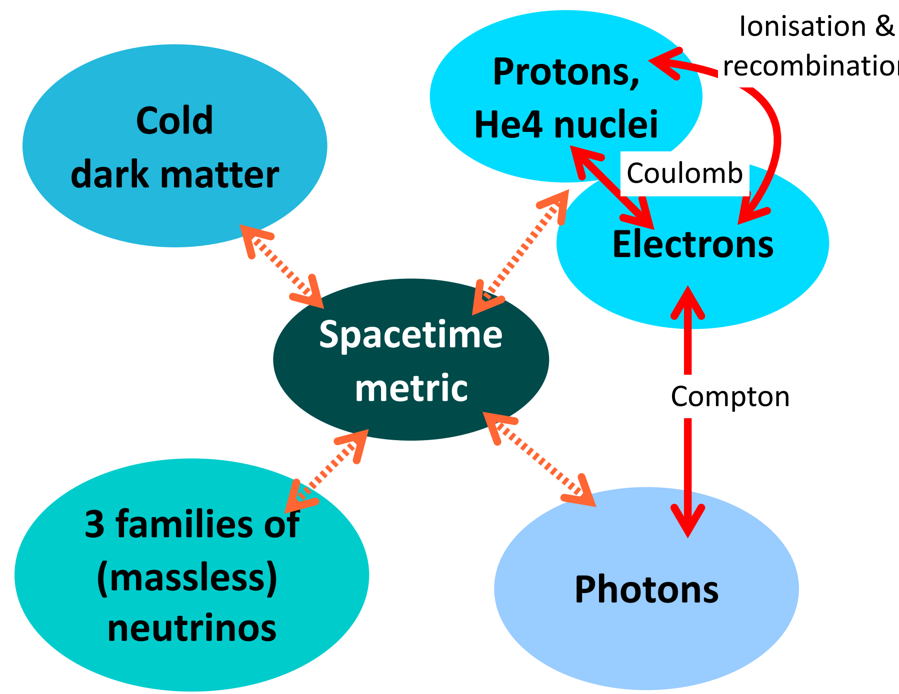
Tracking perturbations in $T_{\mu\nu}$...

In standard inflationary ΛCDM, we track 4 forms of matter/energy.



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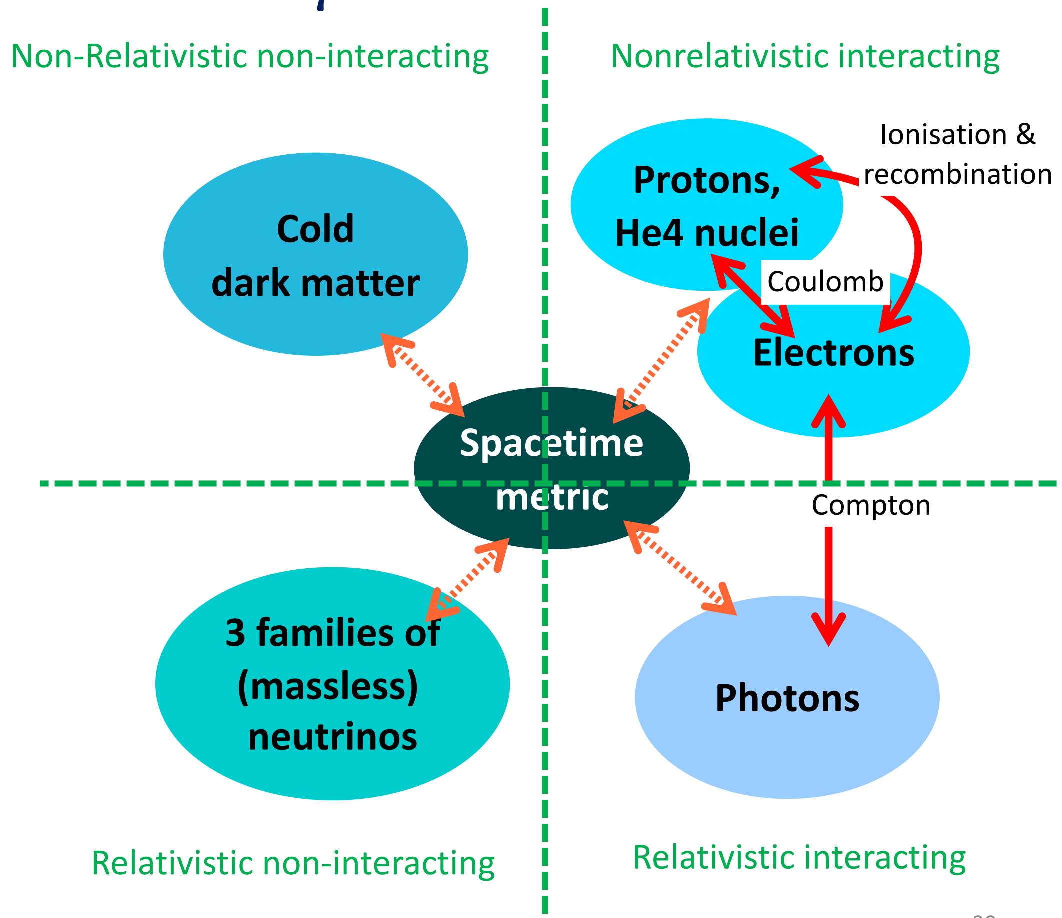

Boltzmann equation…

We use the Boltzmann equation to track the 1-particle phase space density $f_1\left(x^\alpha, P^i\right)$:

$$P^\alpha \frac{\partial f_1}{\partial x^\alpha} - \Gamma^i_{\alpha\beta} P^\alpha P^\beta \frac{\partial f_1}{\partial P^i} = C[f_1]$$

Collision term (Lorentz-invariant); "short range" interactions

Gravity goes in here; "long range" interactions

- $f_1(x^\alpha, P^i)$ is defined such that the number of particles 1 in a phase space volume $dx^1 dx^2 dx^3 dP_1 dP_2 dP_3$ is

$$dN = f_1(x^\alpha, P^i) dx^1 dx^2 dx^3 dP_1 dP_2 dP_3$$

Boltzmann equation…

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Collision term (Lorentz-invariant); "short range" interactions

- **The collision term** for e.g., $1 + 2 \rightarrow 3 + 4$

9D phase space integral

Energy-momentum conservation

Matrix element

$$C[f_1] = \frac{1}{2} \int \prod_{i=2}^{4} \frac{d^3 p_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^4 (P_1 + P_2 - P_3 - P_4) |M|^2$$
$$\times [f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)]$$

Quantum statistical factors

Boltzmann equation…

We use the Boltzmann equation to track the 1-particle phase space density $f_1(x^\alpha, P^i)$:

$$P^\alpha \frac{\partial f_1}{\partial x^\alpha} - \Gamma^i_{\alpha\beta} P^\alpha P^\beta \frac{\partial f_1}{\partial P^i} = C[f_1]$$

Collision term (Lorentz-invariant); "short range" interactions

- The **stress-energy tensor**:

$$T^{\mu\nu}(x^\alpha) = \int dP_1\, dP_2\, dP_3\, \frac{\sqrt{-g(x)}}{|P_0(x, P^i)|} P^\mu P^\nu f_1(x^\alpha, P^i)$$

2.1 Boltzmann equation

We adopt the dilute gas approximation and study the 1-particle phase space density using the Boltzmann equation:

gravity and/or long-range physics

$$\boxed{p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i} = C[f]}$$

collision integral; short-range physics

where $f(x^\alpha, P^i)$ is defined such that

conjugate to x^α

$$dN = f(x^\alpha, P^i) \underbrace{dx^1 dx^2 dx^3 \, dP_1 \, dP_2 \, dP_3}_{\text{phase space volume element}}$$

is the number of particles in the phase space volume element.

