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Exercise: Derive the Gravitational Wave Equation of motion in a FLRW background

To simplify the calculation, it is convenient to use a relation between the Ricci tensor

$$R_{\mu\nu} = \partial_{[\lambda}\Gamma^{\lambda}_{\mu\nu]} + \Gamma^{\alpha}_{[\alpha\lambda}\Gamma^{\lambda}_{\mu\nu]} \equiv \partial_\lambda\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\alpha}\,, \tag{1}$$

computed from a given metric $g_{\mu\nu}$, and that from another metric $\bar{g}_{\mu\nu}$, related by a conformal factor

$$\bar{g}_{\mu\nu} = e^{2\Omega(x)}g_{\mu\nu}\,. \tag{2}$$

Note that here $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}[g_{\beta(\mu,\nu)} - g_{\mu\nu,\beta}] \equiv \frac{1}{2}g^{\alpha\beta}[g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}]$ are the *Christoffel symbols*.

Exercise 1: Derive the relation given in Eq. (3) (where ; = covariant derivatives)

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + 2\left\{\Omega_{,\mu}\,\Omega_{,\nu} - \Omega_{,\mu;\nu}\right\} - g_{\mu\nu}\left\{2\Omega_{,\alpha}\,\Omega^{,\alpha} + (\Omega^{,\alpha})_{;\alpha}\right\}\,. \tag{3}$$

Let us now consider that $\bar{g}_{\mu\nu}$ is the background metric where matter fields (potential sources of GWs) live. It consists of two pieces: a homogeneous and isotropic spatially flat part (FLRW), plus a small perturbation on top of it,

$$ds^2 = \bar{g}_{\mu\nu}\,dx^\mu dx^\nu = a^2(\eta)\,g_{\mu\nu}\,dx^\mu dx^\nu = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu\,, \tag{4}$$

Here, we consider $|h_{\mu\nu}| \ll 1$. Notice that we have introduced the *conformal time* $d\eta = dt/a(t)$, since this way we will be able to apply the previous relation (3). Identifying

$$a(\eta) = e^{\Omega(x)}\,, \tag{5}$$

then $\Omega(x) = \log a(\eta)$ only depends on time, and hence

$$\Omega_{,i} = 0, \qquad \Omega' = a'/a = \mathcal{H}, \qquad \Omega'' = a''/a - \mathcal{H}^2\,, \tag{6}$$

with ' $\equiv \frac{d}{d\eta}$, $\mathcal{H} \equiv a'/a$. This then leads to

$$\bar{g}_{\mu\nu} = e^{2\Omega} g_{\mu\nu} \;\Rightarrow\; \bar{R}_{\mu\nu} = R_{\mu\nu}[\eta_{\mu\nu} + h_{\mu\nu}(x)] + 2(\Omega_{,\mu}\,\Omega_{,\nu} - \Omega_{,\nu;\mu}) - g_{\mu\nu}(2\Omega^{,\alpha}\Omega_{,\alpha} + \Omega_{;\alpha}^{;\alpha}) \tag{7}$$

where $(2\Omega^{,\alpha}\Omega_{,\alpha} + \Omega_{;\alpha}^{;\alpha}) = 2g^{\alpha\beta}\Omega_{,\alpha}\Omega_{,\beta} + (g^{\alpha\beta}\Omega_{,\beta})_{;\alpha})$ with $g^{\alpha\beta} \equiv \eta^{\alpha\beta} - h^{\alpha\beta}$, $h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$, and we have

$$\Omega_{,\mu} = \Omega'\delta_{\mu 0} \;\Rightarrow\; \Omega_{,\mu}\,\Omega_{,\nu} = \Omega'^2\delta_{\mu 0}\delta_{\nu 0} \tag{8}$$

$$\Omega_{,\mu;\nu} = \Omega_{,\nu;\mu} = \Omega_{,\mu\nu} - \Gamma^{\lambda}_{\mu\nu}\Omega_{\lambda} = \Omega''\delta_{\mu 0}\delta_{\nu 0} - \Gamma^{0}_{\mu\nu}\Omega_{,0} \tag{9}$$

$$(\Omega^{,\alpha})_{;\alpha} = \Omega_{,\alpha}^{;\alpha} + \Gamma^{\alpha}_{\alpha\beta}\Omega^{,\beta} = \Omega''\delta_{\mu 0}\delta_{\nu 0} + \Gamma^{\alpha}_{\alpha\beta}g^{\beta 0}\Omega_{,0}\;. \tag{10}$$

