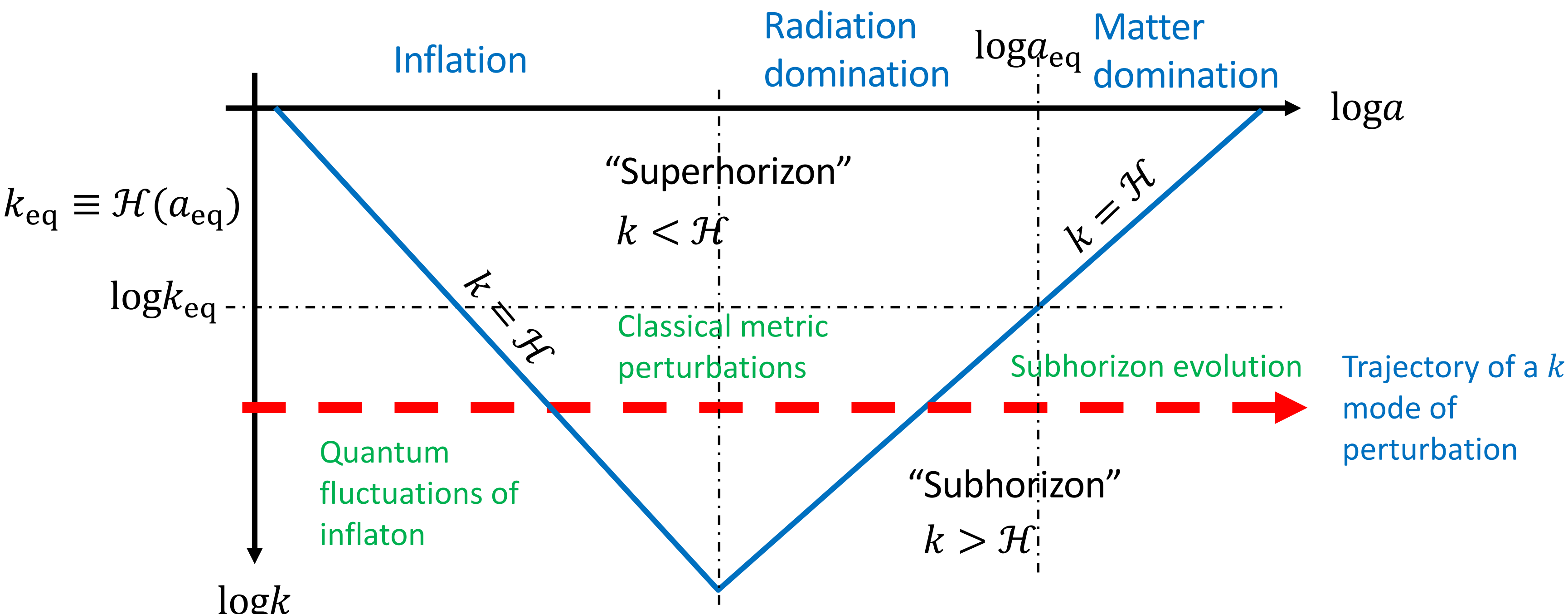
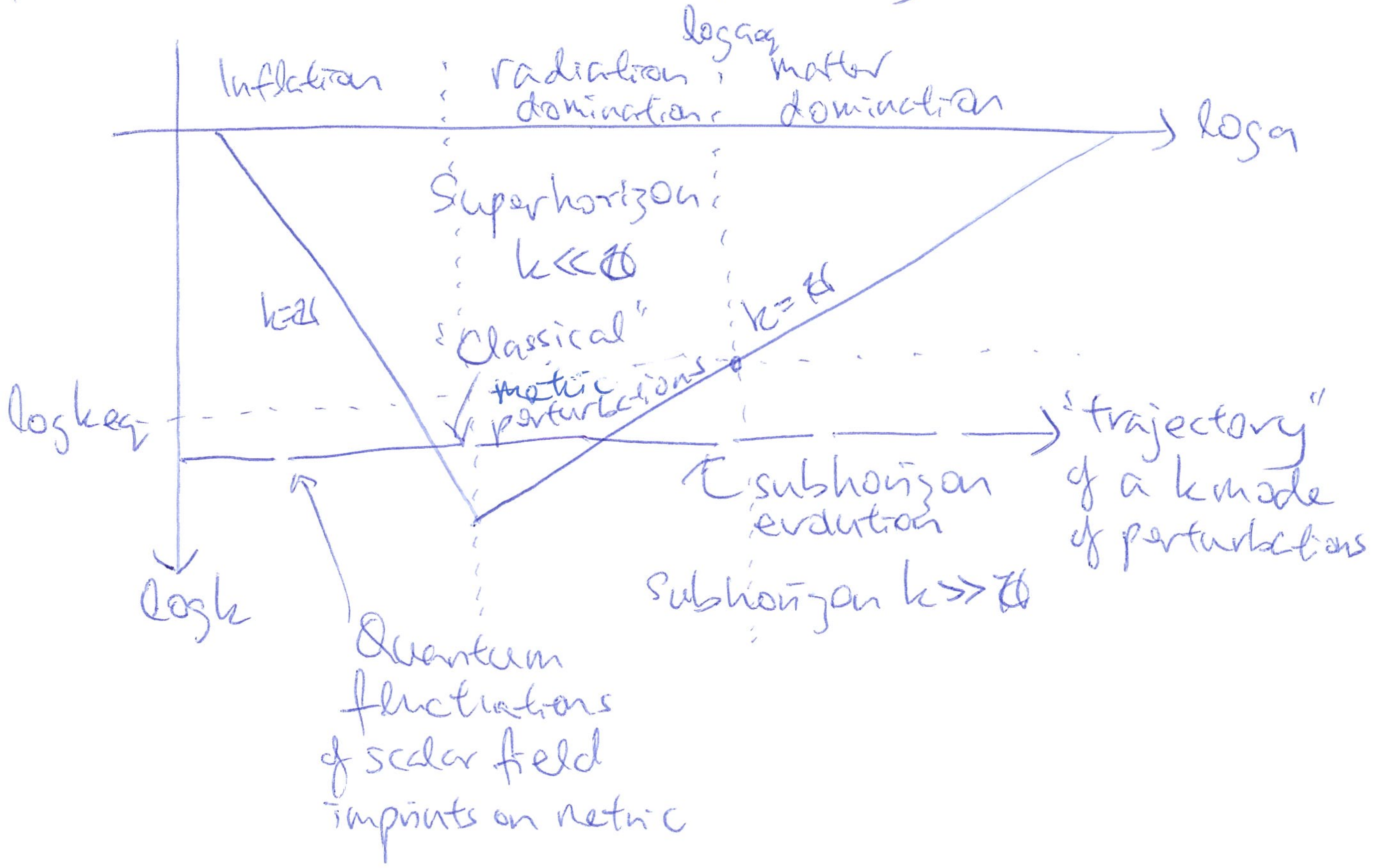
# Initial conditions…

All observables scales are subhorizon today, i.e., $k \gg \mathcal{H} = aH$.



# 3 Initial conditions

All scales that can be observed are subhorizon today, i.e., $k \gg \mathcal{H} = aH$, where $\mathcal{H}^{-1}$ is the comoving Hubble length. $\mathcal{H}$ is a decreasing function of time during RD and MD, but an increasing function of time during inflation. Thus, we have the following situation:




All observed scales today have been superhorizon at some time in the past (by design: inflation must last long enough for this to be true).

Consider a k mode, $k \ll \mathcal{H}$, deep in the radiation era.

The Boltzmann equations in this limit are:

Massless neutrinos $\dot{\delta}_\nu - 4\dot{\Phi} = 0$

Massive neutrinos $\dot{F}_0^{(0)} + \dot{\Phi} \frac{\partial \bar{f}}{\partial \ln q} = 0$

|| Scalar only

Photons: $\dot{\delta}_\gamma - 4\dot{\Phi} = 0$

CDM: $\dot{\delta}_c - 3\dot{\Phi} = 0$

Baryons: $\dot{\delta}_b - 3\dot{\Phi} = 0$

$$\Rightarrow \quad \boxed{\dot{\delta}_\nu = -4\left(\frac{\partial f}{\partial \ln q}\right)^{-1} \dot{F}_0^{(0)} = \dot{\delta}_\gamma = \frac{4}{3}\dot{\delta}_c = \frac{4}{3}\dot{\delta}_b = 4\dot{\Phi}}$$

Two types of solutions are possible:

① Adiabatic: $\boxed{\delta_\nu = \delta_\gamma = \frac{4}{3}\delta_c = \frac{4}{3}\delta_b = \Phi + C}$ ⊛

② Isocurvature:

$$\boxed{\begin{aligned} \delta_{b,c} &= \frac{3}{4}\delta_\gamma + C_{b,c} \\ \delta_\nu &= \delta_\gamma + C_\nu \end{aligned}}$$

← different integration constants

In the adiabatic case,

$$\delta_{b,c} = \frac{\rho_{b,c} - \bar{\rho}_{b,c}}{\bar{\rho}_{b,c}} = \frac{n_{b,c} - \bar{n}_{b,c}}{\bar{n}_{b,c}}$$

$$\delta_{\nu,\gamma} = 4\frac{\Delta T_{\nu,\gamma}}{T_{\nu,\gamma}} = \frac{4}{3}\frac{n_{\nu,\gamma} - \bar{n}_{\nu,\gamma}}{\bar{n}_{\nu,\gamma}}$$

|| If in equilibrium

Thus, the adiabatic initial condition ⊛ are equivalent to

$$\frac{n_i(x)}{n_\gamma(x)} = \frac{\bar{n}_i}{\bar{n}_\gamma}$$

local ratio → ; ← global ratio

i.e., the same ratio of particle number densities everywhere.

Adiabatic initial conditions are a necessary consequence of single-field inflation, in which all perturbations originate from the same inflaton field. Because of their common source, the number density ratios between different fluids must be the same everywhere (determined by equipartition if in equilibrium, or by branching ratios if not.).

If primordial perturbations come from several different sources, then a mixture of adiabatic and isocurvature perturbations is possible. However, if equilibrium is established after the generation of perturbations for <u>all</u> interactions, then the particle number densities must obey either FD or BE statistics locally. This again guarantees that the local number density ratios are the same everywhere and we are back to adiabatic perturbations.