Here, $C[f]$ is the Lorentz-invariant collision integral describing processes that happen at a point x^α. For, e.g., $1+2 \to 3+4$, it takes the form:

$$C[f_1] = \frac{1}{2} \int \prod_{i=2}^{4} \frac{d^3 p_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P_4) |\mathcal{M}|^2$$
$$\times \left[f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \right]$$

The stress energy tensor is given by:

$$\boxed{T^{\mu\nu}(x^\alpha) = \int dP_1 dP_2 dP_3 \frac{\sqrt{-g(x)}}{|P_0(x, P^i)|} P^\mu P^\nu f(x^\alpha, P^i)}$$

We can work in the coordinate basis. However, as in the case of an FLRW universe, it is easier to work in the orthonormal basis of the comoving observer who is at rest with the coordinate system.

Effectively this means instead of P^μ, we use E and $\vec{p}$, i.e., the energy and 3-momentum measured by the comoving observer, so that

$$f(x^\alpha, P^i) \longrightarrow f(x^\alpha, \vec{p})$$

The relation between P^μ and $(E, \vec{p})$ depends on the gauge choice. For a metric of the form

$$ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2(\gamma_{ij} + 2H_{ij})dx^i dx^j$$

i.e., set $B_i = 0$

where

$$H_{ij} = -\Phi\gamma_{ij} + H_{ij}^{(V)} + H_{ij}^{(T)}$$

we have:

$$\boxed{\begin{array}{ll} P^0 = \frac{1}{a}(1-\Psi)E & P^i = \frac{1}{a}n^j(\delta^i_j - H^i_j)|\vec{p}| \\ P_0 = -a(1+\Psi)E & P_i = a\underbrace{n_j}_{\text{momentum direction}}(\delta^j_i + H^j_i)|\vec{p}| \end{array}}$$

to linear order in small parameters

It is also useful to define

$$\boxed{\begin{array}{l} q \equiv a|\vec{p}| \\ \varepsilon \equiv aE = \sqrt{q^2 + a^2 \underbrace{m}_{\text{particle mass}}} \end{array}}$$

1st order:

$$\boxed{\frac{\partial F^{(1)}}{\partial \eta} + \frac{q}{\varepsilon} n^i \frac{\partial F^{(1)}}{\partial x^i} + \frac{dq^{(1)}}{d\eta}\frac{\partial \bar{f}}{\partial q} = \left[\frac{1}{p^0} C[f]\right]^{(1)}}$$

It remains to specify $\frac{dq^{(1)}}{d\eta}$. This can be achieved using the geodesic equation:

$$\frac{dP^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} P^\beta P^\gamma = 0$$

Which gives to linear order:

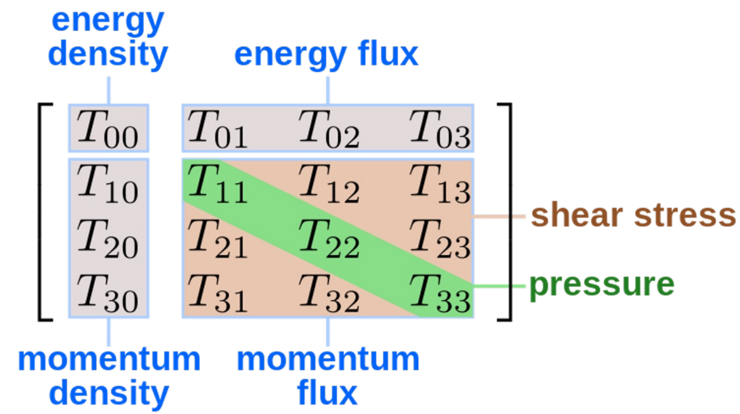
$$\boxed{\frac{dq^{(1)}}{d\eta} = -\varepsilon n^i \partial_i \Psi + q\dot{\Phi} - q n^i n^j \left(\dot{H}^{(V)}_{ij} + \dot{H}^{(T)}_{ij}\right)}$$

for $B_i = 0$

Inserting $f(\eta, x^i, q, n^i)$ into the stress energy tensor, we find in terms of the variables q and ε:

$$\boxed{\begin{aligned}
\delta\rho &= a^{-4}\int d^3q\, \varepsilon F^{(1)} \\
(\bar{\rho}+\bar{P})v^i &= a^{-4}\int d^3q \left(\frac{q}{\varepsilon}\right)\varepsilon n^i F^{(1)} \\
\delta P &= \frac{1}{3} a^{-4}\int d^3q \left(\frac{q}{\varepsilon}\right)^2 \varepsilon F^{(1)} \\
\Pi^i{}_j &= a^{-4}\int d^3q \left(\frac{q}{\varepsilon}\right)^2 \varepsilon \left(n^i n_j - \frac{1}{3}\delta^i_j\right) F^{(1)}
\end{aligned}}$$

Density, pressure, velocity, stress...



Energy density

$$\delta\rho = a^{-4}\int d^3q\ \varepsilon(q)F$$

Energy flux

$$(\bar{\rho}+\bar{P})v^i = a^{-4}\int d^3q\ q\, n^i\, F$$

Pressure

$$\delta P = \frac{1}{3}a^{-4}\int d^3q\ \left(\frac{q}{\varepsilon}\right)^2 \varepsilon\, F$$

trace

Anisotropic stress

$$\Pi^i{}_j = a^{-4}\int d^3q\ \left(\frac{q}{\varepsilon}\right)^2 \varepsilon\left(n^i n_j - \frac{1}{3}\delta^i{}_j\right) F$$

traceless

The q and ε variables are convenient because the effect of expansion largely factors out. Note however that in an inhomogeneous universe $q \neq$ constant! It is only constant in an FLRW universe!

Then, we can also express the 1-particle phase space density as

$$f(x^\alpha, \vec{p}) \longrightarrow f(\eta, x^i, q, n^i)$$

and the corresponding Boltzmann equation reads:

$$\boxed{\frac{\partial f}{\partial \eta} + \frac{dx^i}{d\eta}\frac{\partial f}{\partial x^i} + \frac{dq}{d\eta}\frac{\partial f}{\partial q} + \frac{dn^i}{d\eta}\frac{\partial f}{\partial n^i} = \frac{1}{p^0} C[f]} \quad (*)$$

We split up $f(\eta, x^i, q, n^i)$ into a homogeneous/isotropic part and an inhomogeneous part:

$$f(\eta, x^i, q, n^i) = \underbrace{\bar{f}(\eta, q)}_{\text{unperturbed, homogeneous and isotropic}} + F^{(1)}(\eta, x^i, q, n^i) + \dots$$

e.g., for SM neutrinos, $\bar{f}(q) =$ constant $=$ relativistic Fermi-Dirac distribution.

Expanding the Boltzmann equation $(*)$, we find:

0th order:

$$\boxed{\frac{\partial \bar{f}}{\partial \eta} = \frac{a^2}{\varepsilon} C^{(0)}[\bar{f}] = \frac{a}{E} C^{(0)}[\bar{f}]}$$

But $F^{(1)}$ is a scalar quantity! What happened to the SVT decomposition?

