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Introduction to multi-messenger astrophysics
Exercise sheet 1

1 Fermi acceleration with only head-on collisions

When we derived the spectrum of cosmic rays from first-order Fermi acceleration, we assumed a distribution of scattering angles that favored approximately head-on collisions, *i.e.*, $P(\theta) \propto \cos\theta$. With this, we computed the average relative energy loss,

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{\int_0^{\pi/2} \left(2 v_{\rm shock} v_{\rm CR} + 2 v_{\rm shock}^2\right) \cos\theta \sin\theta \; d\theta}{\int_0^{\pi/2} \cos\theta \sin\theta \; d\theta} \; . \tag{1}$$

Now, repeat this calculation for a scenario in which *every* collision is head-on, $\cos\theta = 1$. What is the cosmic-ray spectrum that results in this case?

Source: Dan Hooper, Particle Cosmology and Astrophysics, 1st edition (2024)

2 Human-made and natural particle accelerators

Consider a test particle of energy E and charge Z in a region containing a magnetic field of intensity B. By equation its Larmor radius, $R_L = E/(ZB)$, to the size of the region in which it is contained, R, we can find the maximum energy to which the particle can be accelerated,

$$E_{\rm max} \approx 10^6 \ {\rm GeV} \cdot Z \left(\frac{B}{\mu{\rm G}}\right) \left(\frac{u}{0.1 \ c}\right) \left(\frac{R}{10 \ {\rm pc}}\right) \tag{2}$$

Recall that 1 pc = 3.26156 light-years.

1. What is the maximum energy, $E_{\rm max}$ to which the Large Hadron Collider (LHC) can accelerate protons? Is your result close to the actual maximum energy of the LHC, about 6.8 TeV? (*Hint:* For the LHC, $B_{\rm LHC} \approx 8.3$ T (1 T = 10^4 G) and $R_{\rm LHC} \approx 4.3$ km. Set $u = c$.)

2. What radius would an LHC-like accelerator, with the same magnetic field intensity as the LHC, would need to have to reach energies comparable to those of ultra-high-energy cosmic rays, *e.g.*,, $E_{\rm max} = 10^{10}$ GeV?

3. Conversely, keeping the radius of the accelerator fixed to that of the LHC, what would be the intensity of the magnetic field needed to accelerate protons to $E_{\rm max} = 10^{10}$ GeV?

3 UHECR $p\gamma$ and pp interactions

1. Derive the expression for the center-of-momentum energy in a $p\gamma$ collision, *i.e.*,

$$E_{\rm CM} \approx 45 \text{ GeV} \left(\frac{E_p}{\rm PeV}\right)^{1/2} \left(\frac{\epsilon_\gamma}{\rm MeV}\right)^{1/2} (1-\cos\theta)^{1/2} \ , \tag{3}$$

   in the limit of $E_{\rm CM} \gg m_p$. (*Hint:* Use the Lorentz invariance of the square of the four-momentum of the particles.)

2. What is the threshold value of the product $E_p\epsilon_\gamma$ needed to produce a Δ resonance via $p+\gamma \to \Delta$. Consider a head-on collision. (*Hint:* Look at the previous question. Also, remember that $m_\Delta = 1232$ MeV.)

3. UHECR protons with energies in excess of 10^{11} GeV hit the atmosphere of the Earth regularly. What is the maximum mass of a particle that can be created in such a pp process? Consider that the target proton in the atmosphere is at rest.

4 "Lossless" propagation of high-energy cosmic particles

Consider the propagation of high-energy particles of cosmic origin that experience no energy losses other than those due to the adiabatic cosmological expansion. Their number density (number of particles per unit energy), Y, evolves with redshift as

$$\partial_z Y(E,z) = -\frac{1}{H(z)(1+z)} \ \partial_E[EH(z)Y(E,z)] \ , \tag{4}$$

where E is the particle energy and H is the Hubble parameter.

This is *not* a good description for the propagation of cosmic rays, but it can describe that of high-energy cosmic neutrinos.

