Dark Matter

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Solutions[1] to exercise sheets

Problem 1: The Liouville operator

a) The effect of curvature only enters in the 1st Friedmann equation, according to which the total energy density is proportional to $(1+\kappa/\dot{a}^2)$; since $\dot{a}$ is monotonically decreasing with time, a non-vanishing curvature ($\kappa \neq 0$) thus necessarily had a smaller effect on the evolution of the early universe than it has today. Measurements 'today' (since CMB times) are consistent with a flat universe, so we can set $\kappa = 0$ for all epochs in standard ΛCDM cosmology. Expressing the line element in Cartesian coordinates, rather than spherical (or polar) coordinates, then gives

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,. \tag{1}$$

In these coordinates, we thus have $\partial_\rho g_{\mu\nu} = 0$ for $\rho \neq 0$, and hence

$$\Gamma^0_{\rho\sigma} = \frac{1}{2}g^{0\nu}\left(g_{\rho\nu,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}\right) = -\frac{1}{2}\left(g_{\rho 0,\sigma} + g_{0\sigma,\rho} - \partial_t g_{\rho\sigma}\right) = \frac{1}{2}\partial_t g_{\rho\sigma}\,, \tag{2}$$

$$\Gamma^i_{\rho\sigma} = \frac{1}{2}g^{i\nu}\left(g_{\rho\nu,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}\right) = \frac{1}{2}a^{-2}\left(g_{\rho i,\sigma} + g_{i\sigma,\rho}\right). \tag{3}$$

Here we further used $\partial_\rho g_{0\nu} = 0$ and that the metric is diagonal ($g_{\mu\nu} = 0$ for $\mu \neq \nu$). The first term is only non-vanishing for $\rho = \sigma = i$, giving $\Gamma^0_{ij} = a^2 H\delta_{ij}$, where $H = \dot{a}/a$. For the second term non-vanishing contributions can only appear when one of the indices ρ, σ equals i (because the metric is diagonal) and the other one equals 0 (because all other derivatives give zero); this gives $\Gamma^i_{j0} = \Gamma^i_{0j} = H\delta^i_j$, i.e. $u(a) = H$.

b) From the line element in free-fall coordinates, we can deduce the line element in arbitrary coordinates as

$$ds^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu = \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x_\rho}\frac{\partial\xi^\nu}{\partial x_\sigma}dx^\rho dx^\sigma \equiv g_{\rho\sigma}dx^\rho dx^\sigma \,. \tag{4}$$

This gives

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{\partial\xi^0}{\partial x^0}\right)^2 dt^2 + \delta_{ij}\frac{\partial\xi^i}{\partial x^\mu}\frac{\partial\xi^j}{\partial x^\nu}dx^\mu dx^\nu \tag{5}$$

Direct comparison to $g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2\delta_{ij}dx^i dx^j$ thus gives $\partial\xi^0/\partial x^0 = 1$ as well as $\partial\xi^i/\partial x^j = a\delta^i_j$. In other words, we have $v(a) = 1$ and $w(a) = a$. p^μ is a 4-vector, so it transforms as $p^i = (\partial\xi^i/\partial x^\mu)\overline{p}^\mu = w(a)\overline{p}^i$ and $p^0 = (\partial\xi^0/\partial x^\mu)\overline{p}^\mu = v(a)\overline{p}^0$. Or, more compactly, $p^\mu = (\overline{p}^0, a\overline{\mathbf{p}})$.

[1] A huge thanks to Frederik Depta for LaTeX solutions to older versions of some of these problems!