Expanding to first order in $h_{\mu\nu}$,

$$\Gamma^{0}_{\mu\nu} = \frac{1}{2}\eta^{0\alpha}(h_{\alpha(\mu,\nu)} - h_{\mu\nu,\alpha}) = \frac{1}{2}(h'_{\mu\nu} - h_{0(\mu,\nu)}) + O(h^2_{**})\,, \tag{11}$$

$$g^{\beta 0}\Gamma^{\alpha}_{\alpha\beta} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu}g^{\mu 0} = \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta,\mu}\eta^{\mu 0} = -\frac{1}{2}\eta^{\alpha\beta}h'_{\alpha\beta} + O(h^2_{**})\,, \tag{12}$$

the *rhs* terms of (7) are

$$\Omega_{,\mu}\,\Omega_{,\nu} - \Omega_{,\nu;\mu} = (\Omega'^2 - \Omega'')\delta_{\mu 0}\delta_{\nu 0} + \frac{1}{2}(h'_{\mu\nu} - h_{0(\mu,\nu)})\Omega' \tag{13}$$

$$g_{\mu\nu}(2\Omega^{,\alpha}\Omega_{,\alpha} + \Omega_{;\alpha}^{;\alpha}) = (2\Omega'^2 + \Omega'')\delta_{\alpha 0}\delta_{\beta 0}(\eta_{\mu\nu}\eta^{\alpha\beta} + h_{\mu\nu}\eta^{\alpha\beta} - \eta_{\mu\nu}h^{\alpha\beta}) - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h'_{\alpha\beta}\Omega'\,. \tag{14}$$

Take now e.g. the *Synchronous Gauge*

$$h_{\mu\nu} = h^{*}_{\mu\nu} + \xi_{[\mu;\nu]}\,, \qquad \text{with } h^{*}_{0\mu} = 0\,, \tag{15}$$

just to simplify the upcoming expressions. From now on we omit the $*$ mark, as we will consider a perturbation in (4) such that $h_{0\mu} = 0$. Using this fact and putting together (8),(9),(11),(13) and (14), then

$$\bar{R}_{\mu\nu} = R_{\mu\nu}[\eta_{\mu\nu} + h_{\mu\nu}] + 2\left(2\mathcal{H}^2 - \frac{a''}{a}\right)\delta_{\mu 0}\delta_{\nu 0} + \mathcal{H}h'_{\mu\nu} + (\eta_{\mu\nu} + h_{\mu\nu})\left(\mathcal{H}^2 + \frac{a''}{a}\right) + \frac{1}{2}\eta_{\mu\nu}h'\mathcal{H}\,, \tag{16}$$

where $h = h^i_i (= h^\mu_\mu)$ is the trace of the perturbation, and $R_{\mu\nu}$ the Ricci tensor of a perturbed Minkowski space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$,

$$R_{\mu\nu}[\eta_{\mu\nu} + h_{\mu\nu}] = \frac{1}{2}(h'^{\alpha}_{\alpha(\mu,\nu)} - h_{,\mu\nu} - h'^{\alpha}_{\mu\nu,\alpha})\,. \tag{17}$$

Exercise 2: Derive Eq. (16)

Exercise 3: Derive Eq. (17)

The Einstein field eqs. for the total metric $\bar{g}_{\mu\nu} = a^2(\eta)g_{\mu\nu}$, are

$$\bar{R}_{\mu\nu} = 8\pi G\bar{S}_{\mu\nu}\,, \qquad \text{with} \qquad \bar{S}_{\mu\nu} = \bar{T}_{\mu\nu} - \frac{1}{2}\bar{T}\bar{g}_{\mu\nu}\,, \tag{18}$$