We can decompose $F^{(1)}(\eta, x^i, q, n^i)$ according to the alignment of the momentum direction n^i relative to the wavevector k^i at each mode:

$$F^{(1)}(\eta, x^i, q, n^i) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \underbrace{(-i)^l (2l+1)}_{\text{convention.}} F_l^{(m)}(\eta, k, q)\, G_l^m(k^i, x^i, n^i)$$

where

$$G_l^m(k^i, x^i, n^i) = n^{i_1} \cdots n^{i_{|m|}} \underbrace{Q^{(m)}_{i_1 \cdots i_{|m|}}}_{\substack{\text{eigenfunctions of} \\ \text{the Laplacian}}}$$

For $K=0$ (flat space), these are just the spherical harmonics times a plane wave:

$$G_l^m(\underset{\substack{\text{wavevector} \\ \text{of perturbation}}}{k^i}, x^i, \underset{\substack{\text{momentum} \\ \text{direction}}}{n^i}) \propto \underbrace{Y_l^m(n^i)}_{\substack{\text{spherical} \\ \text{harmonics}}} \exp(i \vec{k} \cdot \vec{x})$$

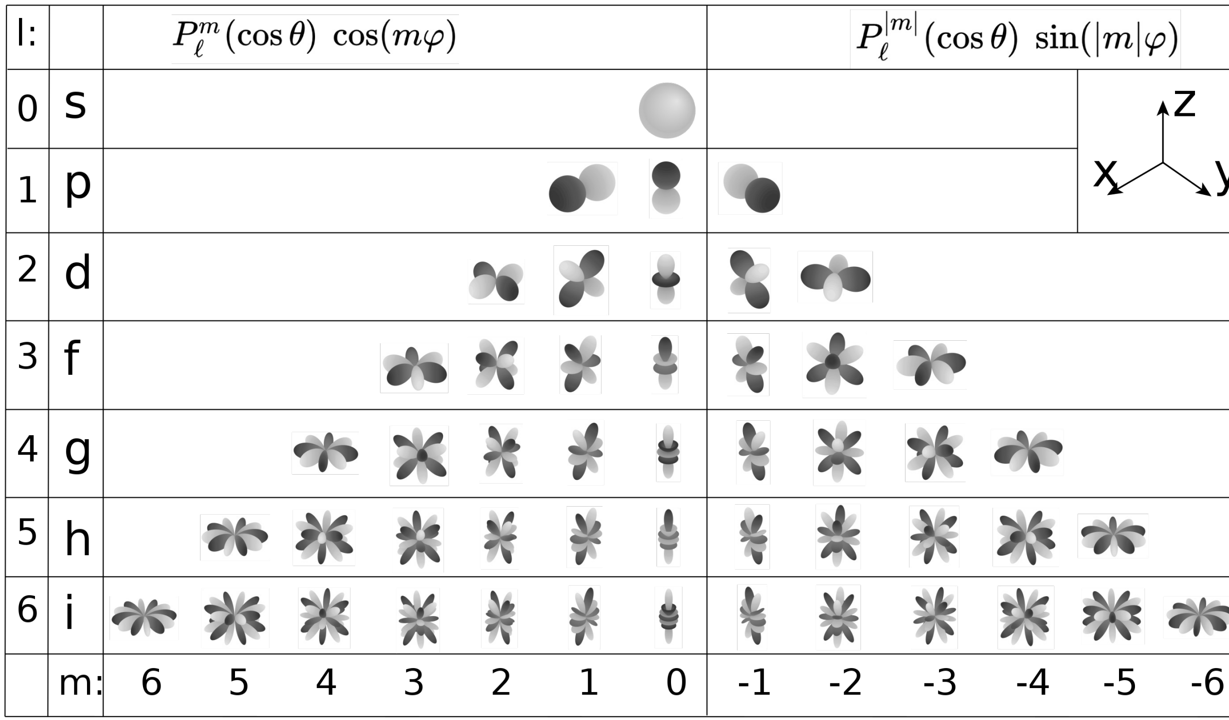
Applying the decomposition to the 1st order Boltzmann equation will automatically pick out the S, V, and T metric perturbations in the $\frac{dq^{(1)}}{d\eta}$ term for $m = 0, \pm 1, \pm 2$ respectively. That is:

$$F_l^0(\eta, k, q) \implies \frac{dq^{(1)}}{d\eta} = -\varepsilon n^i \partial_i \Phi + q \dot{\Phi}$$

depends on $k^i n_i$ alone

scalar perturbations

Spherical harmonics…



$F^{(\pm 1)}(\eta, k, q) \Rightarrow \frac{dq^{(1)}}{d\eta} = -q n^i n^j \dot{H}_{ij}^{(V)}$

depends on components $q n^i$ transverse to k^i

vector perturbation.

and similarly for $F^{(\pm 2)}$.

Notes

① Different m modes do not mix, i.e., S, V, T evolve independently.

② $l=0$ is scalar only; $l=1$ has S and V components; $l \geq 2$ has all three S, V, T.

③ The l moments are physically meaningful:

$$\boxed{\begin{aligned} \delta\rho &= a^{-4} \int d^3q \, \varepsilon F^{(0)}_{l=0} \\ \delta P &= \frac{1}{3} a^{-4} \int d^3q \left(\frac{q}{\varepsilon}\right)^2 \varepsilon F^{(0)}_{l=0} \\ (\bar{\rho} + \bar{P}) v^i &= a^{-4} \int d^3q \, \varepsilon \left(\frac{q}{\varepsilon}\right) \sum_{m=-1}^{1} F^{(m)}_{l=1} Q^{(m)i} \\ \Pi^i_j &= a^{-4} \int d^3q \left(\frac{q}{\varepsilon}\right)^2 \varepsilon \sum_{m=-2}^{2} F^{(m)}_{l=2} Q^{(m)i} \end{aligned}}$$

④ The gradient term in the Boltzmann equation $\frac{\partial F^{(1)}}{\partial x^i}$ couples F_l to F_{l-1} and F_{l+1}.

Scalar Boltzmann hierarchy (Newtonian gauge):

$$\boxed{\begin{aligned}
\dot{F}_0^{(0)} &= -\frac{qk}{\varepsilon} F_1^{(0)} - \dot{\Phi} \frac{\partial \bar{f}}{\partial \ln q} \\
\dot{F}_1^{(0)} &= -\frac{qk}{3\varepsilon}\left(F_0^{(0)} - 2F_2^{(0)}\right) - \frac{\varepsilon k}{3q} \Psi \frac{\partial \bar{f}}{\partial \ln q} \\
\dot{F}_{\ell\geq 2}^{(0)} &= \frac{qk}{(2\ell+1)\varepsilon}\left(\ell F_{\ell-1}^{(0)} - (\ell+1) F_{\ell+1}^{(0)}\right)
\end{aligned}}$$

Infinite hierarchy: choice of l_{max} depends on desired accuracy, and how long you need to integrate the differential equations.

Density, pressure, velocity, stress...

Monopole = density, pressure
Dipole = velocity
Quadruple = anisotropic stress

Energy density

$$\delta\rho = a^{-4}\int d^3q\ \varepsilon(q)F \qquad = a^{-4}\int d^3q\ \varepsilon(q)F^{(0)}_{\ell=0}$$

Energy flux

$$(\bar{\rho}+\bar{P})v^i = a^{-4}\int d^3q\ q\,n^i\,F \qquad = a^{-4}\int d^3q\ q\sum_{m=-1}^{1} F^{(m)}_{\ell=1}\,Q^{(m)i}$$

Pressure

$$\delta P = \frac{1}{3}a^{-4}\int d^3q\ \left(\frac{q}{\varepsilon}\right)^2 \varepsilon\,F \qquad = \frac{1}{3}a^{-4}\int d^3q\ \left(\frac{q}{\varepsilon}\right)^2 \varepsilon\,F^{(0)}_{\ell=0}$$

Anisotropic stress

$$\Pi^i{}_j = a^{-4}\int d^3q\ \left(\frac{q}{\varepsilon}\right)^2 \varepsilon\left(n^i n_j - \frac{1}{3}\delta^i{}_j\right)F \quad = a^{-4}\int d^3q\ \left(\frac{q}{\varepsilon}\right)^2 \varepsilon\sum_{m=-2}^{2} F^{(m)}_{\ell=2}\,Q^{(m)i}$$

1^{st} order Boltzmann equation...

