Phase transitions solutions I

1. Starting from $[\partial_\mu]=1$, $[d^dx]=-d$, $[S]=0$,

$$[\phi]=[A_\mu]=\frac{d-2}{2}$$

$$[m^2]=2,$$

$$[g^2]=[\lambda]=4-d,$$

So, the gauge coupling is:

- superrenormalisable in 3d,
- marginal in 4d,
- nonrenormalisable in 5d.

2. First introduce Schwinger's proper time,

$$\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} = \sum\!\!\!\!\!\!\int_Q \int_0^\infty dt\, e^{-tQ^2}$$

$$= \sum_{n\in\mathbb{Z}} T\frac{\Omega_d}{(2\pi)^d}\underbrace{\int_0^\infty dq\cdot q^{d-1}e^{-tq^2}}_{u=tq^2}\int_0^\infty dt\, e^{-t\omega_n^2}$$

introduce radial momentum

$$= \sum_n T\frac{\Omega_d}{(2\pi)^d}\cdot\frac{1}{2}\underbrace{\int_0^\infty du\, u^{\frac{d}{2}-1}e^{-u}}_{\Gamma(\frac{d}{2})}\underbrace{\int_0^\infty dt\cdot t^{-\frac{d}{2}+2}e^{-t\omega_n^2}}_{t\omega_n^2=v}$$

$$=\sum_n T\underbrace{\Omega_d}_{\Omega_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}}\frac{1}{(2\pi)^d}\cdot\frac{1}{2}\Gamma(\tfrac{d}{2})\underbrace{\omega_n}_{2\pi T n}{}^{d-2}\underbrace{\int_0^\infty dv\, v^{(-\frac{d}{2}+1)+1}e^{-v}}_{\Gamma(-\frac{d}{2}+1)=\frac{1}{(-\frac{d}{2}+1)}\Gamma(-\frac{d}{2}+2)}$$

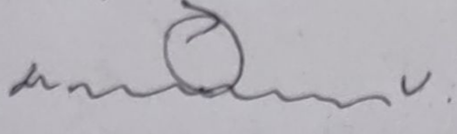
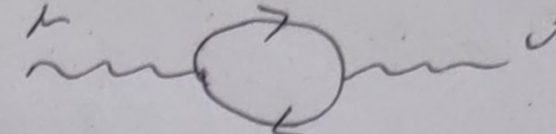
$$=\frac{(2\pi)^{d-2}}{\cancel{(2\pi)^d}}\cdot\frac{\cancel{2}\cdot\pi^{d/2}}{\cancel{2}\cancel{\Gamma(\frac{d}{2})}}\frac{\cancel{\Gamma(\frac{d}{2})}}{(-\frac{d}{2}+1)}\Gamma(-\tfrac{d}{2}+2)\cdot\sum_{n=1}^{\infty}2\cdot n^{d-2}\cdot T^{d-1}$$

$$d=3-2\varepsilon\left\{=\frac{2\cdot\pi^{3/2}\cdot\pi^{1/2}T^2}{(2\pi)^2(-\frac{1}{2})}\underbrace{\zeta(-1+2\varepsilon)}_{-\frac{1}{12}+\mathcal{O}(\varepsilon)}=\frac{T^2}{12}.\right.$$

3. As before, there are no UV parts of the loop integrals in the 3d EFT, so

$$\Gamma^{A_\mu A_\nu}_{EFT}(0) = m_D^2\, \delta_{\mu 0}\, \delta_{\nu 0}.$$

In the 4d theory, there is no tree-level diagram. At one-loop, we have:

 + 

The first diagram is easier,

 $= \underbrace{-2g^2\delta_{\mu\nu}}_{\text{vertex}} \sum\!\!\!\!\!\!\int \frac{1}{Q^2} = -\frac{g^2T^2}{6}\delta_{\mu\nu}.$

symmetry factor is 1.

The second diagram reads,

 $= +g^2 \sum\!\!\!\!\!\!\int \frac{(2Q_\mu)(2Q_\nu)}{Q^4} \equiv 4g^2 I_{\mu\nu}$

Consider $T^2\frac{d}{dT^2}\sum\!\!\!\!\!\!\int \frac{1}{Q^2} = \frac{1}{2}\sum\!\!\!\!\!\!\int \frac{1}{Q^2} - \sum\!\!\!\!\!\!\int \frac{Q_0^2}{Q^4} = T^2\frac{d}{dT^2}\left(\frac{T^2}{12}\right) = \frac{T^2}{12}$

$$\Rightarrow \sum\!\!\!\!\!\!\int \frac{Q_0^2}{Q^4} = -\frac{T^2}{24}.$$

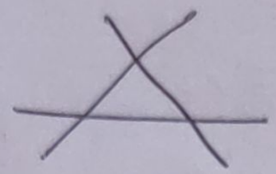
$$\delta_{ij}\sum\!\!\!\!\!\!\int \frac{q_i q_j}{Q^4} = \sum\!\!\!\!\!\!\int \frac{(Q^2-Q_0^2)}{Q^4} = \frac{T^2}{12} + \frac{T^2}{24} = 3\cdot\frac{T^2}{24}$$

$$\Rightarrow I_{\mu\nu} = \frac{T^2}{24}\left(-\delta_{\mu 0}\delta_{\nu 0} + \delta_{\mu i}\delta_{\nu i}\right)$$

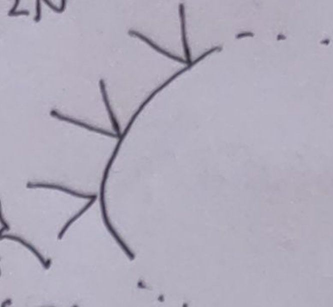
$$\Rightarrow \Gamma^{A_\mu A_\nu}(0) = \frac{g^2T^2}{6}\left[\delta_{\mu\nu} + \delta_{\mu 0}\delta_{\nu 0} - \delta_{\mu i}\delta_{\nu i}\right]$$

$$= \begin{cases} \frac{g^2T^2}{3} & , \mu=\nu=0 \\ 0 & , \text{otherwise.} \end{cases}$$

4. a) $C_6\phi^6$



$C_{2N}\phi^{2N}$



N pairs of legs on one loop

b) $C_6 \sim \frac{\lambda^3}{\pi^4}$, { π's optional

$C_{2N} \sim \frac{\lambda^N T^3}{T^N \pi^{2N-2}}$

N.B. some Ts from loop, others from canonical normalisation.

5. $\bar{\Phi}_{min}^2 = -\frac{m_3^2}{\lambda_3}, 0$, so $V_{EFT}(\bar{\Phi}_{min}) = \frac{-1}{2}\frac{m_3^4}{\lambda_3} + \frac{m_3^4}{4\lambda_3}, 0$

$m_3^2(T_c) = 0.$

$$= -\frac{m_3^4}{4\lambda_3}, 0$$

$$\left.\frac{df}{dT}\right|_{T_c^-} = -\left.\cancelto{0}{\frac{m_3^4}{4\lambda_3}}\right|_{T_c} - \left.\frac{T}{2}\cancelto{0}{\frac{m_3^2}{\lambda_3}}\frac{dm_3^2}{dT}\right|_{T_c} + \left.\frac{T}{4}\cancelto{0}{\frac{m_3^4}{\lambda_3^2}}\frac{d\lambda_3}{dT}\right|_{T_c} = 0$$

$$\left.\frac{d^2f}{dT^2}\right|_{T_c^-} = -\left.\frac{T}{2\lambda_3}\left(\frac{dm_3^2}{dT}\right)^2\right|_{T_c} \neq 0.$$