Let us split these equations into the background $R^{(0)}_{\mu\nu} = 8\pi G\, S^{(0)}_{\mu\nu}$ and the perturbed $\delta R_{\mu\nu} = 8\pi G\, \delta \mathrm{S}_{\mu\nu}$ equations, the latter computed to first order in $h_{\mu\nu}$. Thus, we should decompose the (trace reversed) energy-momentum tensor $S_{\mu\nu}$ into *background* and *perturbative* parts,

$$\bar{S}_{\mu\nu} = \left\langle S^{\rm FLRW}_{\mu\nu}\right\rangle + \left\langle \delta S^{\rm FLRW}_{\mu\nu}\right\rangle + \Pi_{\mu\nu}\,, \tag{19}$$

where $\left\langle S^{\rm FLRW}_{\mu\nu}\right\rangle$ should be understood as a spatial average, with the tensor $S^{\rm FLRW}_{\mu\nu}$ computed in a FLRW background. Alternatively, one can think of a $H\&I$ perfect-fluid with energy-momentum tensor $T_{\mu\nu} = (\rho + p)a^2u_\mu u_\nu + pg_{\mu\nu}$, with 4-velocity $u_\mu = (1,0,0,0)$ and background energy and pressure densities ρ and p. From there one can build up the (trace-reversed) tensor as $S^{H\&I}_{\mu\nu} = (\rho+p)a^2u_\mu u_\nu + \frac{1}{2}(\rho-p)g_{\mu\nu}$, and then identify $S^{H\&I}_{\mu\nu} = \left\langle S^{\rm FLRW}_{\mu\nu}\right\rangle$. On the other hand, $\left\langle \delta S^{\rm FLRW}_{\mu\nu}\right\rangle$ should be understood as the perturbation of $S^{H\&I}_{\mu\nu}$, due to the perturbations of background densities, as well as of the metric, whereas $\Pi_{\mu\nu}$ should be understood as an *anisotropic* stress representing an additional perturbation over the background energy-momentum tensor, but unrelated to a perturbation of the metric or of the background densities. Thus, one obtains the identification $\left\langle S^{\rm FLRW}_{00}\right\rangle = a^2(\rho+3p)/2$ and $\left\langle \sum_i S^{\rm FLRW}_{ii}\right\rangle = 3a^2(\rho-p)/2$, and e.g. $\left\langle \delta S^{\rm FLRW}_{ij}\right\rangle = \frac{1}{2}a^2(\rho-p)h_{ij} + \frac{1}{2}a^2(\delta\rho-\delta p)\delta_{ij}$. We will drop from now on, for the shake of clarity, the label $^{\rm FLRW}$ from the terms spatially averaged, though one should bare in mind that the averages $\langle ...\rangle$ will be always taken over $S^{(i)}_{\mu\nu}$ tensors computed in a FLRW background.

Using $\bar{R}_{\mu\nu}$ (16) written in terms of h_{ij} and $a(\eta)$, then the Einstein eqs., component by component, read

$$00: \quad \frac{1}{2}(-h'' + \mathcal{H}h') + 3(\mathcal{H}^2 - \frac{a''}{a}) = 8\pi G(\langle S_{00}\rangle + \langle \delta S_{00}\rangle + \Pi_{00}) \tag{20}$$

$$0i: \quad h'_{ki,k} - \frac{1}{2}h,'_i = 8\pi G(\langle S_{0i}\rangle + \langle \delta S_{0i}\rangle + \Pi_{0i}) \tag{21}$$

$$ij: \quad \frac{1}{2}(h'^{k}_{k(i,j)} - h,_{ij} + h''_{ij} - h_{ij,kk}) + \mathcal{H}h'_{ij} + (\delta_{ij} + h_{ij})\left(\mathcal{H}^2 + \frac{a''}{a}\right) + \frac{1}{2}\mathcal{H}h'\delta_{ij} = 8\pi G(\langle S_{ij}\rangle + \langle \delta S_{ij}\rangle + \Pi_{ij}) \tag{22}$$

Exercise 4: Derive Eqs. (20)-(22)

Appealing to isotropy in the FLRW Universe, then

$$\langle \mathrm{S_{ij}}\rangle = \frac{1}{3}\delta_{\rm ij}\sum_k \langle S_{kk}\rangle \tag{23}$$

and hence $S_{0i} = 0$. Thus, the background parts of (20)-(22), which describe the evolution of the flat FLRW Universe, will be

$$00: \quad 3\left(\mathcal{H}^2 - \frac{a''}{a}\right) = 8\pi G\,\langle S_{00}\rangle \tag{24}$$

$$0i: \quad 0 = 0 \tag{25}$$

$$ij: \quad (\mathcal{H}^2 + \frac{a''}{a}) = \frac{8\pi G}{3}\sum_k \langle S_{kk}\rangle\,, \tag{26}$$

Exercise 5: Derive Eqs. (24)-(26)

As said before, identifying the energy and pressure densities ρ and p of a $H\&I$ perfect-fluid as $\langle S_{00}\rangle = a^2(\rho+3p)/2$ and $\sum_k\langle S_{kk}\rangle = 3a^2(\rho-p)/2$, it can be easily shown that eqs. (24),(26) are indeed equivalent to linear combinations of the *Friedmann Equations*, expressed in conformal time.