Gravitational source term

Collision term

$$\frac{\partial F}{\partial \eta} + \frac{q}{\epsilon} n^i \partial_i F + \frac{dq}{d\eta}^{(1)} \frac{\partial \bar{f}}{\partial q} = \left(\frac{1}{P^0} C[f]\right)^{(1)}$$

$$\frac{dq}{d\eta}^{(1)} = -\epsilon n^i \partial_i \Psi + q\dot{\Phi} - q n^i n^j \left(\dot{H}_{ij}^{(V)} + \dot{H}_{ij}^{(T)}\right)$$

- Different m modes $m = 0, \pm 1, \pm 2$ **decouple**.
- However, the $n^i \partial_i F$ term **couples** F_ℓ to $F_{\ell-1}$ and $F_{\ell+1}$.

Metric

$$ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2[(1-2\Phi)\gamma_{ij} + 2H_{ij}^{(V)} + 2H_{ij}^{(T)}]dx^i dx^j$$

Scalar Boltzmann hierarchy ($K = 0$ case)...

Newtonian gauge

An **infinite hierarchy** of coupled equations for $m = 0$:

$$\dot{F}_0^{(0)} = -\frac{qk}{\varepsilon} F_1^{(0)} - \dot{\Phi}\frac{\partial \bar{f}}{\partial \ln q} + \left(\frac{1}{P^0} C[f]\right)^{(1)}_{\ell=0,m=0}$$

$$\dot{F}_1^{(0)} = \frac{qk}{3\varepsilon}\left(F_0^{(0)} - 2F_2^{(0)}\right) - \frac{\varepsilon k}{3q}\Psi\frac{\partial \bar{f}}{\partial \ln q} + \left(\frac{1}{P^0} C[f]\right)^{(1)}_{\ell=1,m=0}$$

$$\dot{F}_{\ell\geq 2}^{(0)} = \frac{qk}{(2\ell+1)\varepsilon}\left(\ell F_{\ell-1}^{(0)} - (\ell+1)F_{\ell+1}^{(0)}\right) + \left(\frac{1}{P^0} C[f]\right)^{(1)}_{\ell,m=0}$$

$$\dot{F}_\infty^{(0)}$$

Gravitational source terms

Collision terms (to be decomposed)

Tensor Boltzmann hierarchy ($K = 0$ case)...

$$\dot{F}_{\ell\geq 2}^{(\pm 2)} = \frac{qk}{(2\ell+1)\varepsilon}\left(\sqrt{\ell^2-4}F_{\ell-1}^{(\pm 2)} - \sqrt{(\ell+1)^2-4}F_{\ell+1}^{(\pm 2)}\right) - \dot{H}_T^{(\pm 2)} + \left(\frac{1}{P^0}C[f]\right)_{\ell,m=2}^{(1)}$$

$$\downarrow$$

$$\dot{F}_{\infty}^{(\pm 2)}$$

Gravitational source terms

Collision terms (to be decomposed)

Perturbations in $T_{\mu\nu}$...

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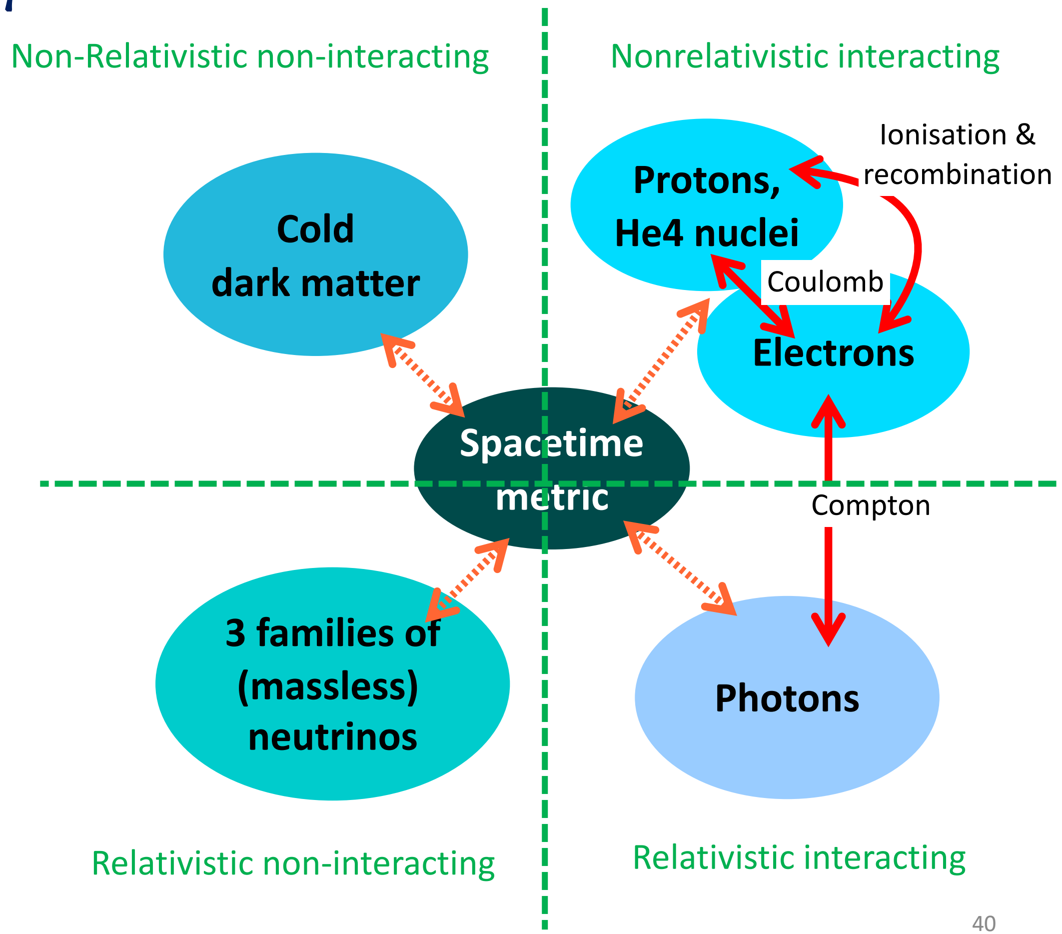


2.2 Boltzmann equation for cold dark matter

Non-relativistic and non-interacting. At 0th order, $\bar{P}=0$ by definition for CDM.

At 1st order, the equivalent condition is to ignore all terms proportional to $(q/\varepsilon)^n$ where $n \geq 2$.

In practice this means $\delta P = \underline{\underline{\Pi}} = 0$, or equivalently, $\underline{\underline{F_{\ell \geq 2} = 0}}$. We also set $\varepsilon = \sqrt{q^2+a^2m^2} \to m$, i.e., rest mass of the CDM particle.

