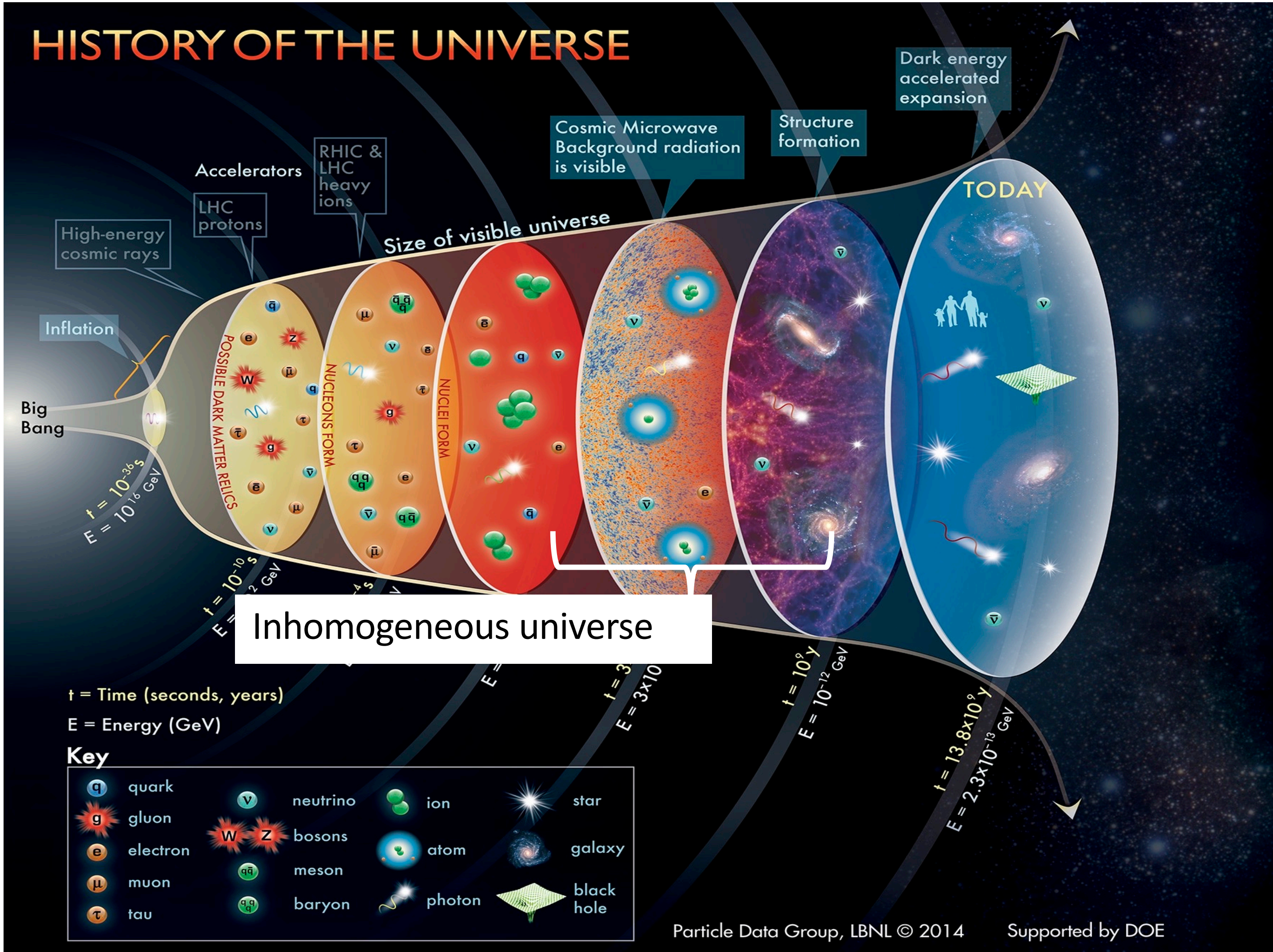
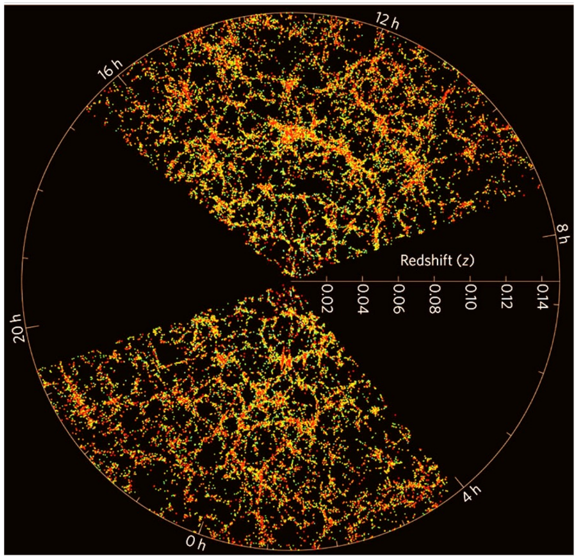
Inhomogeneous universe

(aka what's in those black boxes called CAMB and CLASS)

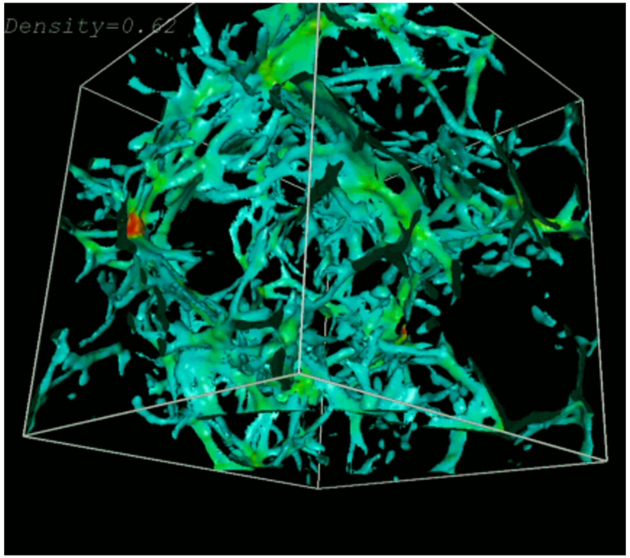

HISTORY OF THE UNIVERSE
Dark energy accelerated expansion
Cosmic Microwave Background radiation is visible
Structure formation
TODAY
Size of visible universe
Accelerators
RHIC & LHC heavy ions
LHC protons
High-energy cosmic rays
Inflation
Big Bang
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
t = 10^-36 s
E = 10^16 GeV
t = 10^-10 s
t = 10^9 y
E = 10^-12 GeV
t = 13.8x10^9 y
E = 2.3x10^-13 GeV
Inhomogeneous universe
t = Time (seconds, years)
E = Energy (GeV)
Key
quark
gluon
electron
muon
tau
neutrino
bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
Particle Data Group, LBNL © 2014
Supported by DOE

Overview…

The distribution of matter in the universe, even on large scales, is **not** exactly homogeneous and isotropic!



