mathcal{D}x\, e^{-S_E[x]}}\right)}$$

Imaginary parts from 'real' integrals?

$$I \equiv \int_{-\infty}^{\infty} dx\, e^{-\beta V(x)}, \quad V(x) = -\frac{x^2}{2} + \frac{x^2}{4}, \quad \beta >> 1.$$

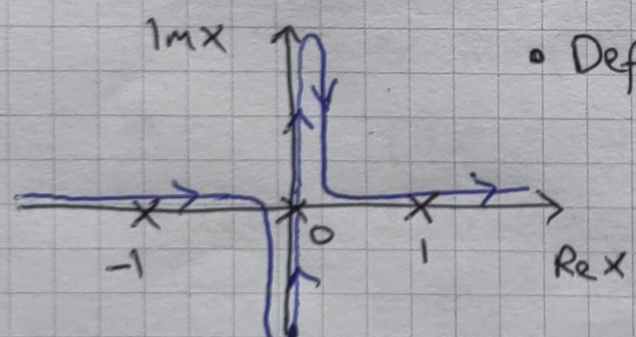


Steepest descent method:

solve extrema $\left\{ V'(x) = -x(1-x^2) \stackrel{!}{=} 0 \right.$
$\Rightarrow x = 0, \pm 1$

quadratic fluct$^{\underline{n}}$s $\left\{ V''(x) = -1+3x^2 \right.$

$$I = I_1 + I_2 + I_3$$



- Deform contour in complex plane to pass through each extremum so that the real part diminishes fastestst away from the extremum.

$$I_1 = I_3 \simeq \int_{-\infty}^{\infty} dx\, e^{\beta(\frac{1}{4}-\frac{3}{2}x^2)} = \sqrt{\frac{2\pi}{3\beta}}\, e^{\beta/4}$$

$$I_2 \approx \int_{-i\infty}^{i\infty} dx\, e^{\beta x^2/2} = -i\int_{-\infty}^{\infty} dy\, e^{-\beta y^2} = -i\sqrt{\frac{2\pi}{\beta}}$$

$$\Rightarrow \quad I \approx \sqrt{\frac{2\pi}{3\beta}}\, e^{\beta/4} - i\sqrt{\frac{2\pi}{\beta}} + \sqrt{\frac{2\pi}{3\beta}}\, e^{\beta/4}$$

exponentially subdominant

- I_2 should be dropped at this order. I is real, and all imaginary parts must cancel between the saddles in a complete calculation.
- In Γ, the $\delta(\tau_B[x])$ removes the dominant saddles, leaving us with a genuine imaginary part.

Decay rate in QFT

$$\Gamma = \lim_{\tau \to \infty} 2\,\mathrm{Im}\left(\frac{\int D\phi\, e^{-S_E}\, \delta(\tau_\Sigma[\phi])}{V \int D\phi\, e^{-S_E}} \right)$$

$\Sigma \equiv$ co-dimension 1 surface in field configuration space separating metastable and stable minima.

Steepest descent:

solve extrema: $\left.\dfrac{\delta S_E}{\delta \phi}\right|_{\tilde\phi} = 0$, $\quad \tilde\phi(\pm\infty) = \phi_a \leftarrow$ metastable minimum

$\tilde\phi(\tau_0 = 0) \in \Sigma$ in numerator.

$-\partial_\mu\partial_\mu + V''(\phi_b)$ quadratic fluctuations: $\left.\dfrac{\delta^2 S_E}{\delta\phi\,\delta\phi}\right|_{\tilde\phi} f_\alpha = \lambda_\alpha f_\alpha$

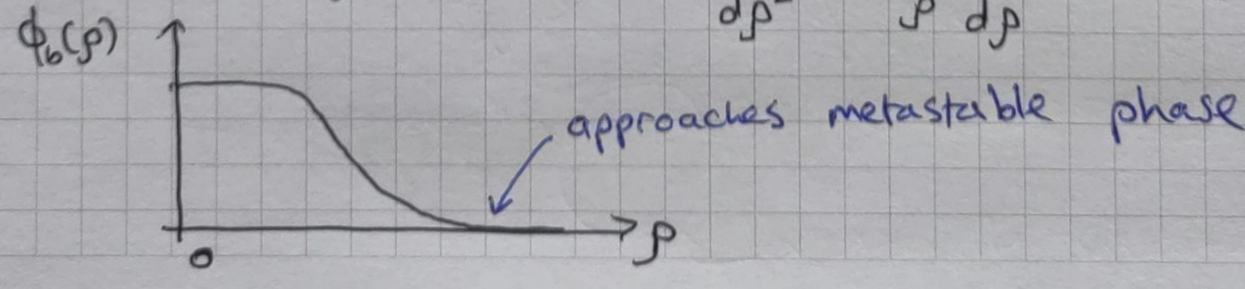
If there is a single negative eigenvalue, $\lambda_- < 0$

