MITP Summer School 2024: "Crosslinks of Early Universe Cosmology"

Dark Matter

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Exercise sheet 1

Problem 1: The Liouville operator

In a Friedman-Robertson-Waker spacetime, the line element can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right]. \tag{1}$$

a) Assume for simplicity that $\kappa = 0$ (why is this ok?) and transform to Cartesian coordinates. Show that the only non-vanishing Christoffel symbols are given by $\Gamma^0_{ij} = a^2 f(a)\delta_{ij}$ and $\Gamma^i_{j0} = \Gamma^i_{0j} = u(a)\delta^i_j$, stating the function $u(a)$ explicitly !

b) Locally, one can always choose coordinates ξ^μ such that the line element is that of flat space, i.e. $ds^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu$ in Cartesian coordinates. Show that the coordinate transformation relating the free-fall coordinates to the flat FRW metric is of the form $\partial\xi^0/\partial x^0 = v(a)$, $\partial\xi^j/\partial x^i = w(a)\delta^i_j$ and state $v(a)$ and $w(a)$ explicitly. How does thus the 'physical momentum' $p^i \equiv m\,d\xi^i/d\tau$ of some particle with mass m, as measured by a freely-falling observer, relate to the 'coordinate momentum' $\bar{p}^i \equiv m\,dx^i/d\tau$ observed in the cosmic rest frame ?

c) The phase-space distribution $f(\xi^\mu, p^i)$ of some particle species is typically stated in terms of local (free-fall) coordinates and the conjugate momenta. The Liouville operator $L[f]$ that appears on the l.h.s. of the Boltzmann equation in general takes the form $L[f] = df/d\tau$. Evaluate it for the case of a flat FRW spacetime!
[Hint: Start by expanding the total derivative, then change variables from p to $\bar{p}$ (while still treating f as an explicit function of t and p^i). For the next step you will find the result from b) useful. Finally, convert everything back to p^i.

Problem 2: The free Boltzmann equation

a) Show that the free Boltzmann equation $\partial f/\partial t - Hp\,\partial f/\partial p = 0$ is solved by $f(t,p) = g(p\,a(t))$, where a is the scale factor and g is an arbitrary function.

b) In which cases is the Fermi-Dirac or Bose-Einstein distribution function a solution of the free Boltzmann equation?
[Hint: For these functions to be a solution, there must exist a function h such that $(E-\mu)/T = h(p\,a(t))$, where μ and T are functions of t, but not of p. Consider the ultra-relativistic and the non-relativistic limits with the first order including p. What happens when you include higher-order terms?]

Problem 3: Collision operator for the number density

The Boltzmann equation for a particle χ with momentum $\mathbf{p}$ and energy $E = \sqrt{\mathbf{p}^2 + m_\chi^2}$ is given by $L[f(\mathbf{p})] = C[f(\mathbf{p})]$. You derived the form of the Liouville operator L in problem 1. The collision operator C is the sum of all terms describing processes that contain a particle χ in the initial or final state.[1] The Boltzmann equation for the number density is then obtained by integration over all momenta and reads

$$\dot{n}_\chi + 3Hn_\chi = g_\chi \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{C[f_\chi]}{E} \equiv C_n[f_\chi]\,, \tag{4}$$

a) Show that elastic scatterings of the form $\chi + \psi \to \psi + \chi$ for some other particle ψ do not contribute to $C_n[f_\chi]$.

b) Assume that χ is self-conjugate and there are annihilation reactions $\chi\chi \leftrightarrow \psi_1\psi_2$ with $|\mathcal{M}_{\chi\chi\to\psi_1\psi_2}|^2 = |\mathcal{M}_{\psi_1\psi_2\to\chi\chi}|^2 = |\mathcal{M}|^2$ into particles ψ_1 and ψ_2, which are part of a heat bath with temperature T and vanishing chemical potential. Further assume that for all relevant times, elastic scatterings efficiently enforce *kinetic equilibrium*, i.e. $f_\chi = [(E-\mu_\chi)/T \pm 1]^{-1}$. Derive the Boltzmann equation for the number density for $m_\chi \gg T$!

[Hint: Step by step, you want to

1. *Convince yourself that the starting point is*

$$C_n = S_\psi \int \frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3k_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3k_2}{(2\pi)^3 2\omega_2} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)$$
$$\times |\mathcal{M}|^2 (f_{\psi_1} f_{\psi_2} (1 \pm f_{\chi,1})(1 \pm f_{\chi,2}) - f_{\chi,1} f_{\chi,2} (1 \pm f_{\psi_1})(1 \pm f_{\psi_2}))\,, \tag{5}$$

where the symmetry factor S_ψ takes into account whether ψ_1 and ψ_2 are identical particles ($S_\psi = 1/2$) or not ($S_\psi = 1$), and all integrations are over the entire $\mathbb{R}^3$.