c) We follow the hint and first expand, then change variables:

$$\begin{aligned} L[f] &= \frac{df(\xi^\mu, p^i)}{d\tau} = \frac{d\xi^\mu}{d\tau}\frac{\partial f}{\partial \xi^\mu} + \frac{dp^i}{d\tau}\frac{\partial f}{\partial p^i} && (6)\\ &= \frac{1}{m}p^0 \frac{\partial x^0}{\partial \xi^0}\frac{\partial f}{\partial t} + \frac{1}{m}p^i\frac{\partial f}{\partial \xi^i} + \frac{dt}{d\tau}\frac{\partial[w(a)\overline{p}^i])}{\partial t}\frac{\partial f}{\partial p^i} && (7)\\ &= \frac{\overline{p}^0}{m}\dot{f} + \frac{w}{m}\overline{p}^i\frac{\partial f}{\partial \xi^i} + \left(w\frac{d\overline{p}^i}{d\tau} + \frac{\overline{p}^0}{m}\overline{p}^i w'\dot{a}\right)\frac{\partial f}{\partial p^i}\,, && (8) \end{aligned}$$

where $\dot{}$ ($'$) denotes a derivative with respect to t (a). Note that the transformations leading to the second line hold because of the relations in b); in particular, there is a unique relation between ξ^0, $x^0 = t$ and a. Due to the homogeneity of the FRW spacetime, f cannot depend on ξ^i, so the second term vanishes. For the time-derivative of $\overline{p}$, we can use the geodesic equation (following the second hint) since the Liouville operator describes the way particles evolve in time in the absence of any interactions:

$$\frac{d\overline{p}^i}{d\tau} = -\frac{1}{m}\Gamma^i_{\mu\nu}\overline{p}^\mu\overline{p}^\nu \overset{a)}{=} -\frac{2}{m}u\overline{p}^i\overline{p}^0\,. \tag{9}$$

In total, we thus find

$$\begin{aligned} L[f] &= \frac{\overline{p}^0}{m}\dot{f} + \frac{\overline{p}^0}{m}\overline{p}^i\left(-2wu + w'\dot{a}\right)\frac{\partial f}{\partial p^i} && (10)\\ &= v^{-1}\frac{p^0}{m}\left(\dot{f} + \frac{w'}{w}\dot{a}p^i\frac{\partial f}{\partial p^i} - 2up^i\frac{\partial f}{\partial p^i}\right) && (11)\\ &= \frac{p^0}{m}\left(\frac{\partial}{\partial t} - Hp^i\frac{\partial}{\partial p^i}\right)f\,. && (12) \end{aligned}$$

Problem 2: The free Boltzmann equation

a) Simply plugging in immediately gives

$$\frac{\partial f(t,p)}{\partial p} = p\,a(t)\frac{\mathrm{d}g(p\,a(t))}{\mathrm{d}(p\,a(t))}\,, \tag{13}$$

$$\frac{\partial f(t,p)}{\partial t} = p\,\dot{a}(t)\frac{\mathrm{d}g(p\,a(t))}{\mathrm{d}(p\,a(t))} = p\,H(t)\frac{\partial f(t,p)}{\partial p}\,. \tag{14}$$

b) The Fermi-Dirac (-) and Bose-Einstein (+) distribution function are given by

$$f(t,p) = \frac{1}{\exp([E-\mu]/T) \pm 1}\,, \tag{15}$$

where neither μ nor T are functions of p and may only be functions of t. As we have just seen, these functions only solve the free Boltzmann equation if there exists a function g such that $f(t,p) = g(p\,a(t))$. This is equivalent to the existence of a function h such that

$$h(p\,a(t)) = \frac{E-\mu}{T}\,. \tag{16}$$

1. Ultra-relativistic limit:

$$E \simeq p \Rightarrow \frac{p}{T} - \frac{\mu}{T} = h(p\,a(t))\,. \tag{17}$$

This can only be fulfilled if

$$T \propto 1/a(t) \quad \text{and} \quad \mu \propto T \propto 1/a(t)\,. \tag{18}$$

Taking into account the next order in the expansion of E we find

$$E \simeq p + \frac{m^2}{2p} \Rightarrow \frac{p}{T} + \frac{m^2}{2pT} - \frac{\mu}{T} = h(p\,a(t))\,. \tag{19}$$

Here, it is not possible to find a t-dependence of T and μ such that the first two terms *only* depend on $p\,a(t)$ and otherwise have no p- or t-dependence.