$\Rightarrow$ steepest descent contour gives 'i'.

For a first-order phase transition there are two relevant extrema:

- $\tilde\phi = \phi_a$ relevant for denominator
- $\tilde\phi = \phi_{\text{bounce}}(\rho)$ relevant for numerator

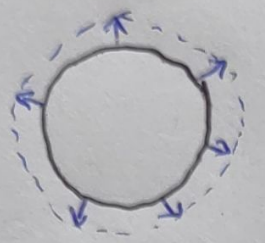
O(4) symmetric sol$^{\underline{n}}$: $\dfrac{d^2\phi_b}{d\rho^2} + \dfrac{3}{\rho}\dfrac{d\phi_b}{d\rho} - V'(\phi_b) = 0.$



$\phi_b(\rho) \in \Sigma \iff \exists$ small perturbations which lower action towards both metastable and stable phases $=$ negative eigenvalue

Negative eigenmode is growing/shrinking, roughly.

$$\phi_b(\rho) \longrightarrow \phi_b(\rho) + \underbrace{\varepsilon \rho \frac{\partial \phi_b(\rho)}{\partial \rho}}_{\text{approximate negative eigenmode}}$$



In addition there are zero modes corresponding to symmetries broken by the solⁿ (Goldstone modes), most notably translations.

$$\partial_\nu \left(\partial_\mu \partial_\mu \phi_b - V'(\phi_b) \right) = 0.$$

$$\Rightarrow \left(\partial_\mu \partial_\mu - V''(\phi_b) \right) \partial_\nu \phi_b = 0.$$

$\Rightarrow \partial_\nu \phi_b$ are zero eigenmodes.

In total, write,

$$\int_x f_n f_m = \delta_{nm}$$

$$\phi = \phi_b(x - \bar{x}) + \sum_n{}' c_n f_n(x - \bar{x}),$$

want to integrate over $\bar{x}_\nu$ rather than $\partial_\nu \phi_b$.

$$\| \delta\phi \|^2 = \int_x \left(\phi_b(x - \delta\bar{x}) + \sum_n{}' c_n f_n(x - \delta\bar{x}) - \phi_b(x) \right)^2$$

$$= \int_x \left(\delta\bar{x}_\mu \partial_\mu \phi_b + \sum_n{}' c_n f_n \right)^2$$

$$= \delta\bar{x}_\mu \delta\bar{x}_\nu \int_x \partial_\mu \phi_b \partial_\nu \phi_b + \sum_n{}' c_n^2$$

$$= \delta\bar{x}_\mu \delta\bar{x}_\mu \underbrace{\frac{1}{d} \int_x \partial_\mu \phi_b \partial_\mu \phi_b}_{J^2} + \sum_n{}' c_n^2 .$$

$$\Rightarrow \int D\phi = \left(\prod_n{}' \frac{dc_n}{\sqrt{2\pi}} \right) \cdot J^d \frac{d^d x}{(2\pi)^{d/2}}$$

$$S_E[\phi] \simeq S_E[\phi_b] + \frac{1}{2}{\sum_n}' \lambda_n c_n^2 ,$$

And now putting it all together,

$$\int D\phi\, e^{-S_E[\phi]} \delta(\tau_\Sigma[\phi])$$

$$\simeq \int \left({\prod_n}' \frac{dc_n}{\sqrt{2\pi}}\right) J^4 \int \frac{d^4x}{(2\pi)^2} \exp\left(-S_E[\phi_b] + \frac{1}{2}|\lambda_-| c_-^2 - \frac{1}{2}\sum_n^{+} \lambda_n c_n^2\right) \delta(\tau_\Sigma)$$

$$= e^{-S_E[\phi_b]} \cdot \frac{J^4}{(2\pi)^2} \cdot V \left(\prod_n^{+} \left(\frac{1}{\lambda_n}\right)^{\frac{1}{2}}\right) \cdot \frac{1}{2} \frac{i}{|\lambda_-|^{1/2}}$$