1. Assume that particles are injected at a single redshift, z_1. This is a rough approximation of the reality, where most purported cosmic accelerators are at $z \approx 1$. The injected particle spectrum $Y(E,z_1) = f(E,z_1)$. Show that, in this case, the particle density at $z \le z_1$ is

$$Y(E,z) = \left(\frac{1+z_1}{1+z}\right) f\left(\frac{1+z_1}{1+z}, z_1\right) \ . \tag{5}$$

2. Now consider that the particle spectrum is a power law in energy, *i.e.*,

$$f(E,z_1) = N_{\rm src}(z_1) F \left(\frac{E}{E^*}\right)^{-\gamma} \ , \tag{6}$$

   where $N_{\rm src}(z_1)$ is the number of particle-injecting sources at redshift z_1, F is a normalization factor that represents the number density of particles with energy E^* injected by a single source, E^* is an arbitrary choice of energy (for high-energy neutrinos, this is typically

100 TeV), and γ is the spectral index of the power-law spectrum of the particles. (The value of γ is fixed by the particle-generation mechanism at the sources, but we will keep the derivation generic here.)

Plugging Eq. (6) into the propagation equation, Eq. (4), find the flux (number of particles per unit energy, area, time, and soldi angle), J, of particles that reaches the Earth (at $z = 0$). (*Hint:* Remember that the particle flux and density are connected by the factor $c/(4\pi)$.)

3. What units must the normalization factor, F in Eq. (6), have?

4. What does the flux at Earth look like if particles are injected instead at two redshifts, z_1 and z_2? Assume that sources at the two redshifts inject spectra with the same shape as above, Eq. (6), but that their abundance is different, *i.e.*, $N_{\rm src}(z_1)$ and $N_{\rm src}(z_2)$. (*Hint:* Ask yourself if you can treat each particle injection—and the ensuing propagation—independently of one another.)

5. What does the flux at Earth look like if particles are injected at a finite number, N_z, of different redshifts?

6. Let's make the problem more realistic by considering instead a population of sources injecting particles continuously from $z = 0$ to $z = z_{\rm max}$. Assume each source in the population is identical to each other and injects particles with the same shape as before, Eq. (6), but now the differential number density of sources, $dN_{\rm src}/dz$, is a continuous function of redshift. Find the particle flux at Earth in this case.

5 Maximum shower depth of a proton-initiated air shower

Recall that the atmospheric depth (g cm^{-2}) at which a photon-initiated air shower reaches its maximum size is

$$X_{\rm max}^{\gamma} = \lambda_\Gamma \ln(E_0^\gamma, E_{\rm C}^e) \ , \tag{7}$$

where E_0^γ is the energy of the primary photon, λ_Γ is the photon interaction length, E_0^γ is the energy of the primary photon, and $E_{\rm C}^e$ is the energy that each particle in the shower has at shower maximum. Show that, for a proton-initiated air shower, the maximum shower depth is

$$X_{\rm max}^{p} = X_1 + \lambda_\Gamma \ln\left(\frac{E_0^p}{3N_{\rm ch}E_{\rm C}^e}\right) \ , \tag{8}$$

where E_0^p is the energy of the primary proton, $X_1 = \lambda_I \ln 2$ is the depth of the first interaction (*i.e.*, where the primary proton interacts), and λ_I is the proton-air interaction length.

To do this, consider the following:

- When the primary proton interacts, 2/3 of its energy go to the bulk of charged pions and 1/3, to the bulk of neutral pions.

- The multiplicity of charged pions in a proton-air interaction (*i.e.*, the number of charged pions created in the interaction) is $N_{\rm ch}$. The multiplicity of neutral pions is $N_{\rm ch}/2$.

- Each neutral pion decays into two photons, $\pi^0 \to \gamma + \gamma$.
- Consider that each photon from a π^0 decay starts a shower with primary energy $(E_0^p/3)/N_{\mathrm{ch}}$.
- Use the expression for the maximum shower depth of a photon-initiated shower, Eq. (7), with $E_0^\gamma = (E_0^p/3)/N_{\mathrm{ch}}$.

For more details, see J. Matthews, *Astropart. Phys.* 22, 387 (2005).