On the other hand, the perturbed Einstein equations, which describe the evolution of the metric perturbations, are

$$00: \quad -h'' + \mathcal{H}h' = 16\pi G(\Pi_{00} + \langle\delta S_{00}\rangle) \tag{27}$$

$$0i: \quad 2h'_{ik,k} - h'_{,i} = 16\pi G(\Pi_{0i} + \langle\delta S_{0i}\rangle) \tag{28}$$

$$ij: \quad h'^{k}_{(i,j),k} - h_{,ij} + h''_{ij} - h_{ij,kk} + 2\mathcal{H}h'_{ij} + 2h_{ij}(\mathcal{H}^2 + a''/a) + \mathcal{H}h'\delta_{ij} = 16\pi G(\Pi_{ij} + \langle\delta S_{ij}\rangle) \tag{29}$$

In general, a (spatial-spatial) metric perturbation h_{ij} has six independent degrees of freedom, whose contributions can be split into scalar, vector and tensor metric perturbations as

$$h_{ij} = \psi\,\delta_{ij} + E_{,ij} + F_{(i,j)} + h^{TT}_{ij}\,, \tag{30}$$

with $\partial_i F_i = \partial_i h^{TT}_{ij} = h^{TT}_{ii} = 0$. The two scalars ($\psi$ and E) plus one transverse vector (F_i) plus a transverse-traceless tensor (h^{TT}_{ij}), account for the required $1+1+2+2 = 6$ dof, as it should. Let us introduce such a decomposition into the equations (27),(28),(29), and keep only the *Transverse-Traceless* part h^{TT}_{ij} of the perturbations. Notice that in the 00- and $0i-$equations (27),(28) there cannot be any TT part surviving, so we can focus only in the ij-equations (29). Keeping only the TT perturbation in equations (29), leads to the equation for the TT perturbations as

$$h^{TT\,''}_{ij}(\eta,\mathbf{x}) + 2\mathcal{H}h^{TT\,'}_{ij}(\eta,\mathbf{x}) - h^{TT}_{ij,kk}(\eta,\mathbf{x}) + 2h^{TT}_{ij}\left(\mathcal{H}^2 + \frac{a''}{a}\right) = 16\pi G(\Pi_{ij} + \langle\delta S_{ij}\rangle)^{TT}(\eta,\mathbf{x})\,. \tag{31}$$

It is remarkable that only the TT metric components obey a wave-like operator (you can see this explicitly by looking at the equations of scalar and vector parts of the perturbations, but we will skip that here). Therefore, only the h^{TT}_{ij} metric perturbations — the transverse-traceless dof — characterize the radiative dof in the space-time. Those are the only dof carring energy in the form of GW.

Exercise 6: Derive Eq. (31) by keeping only TT dof in Eq.(29).

The tensor $\Pi^{TT}_{ij}(\eta,\mathbf{x})$ is the TT part of the spatial-spatial components of the anisotropic stress-tensor Π_{ij}, and thus

$$\partial_i \Pi^{TT}_{ij} = \Pi^{TT}_{ii} = 0. \tag{32}$$

Therefore, in order to solve (31), we need to obtain Π_{ij} from the matter fields that generate the GWs, and then take its TT part $\Pi^{\rm TT}_{ij}$. At the same time, we also have

$$[\langle\delta S_{ij}\rangle]^{\rm TT} \equiv \frac{1}{2}a^2[(\rho-p)h_{ij} + (\delta\rho - \delta p)\delta_{ij}]^{\rm TT} = \frac{1}{2}(\rho-p)a^2 h^{TT}_{ij}\,. \tag{33}$$

Using the Friedman Equations (24),(26), we arrive finally at

$$h^{TT\,''}_{ij} + 2\mathcal{H}h^{TT\,'}_{ij} - \nabla^2 h^{TT}_{ij} = 16\pi G\Pi^{TT}_{ij}\,, \tag{34}$$

with Π_{ij} the anisotropic stress-tensor of some fields, and Π_{ij}^{TT} its TT-part sourcing GWs.

Exercise 7: Derive Eq. (34)

If you arrived here... Congratulations! You have derived the equation of motion for the propagation and creation of GWs in a FLRW background!