2. Non-relativistic limit:

$$E \simeq m + \frac{p^2}{2m} \Rightarrow \frac{p^2}{2mT} + \frac{m-\mu}{T} = h(p\,a(t)) \tag{20}$$

This can only be fulfilled if

$$T \propto 1/a^2(t) \quad \text{and} \quad m - \mu \propto T \propto 1/a^2(t)\,. \tag{21}$$

Taking into account the next order in the expansion of E we find

$$E \simeq m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \Rightarrow \frac{p^2}{2mT} - \frac{p^4}{8m^3T} + \frac{m-\mu}{T} = h(p\,a(t)) \tag{22}$$

Again, it is not possible to find a t-dependence of T and μ such that the first two terms *only* depend on $p\,a(t)$ and otherwise have no p- or t-dependence.

In other words, the Fermi-Dirac and Bose-Einstein distributions are only solutions of the free Boltzmann equation in the fully ultra- or non-relativistic limit.

Problem 3: Collision operator for the number density

a) The matrix element in the case of elastic scatterings is equal for the reaction and inverse reaction by crossing symmetry. The contribution of elastic scatterings to the collision operator for the number density can therefore be written as (with p_1 and p_2 referring to DM, k_1 and k_2 referring to heat bath particles)

$$\begin{aligned} C_n[f_\chi] = \int & \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2\omega_2} (2\pi)^4 \delta^{(4)}(p_1 + k_1 - p_2 - k_2) \\ & \times |\mathcal{M}|^2 \Big\{ f_\chi(p_1) f_\psi(k_1)\,[1 \pm f_\chi(p_2)]\,[1 \pm f_\psi(k_2)] - f_{\chi,2} f_{\psi,2} (1 \pm f_{\chi,1})(1 \pm f_{\psi_1}) \Big\}\,. \end{aligned} \tag{23}$$

This evaluates to zero as the two terms in the curly brackets are equal in magnitude but opposite in sign after integration (where I used a convenient short-hand notation for the second term).

b) Following the hints step by step,

1. This expression for C_n adopts the same short-hand notation as introduced in 3a), so the only potential issue is the factor S_ψ. It results from realizing that the '$\int d^3p/[(2\pi)^3 2E]$' prescription is supposed to sum over all *physically distinct* configurations. In the c.o.m. system, e.g., it is easy to see that there are only $\Delta\Omega = 2\pi$ independent directions in which $\mathbf{p}_1$ can point – as opposed to the entire $\Delta\Omega = 4\pi$ over which we integrate – because the DM particles are identical. In other words, we are '*over*-counting' by a factor of 2 when integrating p_1 and p_2 over the entire $\mathbb{R}^3$, and another factor of 2 if $\psi_1 = \psi_2$ (when integrating over k_1 and k_2). On the other hand, we always need to *multiply* by a total factor factor of 2 since every reaction changes the number of DM particle by *two* units. Taken together, this results in the stated value of S_ψ.

2. As long as detailed balance of the annihilations holds, one has $\mu_\chi = 0$ since ψ_1 and ψ_2 have vanishing chemical potential. Since we assume that this is true for $T \gtrsim m_\chi$, i.e. annihilations only fall out of equilibrium for $T \ll m_\chi$, it is clear that $(m_\chi - \mu_\chi)/T \gg 1$. Inserting into the BE/FD distributions gives

$$f_\chi = \frac{1}{\exp[(E_\chi - \mu_\chi)/T)] \pm 1} \simeq \exp[-(E_\chi - \mu_\chi)/T] \,. \tag{24}$$

Note that this directly also implies $f_\chi \ll 1$. The number density is given by

$$n_\chi = g_\chi \int \frac{\mathrm{d}^3 p_\chi}{(2\pi)^3} f_\chi \simeq \mathrm{e}^{\mu_\chi/T} \frac{g_\chi}{2\pi^2} m_\chi^2 T K_2(m_\chi/T) \,, \tag{25}$$

with K_2 the modified Bessel function of second type and second order. The number density for zero chemical potential is thus simply related to this expression as $n_{\chi,\mathrm{eq}} = \exp(-\mu_\chi/T) n_\chi$, allowing us to write

$$f_\chi \simeq \frac{n_\chi}{n_{\chi,\mathrm{eq}}} \mathrm{e}^{-E_\chi/T} \,. \tag{26}$$