## 3.1 Adiabatic initial conditions

Einstein's equation during radiation domination and in the $k \ll \mathcal{H}$ limit:

$$3\mathcal{H}[\dot{\Phi} + \mathcal{H}\Psi] = -16\pi G a^2 [\bar{\rho}_\gamma \Theta_0^{(0)} + \bar{\rho}_\nu \Delta_0^{(0)}]$$ || using $\bar{\rho}_{\nu,\gamma} \gg \bar{\rho}_{b,c}$

use Friedmann equation $$= -6\mathcal{H}^2 [f_\gamma \Theta_0^{(0)} + f_\nu \Delta_0^{(0)}]$$ $f_\gamma + f_\nu = 1$

adiabatic condition $$= -6\mathcal{H}^2 \Theta_0^{(0)}$$ || $\Theta_0^{(0)} = \Delta_0^{(0)}$

During radiation domination, $\mathcal{H} = \frac{1}{\eta}$. Thus, we find

$$\dot{\Phi}\eta + \Psi = -2\Theta_0^{(0)} \quad (*)$$

$$\Rightarrow \ddot{\Phi}\eta + \dot{\Phi} + \dot{\Psi} = -2\dot{\Theta}_0^{(0)} = -2\dot{\Phi}$$

|| $k \ll \mathcal{H}$: $\dot{\delta}_\gamma - 4\dot{\Phi} \simeq 0$, $\dot{\delta}_\gamma = 4\dot{\Theta}_0$

Assuming $\Phi \simeq \Psi$:

$$\Rightarrow \ddot{\Phi}\eta + 4\dot{\Phi} = 0$$

$$\Rightarrow \Phi = \begin{cases} \text{constant} \\ \eta^{-3} \end{cases}$$

decaying solution. $\Rightarrow$ irrelevant after a long time

Thus the attractor solution is

$$\boxed{\Phi(k \ll \mathcal{H}, \eta \ll \eta_{eq}) = \text{time constant} \equiv \Phi_p(k)}$$ ($p$: primordial)

and from $(*)$

$$\boxed{\Theta_0^{(0)} = \Delta_0^{(0)} = \frac{1}{3}\delta_c = \frac{1}{3}\delta_b = -\frac{1}{2}\Phi_p(k)}$$

Similarly, adiabatic condition implies

$$\Theta_1^{(0)} = \Delta_1^{(0)} = \frac{v_b^{(0)}}{3} + \frac{v_c^{(0)}}{3}$$

and we can use

$$\dot{\Phi} + \mathcal{H}\Psi = 4\pi G a^2 [(\bar{\rho}+\bar{P}) v^{(0)}/k]$$

$\dot{\Phi} \sim 0$, $\mathcal{H}\Psi = \Phi_p$:

$$\Phi_p = \frac{3}{2}\mathcal{H}^2 (1+w) v^{(0)}/k$$

to establish

$$v^{(0)} = \frac{2}{3}\frac{1}{(1+w)}\frac{k}{\mathcal{H}}\Phi_p$$
$$= \frac{1}{2}\frac{k}{\mathcal{H}}\Phi_p \quad \text{for RD } w=\frac{1}{3}$$

Also:
$$\Pi_\nu^{(0)} = \left(\frac{k}{\mathcal{H}}\right)^2 \frac{\Phi_p - \Psi_p}{6 f_\nu}$$

Photon anisotropic stress is zero because during RD photons and electrons are tightly-coupled.

The fact that $\Pi_\nu^{(0)}$ is suppressed by $\left(\frac{k}{\mathcal{H}}\right)^2$ also means that $\Phi - \Psi$ will remain small, justifying $\Phi \simeq \Psi$.

What is $\Phi_p(k)$? This is set by inflation. However, conventionally, perturbations from inflation is specified by the curvature perturbation:

$$\text{(A)} \quad \zeta \equiv -\Phi + \frac{H_T^{(0)}}{3} + \frac{k^i k^{-2}\mathcal{H} T^0{}_i}{\bar{\rho}+\bar{P}}$$

'gauge-invariant'

The advantage of the $\zeta$ variable is that for adiabatic initial conditions, it remains constant in time once the superhorizon condition $k \ll \mathcal{H}$ is satisfied, even after transition to RD (or MD). Then, to translate $\zeta$ from inflation to our $\Phi_p$, we simply evaluate (✱) in the Newtonian gauge, noting that

$$H_T^{(0)} = 0$$
$$T^0{}_i = -(\bar{\rho} + \bar{P}) v_i \qquad \text{|| Newtonian gauge.}$$

$$\Rightarrow \zeta = -\Phi_p - \underbrace{k^i v_i}_{= k v^{(0)}} k^{-2} \mathcal{H}$$

$$= -\Phi_p - \frac{\mathcal{H}}{k} v^{(0)}$$

$$= -\Phi_p - \frac{2}{3} \frac{1}{(1+w)} \Phi_p \qquad \text{|| initial conditions}$$

$$= -\frac{5+3w}{3+3w} \Phi_p$$

$$\Rightarrow \boxed{\Phi_p = -\frac{3+3w}{5+3w} \zeta}$$

$$= -\frac{2}{3} \zeta \quad \text{for RD } w = \frac{1}{3}.$$

# Curvature perturbation…

Conventionally, inflationary perturbations are specified by the curvature perturbation:

$$\zeta \equiv -\Phi + \frac{H_T^{(0)}}{3} + \frac{k^i k^{-2} \mathcal{H} T^0_{\ \cdot\ i}}{\bar{\rho} + \bar{P}}$$

"Gauge-invariant"

- For adiabatic initial conditions, $\zeta$ is **constant on superhorizon scales**.
- Mapping to the Newtonian gauge:

$$\Phi_p(k,\eta) = -\frac{3+3w}{5+3w}\zeta(k)$$

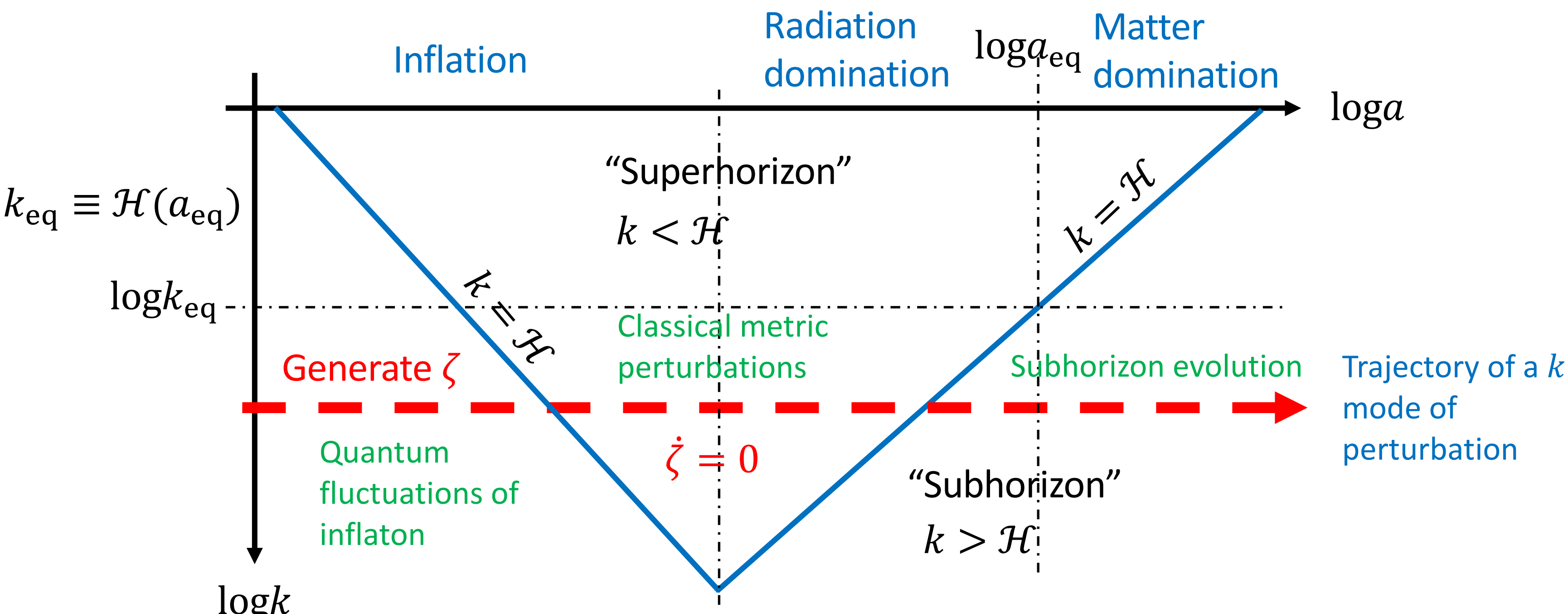
Radiation domination: $w = 1/3$

$$\Phi(k,\eta) = -\frac{2}{3}\zeta(k)$$

$w$ = equation of state of the universe

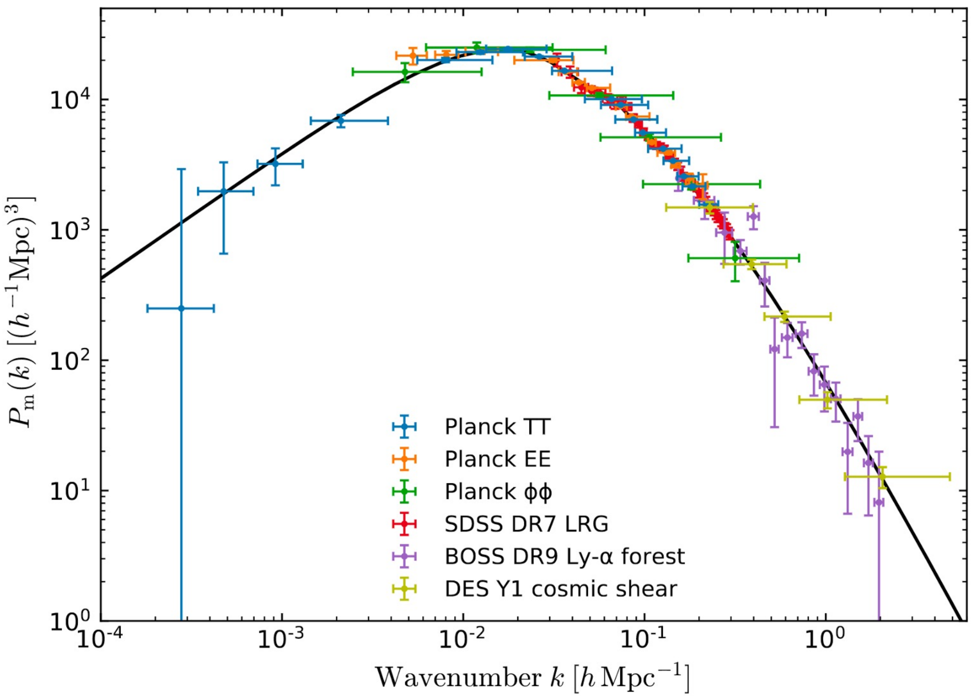
# Initial conditions…

All observables scales are subhorizon today, i.e., $k \gg \mathcal{H} = aH$.



# Deconstructing the matter power spectrum…

**Why does the matter power spectrum look like this?**




$$\langle \delta_m(\vec{k})\delta_m(\vec{k}')\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')P_m(k)$$

$m$ = matter

- We can use a **simplified system** of equations understand the qualitative features.

