1. Horizons

The *particle horizon* is the largest comoving distance from which light emitted in the past could reach the observer at a given time. One could also define a *cosmological event horizon* in an expanding Universe, which corresponds to the largest comoving distance from which light emitted now can ever reach the observer in the future.

a) Assuming the scale factor obeys $a(t) = (t/t_0)^n$, which horizon exists?

b) And what if the scale factor is given by $a(t) = \exp(Ht)$?

c) Are there FLRW metrics that exhibit neither horizon? Or both horizons?

2. Einstein's static universe

The cosmological constant was originally introduced by Einstein to make the universe static, i.e., not expanding.

a) Consider a universe filled with a perfect fluid with $\rho > 0$ and $P \geq 0$. Show that no static solution to the Friedmann equations exists in this case.

b) Now consider a universe filled with pressureless matter ($P_{\rm m} = 0$) and a cosmological constant ($P_\Lambda = -\rho_\Lambda$). Show that in this case it is possible to find a static solution and find the ratio $\xi_{\rm static} \equiv \rho_{\rm m}/\rho_\Lambda$.

c) Now imagine there were small density fluctuations in the matter component of this universe. Describe qualitatively what would happen for

(i) a small matter overdensity, $\xi > \xi_{\rm static}$,

(ii) and a small matter underdensity, $\xi < \xi_{\rm static}$.

3. Conformal time

Consider a flat FLRW universe. Define the conformal time η

$$d\eta \equiv \frac{dt}{a}.$$

What is the functional dependence of the scale factor a on η during

a) matter domination?

b) radiation domination?

c) Λ domination?

4. Neutrinos

Standard model neutrinos decouple from cosmic plasma at temperatures of ~ 1 MeV. Since laboratory experiments have shown that neutrino masses are sub-eV, it is clear that these neutrinos must be ultra-relativistic when decoupling happens. If the decoupling is instantaneous, the phase space density of neutrinos at the time of decoupling takes the relativistic Fermi-Dirac form to a very good approximation, i.e.,

$$f(p) = \frac{1}{\exp(p/T_\nu) + 1} \tag{1}$$

a) Argue why the phase space distribution $f(p)$ must remain the same form, with $T_\nu \propto a^{-1}$, *even after* the temperature T_ν drops below the neutrino rest mass.

b) Suppose one species of neutrinos has a mass of m, where $m \gg T_{\nu,0} \simeq 10^{-5}$ eV and $T_{\nu,0} \simeq 1.95$ K is the present-day neutrino temperature. Show that the present-day reduced energy density of these neutrinos is given by

$$\Omega_\nu h^2 = \frac{m}{93 \text{ eV}}. \tag{2}$$

c) The present-day neutrino temperature is related to the present-day CMB temperature by

$$T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,0}, \tag{3}$$

where $T_{\gamma,0} = 2.725$. If the Fermi coupling constant G_F were a factor of 100 smaller, at what temperature would you expect the neutrinos to decouple? Would the relation given in equation (3) still apply to the present-day neutrino temperature in this case?

5. Light thermal relics

Suppose we extend the standard model (SM) of particle physics with one extra massless scalar particle ϕ. This particle interacts with the SM particles and was once in a state of thermodynamic equilibrium in the early universe. At some temperature $T_{\text{dec}(\phi)}$, ϕ decouples from the rest of the cosmic plasma.

a) Suppose ϕ decouples before neutrino decoupling, i.e., $T_{\text{dec}(\phi)} \gg O(1)$ MeV. Assuming as usual instantaneous decoupling, show using entropy conservation arguments that the present-day temperature of ϕ, $T_{\phi,0}$, is related to the present-day neutrino temperature $T_{\nu,0}$ via

$$T_{\phi,0} = \left[\frac{43}{4}\frac{1}{g_{*s}(T_{\text{dec}(\phi)}) - 1}\right]^{1/3} T_{\nu,0}, \tag{4}$$

where g_{*s} is the effective massless entropy degrees of freedom, and three generations of neutrinos have been assumed. How does $T_{\phi,0}$ relate to the present-day photon temperature $T_{\gamma,0}$?

b) Now suppose ϕ decoupling happens after e^+e^- annihilation ($T \sim 0.5$ MeV). How does $T_{\phi,0}$ now relate to $T_{\nu,0}$ and to $T_{\gamma,0}$

c) Compared with the standard model, how does the presence of this hypothetical particle ϕ affect the expansion rate and hence the neutrino decoupling temperature?

6. Boltzmann equation

The integrated Boltzmann equation for the physical number density of a particle species i is given by

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = \frac{g}{(2\pi)^3}\int d^3p \frac{1}{E} C[f], \tag{5}$$

where $C[f]$ is the collision integral. Assuming an interaction

$$i + j \leftrightarrow k + l \tag{6}$$

and CP invariance, verify that the r.h.s. of equation (5) evaluates to exactly zero when the interaction (6) is in equilibrium.

7. Scalar fields

Begin with the Lagrangian density for a real scalar field $\phi(x^\alpha)$,

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi). \tag{7}$$

(metric convention: $(-,+,+,+)$).

a) Using

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = g^{\mu\nu}\mathcal{L} - 2\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}, \tag{8}$$

where g is the determinant of the metric, derive the stress-energy tensor $T_{\mu\nu}$ for ϕ.

b) Suppose the field is homogeneous, i.e., $\phi = \phi(t)$. Show that the energy density ρ_ϕ and the pressure P_ϕ of $\phi(t)$ are given by

$$\begin{aligned} \rho_\phi(t) &= \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi), \\ P_\phi(t) &= \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi). \end{aligned} \tag{9}$$

c) Consider the scalar field ϕ described by equation (7), and suppose again it is homogeneous, i.e., $\phi = \phi(t)$. Show that in an FLRW universe, the time evolution of $\phi(t)$ is governed by the Klein-Gordon equation:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{10}$$

You can start the derivation from first principles, i.e., minimise the Lagrangian. But you can also arrive at the same point via the continuity equation, $\mathrm{d}\rho/\mathrm{d}t + 3H(\rho + P) = 0$. You should try both.