Then, the only terms that remain are:

$$\boxed{\begin{aligned} \delta\rho &= a^{-3} m \int d^3q \; F_0^{(0)} \\ \bar{\rho} v^i &= a^{-4} \int d^3q \, m \left(\frac{q}{m}\right) n^i \sum_{m=-1}^{1} F_1^{(m)} Q^{(m)i} \end{aligned}} \quad (*)$$

and we only need to compute $F_0^{(0)}$ and $F_1^{(m)}$. Thus, instead of using the Boltzmann hierarchy, it is common and convenient to track directly the integrated $\delta = \frac{\delta\rho}{\bar{\rho}}$ and v^i. Integrating the Boltzmann hierarchy as per $(*)$ and keeping terms up to $\left(\frac{q}{m}\right)^n$ $n>2$, we can construct equations of motion for δ and v^i directly.

$$\boxed{\dot{\delta} + k v^{(s)} - 3\dot{\Phi} = 0}$$

Newtonian gauge
Continuity equation

$$\boxed{\dot{v}^{(s)} + \mathcal{H} v_c^{(s)} - k\Psi = 0}$$

Euler equation
Scalar part.

and

$$\boxed{\dot{v}^{(V)} + \mathcal{H}\, v^{(V)} = 0}$$

Euler equation vector part.

There is no source term for the vector part $\Rightarrow v^{(V)}$ decays away with the expansion of space.

<u>Notes</u>

If we had not set $\left(\frac{q}{\varepsilon}\right)^{n \geq 2}$ to zero, then we would expect extra terms on the RHS of the Euler equation. For the scalar Euler equation for example, we have:

$$-\frac{1}{\bar{\rho}}\left(\delta P - \frac{2}{3} k^2 \Pi^{(S)}\right)$$

Perfect fluid assumption allows us to set $\Pi = 0$. But we can still close the equations by assuming $\delta P = c_s^2 \delta\rho$, where c_s = sound speed in the fluid.

2.3 Boltzmann equation for massless neutrinos

Non-interacting $C[f]=0$. Massless means $q=\varepsilon$.
Then, the Boltzmann equation becomes:

$$\frac{\partial F}{\partial \eta} + n^i \partial_i F + \frac{d\ln q}{d\eta}\frac{\partial \bar{f}}{\partial \ln q} = 0$$

where

$$\frac{d\ln q}{d\eta} = \underbrace{-n^i\partial_i \Psi + \dot{\Phi} - n^i n^j \left(\dot{H}_{ij}^{(V)} + \dot{H}_{ij}^{(T)}\right)}_{\text{independent of } q}$$

Then, it is also possible to integrate F in momentum and track directly δ, v, Π, etc. Equivalently, we use

$$\boxed{F(\eta, x^i, q, n^i) = -\frac{\partial \bar{f}}{\partial \ln q}\Delta(\eta, x^i, n^i)}$$

where $\Delta \equiv \frac{\delta T}{T}$ is the temperature fluctuation.

$$\Rightarrow \boxed{\frac{\partial \Delta}{\partial \eta} + n^i\partial_i \Delta - \frac{d\ln q}{d\eta} = 0}$$

1st order Boltzmann equation

$\Delta(\eta, x^i, n^i)$ inherits the decomposition of $F(\eta, x^i, q, n^i)$ into spherical harmonics:

$$\delta = \frac{\delta\rho}{\rho} = 4\Delta_{\ell=0} \quad \| \quad \rho \propto T^4 \text{ for massless particles}$$

$$v = 3\Delta_{\ell=1}$$

$$\vdots$$

But aren't neutrinos massive? Yes. In that case, neither the CDM nor the massless particle approximations apply, and we must solve the Boltzmann hierarchy in its entirety...

① There is one hierarchy for each distinct mass.

② Can be very time-consuming. In practice, there are approximations one can put in at low redshifts.

③ Same reasoning applies to warm dark matter.

Scalar Boltzmann hierarchy for massless ν...

Newtonian gauge

$$\dot{\delta}_\nu = -\frac{4k}{3} v_\nu^{(0)} + 4\dot{\Phi}$$

Gravitational source terms

No collision terms

$$\dot{v}_\nu^{(0)} = k\left(\frac{1}{4}\delta_\nu - 2\Delta_2^{(0)} + \Psi\right)$$

$$\dot{\Delta}_2^{(0)} = \frac{k}{5}\left(\frac{2}{3} v_\nu^{(0)} - 3\Delta_3^{(0)}\right)$$

$$\dot{\Delta}_{\ell\geq 2}^{(0)} = \frac{k}{(2\ell+1)}\left(\ell\Delta_{\ell-1}^{(0)} - (\ell+1)\Delta_{\ell+1}^{(0)}\right)$$

$$\Big\downarrow$$

$$\dot{\Delta}_\infty^{(0)}$$

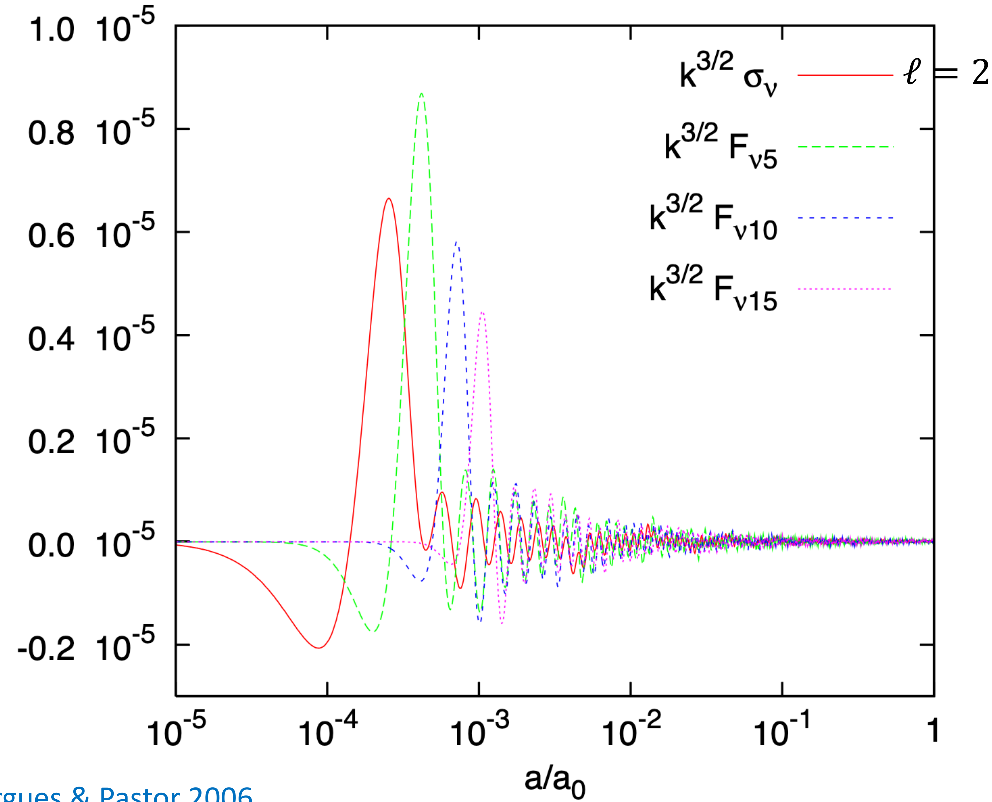
Compare with CDM

$$\dot{\delta}_c = -k v_c^{(0)} + 3\dot{\Phi}$$

$$\dot{v}_c^{(0)} = -\mathcal{H} v_c^{(0)} + k\Psi$$

Free-streaming in inhomogeneities…

- Gravitational source terms for $\ell = 0{,}1$.
- Evolution causes power to be transferred from the low multipoles to the high multipoles.