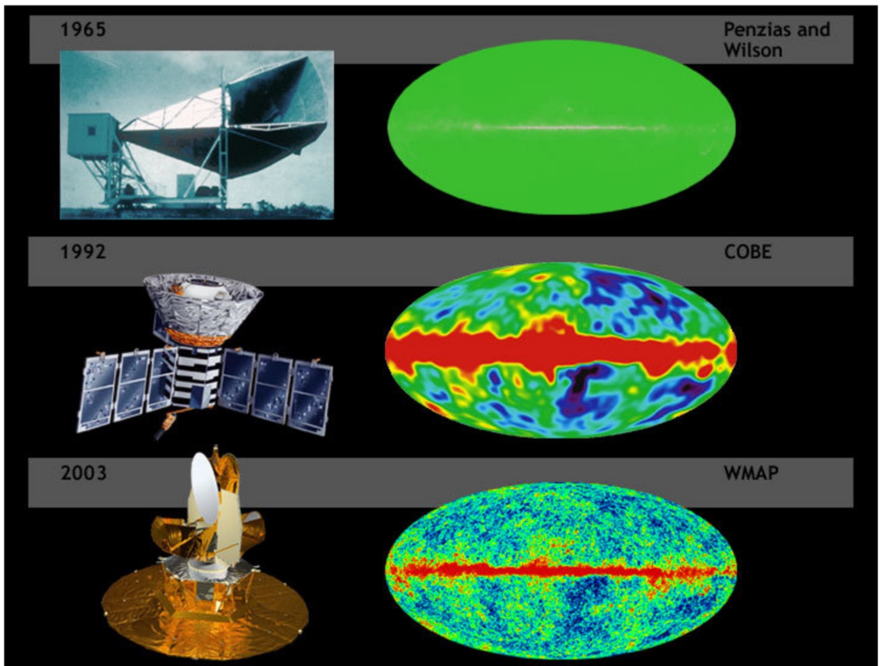
Red galaxies observed by the Sloan Digital Sky Survey



Intergalactic hydrogen clouds (simulations)

Overview…

The cosmic microwave background radiation is anisotropic.



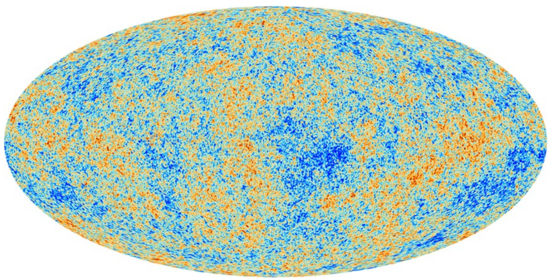
2.73 K background

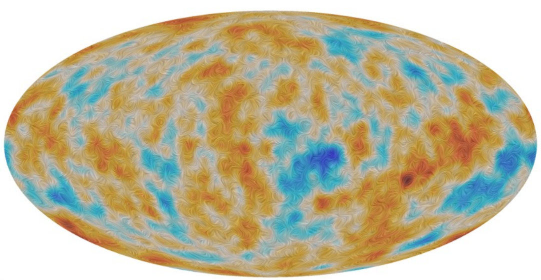
~ 10^{-5} temperature fluctuations
(~ 7 deg resolution)

~ 0.2 deg resolution

Overview…

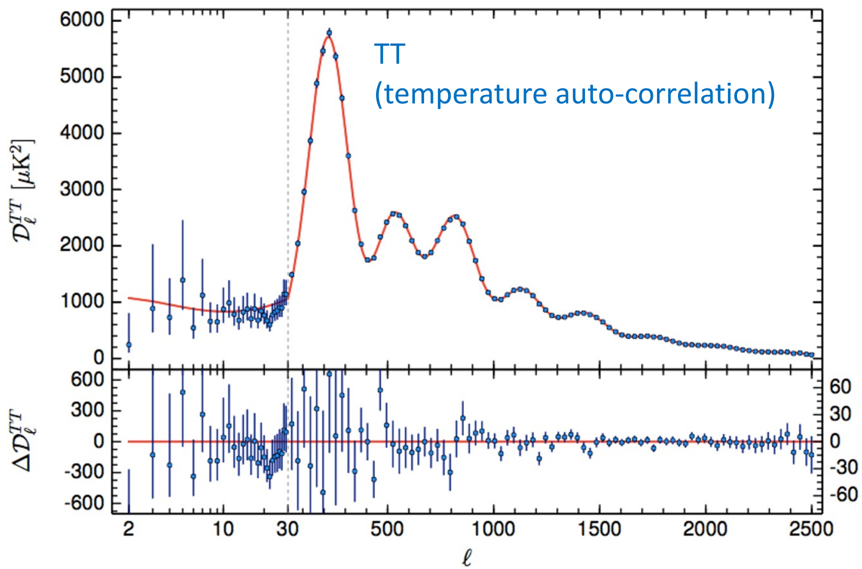
The cosmic microwave background radiation is anisotropic.

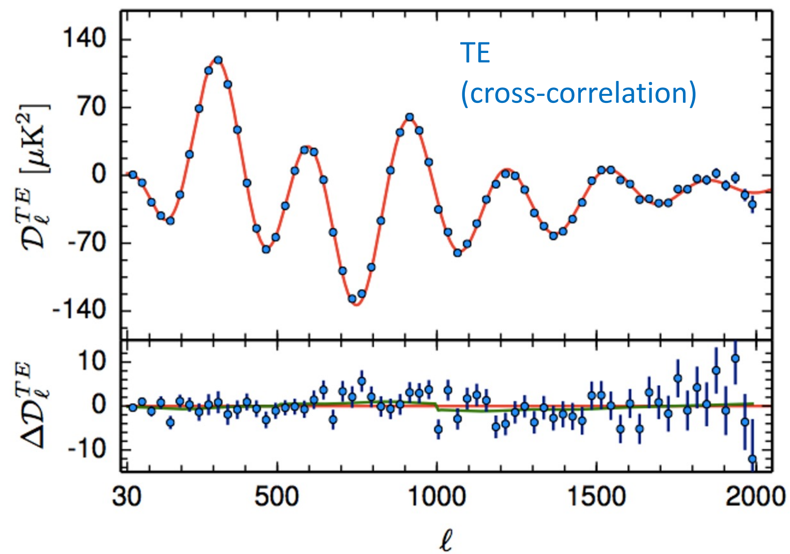


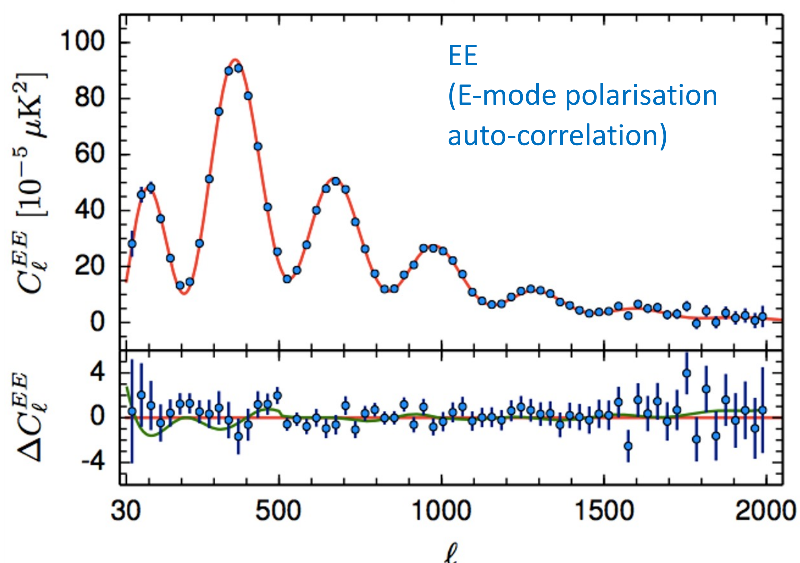


State-of-the-art:

- Temperature and polarisation fluctuations from Planck 2018
- > 0.2 deg: WMAP (9-year data public)
- < 0.2 deg: DASI, CBI, ACBAR, Boomerang, VSA, QuaD, QUIET, BICEP, ACT, SPT, etc.