3. Energy conservation implies $\omega_1 + \omega_2 = E_1 + E_2 \geq 2m_\chi \gg T$. In the phase-space integral, furthermore, the dominant contribution of energies satisfying $\omega_1 + \omega_2 > 2m_\chi$ must come from the integration region with $\omega_1 \sim \omega_2 \sim m_\chi$, with respect to which $f_\psi(\omega_{1,2} \gg m_\chi)$ is exponentially suppressed. With $\mu_{\psi_1} = \mu_{\psi_2} = 0$, one therefore has

$$f_{\psi_{1,2}} = \frac{1}{\exp(\omega_{1,2}/T) \pm 1} \simeq \exp(-\omega_{1,2}/T) \ll 1 \tag{27}$$

inside the integral for C_n for all heat bath energies that are kinematically relevant. This gives

$$f_{\psi_1} f_{\psi_2} \simeq \exp(-[\omega_1 + \omega_2]/T) = e^{-(E_1+E_2)/T} \,, \tag{28}$$

we directly arrive at the expression quoted in the problem.

4. (With the stated steps, this is relatively straight-forward algebra).

5. We can choose coordinates with $\mathbf{p}_1 = (0,0,p_1)$, $\mathbf{p}_2 = (\sin\theta, 0, \cos\theta)p_2$, such that

$$\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3} e^{-(E_1+E_2)/T}\,\sigma v = \frac{1}{8\pi^4}\int_0^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_2 \int_{-1}^1 \mathrm{d}\cos\theta\, p_1^2 p_2^2\, e^{-(E_1+E_2)/T}\,\sigma v$$
$$= \frac{1}{8\pi^4}\int_{m_\chi}^\infty \mathrm{d}E_1 \int_{m_\chi}^\infty \mathrm{d}E_2 \int_{-1}^1 \mathrm{d}\cos\theta\, E_1 p_1 E_2 p_2\, e^{-(E_1+E_2)/T}\sigma v\,. \tag{29}$$

We now change integration variables to $E_\pm \equiv E_{\chi,1} \pm E_2$ and $s \equiv 2m_\chi^2 + 2E_1E_2 - 2p_1p_2\cos\theta$. The integration measure then transforms as (the "−" sign switches integration direction such that s is increasing)

$$\mathrm{d}E_1\,\mathrm{d}E_2\,\mathrm{d}\cos\theta = -\frac{1}{4p_1p_2}\mathrm{d}E_+\,\mathrm{d}E_-\,\mathrm{d}s\,, \tag{30}$$

and the integration regions become $s \geq 4m_\chi^2$, $|E_-| \leq \sqrt{1-4m_\chi^2/s}\sqrt{E_+^2 - s} \equiv E_{-,\mathrm{max}}$ and $E_+ \geq \sqrt{s}$. In summary,

$$\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_1}{(2\pi)^3} e^{-(E_1+E_2)/T}\,\sigma v$$
$$= \frac{1}{32\pi^4}\int_{4m_\chi^2}^\infty \mathrm{d}s \int_{\sqrt{s}}^\infty \mathrm{d}E_+ \int_{E_{-,\mathrm{max}}}^{E_{-,\mathrm{max}}} \mathrm{d}E_-\; E_1E_2\, e^{-E_+/T}\,\sigma v\,. \tag{31}$$

By the definition of the Møller velocity, we have $vE_1E_2 = \sqrt{(p_1\cdot p_2)^2 - m_\chi^4} = \frac{1}{2}\sqrt{s^2 - 4sm_\chi^2}$.[2] With this, the integration over E_- becomes trivial:

$$\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_1}{(2\pi)^3} e^{-(E_1+E_2)/T}\,\sigma v = \frac{1}{32\pi^4}\int_{4m_\chi^2}^\infty \mathrm{d}s(s-4m_\chi^2)\sigma \int_{\sqrt{s}}^\infty \mathrm{d}E_+\sqrt{E_+^2 - s}\,e^{-\frac{E_+}{T}}\,. \tag{32}$$

