Technical exercises related to lectures on Thermal field theory / Mikko Laine / July 2024

Lecture 1: Basics on "fields" in cosmology

Exercise 1: Verify that

$$\frac{1}{\epsilon_k}\left[\frac{1}{2}+\underbrace{\frac{1}{e^{\epsilon_k/T}-1}}_{\equiv\, n_{\rm B}(\epsilon_k)}\right] = T\sum_{\omega_n}\frac{1}{\omega_n^2+\epsilon_k^2}\;, \quad \omega_n \;\equiv\; 2\pi n T\;, \quad n\in\mathbb{Z}\;. \tag{0.1}$$

Hint: you may consider the contour integral $\oint \frac{{\rm d}\omega}{2\pi i}\frac{i n_{\rm B}(i\omega)}{\omega^2+\epsilon_k^2}$, where the contour wraps around the x-axis, and then use the residue theorem in two different ways, inwards or outwards.

Exercise 2: Consider the metric ${\rm d}s^2 = {\rm d}t^2 - a^2(t){\rm d}\mathbf{x}^2$. We define local Minkowskian coordinates as ${\rm d}\mathbf{y} = a(t){\rm d}\mathbf{x}$, and write the action in this system as

$$\mathcal{S} = \int {\rm d}t\,{\rm d}^3\mathbf{y}\left[\frac{1}{2}\dot\phi^2 - \frac{1}{2}(\nabla_{\mathbf{y}}\phi)^2 - V(\phi)\right]. \tag{0.2}$$

After going from $\mathbf{y}$ to $\mathbf{x}$, derive the Euler-Lagrange equations, and show that

$$\left(\partial_t^2 + 3H\partial_t - \frac{\nabla_{\mathbf{x}}^2}{a^2}\right)\phi + V'(\phi) = 0\;, \tag{0.3}$$

where $H \equiv \dot a/a$ is the Hubble rate (or "Hubble friction").

Exercise 3: Consider the Langevin equation

$$\big(\,\partial_t^2 - \nabla^2 + \Gamma\partial_t + m^2\,\big)\,\phi = \xi\;, \quad \langle\xi\rangle = 0\;, \quad \langle\,\xi(\mathcal{X})\xi(\mathcal{Y})\,\rangle \;=\; \Omega\,\delta^{(4)}(\mathcal{X}-\mathcal{Y})\;, \tag{0.4}$$

where $\mathcal{X}\equiv(t,\mathbf{x})$. After solving the equation with a Green's function, compute the equal-time 2-point correlator of ϕ, and show that

$$\langle\,\phi(t,\mathbf{x})\,\phi(t,\mathbf{y})\,\rangle = \frac{\Omega}{2\Gamma}\int_{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\epsilon_k^2}\;. \tag{0.5}$$

Lecture 2: Basics on "particles" in cosmology

Exercise 4: We consider

$$\Pi(K) \;\equiv\; g^2 \SumInt_P \frac{1}{[(\omega_n+p_n)^2+\epsilon_{kp}^2][p_n^2+\epsilon_p^2]}\;, \quad \epsilon_{kp}^2 \;\equiv\; (\mathbf{k}+\mathbf{p})^2+m_1^2\;, \quad \epsilon_p^2 \;\equiv\; p^2+m_2^2\;, \tag{0.6}$$

where $K\equiv(\omega_n,\mathbf{k})$ and $\SumInt_P \equiv T\sum_{p_n}\int_{\mathbf{p}}$, with $p_n \equiv 2\pi n T$, $n\in\mathbb{Z}$. The task is to carry out the sum over p_n and then take the "cut",

$$\Gamma(\mathcal{K}) \;\equiv\; \frac{1}{\omega}\,{\rm Im}\,\Pi(K)\big|_{\omega_n\to -i[\omega+i0^+]}\;, \quad \mathcal{K} \;\equiv\; (\omega,\mathbf{k})\;. \tag{0.7}$$