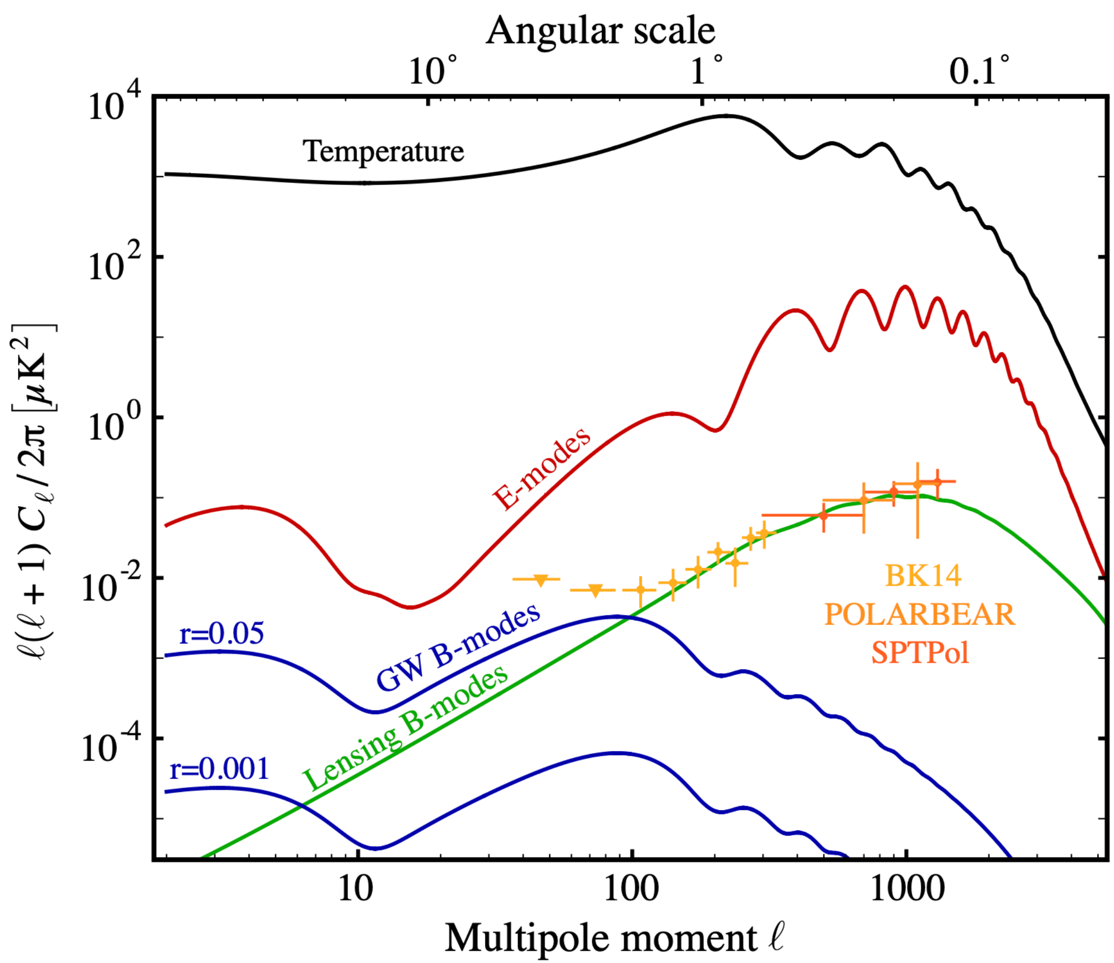




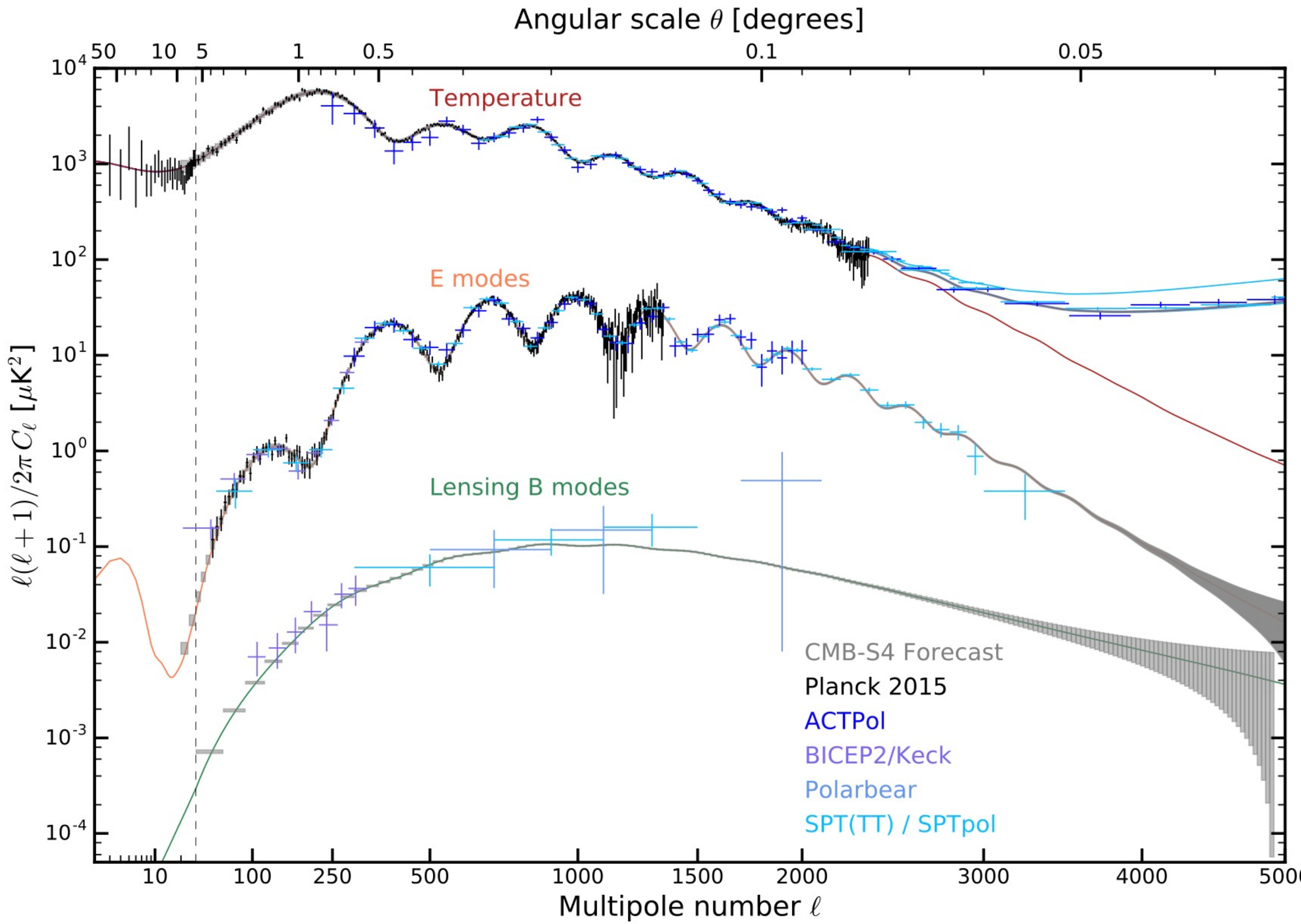
← Large angular separation

Small angular separation →

Ade et al. [Planck collaboration] 2015



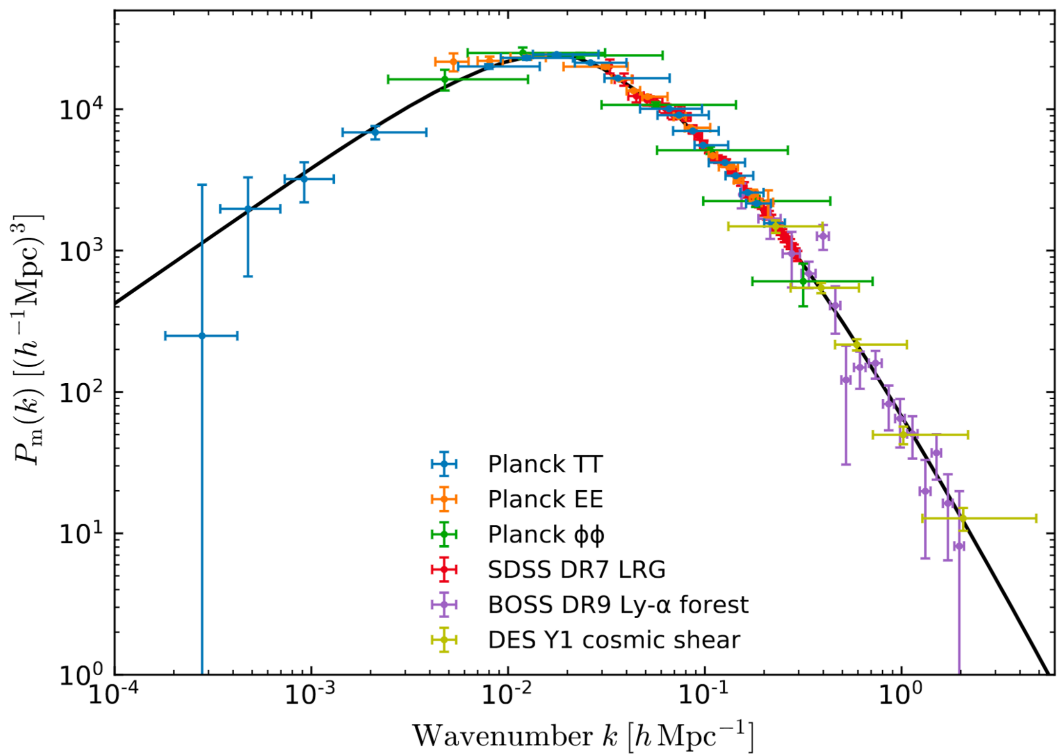
CMB-S4 Science Book 2016



CMB-S4 Science Book 2016

Matter power spectrum…

Measurements circa 2018



Akrami et al. 2018

Six-parameter "vanilla" fit:

Optical depth to reionisation, CMB only

$$P_{\text{vanilla}} = (\Omega_c h^2, \Omega_b h^2, h, n_s, A_s, \tau)$$

Energy content: Cold dark matter density, Baryon density,

Expansion rate

Initial conditions

Assumes flat geometry

$$\sum_i \Omega_i = 1$$

$\Rightarrow \Omega_\Lambda$

Fixes dark energy density

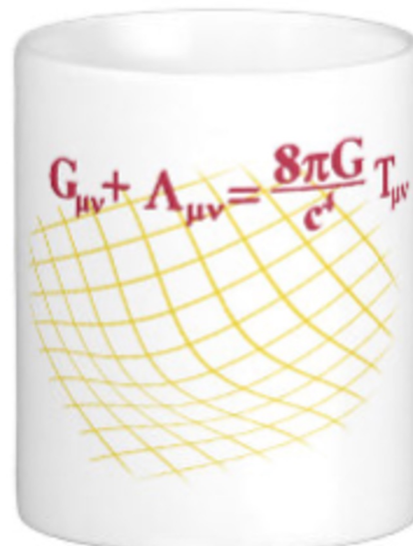
Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10} A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}] . .	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54	67.66 ± 0.42
Ω_Λ	0.679 ± 0.013	0.699 ± 0.012	$0.711^{+0.033}_{-0.026}$	0.6834 ± 0.0084	0.6847 ± 0.0073	0.6889 ± 0.0056
Ω_m	0.321 ± 0.013	0.301 ± 0.012	$0.289^{+0.026}_{-0.033}$	0.3166 ± 0.0084	0.3153 ± 0.0073	0.3111 ± 0.0056

Aghanim et al. [Planck collabroation] 2018

Plan…

- Theoretical framework to describe inhomogeneities
- Deconstructing the matter power spectrum
- Deconstructing the CMB temperature power spectrum
- Where cosmological parameter constraints come from
- CMB polarisation

1. Theoretical framework...



Overview…

Our current understanding of the inhomogeneous universe:

Inflation: Quantum fluctuations on a scalar field
→ initial perturbations on spacetime metric
→ initial conditions for CMB & large-scale structure

Neutrino decoupling

Matter-radiation equality

Photon decoupling

time

Gravity competes with pressure to amplify primordial inhomogeneities

CMB anisotropies: Perturbations in photon energy density and polarisation

Structure formation: Perturbations in nonrelativistic matter

GR + Boltzmann

Strategy…

Unperturbed FLRW

$$ds^2 = a^2(\eta)[-d\eta^2 + \gamma_{ij}dx^i dx^j]$$

We study large-scale inhomogeneities by **perturbing** around the FLRW spacetime geometry and stress-energy tensor:

$\bar{g}_{\mu\nu}$ = Unperturbed FLRW metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

Gravity

$\bar{T}_{\mu\nu}$ = Homogeneous and isotropic

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \qquad |\delta T_{\mu\nu}| \ll \bar{\rho}$$

Energy-momentum of the "stuff" in the universe

For what we're interested in, linear perturbation theory suffices

- **Einstein's equation** → How $h_{\mu\nu}$ evolves due to $\delta T_{\mu\nu}$
- **Boltzmann equation** → How $\delta T_{\mu\nu}$ evolves due to $h_{\mu\nu}$ and the properties of the "stuff"

1.1 Introduction

We assume inhomogeneities grow out of small variations in the spacetime geometry and in the stress energy tensor:

Metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}$ $\quad || \quad |h_{\mu\nu}| \ll 1$

($a^2 h_{\mu\nu}$ ← perturbations)

($\bar{g}_{\mu\nu}$ ↑) FLRW $ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}dx^i dx^j\right]$

Stress-energy tensor:

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \qquad |\delta T_{\mu\nu}| \ll \bar{\rho}$$

($\bar{T}_{\mu\nu}$ ↑) homogeneous & isotropic $\bar{T}^{\mu}{}_{\nu} = \mathrm{diag}(-\bar{\rho}, \bar{P}, \bar{P}, \bar{P})$

Our goal:

① To find the relation between $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ using the Einstein equation.

② To find evolution equations for $\delta T_{\mu\nu}$ using the Boltzmann equation.

1.2 Harmonic decomposition

FLRW has 3 translational and 3 rotational symmetries. The small perturbations also inherit these symmetries. We therefore decompose a (spatial) tensor field into components that transform irreducibly under translation and rotation

Translation symmetry implies decomposition into eigenfunctions of the Laplacian:

$$\nabla^2 Q_{\vec{k}} \equiv \gamma^{ij}\nabla_i \nabla_j Q_{\vec{k}} = -k^2 Q_{\vec{k}}$$

In flat space $(K=0)$:

$$Q_{\vec{k}}(\vec{x}) \propto e^{i\vec{k}\cdot\vec{x}}$$

|| plane waves
$\Rightarrow$ Fourier decomposition

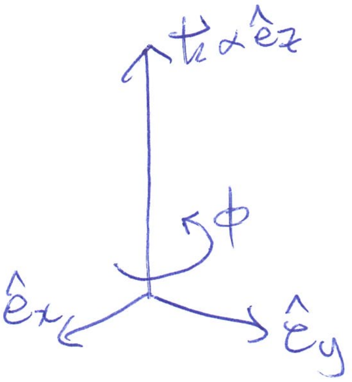
Rotation symmetry means $Q_{\vec{k}}$ can be further decomposed into irreducible scalars, vectors, and tensors:

