1. **Charge conjugation and seesaw mechanism**
   Proof the following relations:
   (a) $(\psi^c)^c = \psi$,
   (b) $\overline{\psi_1^c}\psi_2 = \overline{\psi_2^c}\psi_1$,
   (c) Show the following identify from the lecture:
   $$-\frac{1}{2}\overline{n^c}Mn + h.c. = -m_D\overline{\nu_L}N_R - \frac{1}{2}m_M\overline{(N_R)^c}N_R + h.c.\,. \tag{1}$$
   Here, $n = (\nu_L, (N_R)^c)^T$ and
   $$M = \begin{pmatrix} 0 & m_D \\ m_D & m_M \end{pmatrix}\,. \tag{2}$$
   (d) Compute the eigenvalues and eigenvectors of M approximately and use them to prove that, indeed, an effective mass term of the form $-\frac{1}{2}m_\nu\overline{(\nu'_L)^c}\nu'_L$, with $m_\nu = m_D^2/m_M$, is generated.

2. **Majorana mass term**
   Why is a Majorana mass term for neutrinos, i.e. a term of the form
   $$\mathcal{L} \supset \frac{1}{2}m\,\overline{(\nu_L)^c}\,\nu_L + h.c.\,, \tag{3}$$
   forbidden in the Standard Model?

3. **Neutrinoless double electron capture**
   Neutrinoless double electron capture has been proposed as an alternative to neutrinoless double beta decay for measuring (Majorana) neutrino mass. (Sujkowski Wycech, arXiv:hep-ph/0312040)
   (a) Draw the Feynman diagram corresponding to neutrinoless double electron capture.
   (b) What would be the experimental signature?
   (c) Discuss how a coincidence trigger can help to reduce backgrounds.

4. **Neutrino Brain Teasers**
   (a) Imagine a world in which neutrinos are massive, but charged leptons are massless. Will neutrinos oscillate in such a world?
   (b) Do neutrinos produced in the decay $Z^0 \to \bar{\nu}\nu$ oscillate? If so, describe a gedankenexperiment in which these oscillations could be observed.

5. **Neutrino oscillations in matter**
   (a) Diagonalize the 2-flavor neutrino Hamiltonian in matter,
   $$\hat{H}_{\text{eff}} = -\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\frac{\Delta m^2}{4p} & 0 \\ 0 & \frac{\Delta m^2}{4p} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} - \begin{pmatrix} \sqrt{2}G_F n_e & 0 \\ 0 & 0 \end{pmatrix}, \tag{4}$$

to show that the effective mass squared difference and mixing angle in matter are given by

$$\frac{\Delta m^2_{\text{eff}}}{2E} = \sqrt{\Big(\sqrt{2} G_F n_e - \frac{\Delta m^2}{2p} \cos 2\theta\Big)^2 + \Big(\frac{\Delta m^2}{2p}\Big)^2 \sin^2 2\theta} \tag{5}$$

$$\sin 2\theta_{\text{eff}} = \frac{\sin 2\theta \, \frac{\Delta m^2}{2p}}{\sqrt{\big(\sqrt{2} G_F n_e - \frac{\Delta m^2}{2p} \cos 2\theta\big)^2 + \big(\frac{\Delta m^2}{2p}\big)^2 \sin^2 2\theta}} \tag{6}$$

(Note that, in $\hat{H}$, we have subtracted a term $\Delta m^2/(4p) \times \mathbb{1}_{2\times 2}$, which makes the calculation a bit simpler. We can always do this as flavor-universal terms do not affect oscillations.

(b) Consider now neutrino oscillations in spatially *inhomogeneous* matter of density $n_e = n_e(x)$. This is relevant for instance for solar neutrinos propagating out of the core of the Sun. Neutrino evolution is described by the Schrödinger-like equation

$$-i \frac{d}{dx} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\text{eff}} & \sin\theta_{\text{eff}} \\ -\sin\theta_{\text{eff}} & \cos\theta_{\text{eff}} \end{pmatrix} \begin{pmatrix} -\frac{\Delta m^2_{\text{eff}}}{4o} & 0 \\ 0 & \frac{\Delta m^2_{\text{eff}}}{4p} \end{pmatrix} \begin{pmatrix} \cos\theta_{\text{eff}} & -\sin\theta_{\text{eff}} \\ \sin\theta_{\text{eff}} & \cos\theta_{\text{eff}} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \tag{7}$$

Rewrite this equation in the basis of *matter eigenstates*, i.e. states of definite energy and momentum in matter: $(\nu_A, \nu_B) = U^\dagger_{\text{eff}}(x)\,(\nu_e, \nu_\mu)$, where U_{eff} is the effective mixing matrix in matter. You should find

$$-i \frac{d}{dx} \begin{pmatrix} \nu_A \\ \nu_B \end{pmatrix} = \begin{pmatrix} p_A & i\frac{d\theta}{dx} \\ -i\frac{d\theta}{dx} & p_B \end{pmatrix} \begin{pmatrix} \nu_A \\ \nu_B \end{pmatrix} , \tag{8}$$

where p_A, p_B are the momentum eigenvalues in matter.

(c) For slowly varying matter density, $d\theta/dx \ll |p_A - p_B|$, the off-diagonal terms on the right hand side of eq. (8) are negligible. Solve the equation to show that the survival probability of solar neutrinos $P(\nu_e \to \nu_e)$ is given by

$$P(\nu_e \to \nu_e) = \frac{1}{2}\bigg(1 + \cos 2\theta_i \cos 2\theta_f + \sin 2\theta_i \sin 2\theta_f \cos\bigg[\int_{x_i}^{x_f} dx \, \frac{\Delta m^2_{\text{eff}}(x)}{2p}\bigg]\bigg) . \tag{9}$$

Here θ_i and θ_f are the effective mixing angles corresponding to the center of the Sun and its exterior, respectively. The integral in the last term runs along the neutrino trajectory from its production point x_i to its detection point x_f.

(d) What is the maximum conversion probability in the case of small vacuum mixing angle $\theta \ll 1$? Consider the case that the matter density at the center of the Sun lies well above the MSW resonance, whereas the density outside the Sun is far below.

Cartoon by Jack Pairin, Harvard