Lesgourgues & Pastor 2006

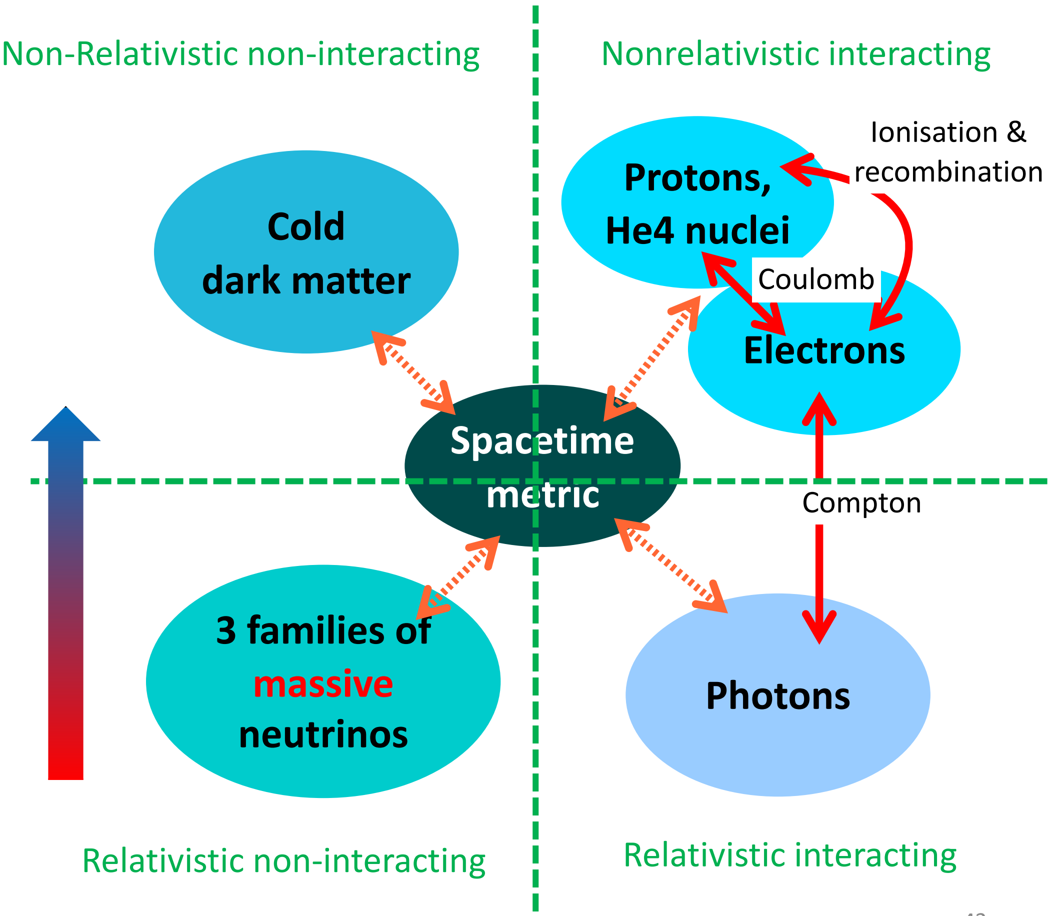
Real neutrinos are massive…

For sub-eV masses, relativistic-to-NR transition happens at redshifts $z = O(100) - O(1000)$.

→ Technically NR today **but is not be totally "cold"**.

→ Spend a substantial amount of time in the CMB/structure formation epoch as relativistic particles.

→ Need to solve the **full momentum-dependent Boltzmann hierarchy**.

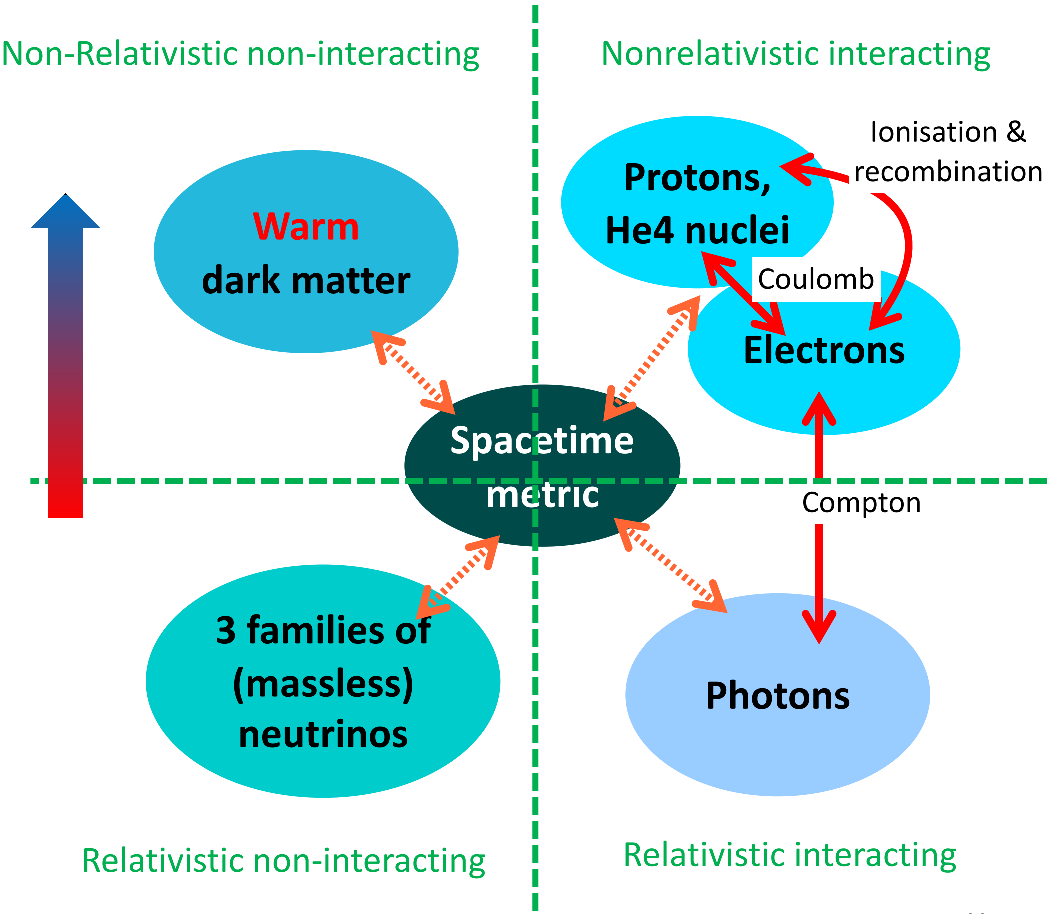


The same applies to warm dark matter...

Relativistic-to-NR transition for keV-mass WDM happens at much higher redshifts $z = O(10^6)$ than for SM neutrinos.

→ Technically NR but still has a lot of free-streaming.

→ Again, need to solve the **full momentum-dependent Boltzmann hierarchy**.



2.4 Boltzmann equation for photons

Massless means $q = \varepsilon$. But now we need to also take into account Compton scattering:

$$e^- + \gamma \rightarrow e^- + \gamma$$

free electron

We are particularly interested in $T \sim O(1)\,\mathrm{eV}$ and below, where $E_\gamma \ll m_e$, and the Thomson limit applies. This means the scattering changes only the direction of the incoming photon, while preserving its energy. Defining

$$\boxed{F_\gamma(\eta, x^i, q, n^i) = -\frac{\partial \bar{f}_\gamma}{\partial \ln q}\,\Theta(\eta, x^i, n^i)}$$

the Boltzmann equation is

$$\boxed{\frac{\partial \Theta}{\partial \eta} + n^i \partial_i \Theta - \frac{d \ln q}{d\eta} = -\left(\frac{\partial \bar{f}_\gamma}{\partial \ln q}\right)^{-1} \underbrace{\left(\frac{df}{d\eta}\right)_{\text{collision}}}_{\substack{\text{photon collision term}\\ \text{(changes } \gamma \text{ direction but}\\ \text{preserves energy).}}}}$$

What is $\left(\frac{df}{d\eta}\right)_{\text{collision}}$? In the rest frame of the electron, the differential Thomson cross section is

$$\sigma_T = 6.6524 \times 10^{-25}\,\mathrm{cm}^2$$

directions of incoming and outgoing photons

$$\frac{d\sigma}{d\Omega} = \frac{\sigma_T}{4\pi}\,\omega(\hat{n}, \hat{n}') = \frac{3\sigma_T}{16\pi}\left[1 + (\hat{n}\cdot\hat{n}')^2\right]$$

averaged over incoming and summed over final photon polarisation.