S: 1 dof $\quad \nabla^2 Q^{(0)} = -k^2 Q^{(0)}$

V: 3−1 = 2 dof $\quad \nabla^2 Q_i^{(\pm 1)} = -k^2 Q_i^{(\pm 1)}$ || divergence free $\nabla^i Q_i^{(\pm 1)} = 0$ (1 constraint)

T: 6−3−1 = 2 dof $\quad \nabla^2 Q_{ij}^{(\pm 2)} = -k^2 Q_{ij}^{(\pm 2)}$ || transverse: $\nabla^i Q_{ij}^{(\pm 2)} = 0$ (3 cons); traceless: $\gamma^{ij} Q_{ij}^{(\pm 2)} = 0$ (1 cons)

$\Downarrow$



$$Q^{(0)} \propto e^{i\vec{k}\cdot\vec{x}}$$

$$Q_i^{(\pm 1)} \propto (\hat{e}_x \pm i\hat{e}_y)_i e^{i\vec{k}\cdot\vec{x}}$$

$$Q_{ij}^{(\pm 2)} \propto (\hat{e}_x \pm i\hat{e}_y)_i (\hat{e}_x \pm i\hat{e}_y)_j e^{i\vec{k}\cdot\vec{x}}$$

up to overall constants

$K=0$

Rotating the plane perpendicular to $\vec{k}$, $Q^{(\pm m)}$ transforms as

$$Q'^{(\pm m)} = Q^{(\pm m)} e^{\mp i m \phi}$$

We can construct vector and tensor objects out of irreducible scalar and vector eigenfunctions:

Vector: $Q_i^{(0)} \equiv -k^{-1} \nabla_i Q^{(0)}$

Tensor: (symmetric in i and j)

$$Q_{ij}^{(0)} \equiv (k^{-2} \nabla_i \nabla_j + \frac{1}{3} \gamma_{ij}) Q^{(0)}$$ || traceless

$$Q_{ij}^{(\pm 1)} \equiv -\frac{1}{2k} [\nabla_j Q_i^{(\pm 1)} + \nabla_i Q_j^{(\pm 1)}]$$

Then, the kth eigenmode of any vector can be written as

$$B_i(\vec{x}) = \sum_{m=-1}^{1} V^{(m)}(k) Q_i^{(m)}$$

|| 3 dof
1 scalar
1 vector (2 dof)

and a symmetric tensor is:

$$H_{ij}(\vec{x}) = H_L(k) Q^{(0)} \gamma_{ij} + \sum_{m=-2}^{2} H_T^{(m)}(k) Q_{ij}^{(m)}$$

|| T = transverse

↑ trace
1 dof
L = longitudinal

↑ 1 scalar
1 vector (2 dof)
1 tensor (2 dof

A scalar is simply:

$$A(\vec{x}) = A(k) Q^{(0)}$$

Metric perturbations…

Unperturbed FLRW

$$ds^2 = a^2(\eta)[-d\eta^2 + \gamma_{ij}dx^i dx^j]$$

It is common to see the metric perturbations parameterised thus:

$$h_{\mu\nu}dx^\mu dx^\nu = -2A d\eta^2 - 2B_i d\eta dx^i + 2H_{ij}dx^i dx^j$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}$$

Scalar

Scalar + Vector

Trace: Scalar

Traceless: Scalar + Vector + Tensor

4 scalars: $A, B^{(0)}, H_L, H_T^{(0)}$ 4 x 1 d.o.f.

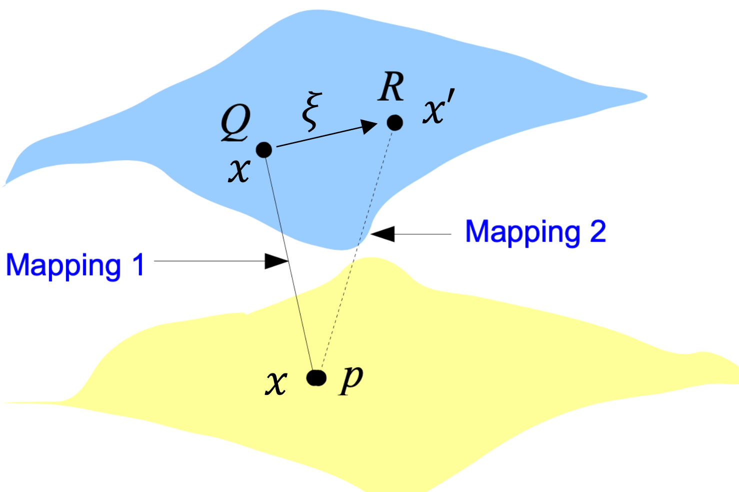
2 vectors: $B^{(0)}, H_T^{(\pm 1)}$ 2 x 2 d.o.f.

1 tensor: $H_T^{(\pm 2)}$ 1 x 2 d.o.f.

= 10 d.o.f.

Gauge modes...

- **Only 6 d.o.f. are physical**.
- 4 are gauge modes, because we are free to choose the mapping between the **real inhomogeneous** spacetime to the **fictitious FLRW** spacetime on which we build the perturbation theory.



Mapping 1 $g_{\mu\nu}(Q) = \bar{g}_{\mu\nu}(P) + a^2 h_{\mu\nu}(P)$

Mapping 2 $g'_{\mu\nu}(R) = \bar{g}_{\mu\nu}(P) + a^2 h'_{\mu\nu}(P)$

$$h_{\mu\nu}(P) \neq h'_{\mu\nu}(P)$$

1.3 Metric perturbations

The perturbed metric has the form:

$$g_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\text{FLRW}} + a^2 h_{\mu\nu}$$

Where $h_{\mu\nu}$ can be parameterised as

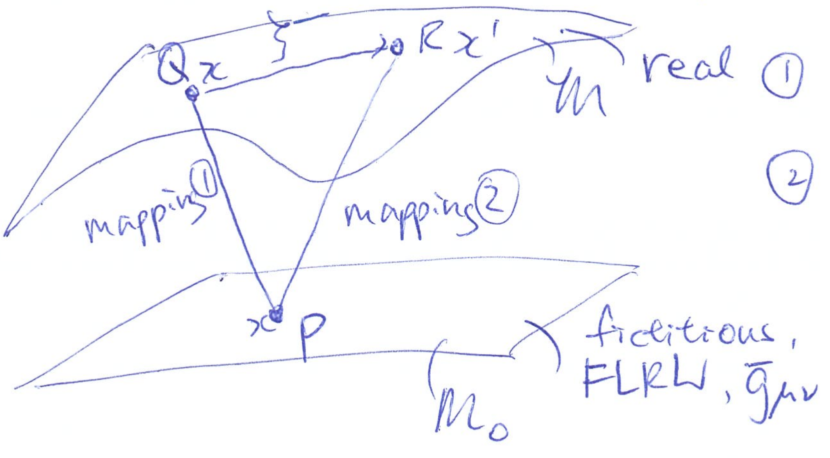
$$\boxed{h_{\mu\nu} dx^\mu dx^\nu = -2A d\eta^2 - 2B_i d\eta dx^i + 2H_{ij} dx^i dx^j}$$

A is a scalar; B_i and H_{ij} can be SVT-decomposed.

$h_{\mu\nu}$ has altogether 10 independent components:

4 scalars:	$A, B^{(0)}, H_L, H_T^{(0)}$	4×1 d.o.f.
2 vectors:	$B^{(\pm 1)}, H_T^{(\pm 1)}$	2×2 d.o.f.
1 tensor:	$H_T^{(\pm 2)}$	1×2 d.o.f
		10 d.o.f

However, only 6 d.o.f. are physical. The other 4 are gauge modes, arising from the fact that we are free to choose the mapping between the real inhomogeneous spacetime manifold to the fictitious FLRW manifold on which we build the perturbation theory.