Verify that the answer can be written in a Boltzmann-equation like form,

$$\begin{aligned}\Gamma &= \frac{g^2}{2\omega}\int_{\mathbf{p},\mathbf{q}}\frac{1}{4\epsilon_p\epsilon_q}\Big\{(2\pi)^4\delta^{(4)}\big(\mathcal{K}-\mathcal{P}-\mathcal{Q}\big)\big[\,(1+n_p)(1+n_q)-n_p n_q\,\big] \\ &\quad + (2\pi)^4\delta^{(4)}\big(\mathcal{K}+\mathcal{P}-\mathcal{Q}\big)\big[\,n_p(1+n_q)-n_q(1+n_p)\,\big] \\ &\quad + (2\pi)^4\delta^{(4)}\big(\mathcal{K}-\mathcal{P}+\mathcal{Q}\big)\big[\,n_q(1+n_p)-n_p(1+n_q)\,\big]\Big\}\,, \end{aligned} \tag{0.8}$$

where $n_p \equiv n_{\rm B}(\epsilon_p)$ and ϵ_q corresponds to the previous ϵ_{kp}.
Hint: $T\sum_{p_n}\frac{e^{ip_n\tau}}{p_n^2+\epsilon_p^2} = \frac{n_{\rm B}(\epsilon_p)}{2\epsilon_p}\Big[e^{(\beta-\tau)\epsilon_p}+e^{\tau\epsilon_p}\Big]\,,\quad 0<\tau<\beta.$

Lecture 3: Resummation and effective field theories

Exercise 5: Spectral functions are defined as

$$\rho^{\rm T,E}_{\mathcal{P}} \;\equiv\; {\rm Im}\bigg\{\frac{1}{p_n^2+p^2+\Pi^{\rm T,E}_{P}}\bigg\}_{p_n\to -i[p_0+i0^+]}\,,\quad \mathcal{P}\;\equiv\;(p_0,\mathbf{p})\,,\quad p\;\equiv\;|\mathbf{p}|\,. \tag{0.9}$$

We assume the properties

$$\Pi^{\rm T,E}_{(-p_n,\mathbf{p})} = \Pi^{\rm T,E}_{(p_n,\mathbf{p})}\,,\quad \Pi^{\rm T}_{(0,\mathbf{p})}\;=\;0\,,\quad \Pi^{\rm E}_{(0,\mathbf{p})}\;=\;m_{\rm E}^2\,,\quad \Pi^{\rm T}_{(p_n,\mathbf{0})}\;=\;\Pi^{\rm E}_{(p_n,\mathbf{0})}\;=\;\frac{m_{\rm E}^2}{3}\,. \tag{0.10}$$

Verify the sum rule

$$\int_{-\infty}^{+\infty}\frac{{\rm d}p_0}{\pi}\left[\frac{\rho^{\rm T}_{(p_0,\mathbf{p}_\perp+p_0\mathbf{e}_\parallel)}}{p_0}-\frac{\rho^{\rm E}_{(p_0,\mathbf{p}_\perp+p_0\mathbf{e}_\parallel)}}{p_0}\right]\frac{p_\perp^2}{p_\perp^2+p_0^2}\;=\;\frac{1}{p_\perp^2}-\frac{1}{p_\perp^2+m_{\rm E}^2}\,. \tag{0.11}$$

Lecture 4: Example: thermal production of gravitational waves

Exercise 6: Let $\mathcal{K}=(k,\mathbf{k})$ and $\mathcal{Q}=(q,\mathbf{q})$ be lightlike 4-vectors, and $\mathcal{P}=(p_0,\mathbf{p})$ be a general 4-vector. Show that the phase-space integral for a $2\to 1$ process $\mathcal{P}+\mathcal{Q}\to\mathcal{K}$ can be written as

$$\begin{aligned} I[f(\mathcal{P})] &\equiv \int\frac{{\rm d}p_0}{2\pi}\int_{\mathbf{p},\mathbf{q}}\frac{(2\pi)^4\delta^{(4)}(\mathcal{P}+\mathcal{Q}-\mathcal{K})}{2q}\,f(\mathcal{P}) \\ &= \int_{-\infty}^{+\infty}{\rm d}p_0\int_0^\infty {\rm d}p\,p\,\frac{\theta(|p-k|<k-p_0<p+k)}{8\pi^2 k}\,f(\mathcal{P})\,, \end{aligned} \tag{0.12}$$

where $\theta(\text{true})\equiv 1$. Sketch the integration domain in the (p_0,p) plane and show that $p^2-p_0^2>0$, i.e. that $\mathcal{P}$ is spacelike. If we consider a large momentum k, and write $p^2-p_0^2\equiv p_\perp^2$, show that the IR part of the measure can be written as

$$I[f(\mathcal{P})]\supset\int{\rm d}p_0\int_0^{(2k)^2}\frac{{\rm d}p_\perp^2}{16\pi^2 k}\,f(\mathcal{P})\,. \tag{0.13}$$