# Simplified system…

Ignoring:
1. Neutrinos
2. Baryons
3. Anisotropic stress, $\Phi = \Psi$

Assuming adiabatic initial conditions

- Boltzmann equation for **dark matter** (exact):

$$\dot{\delta}_c + k v_c^{(0)} - 3\dot{\Phi} = 0$$
$$\dot{v}_c^{(0)} + \mathcal{H} v_c^{(0)} - k\Phi = 0$$

- Einstein equation for **scalar perturbations**:

$$k^2\Phi + 3\mathcal{H}\left(\dot{\Phi} + \mathcal{H}\Phi\right) = -4\pi G a^2\left(\bar{\rho}_c\delta_c + \bar{\rho}_\gamma\delta_\gamma\right)$$

or

$$\dot{\Phi} + \mathcal{H}\Phi = 4\pi G a^2\left[\bar{\rho}_c v_c^{(0)} + \frac{4}{3}\bar{\rho}_\gamma v_\gamma^{(0)}\right]/k$$

- Boltzmann equation for **photons**:

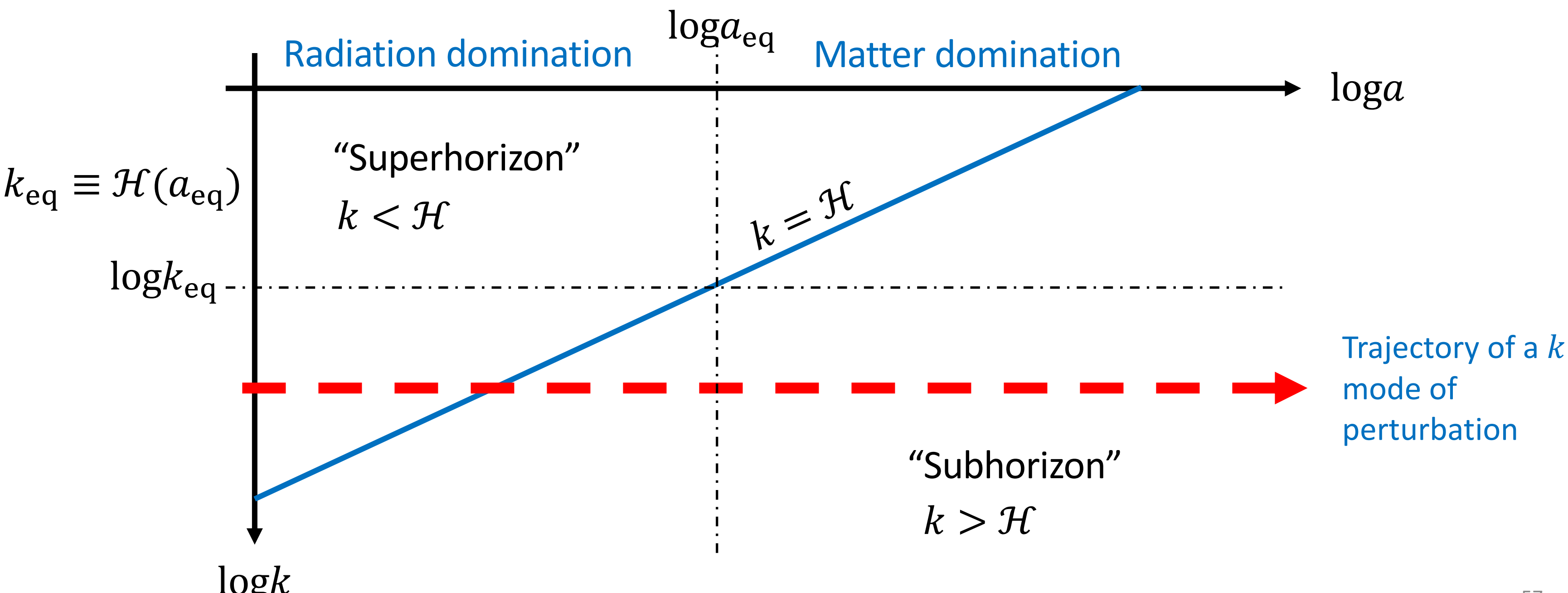
$$\dot{\delta}_\gamma + \frac{4}{3} k v_\gamma^{(0)} - 4\dot{\Phi} = 0$$
$$\dot{v}_\gamma^{(0)} - k\left[\frac{1}{4}\delta_\gamma + \Phi\right] = 0$$

Truncated; we're ignoring all multipoles $\ell \geq 2$, which is OK pre-photon decoupling in the tightly- coupled limit where $\dot{\kappa} \gg \mathcal{H}$.

# Three stages of evolution…

**Trajectory** of a $k$ mode: superhorizon → horizon crossing → subhorizon

- **Crucial point**: When? During RD of MD?

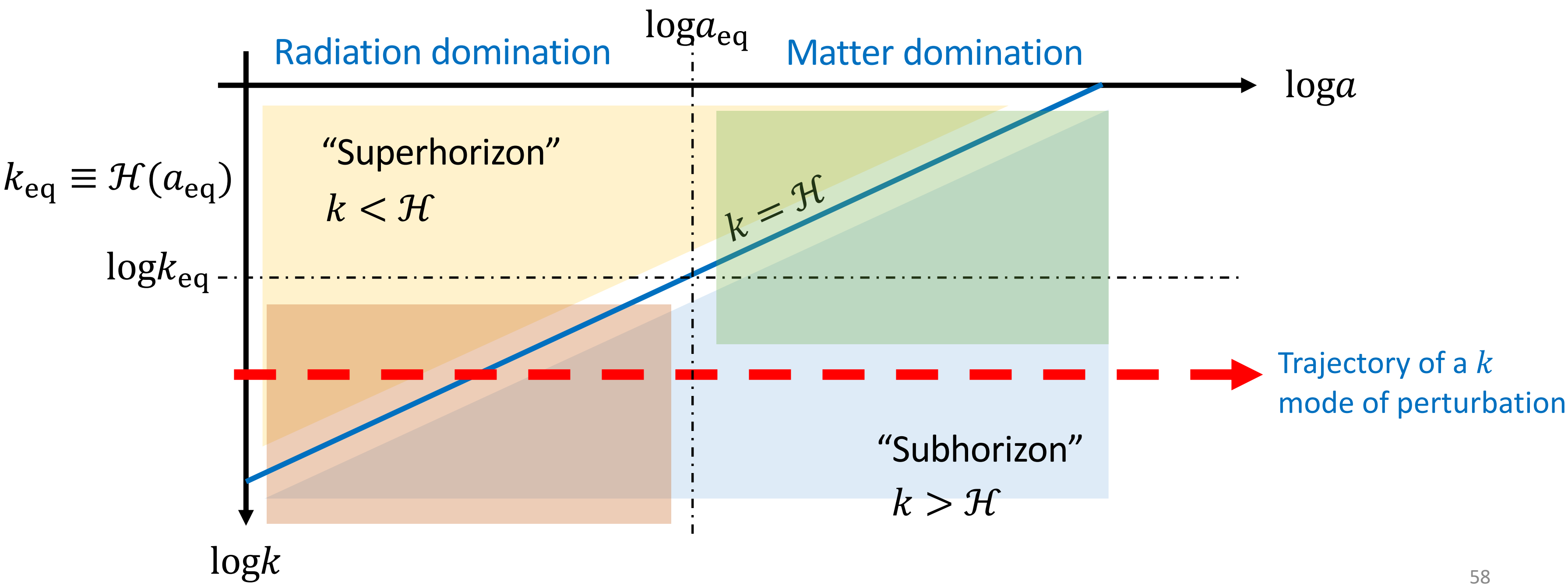


# Three stages of evolution…

Shaded regions = approximate analytical solutions exist for the simplified system

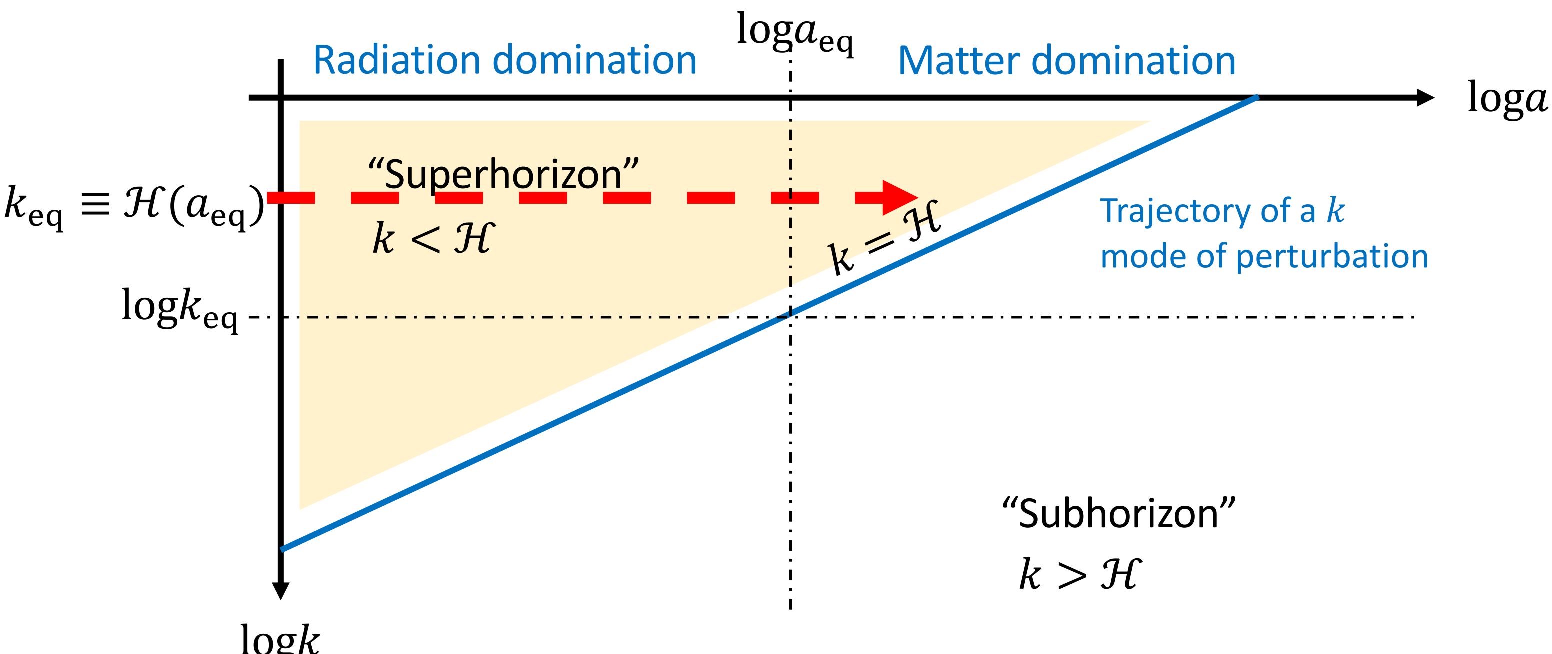
**Trajectory** of a $k$ mode: superhorizon → horizon crossing → subhorizon

- **Crucial point**: When? During RD of MD?



# Superhorizon evolution…

Consider a $k$ mode that remains entirely superhorizon as we transition from RD to MD.