① $g_{\mu\nu}(Q) = \bar{g}_{\mu\nu}(P) + a^2 h_{\mu\nu}(P)$

② $g'_{\mu\nu}(R) = \bar{g}_{\mu\nu}(P) + a^2 h'_{\mu\nu}(P)$

Different mappings:

$\Rightarrow h_{\mu\nu}(P) \neq h'_{\mu\nu}(P)$

Picking a mapping is called choosing a gauge. Going between mappings is called gauge transformation.

Gauge transformation

You can think about it as if it was a coordinate transformation on $\mathcal{M}$, but not on $\mathcal{M}_0$.

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x) \qquad \xi = \text{infinitesimal}.$$

where $\xi^{\alpha} = (T, L^i)$; $L_i = \delta_{ij} L^j$

$$\text{SVT-decompose: } = \sum_{m=-1}^{1} L_i^{(m)} Q_i^{(m)}$$

The metric transforms as: $\to (\delta^{\delta}_{\beta} - \partial_{\beta}\xi^{\delta})$

$$g'_{\alpha\beta}(x') = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta}(x) \quad \to = \bar{g}_{\gamma\delta}(x) + a^2 h_{\gamma\delta}(x)$$

$$= g'_{\alpha\beta}(x) + a^2 \xi^{\gamma} \partial_{\gamma} g'_{\alpha\beta}(x) + \ldots$$

$$= \bar{g}_{\alpha\beta}(x) + a^2 h'_{\alpha\beta}(x) + a^2 \xi^{\gamma} \partial_{\gamma} \bar{g}'_{\alpha\beta}(x) + \ldots$$

$$\Rightarrow \boxed{a^2 h'_{\alpha\beta}(x) = a^2 h_{\alpha\beta}(x) - \bar{g}_{\alpha\delta}(x)\partial_{\beta}\xi^{\delta} - \bar{g}_{\gamma\beta}(x)\partial_{\alpha}\xi^{\gamma} - \xi^{\gamma}\partial_{\gamma}\bar{g}_{\alpha\beta}(x)}$$

or

$$A' = A - \frac{\partial T}{\partial \eta} - aHT \qquad H_L' = H_L - \frac{k}{3} L^{(0)} - aHT$$

$$B^{(0)\prime} = B^{(0)} + kT + \frac{\partial L^{(0)}}{\partial \eta} \qquad H_T^{(0,\pm1)\prime} = H_T^{(0,\pm1)} + kL^{(0,\pm1)}$$

$$B^{(\pm1)\prime} = B^{(\pm1)} + \frac{\partial L^{(\pm1)}}{\partial \eta} \qquad H_T^{(\pm2)\prime} = H_T^{(\pm2)}$$

Gauge transformation

Gauge transformation after SVT…

$$A' = A - \left(\frac{\partial}{\partial \eta} + aH\right) T$$

$$B^{(0)'} = B^{(0)} + kT + \frac{\partial L^{(0)}}{\partial \eta}$$

$$B^{(\pm1)'} = B^{(\pm1)} + \frac{\partial L^{(\pm1)}}{\partial \eta}$$

$$H_L{}' = H_L - \frac{k}{3} L^{(0)} - aHT$$

$$H_T^{(0,\pm1)'} = H_T^{(0,\pm1)} + kL^{(0,\pm1)}$$

$$H_T^{(\pm2)'} = H_T^{(\pm2)}$$

- **Tensor** modes are gauge-invariant.
- For **Scalar and Vector** modes, 2 (S) and 2 (V) are gauge dof, i.e., we can set 2 (S) and 2 (V) perturbation variables to zero.
- **Gauge choices** can be governed by
    - Simplicity of the equations
    - Numerical stability of the solutions
    - Newtonian intuition
    - …
- Different gauges are useful at different stages of the universe's evolution.

Common gauge choices: scalars...

$$h_{\mu\nu}dx^\mu dx^\nu = -2A d\eta^2 - 2B_i d\eta dx^i + 2H_{ij}dx^i dx^j$$

- **Conformal Newtonian**: $H_T^{(0)'} = B^{(0)'} = 0$
  - Completely fixed
  - Physically intuitive

$$ds^2 = a^2(\eta)[-(1+2A)d\eta^2 + (1+2H_L)\gamma_{ij}dx^i dx^j]$$

$A \to \Psi$, $H_L \to -\Phi$

- **Spatially flat**: $H_T^{(0)'} = H_L^{(0)'} = 0$
  - Completely fixed
  - Can sometimes lead to simpler equations
- **Synchronous**: $A' = B^{(0)'} = 0$
  - Not completely fixed (unfixed integration constants)
  - Numerical stability: commonly used in CMB codes like CAMB and CLASS.

Conformal Newtonian gauge

Set $H_T^{(0)'} = 0 \Rightarrow L^{(0)} = -H_T^{(0)}/k$

$B^{(0)'} = 0 \Rightarrow T = -B^{(0)}/k - \frac{1}{k}\frac{\partial L^{(0)}}{\partial \eta}$

$= -B^{(0)}/k + 1/k^2 \frac{\partial H_T^{(0)}}{\partial \eta}$

$L^{(0)}$ and T are fixed.

Substitute $L^{(0)}$ and T back into gauge transformation relations, we find:

$$A' = A - \left(\frac{\partial}{\partial \eta} + aH\right)\left[-B^{(0)}/k + 1/k^2 \frac{\partial H_T^{(0)}}{\partial \eta}\right] \equiv \Phi_A$$

$$H_L' = H_L + \frac{1}{3}H_T^{(0)} - aH\left[-B^{(0)}/k + 1/k^2 \frac{\partial H_T^{(0)}}{\partial \eta}\right] \equiv -\Phi_H$$

Φ_A and Φ_H are called the Bardeen potentials

Then, the scalar perturbed metric has the form:

$$ds^2 = a^2(\eta)\left[-(1+2\Phi)d\eta^2 + (1-2\Phi)\gamma_{ij}dx^i dx^j\right]$$

cf the static weak field metric:

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\delta_{ij}dx^i dx^j$$

where Φ is the Newtonian gravitational potential.

Gauge choices: vectors…

- Can choose $H_T^{(\pm 1)'} = 0$
  - Completely fixed
- Or choose: $B^{(\pm 1)'} = 0$
  - Not completely fixed

1.4 Perturbations in the stress-energy tensor

The perturbed stress-energy tensor reads

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu \qquad \Big\| \quad |\delta T^\mu{}_\nu| \ll \bar{\rho}$$

where $\bar{T}^\mu{}_\nu = \mathrm{diag}\,(-\bar{\rho}(\eta), \bar{P}(\eta), \bar{P}(\eta), \bar{P}(\eta))$.

For a perfect fluid,

$$T^\mu{}_\nu = (\rho_{rest} + P_{rest})\, u^\mu u_\nu + g^\mu{}_\nu P_{rest} \qquad \Big\| \quad \text{rest} = \text{rest frame}$$

(u^μ: 4-velocity of the fluid.)

Standard perturbed FLRW cosmology has no large bulk flows, i.e., fluid flows are non-relativistic.