2. *Convince yourself that $(m_\chi - \mu_\chi)/T \gg 1$, implying that χ follows a Maxwell-Boltzmann distribution with $f_\chi \simeq n_\chi/n_{\chi,\mathrm{eq}} \exp(-E_\chi/T)$ and $f_\chi \ll 1$, where $n_{\chi,\mathrm{eq}}$ is the number density of χ for $\mu_\chi = 0$.*

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[1] For a single χ initial state, for example, the contribution from $2 \to 2$ interactions is given by

$$C_{\chi 1\to 23} = -\int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3k_2}{(2\pi)^3 2\omega_2} \frac{\mathrm{d}^3k_3}{(2\pi)^3 2\omega_3} (2\pi)^4 \delta^{(4)}(p + k_1 - k_2 - k_3)\, |\mathcal{M}|^2 f_\chi f_1 (1 \pm f_2)(1 \pm f_3) \tag{2}$$

where $k_a = (\omega_a, \mathbf{k}_a)$ denotes the momenta of some particles $\psi_a \neq \chi$ (with $a = 1,2,3$); $|\mathcal{M}|^2$ is the squared amplitude summed over *both* initial and final states, and the + (−) signs in the second line apply for boson (fermion) final states. For a single χ *final* state, we have (note the overall sign!)

$$C_{\chi 1\to 23} = \int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3k_2}{(2\pi)^3 2\omega_2} \frac{\mathrm{d}^3k_3}{(2\pi)^3 2\omega_3} (2\pi)^4 \delta^{(4)}(p + k_1 - k_2 - k_3)\, |\mathcal{M}|^2 f_2 f_3 (1 \pm f_\chi)(1 \pm f_1)\,. \tag{3}$$

3. *Why can you also approximate* $f_{\psi,1/2} \simeq \exp(-\omega_{1,2}/T)$*? Show that*

$$f_{\psi_1} f_{\psi_2}(1 \pm f_{\chi,1})(1 \pm f_{\chi,2}) - f_{\chi,1} f_{\chi,2}(1 \pm f_{\psi_1})(1 \pm f_{\psi_2}) \simeq \frac{n_{\chi,\mathrm{eq}}^2 - n_\chi^2}{n_{\chi,\mathrm{eq}}^2} e^{-\frac{E_1+E_2}{T}} . \tag{6}$$

4. *Use the definition of the cross-section for* $\chi\chi \to \psi_1\psi_2$,

$$\sigma v \equiv \frac{4g_\chi^2}{E_1 E_2} \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2\omega_2} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)|\mathcal{M}|^2 \tag{7}$$

with the Møller velocity $v \equiv \sqrt{(p_1 \cdot p_2)^2 - m_\chi^4}/(E_1 E_2)$, *to arrive at*

$$\dot{n}_\chi + 3Hn_\chi \simeq \langle \sigma v \rangle (n_{\chi,\mathrm{eq}}^2 - n_\chi^2) . \tag{8}$$

Here, the thermal average is defined as

$$\langle \sigma v \rangle \equiv \frac{g_\chi^2}{n_{\chi,\mathrm{eq}}^2} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \frac{\mathrm{d}^3 p_2}{(2\pi)^3} e^{-\frac{E_1+E_2}{T}} \sigma v . \tag{9}$$

5. *Perform 5 of the 6 phase-space integrals to simplify the thermal average definition to*

$$\langle \sigma v \rangle = \frac{4x}{K_2^2(x)} \int_1^\infty \mathrm{d}\tilde{s}\, (\tilde{s} - 1)\sqrt{\tilde{s}} K_1(2\sqrt{\tilde{s}}x)\sigma , \tag{10}$$

where K_i *is the modified Bessel function of second kind and order* i, $\tilde{s} \equiv /(4m_\chi^2)$ *and* $x \equiv m_\chi/T$.
[Further hints: First choose a coordinate system such that the integrals in Eq. (**??**) *are over* E_1, E_2, *and* $\cos\theta$, *where* $\theta = \angle(\mathbf{p}_1, \mathbf{p}_2)$. *In the next step, transform to* $E_\pm \equiv E_1 \pm E_2$ *and* $s = 2m_\chi^2 + 2E_1E_2 - 2p_1p_2\cos\theta$; *the integration region then becomes* $s \geq 4m_\chi^2$, $E_+ \geq \sqrt{s}$ *and* $|E_-| \leq \sqrt{1 - 4m_\chi^2/s}\sqrt{E_+^2 - s}$.*]*

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c) What changes in the last exercise if χ is not self-conjugate, but instead annihilates with a particle $\bar{\chi}$ in a reaction $\bar{\chi}\chi \leftrightarrow \psi_1\psi_2$?
[Hint: Why does the symmetry factor in Eq. (**??**) *miss a factor* $1/2$ *from* χ *being self-conjugate?]*

Problem 4: Preparing to use DarkSUSY

DarkSUSY is a widely used numerical package to calculate all kinds of dark matter observables, and during the next exercise session we want to explore some of its functionalities for relic density calculations. For this to work in practice, I need you to come prepared – so please do the following **before** Wednesday:

1. Go to `https://www.darksusy.org`, download the most recent version (6.4.0) and follow the instructions on the webpage to install it.

2. Go to /examples/aux, and compile and run the example program oh2_generic_wimp. Plot the output in the $\langle\sigma v\rangle$ vs $m_{\rm DM}$ plane, and interpret it. Try to identify the DarkSUSY functions that *i)* initialize a model with a given set of model parameters and *ii)* calculate the relic density for this parameter point.

3. Do the same with the example program oh2_ScalarSinglet.

4. If time allows, also take a glance at the tutorial at `https://darksusy.hepforge.org/tutorials/TOOLS_2021/DarkSUSY_getting_started.pdf` in order to familiarize yourself with some of the key principles of how the code is organized.

If you encounter problems at any of the steps above, please let me know ASAP for support – either by email or directly on site. Note that for the purpose of the numerical exercises discussed here, it will be sufficient to install the 'light' version of the code.