# Superhorizon evolution…

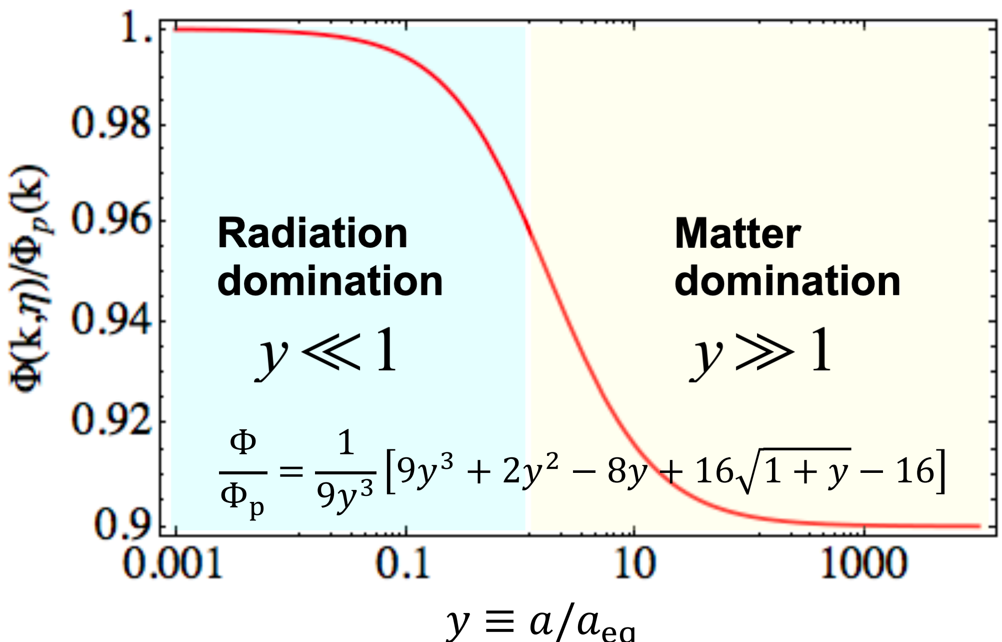
We have seen how the **curvature perturbation** $\zeta$ relates to the superhorizon $\Phi$ in the Newontian gauge for adiabatic initial conditions

$w$ = equation of state

$$\Phi(k,\eta) = -\frac{3+3w}{5+3w}\zeta(k)$$

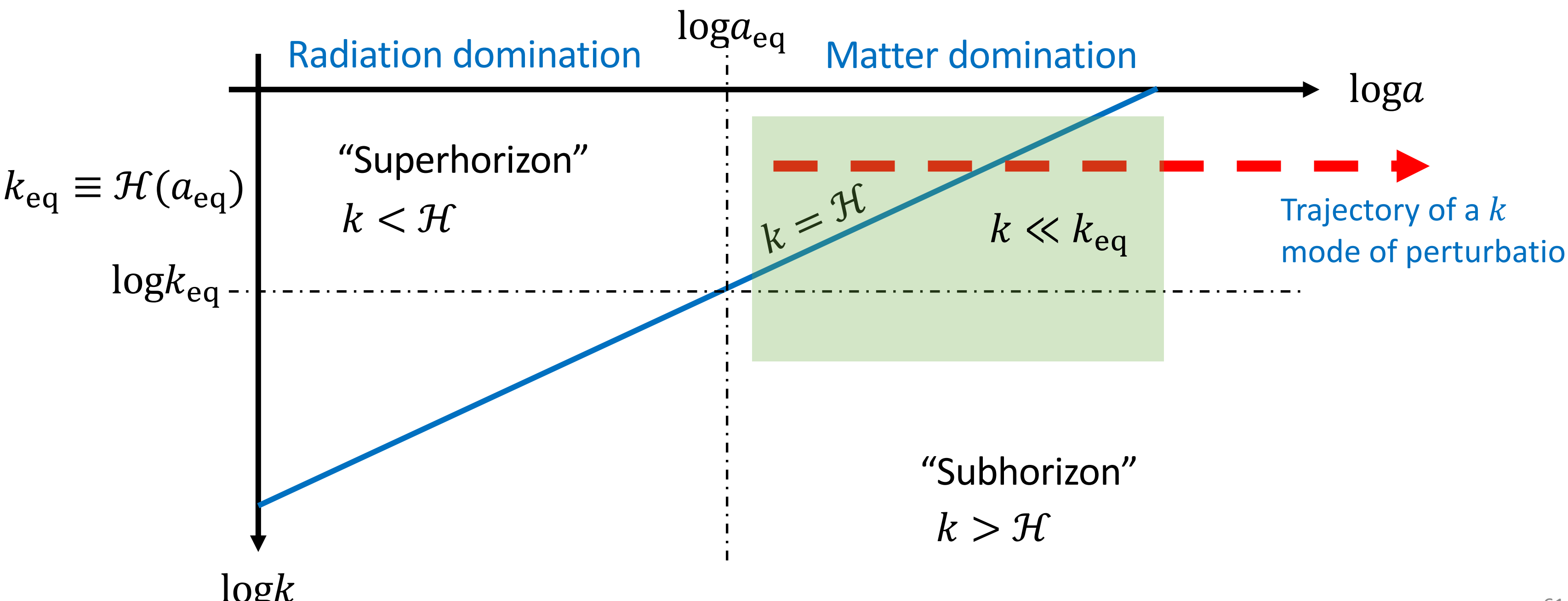
- Crucially, $\zeta$ is constant in time on superhorizon scales.

→ $\Phi$ transitions from $-\frac{2}{3}\zeta$ during RD to $-\frac{3}{5}\zeta$ in MD.



# Horizon crossing during matter domination…

Consider a $k$ mode that goes from superhorizon to subhorizon during matter domination.



# Horizon crossing during matter domination…

During MD, radiation density is unimportant, so we can set $\bar{\rho}_\gamma = 0.$

- The relevant equations are:

$$\dot{\delta}_c + k v_c^{(0)} - 3\dot{\Phi} = 0$$
$$\dot{v}_c^{(0)} + \mathcal{H} v_c^{(0)} - k\Phi = 0$$

CDM

$$k^2\Phi + 3\mathcal{H}\left(\dot{\Phi} + \mathcal{H}\Phi\right) = -4\pi G a^2 \bar{\rho}_c \delta_c = -\frac{3}{2}\mathcal{H}^2\delta_c$$
$$\dot{\Phi} + \mathcal{H}\Phi = 4\pi G a^2 \bar{\rho}_c v_c^{(0)}/k = -\frac{3}{2}\mathcal{H}^2 v_c^{(0)}$$

Scalar Einstein

Growing solution

$$\Phi(k \ll k_{\mathrm{eq}}, \eta) = \frac{9}{10}\Phi_{\mathrm{p}}$$

Subhorizon matter perturbations evolve as:

$$\delta_c \sim a$$

Optimal growth rate. Linear perturbations can **never ever** grow faster than this.

## 4.3 Horizon crossing during matter domination

During matter domination, radiation density is <u>not</u> important $\Rightarrow$ set $\bar{\rho}_\gamma = \delta_\gamma = 0$. Therefore, the relevant equations are

$$\dot{\delta}_c + k v_c^{(0)} - 3\dot{\Phi} = 0$$

$$\dot{v}_c^{(0)} + \mathcal{H} v_c^{(0)} - k\Phi = 0$$

and $$k^2\Phi + 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi) = -4\pi G a^2 \bar{\rho}_c \delta_c = -\frac{3}{2}\mathcal{H}^2 \delta_c$$

$$\Rightarrow \delta_c = -\frac{2}{3\mathcal{H}^2}\left[k^2\Phi + 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi)\right]$$

also: $$\dot{\Phi} + \mathcal{H}\Phi = 4\pi G a^2 \bar{\rho}_c v_c^{(0)}/k = \frac{3}{2}\mathcal{H}^2 v_c^{(0)}/k.$$

$$\Rightarrow v_c^{(0)} = \frac{2}{3}\frac{k}{\mathcal{H}^2}(\dot{\Phi} + \mathcal{H}\Phi)$$

These can be combined to form a 2nd order DE for $\Phi$. In particular:

$$\dot{\delta}_c = \frac{4}{3}\dot{\mathcal{H}}\mathcal{H}^{-3}k^2\Phi + A\dot{\Phi} + B\ddot{\Phi}$$

Thus: (A)

$$\dot{\delta}_c + k v_c^{(0)} - 3\dot{\Phi} = \frac{2}{3}\frac{k^2}{\mathcal{H}}\Phi\underbrace{\left(\frac{2\dot{\mathcal{H}}}{\mathcal{H}^2} - 1\right)} + A'\dot{\Phi} + B'\ddot{\Phi} = 0$$

During matter domination:

$$\mathcal{H}^2 = a^2H^2 = \Omega_m a^{-1} \Rightarrow \frac{d}{d\eta}(\mathcal{H}^2) = 2\mathcal{H}\dot{\mathcal{H}} = -\Omega_m a^{-2}\dot{a} = -\mathcal{H}^3$$

$$\Rightarrow \boxed{\dot{\mathcal{H}} = -\frac{1}{2}\mathcal{H}^2} \quad \Rightarrow \frac{2\dot{\mathcal{H}}}{\mathcal{H}^2} - 1 = 0$$

The full expression for (*) is in fact

$$\ddot{\Phi} + \alpha \dot{\Phi} = 0 \,, \quad \text{where } \alpha = \frac{38 + 7\, k^2/\mathcal{H}^2}{9 + 2\, k^2/\mathcal{H}^2} > 0$$

which has a generic solution of

$$\Phi(\eta) = C_1 + C_2 \underbrace{\int_0^\eta d\eta' \, e^{-\int_0^{\eta'} d\eta'' \alpha(\eta'')}}_{\to 0 \text{ as } \eta \to \infty}$$

Thus, we find

$$\boxed{\Phi(k \ll k_{eq}, \eta) = \text{constant in } \eta = \frac{9}{10} \Phi_p(k)}$$

(from superhorizon solution.)

$\Phi(k, \eta)$ is constant in time during MD, even as a k mode transits from super- to subhorizon.

After a k mode becomes subhorizon ($k \gg \mathcal{H}$), the Einstein equation becomes:

$$k^2 \Phi \simeq -4\pi G a^2 \bar{\rho}_c \delta_c = \text{constant in } \eta$$

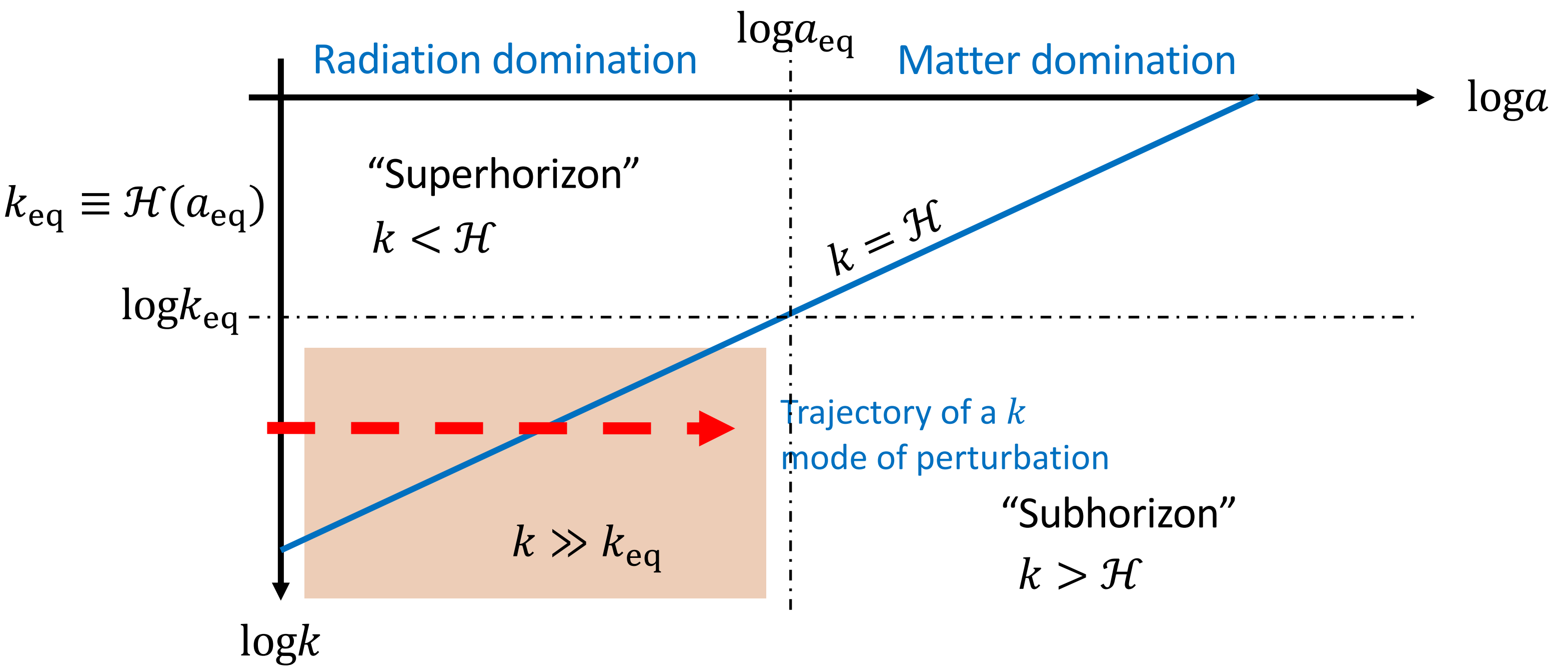
($\bar{\rho}_c \propto a^{-3}$)

$$\Rightarrow \boxed{\delta_c(k \ll k_{eq}, \eta) \sim a}$$

$\Rightarrow$ During MD, CDM density perturbations grow like the scale factor. Optimal growth.