This means:

$$v^i \equiv \frac{dx^i}{d\eta} = \frac{u^i}{u^0} \; ; \; |v^i| \ll 1 \qquad \Big\| \quad \text{coordinate 3-velocity}$$

energy density and pressure in our coord system

$$\begin{cases} \rho(x) = \rho_{rest} + \mathcal{O}(v^2) \\ P(x) = P_{rest} + \mathcal{O}(v^2) \end{cases}$$

Also: $\delta\rho \equiv \rho - \bar{\rho} \ll \bar{\rho} \; ; \; \delta P \equiv P - \bar{P} \ll \bar{P}$

Then, to linear order in the small parameters:

$$\boxed{\begin{aligned} T^0{}_0 &= \bar{T}^0{}_0 + \delta T^0{}_0 = -(\bar{\rho} + \delta\rho) \\ T^i{}_0 &= \delta T^i{}_0 = -(\bar{\rho} + \bar{P})\, v^i \\ T^0{}_i &= \delta T^0{}_i = (\bar{\rho} + \bar{P})(v_i - B_i) \\ T^i{}_j &= \bar{T}^i{}_j + \delta T^i{}_j = (\bar{P} + \delta P)\, \delta^i{}_j \end{aligned}}$$

A fluid should be locally in thermal equilibrium if it is to be described as a perfect fluid. In cosmology, this is not always true. We allow for an imperfect fluid by introducing an anisotropic stress to the stress-energy tensor:

$$T^\mu{}_\nu = (\rho_{rest} + P_{rest}) u^\mu u_\nu + g^\mu{}_\nu P_{rest} + \Sigma^\mu{}_\nu$$

← Not present in FLRW

where

$$\Sigma^\mu{}_\mu = 0 \qquad \text{traceless}$$

$$\Sigma^\mu{}_\nu u^\nu = 0 \qquad \text{flow orthogonal}$$

In the fluid's rest frame:

Not present in a homogeneous and isotropic universe. They are perturbations

$$\Sigma^\mu{}_\nu = \Pi^\mu{}_\nu$$

$$\Pi^{00} = \Pi^{0i} = 0 \qquad \text{spatial tensor}$$

$$\Pi^i{}_i = 0 \qquad \text{traceless.}$$

Then, the perturbations to the stress-energy tensor read:

$$\delta T^0{}_0 = -\delta\rho$$

$$\delta T^i{}_0 = -(\bar\rho + \bar P) v^i$$

$$\delta T^0{}_i = (\bar\rho + \bar P)(v_i - B_i)$$

$$\delta T^i{}_j = (\bar P + \delta P)\delta^i{}_j + \Pi^i{}_j$$

This is obviously not the only way to get an imperfect fluid. But to describe fluids made up of dilute gases of particles, this is sufficient.

Perturbed stress-energy tensor...

For a **perfect fluid**:

$$T^0{}_0 = \bar{T}^0{}_0 + \delta T^0{}_0 = -(\bar{\rho} + \delta\rho)$$

$$T^i{}_0 = \delta T^i{}_0 = -(\bar{\rho} + \bar{P})v^i$$

$$T^0{}_i = \delta T^0{}_i = -(\bar{\rho} + \bar{P})(v_i - B_i)$$

$$T^i{}_j = \bar{T}^i{}_j + \delta T^i{}_j = -(\bar{P} + \delta P)\delta^i{}_j$$

Perturbed stress-energy tensor…

For an **imperfect** **fluid**:

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$$T^i{}_j = \bar{T}^i{}_j + \delta T^i{}_j = -(\bar{P} + \delta P)\delta^i{}_j + \Pi^i{}_j$$

Anisotropic stress $\Pi^i{}_i = 0$ Traceless

- In general, anisotropic stress is **not** the only way to make a fluid imperfect. But if particles in the early universe behave like dilute gases, this is it.

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SVT decomposition

$$\delta T^0{}_0 = -\delta\rho Q^{(0)}$$

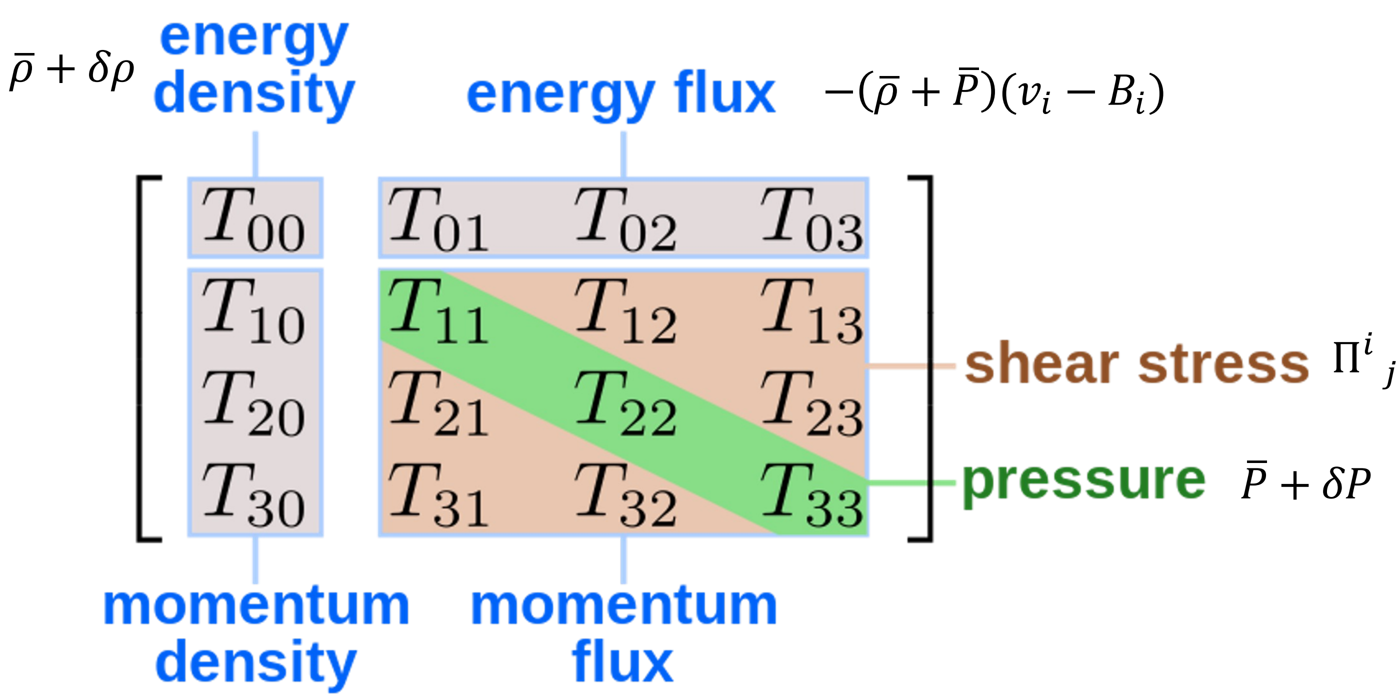
$$\delta T^i{}_0 = -(\bar{\rho} + \bar{P}) \sum_{m=-1}^{1} v^{(m)} Q^{(m)i}$$

$$\delta T^0{}_i = -(\bar{\rho} + \bar{P}) \sum_{m=-1}^{1} (v^{(m)} - B^{(m)}) Q_i^{(m)}$$

$$\delta T^i{}_j = -\delta P \delta^i{}_j Q^{(0)} + \sum_{m=-2}^{2} \Pi^{(m)} Q^{(m)i}{}_j$$

- In general, anisotropic stress is **not** the only way to make a fluid imperfect. But if particles in the early universe behave like dilute gases, this is it.

Perturbed stress-energy tensor...