The collision term in the rest frame of the electron is therefore:

$$\left(\frac{df}{d\eta}\right)^{\text{rest}}_{\text{Collision}} = -a\,\sigma_T n_e \Big[\underbrace{f(p,\hat{n})}_{\substack{\text{scattering out of}\\ (p,\hat{n})\text{ state}}} - \underbrace{\frac{1}{4\pi}\int d\Omega' f(p,\hat{n}')\,\omega(\hat{n},\hat{n}')}_{\substack{\text{Scattering into}\\ (p,\hat{n})\text{ state from}\\ \text{all other directions}}}\Big]$$

Or in terms of $f(p,\hat{n}) = \bar{f}(p) + F(p,\hat{n}) = \bar{f}(p) - \frac{\partial \bar{f}}{\partial \ln q}\Theta(\hat{n})$,

$$\left(\frac{df}{d\eta}\right)^{\text{rest}}_{\text{collision}} = \frac{\partial \bar{f}}{\partial \ln q}\, a\sigma_T n_e \left[\Theta(\hat{n}) - \frac{1}{4\pi}\int d\Omega' \Theta(\hat{n}')\,\omega(\hat{n},\hat{n}')\right]$$

But the electrons may be moving with respect to the comoving observer, and this motion induces a dipole term in the collision term:

$$\left(\frac{df}{d\eta}\right)^{\text{Comoving frame}}_{\text{collision}} = \frac{\partial \bar{f}}{\partial \ln q}\, a\sigma_T n_e \Big[\Theta(\hat{n}) - \frac{1}{4\pi}\int d\Omega' \Theta(\hat{n}')\,\omega(\hat{n},\hat{n}') + \underbrace{v_e^i}_{\substack{\text{coordinate velocity}\\ \text{of the electrons}}} n_i\Big]$$

Thus, the Boltzmann equation becomes:

$$\boxed{\frac{\partial \Theta}{\partial \eta} + n^i \partial_i \Theta - \frac{d\ln q}{d\eta} = -\dot{\kappa}\left[\Theta(\hat{n}) - \frac{1}{4\pi}\int d\Omega' \Theta(\hat{n}')\,\omega(\hat{n},\hat{n}') + v_e^i n_i\right]}$$

Scalar Boltzmann hierarchy for photons...

$$\dot{\delta}_\gamma = -\frac{4k}{3} v_\gamma^{(0)} + 4\dot{\Phi}$$

Coupling to baryons via velocity terms

$$\dot{v}_\gamma^{(0)} = k\left(\frac{1}{4}\delta_\gamma - 2\Theta_2^{(0)} + \Psi\right) - \dot{\kappa}\left(v_\gamma^{(0)} - v_b^{(0)}\right)$$

Damping

$$\frac{d\Theta_\ell^{(0)}}{d\ln a} = \cdots - \frac{\dot{\kappa}}{\mathcal{H}}\Theta_\ell^{(0)}$$

$$\dot{\Theta}_2^{(0)} = \frac{k}{5}\left(\frac{2}{3} v_\gamma^{(0)} - 3\Theta_3^{(0)}\right) - \dot{\kappa}\Theta_2^{(0)}$$

$$\dot{\Theta}_{\ell>2}^{(0)} = \frac{k}{(2\ell+1)}\left(\ell\Theta_{\ell-1}^{(0)} - (\ell+1)\Theta_{\ell+1}^{(0)}\right) - \dot{\kappa}\Theta_\ell^{(0)}$$

$$\dot{\Theta}_\infty^{(0)}$$

Gravitational source terms

Collision terms

Tensor Boltzmann hierarchy for photons…

$$\dot{\Theta}_{\ell\geq 2}^{(\pm 2)} = \frac{k}{(2\ell+1)}\left(\sqrt{\ell^2-4}\,\Theta_{\ell-1}^{(\pm 2)} - \sqrt{(\ell+1)^2-4}\,\Theta_{\ell+1}^{(\pm 2)}\right) - \dot{H}_T^{(\pm 2)} - \dot{\kappa}\Theta_{\ell}^{(\pm 2)}$$

Gravitational source terms

Collision terms

- No coupling to baryons (no $\ell \leq 1$ equations of motion).
- But the same damping structure at $\ell \geq 2$.

Tightly-coupled limit for photons...

$$\dot{\delta}_\gamma = -\frac{4k}{3} v_\gamma^{(0)} + 4\dot{\Phi}$$

$$\dot{v}_\gamma^{(0)} = k\left(\frac{1}{4}\delta_\gamma - \cancel{2\Theta_2^{(0)}} + \Psi\right) - \dot{\kappa}\left(v_\gamma^{(0)} - v_b^{(0)}\right)$$

$$\cancel{\dot{\Theta}_2^{(0)} = \frac{k}{5}\left(\frac{2}{3} v_\gamma^{(0)} - 3\Theta_3^{(0)}\right) - \dot{\kappa}\Theta_2^{(0)}}$$

$$\cancel{\dot{\Theta}_{\ell>2}^{(0)} = \frac{k}{(2\ell+1)}\left(\ell\Theta_{\ell-1}^{(0)} - (\ell+1)\Theta_{\ell+1}^{(0)}\right) - \dot{\kappa}\Theta_\ell^{(0)}}$$

$$\downarrow$$

$$\dot{\Theta}_\infty^{(0)}$$

Coupling to baryons via velocity terms

Damping

$$\frac{d\Theta_\ell^{(0)}}{d\ln a} = \cdots - \frac{\dot{\kappa}}{\mathcal{H}}\Theta_\ell^{(0)}$$

Gravitational source terms

Collision terms

Tightly-coupled limit:

$$\frac{\dot{\kappa}}{\mathcal{H}} \gg 1 \quad \Rightarrow \quad \Theta_{\ell\geq 2}^{(0)} \to 0$$

A good approximation before photon decoupling.

where $\dot{\varkappa} \equiv a \sigma_T n_e$ is the comoving scattering rate such that

$$\varkappa(\eta) = -\int_{\eta}^{\eta_0} d\eta' \, a \bar{n}_e \sigma_T$$

gives the optical depth.

Tightly-coupled limit I

In the tightly-coupled limit, $\frac{\dot{\varkappa}}{\mathcal{H}} \gg 1$ and we expect $\Theta_{\ell \geq 2} \to 0$. Suppose for now that $v_\gamma^{(0)} \simeq v_b^{(0)}$, i.e., photons and baryons move exactly in unison. Then, we can construct a 2nd order DE for δ_γ:

$$\dot{\delta}_\gamma = -\frac{4k}{3} v_\gamma^{(0)} + 4\dot{\Phi}$$

$$\dot{v}_\gamma^{(0)} = k\left(\frac{1}{4}\delta_\gamma + \Psi\right).$$

$$\Rightarrow \quad \ddot{\delta}_\gamma = -\frac{4k}{3}\dot{v}_\gamma^{(0)} + 4\ddot{\Phi}$$

$$= -\frac{4k^2}{3}\left(\frac{1}{4}\delta_\gamma + \Psi\right) + 4\ddot{\Phi}$$

$$\Rightarrow \quad \boxed{\ddot{\delta}_\gamma + \frac{k^2}{3}\delta_\gamma = 4\ddot{\Phi} - \frac{4}{3}k^2\Psi}$$

Driven Harmonic oscillator.