# Horizon crossing during radiation domination...

What about a $k$ mode that goes from superhorizon to subhorizon during radiation domination?



# Horizon crossing during radiation domination...

During RD, **radiation perturbations dominate**.

- Consider first what happens to $\Phi$ in the limit $\bar{\rho}_c = 0$.

$$k^2\Phi + 3\mathcal{H}\left(\dot{\Phi} + \mathcal{H}\Phi\right) = -4\pi G a^2 \bar{\rho}_\gamma \delta_\gamma = -\frac{3}{2}\mathcal{H}^2\delta_\gamma$$

$$\dot{\Phi} + \mathcal{H}\Phi = \frac{16\pi}{3} G a^2 \bar{\rho}_\gamma v_\gamma^{(0)}/k = 2\mathcal{H}^2\, v_\gamma^{(0)}/k$$

Scalar Einstein

$$\dot{\delta}_\gamma + \frac{4}{3} k v_\gamma^{(0)} - 4\dot{\Phi} = 0$$

$$\dot{v}_\gamma^{(0)} - k\left[\frac{1}{4}\delta_\gamma + \Phi\right] = 0$$

Photons

$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0$$

## 4.4 Horizon crossing during radiation domination

During radiation domination, radiation perturbations dominate the metric perturbations. Our strategy here, therefore, is to first solve for $\Phi$ in the limit $\bar{\rho}_c = 0$, and then see how $\delta_c$ responds to $\Phi$ as an external potential.

To solve for $\Phi$, the relevant equations are

$$k^2\Phi + 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi) = -\frac{3}{2}\mathcal{H}^2\delta_\gamma$$
$$\dot{\Phi} + \mathcal{H}\Phi = 2\mathcal{H}^2 v_\gamma^{(0)}/k$$
} Einstein

$$\dot{\delta}_\gamma + \frac{4}{3}k v_\gamma^{(0)} - 4\dot{\Phi} = 0$$
$$\dot{v}_\gamma^{(0)} - k\left[\frac{1}{4}\delta_\gamma + \Phi\right] = 0$$
} Boltzmann for photons.

As with before, these can be combined into a 2nd order DE for $\Phi$:

$$\boxed{\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0} \quad (\star)$$

where we have used

$$\dot{\mathcal{H}} = -\mathcal{H}^2 \quad \| \text{ RD}$$

and

$$a \sim t^{1/2} \sim \eta \quad \text{(cosmic time } t\text{, conformal time } \eta\text{)}$$
$$\mathcal{H} = aH \sim a^{-1} \quad \| \text{ RD}$$
$$\Rightarrow \mathcal{H} = \frac{1}{\eta}$$

(A) has two solutions:

$$\Phi_1 = \frac{C_1}{\eta} j_1(k\eta/\sqrt{3})$$ $j_1$ = Spherical Bessel function of order 1

$$\Phi_2 = \frac{C_2}{\eta} n_1(k\eta/\sqrt{3})$$ $n_1$ = spherical Neumann function of order 1.

As $\eta \to 0$, $n_1(k\eta/\sqrt{3}) \to -\infty \Rightarrow$ unphysical $\Rightarrow$ discard.

Thus, we are left with

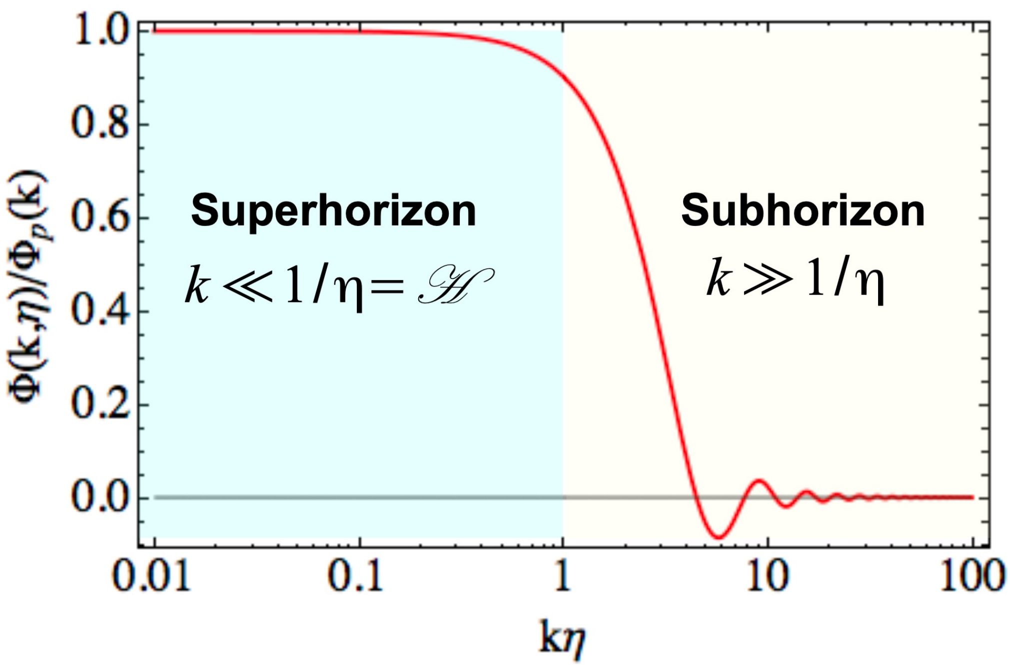
$$\Phi(k \gg k_{eq}, \eta) = 3\,\Phi_p(k) \left( \frac{\sin x - x\cos x}{x^3} \right)$$

where $x \equiv \frac{k\eta}{\sqrt{3}} = \frac{1}{\sqrt{3}} \frac{k}{aH}$

$\Rightarrow$ During RD, $\Phi$ decays away as soon as the k mode enters the horizon because of radiation pressure. Note the oscillations: these are related to acoustic oscillations in the photons.

# Horizon crossing during radiation domination…

Evolution of $\Phi$ across the horizion during RD:




$$\frac{\Phi}{\Phi_p} = 3\left(\frac{\sin x - x\cos x}{x^3}\right)$$

$$x \equiv \frac{k\eta}{\sqrt{3}} = \frac{1}{\sqrt{3}}\frac{k}{\mathcal{H}}$$

- $\Phi$ decays as soon as the $k$ mode enters the horizon because of radiation pressure.
- Oscillations due to **acoustic oscillations** in the photons.

# CDM density perturbations during RD?

We can now feed the solution for $\Phi$ into the equations of motion for CDM:

$$\dot{\delta}_c + k v_c^{(0)} - 3\dot{\Phi} = 0$$

$$\dot{v}_c^{(0)} + \mathcal{H} v_c^{(0)} - k\Phi = 0$$

CDM

$$\ddot{\delta}_c + \frac{1}{\eta}\dot{\delta}_c = k^2 S(k\eta)$$

Source term formed from solution for $\Phi$

$$S(x) = 3\frac{d^2\Phi}{dx^2} + \frac{3}{x}\frac{d\Phi}{dx} - \Phi$$

Radiation pressure hinders growth in the radiation density fluctuations, so the (subdominant) matter density fluctuations can't grow fast either.

Solution at $k\eta \gg 1$ (subhorizon)

$\delta_c \sim \Phi_p \ln(k\eta) \propto \Phi_p \ln(ka)$ during RD

Compared with

$\delta_c \sim a$ during MD

Now we can feed $\Phi(k,\eta) = \Phi(k\eta)$ into the equations of motion for CDM as an external source:

$$\dot{\delta}_c + k v_c^{(0)} - 3\dot{\Phi} = 0$$
$$\dot{v}_c^{(0)} + \mathcal{H} v_c^{(0)} - k\Phi = 0$$
$\mathcal{H} = \frac{1}{\eta}$ during RD

$$\ddot{\delta}_c + \frac{1}{\eta}\dot{\delta}_c = k^2 S(k\eta)\Phi_p \quad (*)$$

where

$$S(x) = 3\frac{d^2\tilde{\Phi}}{dx^2} + \frac{3}{x}\frac{d\tilde{\Phi}}{dx} - \tilde{\Phi}$$
$$\tilde{\Phi} \equiv \frac{\Phi}{\Phi_p}$$

In fact, we can even write $(*)$ as

$$\frac{d^2\delta_c}{dx^2} + \frac{1}{x}\frac{d\delta_c}{dx} = S(x)\Phi_p$$

which has the homogeneous solutions

$$\delta_{c1}(x) = C_1(k) \quad \| \quad C_1 = -\frac{3}{2}\Phi_p \text{ // from superhorizon solution}$$
$$\delta_{c2}(x) = C_2(k)\ln x \quad \| \quad \text{As } \eta \to 0, \ln\eta \to \infty \Rightarrow C_2 = 0$$

Also, the Wronskian is:

$$W(\delta_{c1}, \delta_{c2}) = \delta_{c1}\delta_{c2}' - \delta_{c2}\delta_{c1}' = \frac{1}{x}$$

So that the particular solution is

$$\delta_{cp} = -\delta_{c1}\int\frac{\delta_{c2}S(x)}{W}\Phi_p + \delta_{c2}\int\frac{\delta_{c1}S(x)}{W}\Phi_p$$
$$= -\Phi_p\int_0^x dx'\, x'\ln x'\, S(x') + \Phi_p\ln x\int_0^x dx'\, x'\, S(x')$$

Therefore the formal solution is: $(\star)$

$$\boxed{\delta_c(x) = -\frac{3}{2}\Phi_p - \Phi_p\int_0^x dx'\, x'\ln x'\, S(x') + \Phi_p\ln x\int_0^x dx'\, x'\, S(x')}$$