Gauge transformation

Proceeds in the same way as for $g_{\mu\nu}$. We consider a coordinate transformation

$$x'^\alpha = x^\alpha + \xi^\alpha(x) \qquad \xi^\alpha = (T, L^i)$$

Then,

$$T'^\alpha{}_\beta(x') = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta} T^\mu{}_\nu(x)$$

$$\hookrightarrow = \underbrace{\bar{T}^\mu{}_\nu(x)} + \underbrace{\delta T^\mu{}_\nu(x)}$$

$\downarrow$

$$= T'^\alpha{}_\beta(x) + \xi^\gamma \partial_\gamma T'^\alpha{}_\beta(x) + \ldots$$

$$= \underbrace{\bar{T}^\alpha{}_\beta(x)} + \underbrace{\delta T'^\alpha{}_\beta(x)} + \xi^\gamma \partial_\gamma \bar{T}^\alpha{}_\beta(x) + \ldots$$

$$\Rightarrow \boxed{\delta T'^\alpha{}_\beta(x) = \delta T^\alpha{}_\beta(x) - \bar{T}^\alpha{}_\delta \partial_\beta \xi^\delta + \bar{T}^\delta{}_\beta \partial_\delta \xi^\alpha - \xi^\delta \partial_\delta \bar{T}^\alpha{}_\beta}$$

Or in components:

$$\delta\rho' = \delta\rho - T\frac{\partial\bar{\rho}}{\partial\eta}$$

$$\delta P' = \delta P - T\frac{\partial\bar{P}}{\partial\eta}$$

$$v^{(0,\pm1)'} = v^{(0,\pm1)} + \frac{\partial L^{(0,\pm1)}}{\partial\eta}$$

$$\Pi'^{(0,\pm1,\pm2)} = \Pi^{(0,\pm1,\pm2)}$$

Anisotropic stress is gauge-invariant.

Gauge transformation of $T_{\mu\nu}$...

$$\delta\rho' = \delta\rho - T\frac{\partial\bar{\rho}}{\partial\eta}$$

$$\delta P' = \delta P - T\frac{\partial\bar{P}}{\partial\eta}$$

$$v^{(0,\pm1)\prime} = v^{(0,\pm1)} + \frac{\partial L^{(0,\pm1)}}{\partial\eta}$$

$$\Pi^{(0,\pm1,\pm2)\prime} = \Pi^{(0,\pm1,\pm2)}$$

- **Anisotropic stress** (S, V, T) is gauge-invariant.

1.5 Einstein's equation for perturbations

Einstein's equation can be written as

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = 8\pi G\left(\bar{T}_{\mu\nu} + \delta T_{\mu\nu}\right)$$

from FLRW $\bar{g}_{\mu\nu}$ $\quad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

We have already an SVT-decomposed $\delta T_{\mu\nu} \to \delta T_{\mu\nu}^{(0,\pm1,\pm2)}$

We can also SVT-decompose $\delta G_{\mu\nu} \to \delta G_{\mu\nu}^{(0,\pm1,\pm2)}$,

such that

$$\delta G_{\mu\nu}^{(m)} = 8\pi G\, T_{\mu\nu}^{(m)}$$

$G_{\mu\nu}$ is generally a nonlinear function of $g_{\mu\nu}$. However, because we are only interested in the <u>linear order</u> in small perturbations, $\delta G_{\mu\nu}$ is effectively linear in $h_{\mu\nu}$, such that

$$\delta G_{\mu\nu}^{(0)}\left(A, B^{(0)}, H_L, H_T^{(0)}\right) = 8\pi G\, \delta T_{\mu\nu}^{(0)}\left(\rho, P, v^{(0)}, \Pi^{(0)}\right)$$

$$\delta G_{\mu\nu}^{(\pm1)}\left(B^{(\pm1)}, H^{(\pm1)}\right) = 8\pi G\, \delta T_{\mu\nu}^{(\pm1)}\left(v^{(\pm1)}, \Pi^{(\pm1)}\right)$$

$$\delta G_{\mu\nu}^{(\pm2)}\left(H_T^{(\pm2)}\right) = 8\pi G\, \delta T_{\mu\nu}^{(\pm2)}\left(\Pi^{(\pm2)}\right)$$

Thus, to linear order, it is possible to study S, V, and T perturbations independently of each other.

Scalar Einstein's equation…

From Einstein's equation

$$(-k^2 + 3K)\Phi - 3\mathcal{H}\left(\dot{\Phi} + \mathcal{H}\Psi\right) = 4\pi G a^2 \delta\rho$$

$$\dot{\Phi} + \mathcal{H}\Psi = 4\pi G a^2 [(\bar{\rho} + \bar{P})v^{(0)}/k]$$

$$\ddot{\Phi} - K\Phi + \mathcal{H}\left(\dot{\Psi} + 2\Phi\right) + \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)\Psi - k^2(\Phi - \Psi)/3 = 4\pi G a^2 \delta P$$

$$k^2(\Phi - \Psi) = 8\pi G a^2 \Pi^{(0)}$$

Only two linearly-independent

From local conservation of energy-momentum

$$\left(\frac{\partial}{\partial\eta} + 3\mathcal{H}\right)\delta\rho + 3\mathcal{H}\delta P = -(\bar{\rho} + \bar{P})(kv^{(0)} - 3\dot{\Phi})$$

$$\left(\frac{\partial}{\partial\eta} + 4\mathcal{H}\right)[(\bar{\rho} + \bar{P})v^{(0)}/k] = \delta P - \frac{2}{3}\left(1 - 3\frac{K}{k^2}\right)\Pi^{(0)} + (\bar{\rho} + \bar{P})\Psi$$

Only one linearly-independent

The first two equations can be combined to give

$$(-k^2 + 3K)\Phi = 4\pi G a^2 \left[\delta\rho + \underbrace{3\mathcal{H}(\bar{\rho}+\bar{P})\frac{v^{(0)}}{k}}\right] \qquad \mathcal{H} \equiv aH \qquad (*)$$

The second term on the RHS is $\propto \mathcal{H}/k$, where

$$\frac{\mathcal{H}}{k} = \frac{aH}{k} \sim \frac{\text{comoving perturbation wavelength}}{\text{comoving Hubble length}}.$$

Thus,

$\frac{k}{\mathcal{H}} \gg 1$ "sub horizon"

$\frac{k}{\mathcal{H}} \ll 1$ "super horizon"

The Hubble length is <u>not</u> really a horizon. So this is really bad language.

Therefore, on small, "sub horizon" scales, the RHS of (*) becomes

$$\sim 4\pi G a^2 \delta\rho.$$

On the LHS, we note that

$$|\Omega_K(a)| \equiv \left|\frac{-K}{a^2H^2}\right| \sim O(1)$$

(at most)

Thus, $k \gg \mathcal{H}$ implies immediately that $k^2 \gg K$, and the LHS of (*) becomes $\sim -k^2\Phi$. Combining the results we find

$$\boxed{-k^2\Phi = 4\pi G a^2 \delta\rho}$$

"Newtonian" Poisson eqn.

Vector Einstein's equation...

From Einstein's equation

$$\left(\frac{\partial}{\partial\eta}+2\mathcal{H}\right)\left[kB^{(\pm1)}-\dot{H}_T^{(\pm1)}\right]=-8\pi G a^2\Pi^{(\pm1)}$$

$$\left(1-2\frac{K}{k^2}\right)\left[kB^{(\pm1)}-\dot{H}_T^{(\pm1)}\right]=16\pi G a^2(\bar{\rho}+\bar{P})(v^{(\pm1)}-B^{(\pm1)})/k$$

Four equations: Only two linearly-independent

From local conservation of energy-momentum

$$2\left(\frac{\partial}{\partial\eta}+4\mathcal{H}\right)\left[a^2(\bar{\rho}+\bar{P})(v^{(\pm1)}-B^{(\pm1)})/k\right]=\left(2\frac{K}{k^2}-1\right)\Pi^{(\pm1)}$$

Two equations

Tensor Einstein's equation...

$$\ddot{H}_T^{(\pm 2)} + 2\mathcal{H}\dot{H}_T^{(\pm 2)} + (k^2 + 2K)H_T^{(\pm 2)} = 4\pi G a^2 \Pi^{(\pm 2)}$$

Two equations