The integral over E_+ can be expressed by the modified Bessel function K_1 of second kind and first order (note that σ is only a function of s):

$$\int_{\sqrt{s}}^\infty \mathrm{d}E_+\sqrt{E_+^2 - s}\,e^{-\frac{E_+}{T}} = T\sqrt{s}K_1(\sqrt{s}/T)\,. \tag{33}$$

Inserting the expression for $n_{\chi,\mathrm{eq}}$ from just below Eq. (25) we find

$$\langle\sigma v\rangle =\equiv \frac{g_\chi^2}{n_{\chi,\mathrm{eq}}^2}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3} e^{-\frac{E_1+E_2}{T}}\sigma v \tag{34}$$
$$= \frac{1}{8m_\chi^4 T K_2(m_\chi/T)^2}\int_{4m_\chi^2}^\infty \mathrm{d}s(s-4m_\chi^2)\sqrt{s}K_1(\sqrt{s}/T)\sigma \tag{35}$$
$$= \frac{4x}{K_2^2(x)}\int_1^\infty \mathrm{d}\tilde{s}\,(\tilde{s}-1)\sqrt{\tilde{s}}K_1(2\sqrt{\tilde{s}}x)\sigma\,. \tag{36}$$

[2] Don't get confused – here, $p_{1,2}$ denote 4-momenta, while in the rest of this answer they are just numbers (namely the absolute values of the 3-momenta).

c) As explained above in b.1), in the case with $\chi = \bar{\chi}$ there is a DM symmetry factor 1/2 which is cancelled by the fact that in every process 2 DM particles are created or annihilated. With $\chi \neq \bar{\chi}$, both these factors become unity, with no net effect. In other words, the *same* calculation as above applies[3] and one finds

$$\dot{n}_\chi + 3Hn_\chi \simeq \langle \sigma_{\bar{\chi}\chi\to\psi_1\psi_2} v\rangle (n_{\chi,\mathrm{eq}} n_{\bar{\chi},\mathrm{eq}} - n_\chi n_{\bar{\chi}})\,. \tag{37}$$

Assuming that there is no primordial asymmetry, we have $n_\chi = n_{\bar{\chi}}$. In that case the total DM number density is $n_{\chi,\mathrm{tot}} = n_\chi + n_{\bar{\chi}} = 2n_\chi$, and the Boltzmann equation can be re-written as

$$\dot{n}_{\chi,\mathrm{tot}} + 3Hn_{\chi,\mathrm{tot}} = \frac{\langle \sigma v\rangle}{2} (n^2_{\chi,\mathrm{eq,tot}} - n^2_{\chi,\mathrm{tot}})\,. \tag{38}$$

In this form we can directly see that the value of $\langle \sigma v\rangle$ needed to obtain the observed dark matter relic density for a non-self-conjugate particle ($\bar{\chi} \neq \chi$) is around twice as large as the corresponding value for a self-conjugate particle ($\bar{\chi} = \chi$), as expected from the scaling $\Omega_\chi \propto 1/\langle \sigma v\rangle$. Note that this factor is not exact as there is a logarithmic dependence of the freeze-out temperature on $\langle \sigma v\rangle$ and this also enters into Ω_χ (see 2007.03696 for more details and precise numerical computations).

Problems 4 – 6: DarkSUSY applications

Note that the slides from the DarkSUSY tutorial are available at the school homepage – including solutions to the various tasks brought up during the tutorial. These are very good starting points for exploring the code. On demand I can provide (partially) typed solutions to the more specific problem 5. The final problem, 6, is more an encouragement to start using DarkSUSY for any type of relic density calculation that you may encounter in your own work. Do let me know if you gave it a try and ran into any issues that you need help with!

[3] As long as there is no *extremely* large asymmetry such that $\mu_\chi \gg \mu_{\bar{\chi}}$, or vice versa, such that the approximation of Maxwell-Boltzmann distributions is not valid anymore.