Since $\Phi$ drops to zero at $x = k\eta \gg 1$, we have:

$$\int_0^x dx' x' \ln x' S(x') \simeq \int_0^1 dx' x' \ln x' S(x')$$

$$= \text{constant at } x \gg 1$$

$$\int_0^x dx' x' S(x') \simeq \int_0^1 dx' x' S(x')$$

$$= \text{constant at } x \gg 1$$

Thus, (A) has the form:

$$\delta_c(x) = -\frac{3}{2}\Phi_p - C_1\Phi_p + \Phi_p C_2 \ln x \quad \| \; x \gg 1$$

$$\simeq C_2\Phi_p \ln x$$

$$= C_2\Phi_p \ln(k\eta)$$

$$= C_2\Phi_p \ln(ka) \quad \| \; \text{since } \eta \propto a \text{ during RD.}$$

Thus, in constrast with $\delta_c \propto a$ during MD,

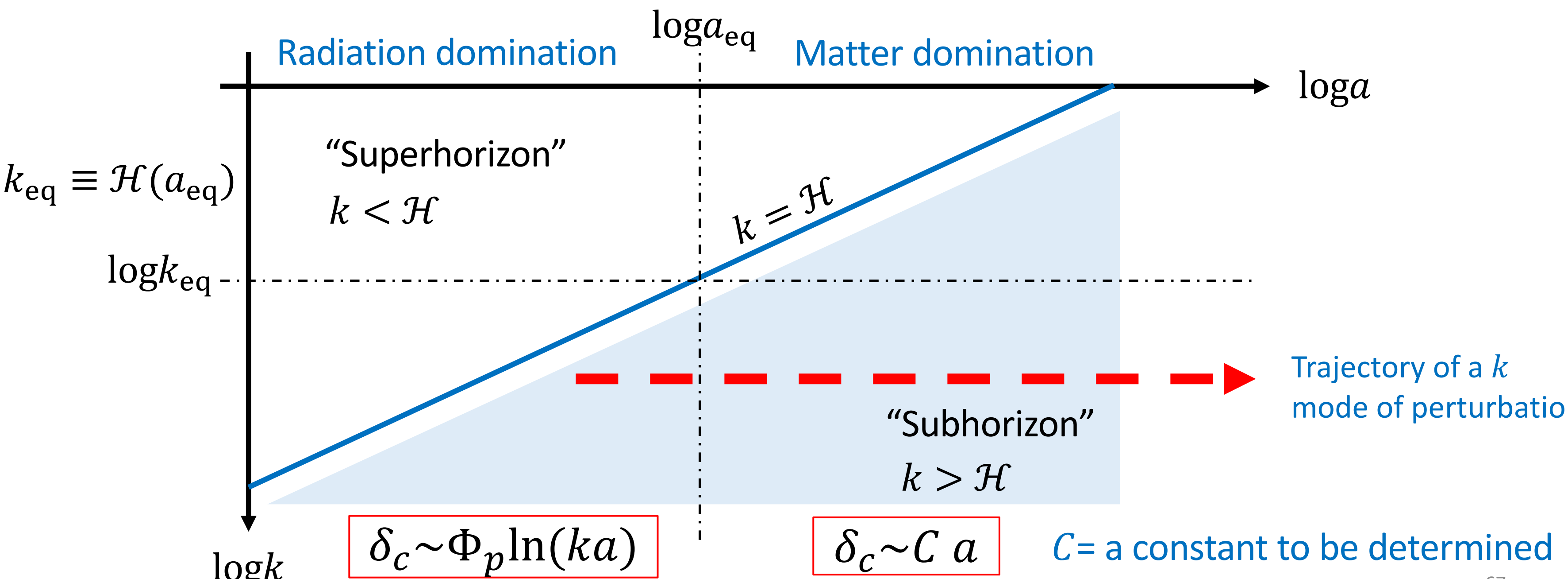
$$\boxed{\delta_c(k\eta \gg 1) \propto \ln(ka)\,\Phi_p}$$

is logarithmically slow. The physical reason is that radiation pressure makes the clumping of matter highly ineffective.

# Subhorizon evolution…

**Trajectory** of a $k$ mode: superhorizon → horizon crossing → subhorizon

- **Crucial point**: When? During RD of MD?




$C$= a constant to be determined

## 4.5 Subhorizon evolution

During radiation domination and after horizon crossing:

$$\delta_c(k,\eta) \sim \ln(k\eta) = \ln(ka)$$

$$\delta_\gamma(k,\eta) \sim \text{oscillatory in } \eta$$

$\Rightarrow$ at some later time, $\bar{\rho}_c\delta_c$ will outgrow $\bar{\rho}_\gamma\delta_\gamma$, even in RD. Therefore, we ignore $\delta_\gamma$ in Einstein's equation:

$$k^2\Phi = -4\pi G a^2 \bar{\rho}\,\delta_c + \mathcal{O}\left(\frac{\mathcal{H}}{k}\right) \leftarrow \text{ignore because in subhorizon limit}$$

$$= -\frac{3}{2}\mathcal{H}^2 \frac{\bar{\rho}_c}{\bar{\rho}_c + \bar{\rho}_\gamma}\delta_c$$

$$= -\frac{3}{2}\mathcal{H}^2 \frac{y}{1+y}\delta_c \quad \Big\| \quad y \equiv \frac{\bar{\rho}_c}{\bar{\rho}_\gamma} = \frac{a}{a_{eq}}$$

The CDM equations are again

$$\dot{\delta}_c + k v_c^{(0)} - 3\dot{\Phi} = 0$$

$$\dot{v}_c^{(0)} + \mathcal{H} v_c^{(0)} - k\Phi = 0$$

Then, combining them all yields 2nd order DE for $\delta_c$:

$$\frac{d^2\delta_c}{dy^2} + \frac{2+3y}{2y(y+1)}\frac{d\delta_c}{dy} - \frac{3}{2y(y+1)}\delta_c = 0$$

which has the formal solution

$$\delta_c(k,\eta) = C_1(k)\,G(\eta) + C_2(k)\,D(\eta)$$

with $G(\eta) = y + 2/3$ growth

$$D(\eta) = (y + 2/3)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 2\sqrt{1+y} \quad \text{decay}$$

The k-dependent coefficients come from matching with the horizon crossing solutions. But if we pick a scale that <u>enters</u> <u>during</u> <u>RD</u> and consider what it should look like <u>during</u> <u>MD</u>, then we need only the growing solution:

$$G(y) \simeq y = \frac{a}{a_{eq}}$$

and

$$\delta_c(k \gg k_{eq}, \eta \gg \eta_{eq}) \sim C(\ln k) \frac{a}{a_{eq}} \Phi_p(k)$$

$C(\ln k)$ ← from RD horizon crossing.

The equivalent $\Phi$ would be:

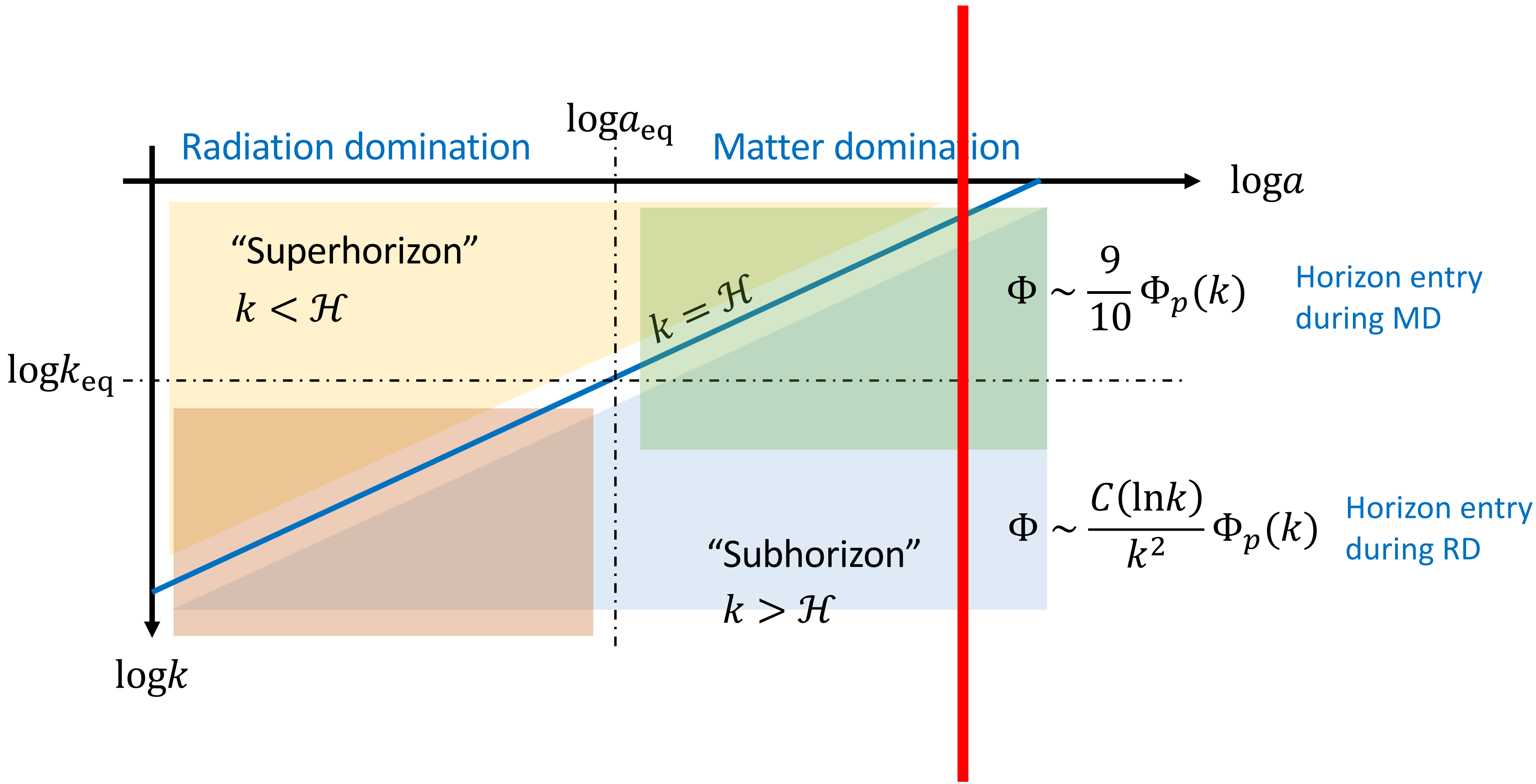
$$\Phi(k \gg k_{eq}, \eta \gg \eta_{eq}) \simeq -\frac{4\pi G a^2}{k^2} \bar{\rho}_c \delta_c \qquad (\bar{\rho}_c \propto a^{-3})$$