And the denominator,

$$V \int D\phi\, e^{-S_E[\phi]} = V \prod_n \left(\frac{1}{\lambda_n^{(0)}}\right)^{\frac{1}{2}}$$

So that,

$$\Gamma = \left(\frac{J}{\sqrt{2\pi}}\right)^4 \cdot \sqrt{\frac{\det(-\partial_\mu\partial_\mu + V''(\phi_a))}{|\det'(-\partial_\mu\partial_\mu + V''(\phi_b))|}}\; e^{-S_E[\phi_b]}$$

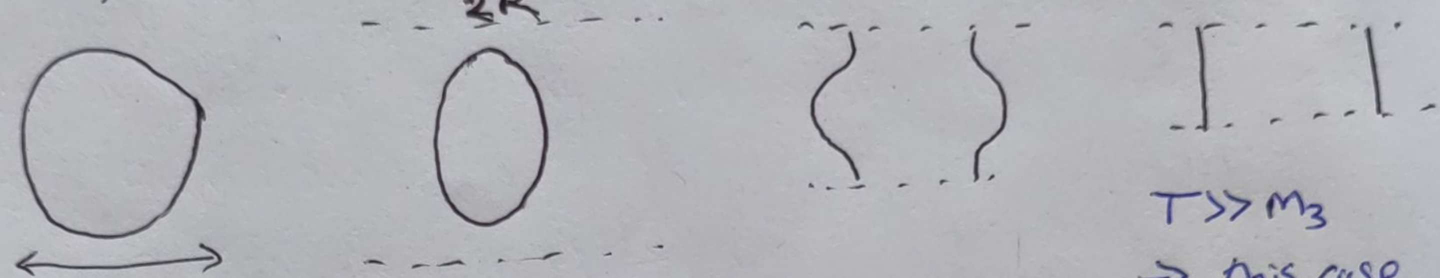
Adding nonzero temperature

Start from a thermal initial state in metastable phase $\hat{\rho}_{meta}$, rather than a pure state $|\phi_a\rangle$.
General structure expected to follow from $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$:

$$\Gamma_T \sim 2\,\mathrm{Im} \left. \frac{\int D\phi\, e^{-S_E[\phi]}\, \delta(\tau_\Sigma[\phi])}{\int D\phi\, e^{-S_E[\phi]}} \right|_{\tau = \frac{1}{T},\ \text{periodic in } \tau}$$

Bounce solutions:

$T=0,\ O(4)$ $\quad$ $T < \frac{1}{2R},\ O(3)$ $\quad$ $T \sim \frac{1}{2R},\ O(3)$ $\quad$ $T > \frac{1}{2R},\ O(3)\times O(2)$



$2R \gtrsim \frac{2}{m_3}$

$T \gg m_3$
$\Rightarrow$ this case

High temperature bounce:

$$\frac{d^2}{dr^2}\phi_b + \frac{2}{r}\frac{d}{dr}\phi_b - V'_{EFT}(\phi_b) = 0$$

potential at tree-level in 3d EFT

First do dimensional reduction, then compute bounce to find, like 3d vacuum decay rate

$$\Gamma_T = A_{dyn} \left(\frac{J_{EFT}}{\sqrt{2\pi}}\right)^3 \underbrace{\sqrt{\frac{\det(-\partial_i\partial_i + V''_{EFT}(\phi_a))}{|\det'(-\partial_i\partial_i + V''_{EFT}(\phi_b))|}}}_{\text{modes} < \Lambda}\, e^{-S_{EFT}[\phi_b]}$$

real-time part associated with $\delta(\tau_\Sigma)$ direction

modes $> \Lambda$ enter through parameters of EFT

$$A_{dyn} \simeq \frac{(\text{initial growth rate of bubble})}{2\pi}$$

$$\simeq \frac{|\lambda_-|^{\frac{1}{2}}}{2\pi}.$$

again, damping modifies this
Langer, Annals of Physics 54 258–275 (1969)