Where the angular frequency of the oscillations is

$$\boxed{\omega = \frac{k}{\sqrt{3}} = k c_s}$$

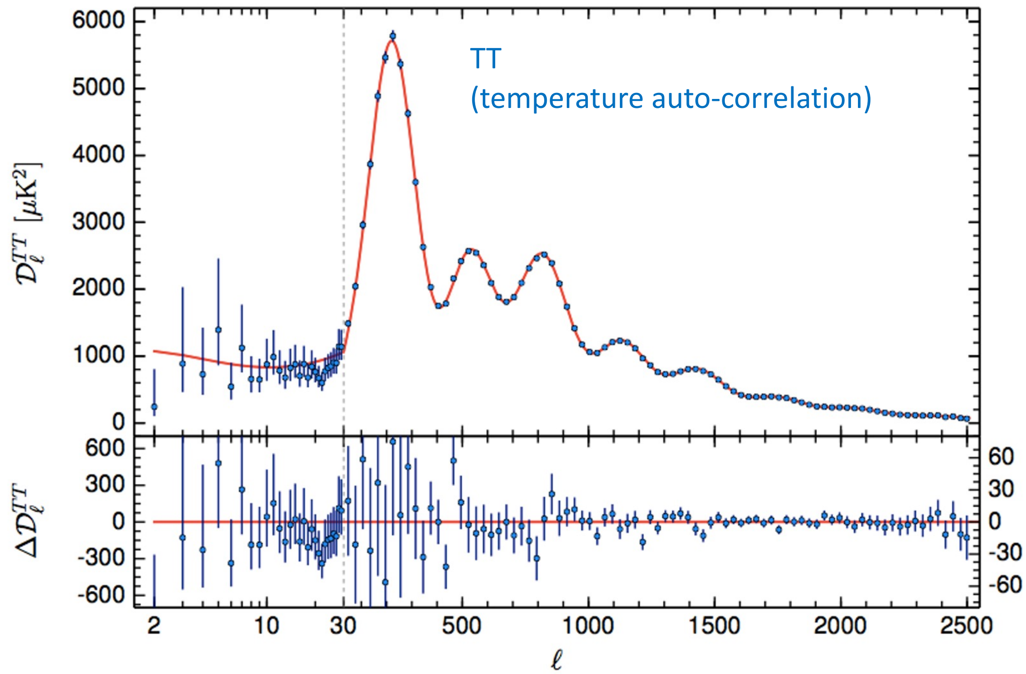
where

$$\boxed{c_s = \frac{1}{\sqrt{3}}}$$

Is the speed of sound waves in an ultrarelativistic fluid.

Acoustic oscillations…

The oscillatory features in the CMB power spectra are primarily the result of acoustic oscillations in the tightly-coupled photon-baryon fluid up to the **last scattering surface**.



Ade et al. [Planck collaboration] 2015

2.5 Boltzmann equation for nuclei / electrons

Nonrelativistic $\Rightarrow$ the same $\left(\frac{q}{\varepsilon}\right)$ expansion as for CDM applies here too. But now we must account for interactions

① Compton between photons and electrons.

② Coulomb between electrons and nuclei, between nuclei themselves, and between electrons.

$$\left(\frac{df_e}{d\eta}\right)_{coll} = \left(\frac{df_e}{d\eta}\right)_{e\gamma\leftrightarrow e\gamma} + \left(\frac{df_e}{d\eta}\right)_{ee\leftrightarrow ee} + \left(\frac{df_e}{d\eta}\right)_{eN\leftrightarrow eN}$$

$$\left(\frac{df_N}{d\eta}\right)_{coll} = \left(\frac{df_N}{d\eta}\right)_{eN\leftrightarrow eN} + \left(\frac{df_N}{d\eta}\right)_{NN\leftrightarrow NN}$$

Generally, their roles are:

① $eN \leftrightarrow eN$ keeps nuclei and electrons moving as one, i.e., at every point in space the net electric charge is zero. This means

$$\delta_e = \delta_N \equiv \delta_b \qquad b = \text{baryons}.$$

$$v_e = v_N \equiv v_b$$

② $NN \leftrightarrow NN$ and $ee \leftrightarrow ee \Rightarrow$ self interactions keep the nuclei and electron fluid locally in equilibrium, such that sound waves can propagate, and compete with gravity. This means for the Euler equation (scalar), we have:

$$\boxed{\dot{v}_b + \mathcal{H} v_b - k\Psi + c_{sb}^2 k\,\delta_b = \begin{matrix} e\gamma \leftrightarrow e\gamma \\ \text{collisions.} \end{matrix}}$$

where $c_{s,b}^2$ is the sound speed in the baryon fluid:

$$c_{s,b}^2 = \frac{\dot{P}_b}{\dot{\rho}_b} = \frac{k_B T_b}{\mu}\left(1 - \frac{1}{3}\frac{d\ln T_b}{d\ln a}\right)$$

μ mean molecular weight

Since no nuclei or electrons are created or destroyed, we must have conservation of number/energy density, which implies

$$\boxed{\dot{\delta}_b + k v_b^{(0)} - 3\dot{\Phi} = 0}$$

same as for CDM.

What is the $e\gamma \leftrightarrow e\gamma$ collision term that appears in the Euler equation?

Since photons and electrons interact only amongst themselves, the total momentum is the e-γ system must be conserved by the interaction, i.e.,

$$\frac{g_b}{a^4(2\pi)^3}\int d^3q\, q^i \left(\frac{df_b}{dq}\right)_{e\gamma\leftrightarrow e\gamma} = -\frac{g_\gamma}{a^4(2\pi)^3}\underbrace{\int d^3q\, q^i \left(\frac{df_\gamma}{dq}\right)_{e\gamma\leftrightarrow e\gamma}}_{=(\bar{\rho}_\gamma + \bar{P}_\gamma)\dot{\kappa}(v_\gamma^i - v_b^i)}$$

$= \frac{4}{3}\bar{\rho}_\gamma$ since $w = \frac{1}{3}$

from $l = 1$

Thus, the Euler equation (scalar) is:

$$(\bar{\rho}_N + \bar{\rho}_e)\left(\dot{v}_b^{(0)} + \mathcal{H} v_b^{(0)} - k\Psi + k c_{s,b}^2 \delta_b\right) = \frac{4}{3}\bar{\rho}_\gamma \dot{\kappa}\left(v_\gamma^{(0)} - v_b^{(0)}\right)$$

$\ll \bar{\rho}_N = \bar{\rho}_b$

$$\Rightarrow \boxed{\dot{v}_b^{(0)} + \mathcal{H} v_b^{(0)} - k\Psi + k c_{s,b}^2 \delta_b = \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b}\dot{\kappa}\left(v_\gamma^{(0)} - v_b^{(0)}\right)}$$

Useful references..

- Hu, Covariant linear perturbation formalism, astro-ph/0402060 (or basically anything you can find on his website at U Chicago)

- Seljak, Lectures on dark matter, *ICTP Lect.Notes Ser.* 4 (2001) 33-77

- Durrer, The Cosmic Microwave Background, Cambridge University Press, 2008

- Dodelson, Modern Cosmology, Academic Press, 2003

Public codes…

Numerical solutions of the linearised Einstein-Boltzmann system are necessary for 0.1% accurate calculations.

- Several publicly available codes out there:
  - CAMB: https://camb.info
  - CLASS: http://class-code.net/

- Mostly optimised for for standard inflationary ΛCDM, but can be fairly easily modified to accommodate "exotic" models.