$$\sim -\frac{1}{k^2} a^{-1} C(\ln k)\, a\, \Phi_p(k)$$

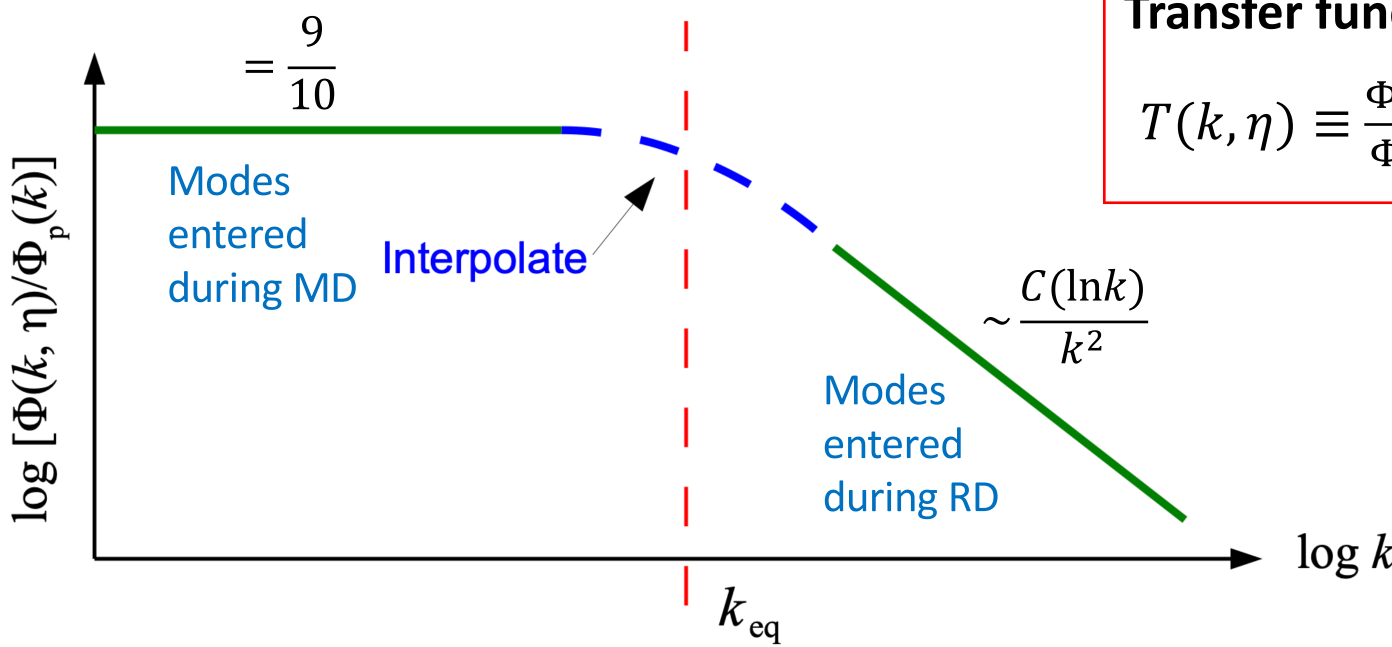
$$\sim \frac{C(\ln k)}{k^2} \Phi_p(k)$$

$\Rightarrow$ constant in time, but suppressed by $\frac{C(\ln k)}{k^2}$ in k space because of horizon entry during RD.

# Putting it all together…



# Transfer function at late times…




**Transfer function:**

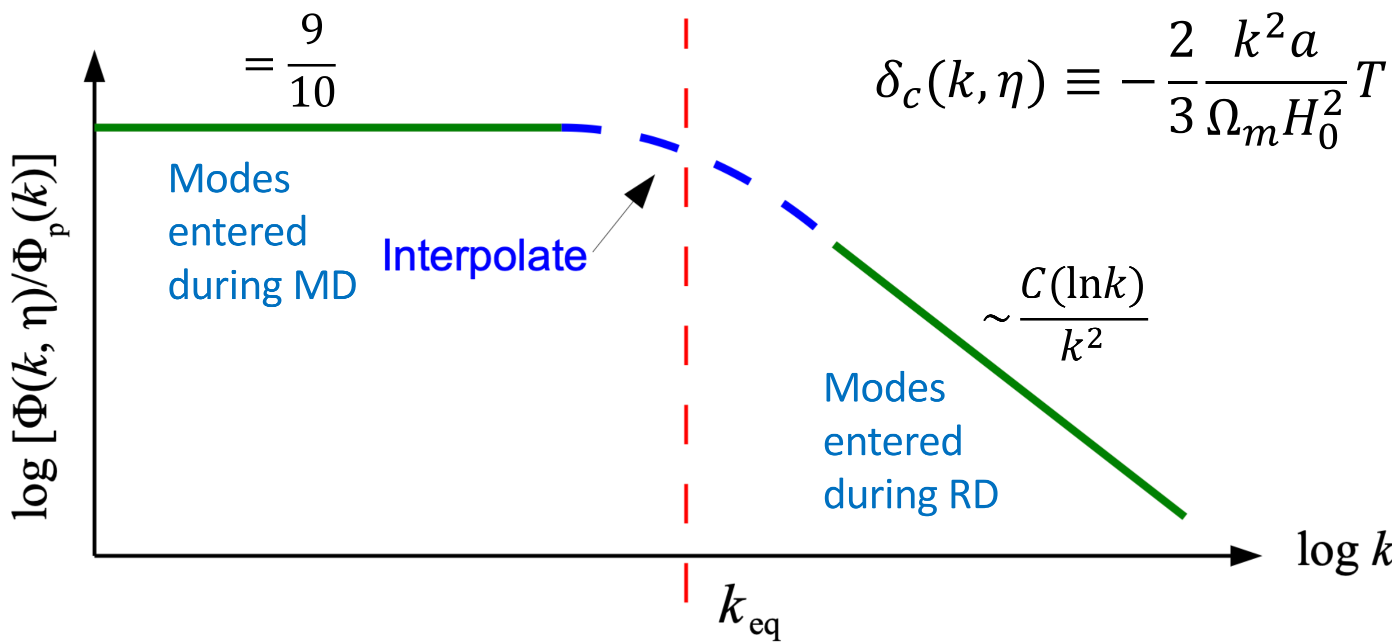
$$T(k,\eta) \equiv \frac{\Phi(k,\eta)}{\Phi_p(k)}$$

- **CDM-type cosmologies** generically all have transfer functions of this shape.

# Transfer function at late times…

**Matter density at late times**

$$\delta_c(k,\eta) \equiv -\frac{2}{3}\frac{k^2 a}{\Omega_m H_0^2} T(k,\eta)\Phi_{\rm p}(k)$$




- **CDM-type cosmologies** generically all have transfer functions of this shape.

# What if we include baryons...

At early times, baryons and photons form a **tightly-coupled** fluid.

→ Like photons, baryon density perturbations oscillate around 0.

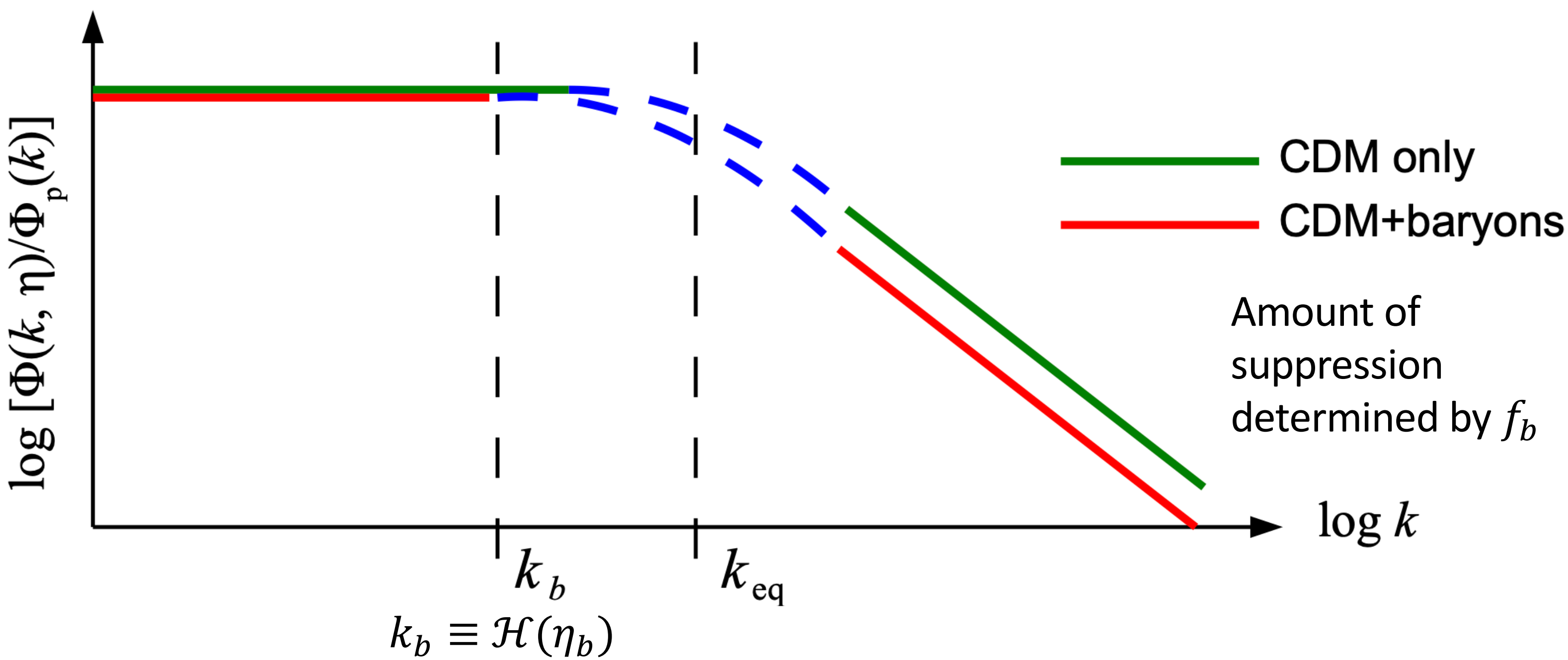
- Replacing some of the CDM with baryons effectively **suppresses the gravitational potential** on subhorizon scales:

Fraction of matter in baryons

$$k^2\Phi = -4\pi G a^2(\bar{\rho}_c\delta_c + \bar{\rho}_b\delta_b) \simeq -4\pi G a^2\bar{\rho}_m(1 - f_b)\delta_c \qquad f_b \equiv \frac{\Omega_b}{\Omega_m}$$

Total matter density

# What if we include baryons…




$\eta_b$ is the time at which baryons decouple from photons (later than photon decoupling from baryons); after decoupling baryons become like CDM.

# Linear matter power spectrum…

**Definition**:

$$\langle \delta_m(\vec{k},a)\delta_m(\vec{k}',a)\rangle = (2\pi)^3\delta^{(3)}\,(\vec{k}+\vec{k}')P_m(k,a)$$

where:

$$P_m(k,a) = \frac{4}{9}\frac{k^4a^2}{\Omega_m^2 H_0^4}T^2(k,a)P_{\Phi_\mathrm{p}}(k)$$

$T$ = Transfer function

$P_{\Phi_\mathrm{p}}$ = Primordial power spectrum predicted by inflation usually parameterised as

$$P_{\Phi_\mathrm{p}}(k) \sim k^{n_s-4}$$

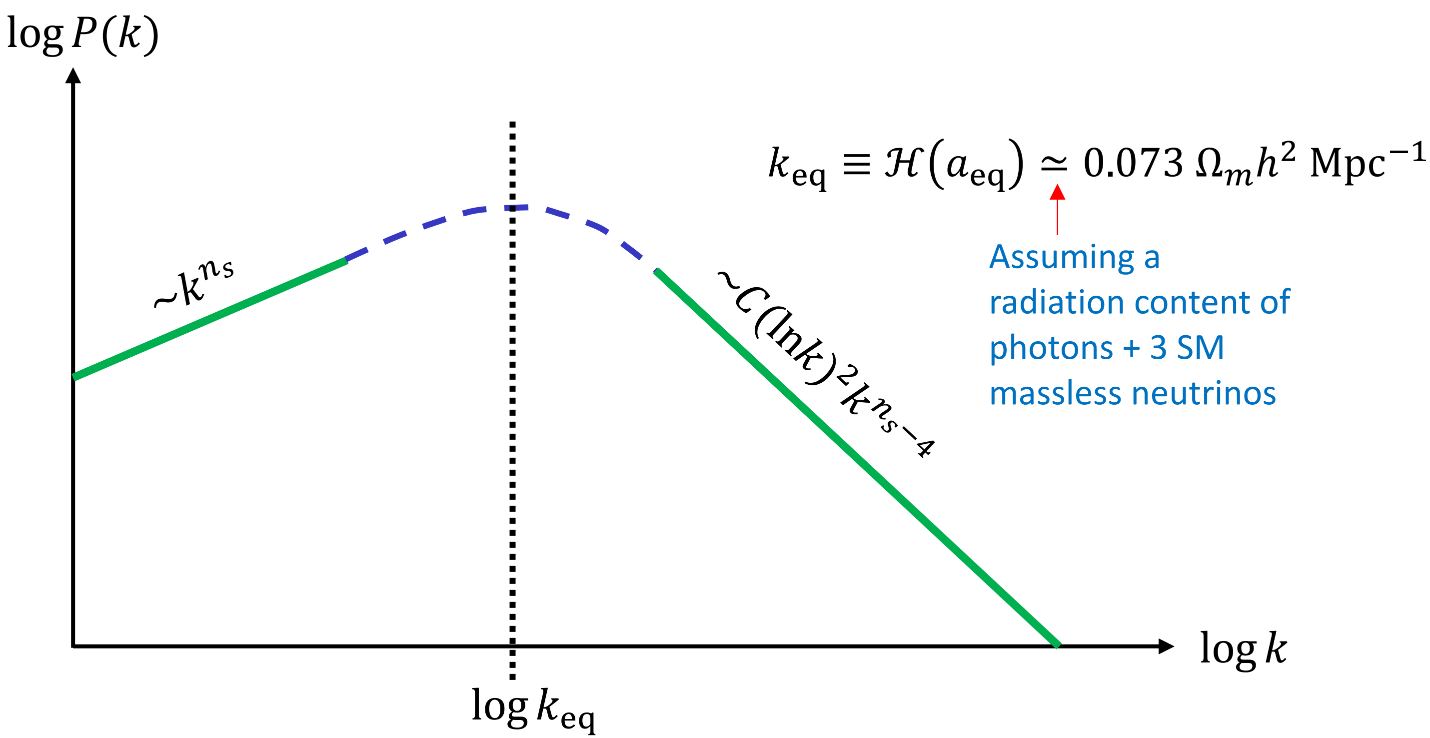
$n_s$ = spectral index

Shape of the power spectrum:

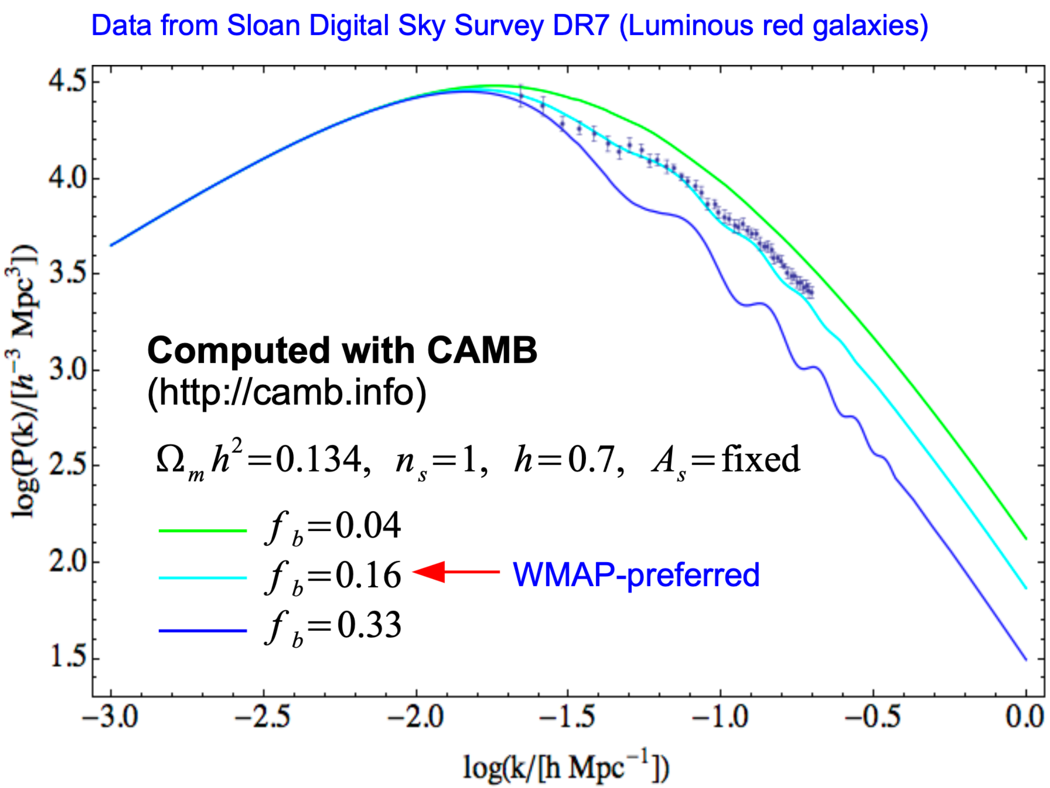
$$P_m(k,a) \sim T^2(k,a)k^{n_s}$$

# Linear matter power spectrum…

$$P_m(k,a) \sim T^2(k,a)k^{n_s}$$



# Linear matter power spectrum…



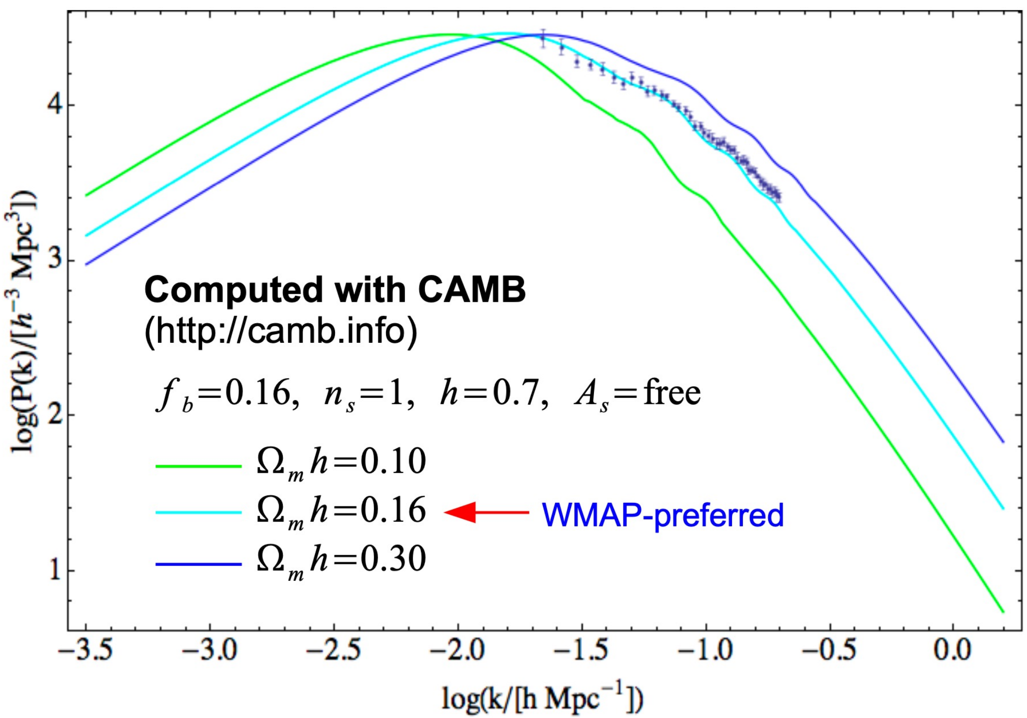

- Power spectrum suppression due to $f_b$.
- Wiggles = baryon acoustic oscillations (same physics as the CMB anisotropies)

# Linear matter power spectrum…

Assuming a radiation content of photons + 3 SM massless neutrinos

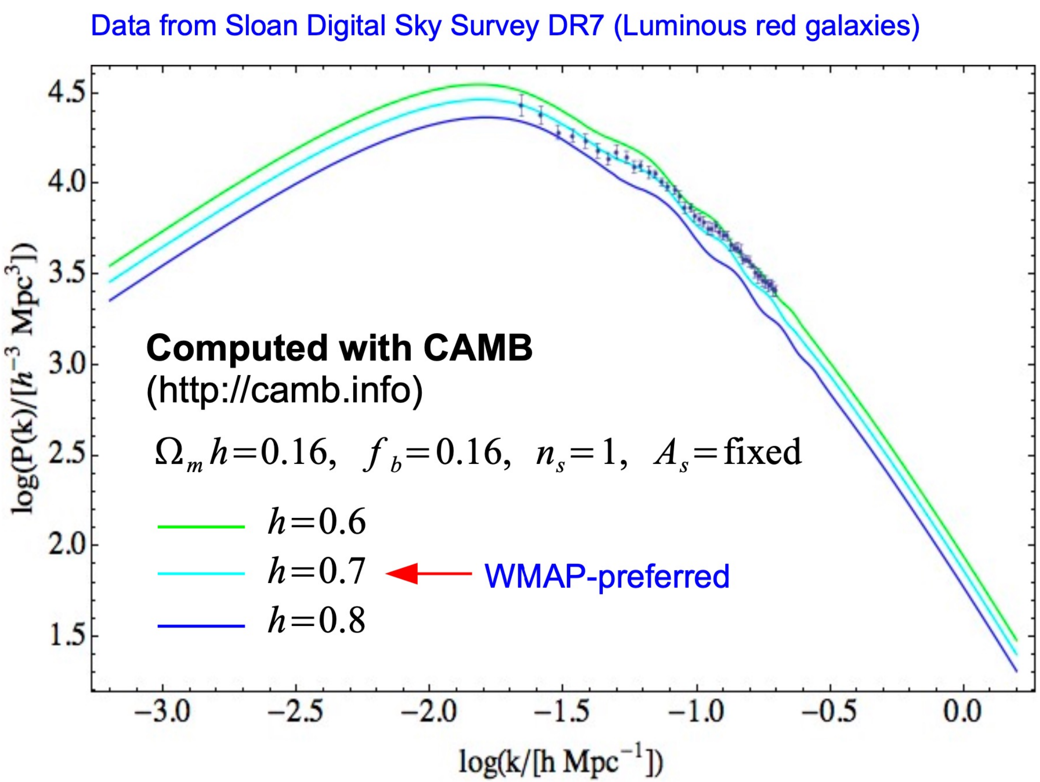
$$k_{\rm eq} \equiv \mathcal{H}(a_{\rm eq}) \simeq 0.073\,\Omega_m h^2 \,{\rm Mpc}^{-1}$$

Data from Sloan Digital Sky Survey DR7 (Luminous red galaxies)




- Shift of the turning power due to different $\Omega_m h$ values.
- Normalisation has been adjusted for better comparison.

# Linear matter power spectrum…




- With $\Omega_m h$ and $f_b$ fixed, changing $h$ does not alter the shape of the power spectrum besides a rescaling of the length scale (already absorbed into the units of $k$).